13

Sequences and Series

Consider the following sum:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^i}+\cdots$$

The dots at the end indicate that the sum goes on forever. Does this make sense? Can we assign a numerical value to an infinite sum? While at first it may seem difficult or impossible, we have certainly done something similar when we talked about one quantity getting "closer and closer" to a fixed quantity. Here we could ask whether, as we add more and more terms, the sum gets closer and closer to some fixed value. That is, look at

$$\begin{aligned}\frac{1}{2}&=\frac{1}{2}\\ \frac{3}{4}&=\frac{1}{2}+\frac{1}{4}\\ \frac{7}{8}&=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\\ \frac{15}{16}&=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\end{aligned}$$

and so on, and ask whether these values have a limit. It seems pretty clear that they do, namely 1. In fact, as we will see, it's not hard to show that

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^i}=\frac{2^i-1}{2^i}=1-\frac{1}{2^i}$$

and then

$$\lim_{i\to\infty} 1-\frac{1}{2^i}=1-0=1.$$

There is one place that you have long accepted this notion of infinite sum without really thinking of it as a sum:

$$0.3333\bar{3}=\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}+\cdots=\frac{1}{3},$$

for example, or

$$3.14159\ldots=3+\frac{1}{10}+\frac{4}{100}+\frac{1}{1000}+\frac{5}{10000}+\frac{9}{100000}+\cdots=\pi.$$

Our first task, then, to investigate infinite sums, called **series**, is to investigate limits of **sequences** of numbers. That is, we officially call

$$\sum_{i=1}^{\infty}\frac{1}{2^i}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^i}+\cdots$$

a series, while

$$\frac{1}{2},\frac{3}{4},\frac{7}{8},\frac{15}{16},\ldots,\frac{2^i-1}{2^i},\ldots$$

is a sequence, and

$$\sum_{i=1}^{\infty}\frac{1}{2^i}=\lim_{i\to\infty}\frac{2^i-1}{2^i},$$

that is, the value of a series is the limit of a particular sequence.

13.1 Sequences

While the idea of a sequence of numbers, $a_1, a_2, a_3, \ldots$ is straightforward, it is useful to think of a sequence as a function. We have up until now dealt with functions whose domains are the real numbers, or a subset of the real numbers, like $f(x)=\sin x$. A sequence is a function with domain the natural numbers $\mathbb{N}=\{1,2,3,\ldots\}$ or the non-negative integers, $\mathbb{Z}^{\geq 0}=\{0,1,2,3,\ldots\}$. The range of the function is still allowed to be the real numbers; in symbols, we say that a sequence is a function $f\colon \mathbb{N}\to\mathbb{R}$. Sequences are written in a few different ways, all equivalent; these all mean the same thing:

$$a_1, a_2, a_3, \ldots$$
$$\{a_n\}_{n=1}^{\infty}$$
$$\{f(n)\}_{n=1}^{\infty}$$

As with functions on the real numbers, we will most often encounter sequences that can be expressed by a formula. We have already seen the sequence $a_i=f(i)=1-1/2^i$,

and others are easy to come by:

$$f(i) = \frac{i}{i+1}$$
$$f(n) = \frac{1}{2^n}$$
$$f(n) = \sin(n\pi/6)$$
$$f(i) = \frac{(i-1)(i+2)}{2^i}$$

Frequently these formulas will make sense if thought of either as functions with domain $\mathbb{R}$ or $\mathbb{N}$, though occasionally one will make sense only for integer values.

Faced with a sequence we are interested in the limit

$$\lim_{i\to\infty} f(i) = \lim_{i\to\infty} a_i.$$

We already understand

$$\lim_{x\to\infty} f(x)$$

when x is a real valued variable; now we simply want to restrict the "input" values to be integers. No real difference is required in the definition of limit, except that we specify, perhaps implicitly, that the variable is an integer. Compare this definition to definition 4.7.4.

DEFINITION 13.1.1 Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence. We say that $\lim_{n\to\infty} a_n = L$ if for every $\epsilon > 0$ there is an $N > 0$ so that whenever $n > N$, $|a_n - L| < \epsilon$. If $\lim_{n\to\infty} a_n = L$ we say that the sequence **converges**, otherwise it **diverges**. □

If $f(i)$ defines a sequence, and $f(x)$ makes sense, and $\lim_{x\to\infty} f(x) = L$, then it is clear that $\lim_{i\to\infty} f(i) = L$ as well, but it is important to note that the converse of this statement is not true. For example, since $\lim_{x\to\infty}(1/x) = 0$, it is clear that also $\lim_{i\to\infty}(1/i) = 0$, that is, the numbers

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$$

get closer and closer to 0. Consider this, however: Let $f(n) = \sin(n\pi)$. This is the sequence

$$\sin(0\pi), \sin(1\pi), \sin(2\pi), \sin(3\pi), \ldots = 0, 0, 0, 0, \ldots$$

since $\sin(n\pi) = 0$ when n is an integer. Thus $\lim_{n\to\infty} f(n) = 0$. But $\lim_{x\to\infty} f(x)$, when x is real, does not exist: as x gets bigger and bigger, the values $\sin(x\pi)$ do not get closer and

closer to a single value, but take on all values between -1 and 1 over and over. In general, whenever you want to know $\lim_{n\to\infty} f(n)$ you should first attempt to compute $\lim_{x\to\infty} f(x)$, since if the latter exists it is also equal to the first limit. But if for some reason $\lim_{x\to\infty} f(x)$ does not exist, it may still be true that $\lim_{n\to\infty} f(n)$ exists, but you'll have to figure out another way to compute it.

It is occasionally useful to think of the graph of a sequence. Since the function is defined only for integer values, the graph is just a sequence of dots. In figure 13.1.1 we see the graphs of two sequences and the graphs of the corresponding real functions.

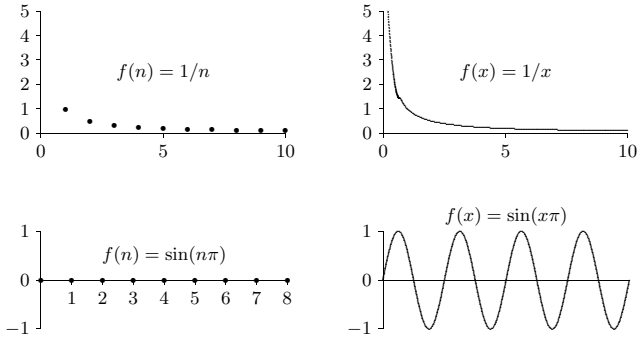

Figure 13.1.1 Graphs of sequences and their corresponding real functions.

Not surprisingly, the properties of limits of real functions translate into properties of sequences quite easily. Theorem 2.3.6 about limits becomes

THEOREM 13.1.2 Suppose that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ and k is some constant. Then

$$\lim_{n\to\infty} ka_n = k \lim_{n\to\infty} a_n = kL$$
$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L + M$$
$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L - M$$
$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = LM$$
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}, \text{ if } M \text{ is not } 0$$

■

Likewise the Squeeze Theorem (4.3.1) becomes

THEOREM 13.1.3 Suppose that $a_n \le b_n \le c_n$ for all $n > N$, for some N. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$. ■

And a final useful fact:

THEOREM 13.1.4 $\lim_{n\to\infty} |a_n| = 0$ if and only if $\lim_{n\to\infty} a_n = 0$. ■

This says simply that the size of a_n gets close to zero if and only if a_n gets close to zero.

EXAMPLE 13.1.5 Determine whether $\left\{\frac{n}{n+1}\right\}_{n=0}^{\infty}$ converges or diverges. If it converges, compute the limit. Since this makes sense for real numbers we consider

$$\lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} 1 - \frac{1}{x+1} = 1 - 0 = 1.$$

Thus the sequence converges to 1. □

EXAMPLE 13.1.6 Determine whether $\left\{\frac{\ln n}{n}\right\}_{n=1}^{\infty}$ converges or diverges. If it converges, compute the limit. We compute

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0,$$

using L'Hôpital's Rule. Thus the sequence converges to 0. □

EXAMPLE 13.1.7 Determine whether $\{(-1)^n\}_{n=0}^{\infty}$ converges or diverges. If it converges, compute the limit. This does not make sense for all real exponents, but the sequence is easy to understand: it is

$$1, -1, 1, -1, 1 \ldots$$

and clearly diverges. □

EXAMPLE 13.1.8 Determine whether $\{(-1/2)^n\}_{n=0}^{\infty}$ converges or diverges. If it converges, compute the limit. We consider the sequence $\{|(-1/2)^n|\}_{n=0}^{\infty} = \{(1/2)^n\}_{n=0}^{\infty}$. Then

$$\lim_{x\to\infty} \left(\frac{1}{2}\right)^x = \lim_{x\to\infty} \frac{1}{2^x} = 0,$$

so by theorem 13.1.4 the sequence converges to 0. □

EXAMPLE 13.1.9 Determine whether $\{(\sin n)/\sqrt{n}\}_{n=1}^{\infty}$ converges or diverges. If it converges, compute the limit. Since $|\sin n| \le 1$, $0 \le |\sin n/\sqrt{n}| \le 1/\sqrt{n}$ and we can use theorem 13.1.3 with $a_n = 0$ and $c_n = 1/\sqrt{n}$. Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0$, $\lim_{n\to\infty} \sin n/\sqrt{n} = 0$ and the sequence converges to 0. □

EXAMPLE 13.1.10 A particularly common and useful sequence is $\{r^n\}_{n=0}^{\infty}$, for various values of r. Some are quite easy to understand: If $r = 1$ the sequence converges to 1 since every term is 1, and likewise if $r = 0$ the sequence converges to 0. If $r = -1$ this is the sequence of example 13.1.7 and diverges. If $r > 1$ or $r < -1$ the terms r^n get large without limit, so the sequence diverges. If $0 < r < 1$ then the sequence converges to 0. If $-1 < r < 0$ then $|r^n| = |r|^n$ and $0 < |r| < 1$, so the sequence $\{|r|^n\}_{n=0}^{\infty}$ converges to 0, so also $\{r^n\}_{n=0}^{\infty}$ converges to 0. converges. In summary, $\{r^n\}$ converges precisely when $-1 < r \le 1$ in which case

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

□

Sometimes we will not be able to determine the limit of a sequence, but we still would like to know whether it converges. In some cases we can determine this even without being able to compute the limit.

A sequence is called **increasing** or sometimes **strictly increasing** if $a_i < a_{i+1}$ for all i. It is called **non-decreasing** or sometimes (unfortunately) **increasing** if $a_i \le a_{i+1}$ for all i. Similarly a sequence is **decreasing** if $a_i > a_{i+1}$ for all i and **non-increasing** if $a_i \ge a_{i+1}$ for all i. If a sequence has any of these properties it is called **monotonic**.

EXAMPLE 13.1.11 The sequence

$$\left\{\frac{2^i - 1}{2^i}\right\}_{i=1}^{\infty} = \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots,$$

is increasing, and

$$\left\{\frac{n+1}{n}\right\}_{i=1}^{\infty} = \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$$

is decreasing. □

A sequence is **bounded above** if there is some number N such that $a_n \le N$ for every n, and **bounded below** if there is some number N such that $a_n \ge N$ for every n. If a sequence is bounded above and bounded below it is **bounded**. If a sequence $\{a_n\}_{n=0}^{\infty}$ is increasing or non-decreasing it is bounded below (by a_0), and if it is decreasing or non-increasing it is bounded above (by a_0). Finally, with all this new terminology we can state an important theorem.

THEOREM 13.1.12 If a sequence is bounded and monotonic then it converges. ■

We will not prove this; the proof appears in many calculus books. It is not hard to believe: suppose that a sequence is increasing and bounded, so each term is larger than the one before, yet never larger than some fixed value N. The terms must then get closer and closer to some value between a_0 and N. It need not be N, since N may be a "too-generous" upper bound; the limit will be the smallest number that is above all of the terms a_i.

EXAMPLE 13.1.13 All of the terms $(2^i-1)/2^i$ are less than 2, and the sequence is increasing. As we have seen, the limit of the sequence is 1—1 is the smallest number that is bigger than all the terms in the sequence. Similarly, all of the terms $(n+1)/n$ are bigger than $1/2$, and the limit is 1—1 is the largest number that is smaller than the terms of the sequence. □

We don't actually need to know that a sequence is monotonic to apply this theorem—it is enough to know that the sequence is "eventually" monotonic, that is, that at some point it becomes increasing or decreasing. For example, the sequence 10, 9, 8, 15, 3, 21, 4, 3/4, 7/8, 15/16, 31/32, ... is not increasing, because among the first few terms it is not. But starting with the term 3/4 it is increasing, so the theorem tells us that the sequence $3/4, 7/8, 15/16, 31/32, \ldots$ converges. Since convergence depends only on what happens as n gets large, adding a few terms at the beginning can't turn a convergent sequence into a divergent one.

EXAMPLE 13.1.14 Show that $\{n^{1/n}\}$ converges.

We first show that this sequence is decreasing, that is, that $n^{1/n} > (n+1)^{1/(n+1)}$. Consider the real function $f(x) = x^{1/x}$ when $x \geq 1$. We can compute the derivative, $f'(x) = x^{1/x}(1-\ln x)/x^2$, and note that when $x \geq 3$ this is negative. Since the function has negative slope, $n^{1/n} > (n+1)^{1/(n+1)}$ when $n \geq 3$. Since all terms of the sequence are positive, the sequence is decreasing and bounded when $n \geq 3$, and so the sequence converges. (As it happens, we can compute the limit in this case, but we know it converges even without knowing the limit; see exercise 1.) □

EXAMPLE 13.1.15 Show that $\{n!/n^n\}$ converges.

Again we show that the sequence is decreasing, and since each term is positive the sequence converges. We can't take the derivative this time, as $x!$ doesn't make sense for x real. But we note that if $a_{n+1}/a_n < 1$ then $a_{n+1} < a_n$, which is what we want to know. So we look at a_{n+1}/a_n:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!} = \frac{(n+1)!}{n!}\frac{n^n}{(n+1)^{n+1}} = \frac{n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \left(\frac{n}{n+1}\right)^n < 1.$$

(Again it is possible to compute the limit; see exercise 2.) □

Exercises 13.1.

1. Compute $\lim_{x\to\infty} x^{1/x}$. ⇒
2. Use the squeeze theorem to show that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.
3. Determine whether $\{\sqrt{n+47} - \sqrt{n}\}_{n=0}^{\infty}$ converges or diverges. If it converges, compute the limit. ⇒
4. Determine whether $\left\{\frac{n^2+1}{(n+1)^2}\right\}_{n=0}^{\infty}$ converges or diverges. If it converges, compute the limit. ⇒
5. Determine whether $\left\{\frac{n+47}{\sqrt{n^2+3n}}\right\}_{n=1}^{\infty}$ converges or diverges. If it converges, compute the limit. ⇒
6. Determine whether $\left\{\frac{2^n}{n!}\right\}_{n=0}^{\infty}$ converges or diverges. ⇒

13.2 Series

While much more can be said about sequences, we now turn to our principal interest, series. Recall that a series, roughly speaking, is the sum of a sequence: if $\{a_n\}_{n=0}^{\infty}$ is a sequence then the associated series is

$$\sum_{i=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

Associated with a series is a second sequence, called the **sequence of partial sums** $\{s_n\}_{n=0}^{\infty}$:

$$s_n = \sum_{i=0}^{n} a_i.$$

So

$$s_0 = a_0, \quad s_1 = a_0 + a_1, \quad s_2 = a_0 + a_1 + a_2, \quad \ldots$$

A series converges if the sequence of partial sums converges, and otherwise the series diverges.

EXAMPLE 13.2.1 If $a_n = kx^n$, $\sum_{n=0}^{\infty} a_n$ is called a **geometric series**. A typical partial sum is

$$s_n = k + kx + kx^2 + kx^3 + \cdots + kx^n = k(1 + x + x^2 + x^3 + \cdots + x^n).$$

We note that

$$\begin{aligned} s_n(1-x) &= k(1+x+x^2+x^3+\cdots+x^n)(1-x) \\ &= k(1+x+x^2+x^3+\cdots+x^n)1 - k(1+x+x^2+x^3+\cdots+x^{n-1}+x^n)x \\ &= k(1+x+x^2+x^3+\cdots+x^n-x-x^2-x^3-\cdots-x^n-x^{n+1}) \\ &= k(1-x^{n+1}) \end{aligned}$$

so

$$s_n(1-x) = k(1-x^{n+1})$$
$$s_n = k\frac{1-x^{n+1}}{1-x}.$$

If $|x|<1$, $\lim_{n\to\infty} x^n = 0$ so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} k\frac{1-x^{n+1}}{1-x} = k\frac{1}{1-x}.$$

Thus, when $|x|<1$ the geometric series converges to $k/(1-x)$. When, for example, $k=1$ and $x=1/2$:

$$s_n = \frac{1-(1/2)^{n+1}}{1-1/2} = \frac{2^{n+1}-1}{2^n} = 2-\frac{1}{2^n} \quad \text{and} \quad \sum_{n=0}^{\infty}\frac{1}{2^n} = \frac{1}{1-1/2} = 2.$$

We began the chapter with the series

$$\sum_{n=1}^{\infty}\frac{1}{2^n},$$

namely, the geometric series without the first term 1. Each partial sum of this series is 1 less than the corresponding partial sum for the geometric series, so of course the limit is also one less than the value of the geometric series, that is,

$$\sum_{n=1}^{\infty}\frac{1}{2^n} = 1.$$

□

It is not hard to see that the following theorem follows from theorem 13.1.2.

THEOREM 13.2.2 Suppose that $\sum a_n$ and $\sum b_n$ are convergent series, and c is a constant. Then

1. $\sum ca_n$ is convergent and $\sum ca_n = c\sum a_n$

2. $\sum(a_n+b_n)$ is convergent and $\sum(a_n+b_n) = \sum a_n + \sum b_n$.

■

The two parts of this theorem are subtly different. Suppose that $\sum a_n$ diverges; does $\sum ca_n$ also diverge if c is non-zero? Yes: suppose instead that $\sum ca_n$ converges; then by the theorem, $\sum(1/c)ca_n$ converges, but this is the same as $\sum a_n$, which by assumption diverges. Hence $\sum ca_n$ also diverges. Note that we are applying the theorem with a_n replaced by ca_n and c replaced by $(1/c)$.

Now suppose that $\sum a_n$ and $\sum b_n$ diverge; does $\sum(a_n+b_n)$ also diverge? Now the answer is no: Let $a_n = 1$ and $b_n = -1$, so certainly $\sum a_n$ and $\sum b_n$ diverge. But $\sum(a_n + b_n) = \sum(1+-1) = \sum 0 = 0$. Of course, sometimes $\sum(a_n+b_n)$ will also diverge, for example, if $a_n = b_n = 1$, then $\sum(a_n+b_n) = \sum(1+1) = \sum 2$ diverges.

In general, the sequence of partial sums s_n is harder to understand and analyze than the sequence of terms a_n, and it is difficult to determine whether series converge and if so to what. Sometimes things are relatively simple, starting with the following.

THEOREM 13.2.3 If $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof. Since $\sum a_n$ converges, $\lim_{n\to\infty} s_n = L$ and $\lim_{n\to\infty} s_{n-1} = L$, because this really says the same thing but "renumbers" the terms. By theorem 13.1.2,

$$\lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0.$$

But

$$s_n - s_{n-1} = (a_0+a_1+a_2+\cdots+a_n) - (a_0+a_1+a_2+\cdots+a_{n-1}) = a_n,$$

so as desired $\lim_{n\to\infty} a_n = 0$. ■

This theorem presents an easy divergence test: if given a series $\sum a_n$ the limit $\lim_{n\to\infty} a_n$ does not exist or has a value other than zero, the series diverges. Note well that the converse is *not* true: If $\lim_{n\to\infty} a_n = 0$ then the series does not necessarily converge.

EXAMPLE 13.2.4 Show that $\sum_{n=1}^{\infty}\frac{n}{n+1}$ diverges.

We compute the limit:

$$\lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0.$$

Looking at the first few terms perhaps makes it clear that the series has no chance of converging:

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\cdots$$

will just get larger and larger; indeed, after a bit longer the series starts to look very much like $\cdots + 1 + 1 + 1 + 1 + \cdots$, and of course if we add up enough 1's we can make the sum as large as we desire. □

EXAMPLE 13.2.5 Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Here the theorem does not apply: $\lim_{n\to\infty} 1/n = 0$, so it looks like perhaps the series converges. Indeed, if you have the fortitude (or the software) to add up the first 1000 terms you will find that

$$\sum_{n=1}^{1000} \frac{1}{n} \approx 7.49,$$

so it might be reasonable to speculate that the series converges to something in the neighborhood of 10. But in fact the partial sums do go to infinity; they just get big very, very slowly. Consider the following:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{16} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{16} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

and so on. By swallowing up more and more terms we can always manage to add at least another $1/2$ to the sum, and by adding enough of these we can make the partial sums as big as we like. In fact, it's not hard to see from this pattern that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} > 1 + \frac{n}{2},$$

so to make sure the sum is over 100, for example, we'd add up terms until we get to around $1/2^{198}$, that is, about $4 \cdot 10^{59}$ terms. This series, $\sum(1/n)$, is called the **harmonic series**. □

Exercises 13.2.

1. Explain why $\sum_{n=1}^{\infty} \frac{n^2}{2n^2+1}$ diverges. ⇒

2. Explain why $\sum_{n=1}^{\infty} \frac{5}{2^{1/n}+14}$ diverges. ⇒

3. Explain why $\sum_{n=1}^{\infty} \frac{3}{n}$ diverges. ⇒

4. Compute $\sum_{n=0}^{\infty} \frac{4}{(-3)^n} - \frac{3}{3^n}$. ⇒

5. Compute $\sum_{n=0}^{\infty} \frac{3}{2^n} + \frac{4}{5^n}$. ⇒

6. Compute $\sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n}$. ⇒

7. Compute $\sum_{n=0}^{\infty} \frac{3^{n+1}}{7^{n+1}}$. ⇒

8. Compute $\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$. ⇒

9. Compute $\sum_{n=1}^{\infty} \frac{3^n}{5^{n+1}}$. ⇒

13.3 The Integral Test

It is generally quite difficult, often impossible, to determine the value of a series exactly. In many cases it is possible at least to determine whether or not the series converges, and so we will spend most of our time on this problem.

If all of the terms a_n in a series are non-negative, then clearly the sequence of partial sums s_n is non-decreasing. This means that if we can show that the sequence of partial sums is bounded, the series must converge. We know that if the series converges, the terms a_n approach zero, but this does not mean that $a_n \geq a_{n+1}$ for every n. Many useful and interesting series do have this property, however, and they are among the easiest to understand. Let's look at an example.

EXAMPLE 13.3.1 Show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

The terms $1/n^2$ are positive and decreasing, and since $\lim_{x\to\infty} 1/x^2 = 0$, the terms $1/n^2$ approach zero. We seek an upper bound for all the partial sums, that is, we want to find a number N so that $s_n \leq N$ for every n. The upper bound is provided courtesy of integration, and is inherent in figure 13.3.1.

The figure shows the graph of $y = 1/x^2$ together with some rectangles that lie completely below the curve and that all have base length one. Because the heights of the rectangles are determined by the height of the curve, the areas of the rectangles are $1/1^2$, $1/2^2$, $1/3^2$, and so on—in other words, exactly the terms of the series. The partial sum s_n is simply the sum of the areas of the first n rectangles. Because the rectangles all lie between the curve and the x-axis, any sum of rectangle areas is less than the corresponding

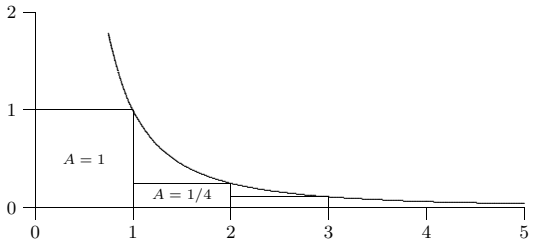

Figure 13.3.1 Graph of $y = 1/x^2$ with rectangles.

area under the curve, and so of course any sum of rectangle areas is less than the area under the entire curve, that is, all the way to infinity. There is a bit of trouble at the left end, where there is an asymptote, but we can work around that easily. Here it is:

$$s_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1 + \int_1^n \frac{1}{x^2}\,dx < 1 + \int_1^\infty \frac{1}{x^2}\,dx = 1 + 1 = 2,$$

recalling that we computed this improper integral in section 11.2. Since the sequence of partial sums s_n is increasing and bounded above by 2, we know that $\lim_{n\to\infty} s_n = L < 2$, and so the series converges to some number less than 2. In fact, it is possible, though difficult, to show that $L = \pi^2/6 \approx 1.6$. □

We already know that $\sum 1/n$ diverges. What goes wrong if we try to apply this technique to it? Here's the calculation:

$$s_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \int_1^n \frac{1}{x}\,dx < 1 + \int_1^\infty \frac{1}{x}\,dx = 1 + \infty.$$

The problem is that the improper integral doesn't converge. Note well that this does *not* prove that $\sum 1/n$ diverges, just that this particular calculation fails to prove that it converges. A slight modification, however, allows us to prove in a second way that $\sum 1/n$ diverges.

EXAMPLE 13.3.2 Consider a slightly altered version of figure 13.3.1, shown in figure 13.3.2.

The rectangles this time are above the curve, that is, each rectangle completely contains the corresponding area under the curve. This means that

$$s_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \int_1^{n+1} \frac{1}{x}\,dx = \ln x\Big|_1^{n+1} = \ln(n+1).$$

As n gets bigger, $\ln(n+1)$ goes to infinity, so the sequence of partial sums s_n must also go to infinity, so the harmonic series diverges. □

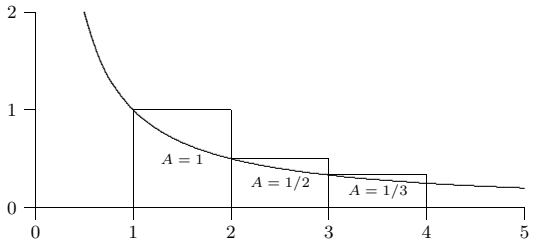

Figure 13.3.2 Graph of $y = 1/x$ with rectangles.

The important fact that clinches this example is that

$$\lim_{n\to\infty} \int_1^{n+1} \frac{1}{x}\,dx = \infty,$$

which we can rewrite as

$$\int_1^\infty \frac{1}{x}\,dx = \infty.$$

So these two examples taken together indicate that we can prove that a series converges or prove that it diverges with a single calculation of an improper integral. This is known as the **integral test**, which we state as a theorem.

THEOREM 13.3.3 Suppose that $f(x) > 0$ and is decreasing on the infinite interval $[k, \infty)$ (for some $k \geq 1$) and that $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^\infty f(x)\,dx$ converges. ■

The two examples we have seen are called p-series; a p-series is any series of the form $\sum 1/n^p$. If $p \leq 0$, $\lim_{n\to\infty} 1/n^p \neq 0$, so the series diverges. For positive values of p we can determine precisely which series converge.

THEOREM 13.3.4 A p-series with $p > 0$ converges if and only if $p > 1$.

Proof. We use the integral test; we have already done $p = 1$, so assume that $p \neq 1$.

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{D\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^D = \lim_{D\to\infty} \frac{D^{1-p}}{1-p} - \frac{1}{1-p}.$$

If $p > 1$ then $1 - p < 0$ and $\lim_{D\to\infty} D^{1-p} = 0$, so the integral converges. If $0 < p < 1$ then $1 - p > 0$ and $\lim_{D\to\infty} D^{1-p} = \infty$, so the integral diverges. ■

EXAMPLE 13.3.5 Show that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

We could of course use the integral test, but now that we have the theorem we may simply note that this is a p-series with $p > 1$. □

EXAMPLE 13.3.6 Show that $\sum_{n=1}^{\infty} \frac{5}{n^4}$ converges.

We know that if $\sum_{n=1}^{\infty} 1/n^4$ converges then $\sum_{n=1}^{\infty} 5/n^4$ also converges, by theorem 13.2.2. Since $\sum_{n=1}^{\infty} 1/n^4$ is a convergent p-series, $\sum_{n=1}^{\infty} 5/n^4$ converges also. □

EXAMPLE 13.3.7 Show that $\sum_{n=1}^{\infty} \frac{5}{\sqrt{n}}$ diverges.

This also follows from theorem 13.2.2: Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = 1/2 < 1$, it diverges, and so does $\sum_{n=1}^{\infty} \frac{5}{\sqrt{n}}$. □

Since it is typically difficult to compute the value of a series exactly, a good approximation is frequently required. In a real sense, a good approximation is only as good as we know it is, that is, while an approximation may in fact be good, it is only valuable in practice if we can guarantee its accuracy to some degree. This guarantee is usually easy to come by for series with decreasing positive terms.

EXAMPLE 13.3.8 Approximate $\sum 1/n^2$ to two decimal places.

Referring to figure 13.3.1, if we approximate the sum by $\sum_{n=1}^{N} 1/n^2$, the error we make is the total area of the remaining rectangles, all of which lie under the curve $1/x^2$ from $x = N$ out to infinity. So we know the true value of the series is larger than the approximation, and no bigger than the approximation plus the area under the curve from N to infinity. Roughly, then, we need to find N so that

$$\int_N^{\infty} \frac{1}{x^2}\,dx < 1/100.$$

We can compute the integral:

$$\int_N^{\infty} \frac{1}{x^2}\,dx = \frac{1}{N},$$

so $N = 100$ is a good starting point. Adding up the first 100 terms gives approximately 1.634983900, and that plus 1/100 is 1.644983900, so approximating the series by the value halfway between these will be at most $1/200 = 0.005$ in error. The midpoint is 1.639983900, but while this is correct to ± 0.005, we can't tell if the correct two-decimal approximation is 1.63 or 1.64. We need to make N big enough to reduce the guaranteed error, perhaps to around 0.004 to be safe, so we would need $1/N \approx 0.008$, or $N = 125$. Now the sum of the first 125 terms is approximately 1.636965982, and that plus 0.008 is 1.644965982 and the point halfway between them is 1.640965982. The true value is then 1.640965982 ± 0.004, and all numbers in this range round to 1.64, so 1.64 is correct to two decimal places. We have mentioned that the true value of this series can be shown to be $\pi^2/6 \approx 1.644934068$ which rounds down to 1.64 (just barely) and is indeed below the upper bound of 1.644965982, again just barely. Frequently approximations will be even better than the "guaranteed" accuracy, but not always, as this example demonstrates. □

Exercises 13.3.

Determine whether each series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{1}{n^{\pi/4}}$ ⇒

2. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ ⇒

3. $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ ⇒

4. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ ⇒

5. $\sum_{n=1}^{\infty} \frac{1}{e^n}$ ⇒

6. $\sum_{n=1}^{\infty} \frac{n}{e^n}$ ⇒

7. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ ⇒

8. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ ⇒

9. Find an N so that $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is between $\sum_{n=1}^{N} \frac{1}{n^4}$ and $\sum_{n=1}^{N} \frac{1}{n^4} + 0.005$. ⇒

10. Find an N so that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ is between $\sum_{n=0}^{N} \frac{1}{e^n}$ and $\sum_{n=0}^{N} \frac{1}{e^n} + 10^{-4}$. ⇒

11. Find an N so that $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ is between $\sum_{n=1}^{N} \frac{\ln n}{n^2}$ and $\sum_{n=1}^{N} \frac{\ln n}{n^2} + 0.005$. ⇒

12. Find an N so that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is between $\sum_{n=2}^{N} \frac{1}{n(\ln n)^2}$ and $\sum_{n=2}^{N} \frac{1}{n(\ln n)^2} + 0.005$. ⇒

13.4 Alternating Series

Next we consider series with both positive and negative terms, but in a regular pattern: they alternate, as in the **alternating harmonic series** for example:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \frac{1}{1} + \frac{-1}{2} + \frac{1}{3} + \frac{-1}{4} + \cdots = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

In this series the sizes of the terms decrease, that is, $|a_n|$ forms a decreasing sequence, but this is not required in an alternating series. As with positive term series, however, when the terms do have decreasing sizes it is easier to analyze the series, much easier, in fact, than positive term series. Consider pictorially what is going on in the alternating harmonic series, shown in figure 13.4.1. Because the sizes of the terms a_n are decreasing, the partial sums s_1, s_3, s_5, and so on, form a decreasing sequence that is bounded below by s_2, so this sequence must converge. Likewise, the partial sums s_2, s_4, s_6, and so on, form an increasing sequence that is bounded above by s_1, so this sequence also converges. Since all the even numbered partial sums are less than all the odd numbered ones, and since the "jumps" (that is, the a_i terms) are getting smaller and smaller, the two sequences must converge to the same value, meaning the entire sequence of partial sums $s_1, s_2, s_3, \ldots$ converges as well.

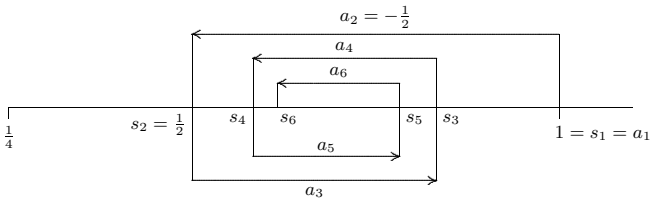

Figure 13.4.1 The alternating harmonic series.

There's nothing special about the alternating harmonic series—the same argument works for any alternating sequence with decreasing size terms. The alternating series test is worth calling a theorem.

THEOREM 13.4.1 Suppose that $\{a_n\}_{n=1}^{\infty}$ is a non-increasing sequence of positive numbers and $\lim_{n\to\infty} a_n = 0$. Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges.

Proof. The odd numbered partial sums, s_1, s_3, s_5, and so on, form a non-increasing sequence, because $s_{2k+3} = s_{2k+1} - a_{2k+2} + a_{2k+3} \le s_{2k+1}$, since $a_{2k+2} \ge a_{2k+3}$. This

sequence is bounded below by s_2, so it must converge, say $\lim_{k\to\infty} s_{2k+1} = L$. Likewise, the partial sums s_2, s_4, s_6, and so on, form a non-decreasing sequence that is bounded above by s_1, so this sequence also converges, say $\lim_{k\to\infty} s_{2k} = M$. Since $\lim_{n\to\infty} a_n = 0$ and $s_{2k+1} = s_{2k} + a_{2k+1}$,

$$L = \lim_{k\to\infty} s_{2k+1} = \lim_{k\to\infty} (s_{2k} + a_{2k+1}) = \lim_{k\to\infty} s_{2k} + \lim_{k\to\infty} a_{2k+1} = M + 0 = M,$$

so $L = M$, the two sequences of partial sums converge to the same limit, and this means the entire sequence of partial sums also converges to L. ∎

Another useful fact is implicit in this discussion. Suppose that

$$L = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$$

and that we approximate L by a finite part of this sum, say

$$L \approx \sum_{n=1}^{N}(-1)^{n-1}a_n.$$

Because the terms are decreasing in size, we know that the true value of L must be between this approximation and the next one, that is, between

$$\sum_{n=1}^{N}(-1)^{n-1}a_n \quad \text{and} \quad \sum_{n=1}^{N+1}(-1)^{n-1}a_n.$$

Depending on whether N is odd or even, the second will be larger or smaller than the first.

EXAMPLE 13.4.2 Approximate the alternating harmonic series to one decimal place.

We need to go roughly to the point at which the next term to be added or subtracted is 1/10. Adding up the first nine and the first ten terms we get approximately 0.746 and 0.646. These are 1/10 apart, but it is not clear how the correct value would be rounded. It turns out that we are able to settle the question by computing the sums of the first eleven and twelve terms, which give 0.737 and 0.653, so correct to one place the value is 0.7. □

We have considered alternating series with first index 1, and in which the first term is positive, but a little thought shows this is not crucial. The same test applies to any similar series, such as $\sum_{n=0}^{\infty}(-1)^n a_n$, $\sum_{n=1}^{\infty}(-1)^n a_n$, $\sum_{n=17}^{\infty}(-1)^n a_n$, etc.

Exercises 13.4.

Determine whether the following series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+5}$ ⇒

2. $\displaystyle\sum_{n=4}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n-3}}$ ⇒

3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{3n-2}$ ⇒

4. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln n}{n}$ ⇒

5. Approximate $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^3}$ to two decimal places. ⇒

6. Approximate $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^4}$ to two decimal places. ⇒

13.5 Comparison Tests

As we begin to compile a list of convergent and divergent series, new ones can sometimes be analyzed by comparing them to ones that we already understand.

EXAMPLE 13.5.1 Does $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2\ln n}$ converge?

The obvious first approach, based on what we know, is the integral test. Unfortunately, we can't compute the required antiderivative. But looking at the series, it would appear that it must converge, because the terms we are adding are smaller than the terms of a p-series, that is,

$$\frac{1}{n^2\ln n}<\frac{1}{n^2},$$

when $n\geq 3$. Since adding up the terms $1/n^2$ doesn't get "too big", the new series "should" also converge. Let's make this more precise.

The series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2\ln n}$ converges if and only if $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n^2\ln n}$ converges—all we've done is dropped the initial term. We know that $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n^2}$ converges. Looking at two typical partial sums:

$$s_n=\frac{1}{3^2\ln 3}+\frac{1}{4^2\ln 4}+\frac{1}{5^2\ln 5}+\cdots+\frac{1}{n^2\ln n}<\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots+\frac{1}{n^2}=t_n.$$

Since the p-series converges, say to L, and since the terms are positive, $t_n<L$. Since the terms of the new series are positive, the s_n form an increasing sequence and $s_n<t_n<L$ for all n. Hence the sequence $\{s_n\}$ is bounded and so converges. □

Sometimes, even when the integral test applies, comparison to a known series is easier, so it's generally a good idea to think about doing a comparison before doing the integral test.

EXAMPLE 13.5.2 Does $\displaystyle\sum_{n=2}^{\infty}\frac{|\sin n|}{n^2}$ converge?

We can't apply the integral test here, because the terms of this series are not decreasing. Just as in the previous example, however,

$$\frac{|\sin n|}{n^2}\leq\frac{1}{n^2},$$

because $|\sin n|\leq 1$. Once again the partial sums are non-decreasing and bounded above by $\sum 1/n^2=L$, so the new series converges. □

Like the integral test, the comparison test can be used to show both convergence and divergence. In the case of the integral test, a single calculation will confirm whichever is the case. To use the comparison test we must first have a good idea as to convergence or divergence and pick the sequence for comparison accordingly.

EXAMPLE 13.5.3 Does $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{n^2-3}}$ converge?

We observe that the -3 should have little effect compared to the n^2 inside the square root, and therefore guess that the terms are enough like $1/\sqrt{n^2}=1/n$ that the series should diverge. We attempt to show this by comparison to the harmonic series. We note that

$$\frac{1}{\sqrt{n^2-3}}>\frac{1}{\sqrt{n^2}}=\frac{1}{n},$$

so that

$$s_n=\frac{1}{\sqrt{2^2-3}}+\frac{1}{\sqrt{3^2-3}}+\cdots+\frac{1}{\sqrt{n^2-3}}>\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=t_n,$$

where t_n is 1 less than the corresponding partial sum of the harmonic series (because we start at $n=2$ instead of $n=1$). Since $\lim_{n\to\infty}t_n=\infty$, $\lim_{n\to\infty}s_n=\infty$ as well. □

So the general approach is this: If you believe that a new series is convergent, attempt to find a convergent series whose terms are larger than the terms of the new series; if you believe that a new series is divergent, attempt to find a divergent series whose terms are smaller than the terms of the new series.

EXAMPLE 13.5.4 Does $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+3}}$ converge?

Just as in the last example, we guess that this is very much like the harmonic series and so diverges. Unfortunately,

$$\frac{1}{\sqrt{n^2+3}}<\frac{1}{n},$$

so we can't compare the series directly to the harmonic series. A little thought leads us to

$$\frac{1}{\sqrt{n^2+3}}>\frac{1}{\sqrt{n^2+3n^2}}=\frac{1}{2n},$$

so if $\sum 1/(2n)$ diverges then the given series diverges. But since $\sum 1/(2n)=(1/2)\sum 1/n$, theorem 13.2.2 implies that it does indeed diverge. □

For reference we summarize the comparison test in a theorem.

THEOREM 13.5.5 Suppose that a_n and b_n are non-negative for all n and that $a_n\le b_n$ when $n\ge N$, for some N.

If $\displaystyle\sum_{n=0}^{\infty}b_n$ converges, so does $\displaystyle\sum_{n=0}^{\infty}a_n$.

If $\displaystyle\sum_{n=0}^{\infty}a_n$ diverges, so does $\displaystyle\sum_{n=0}^{\infty}b_n$.

■

Exercises 13.5.

Determine whether the series converge or diverge.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n^2+3n+5}\Rightarrow$

2. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{2n^2+3n-5}\Rightarrow$

3. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n^2-3n-5}\Rightarrow$

4. $\displaystyle\sum_{n=1}^{\infty}\frac{3n+4}{2n^2+3n+5}\Rightarrow$

5. $\displaystyle\sum_{n=1}^{\infty}\frac{3n^2+4}{2n^2+3n+5}\Rightarrow$

6. $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}\Rightarrow$

7. $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^3}\Rightarrow$

8. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\ln n}\Rightarrow$

9. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{2^n+5^n}\Rightarrow$

10. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{2^n+3^n}\Rightarrow$

13.6 Absolute Convergence

Roughly speaking there are two ways for a series to converge: As in the case of $\sum 1/n^2$, the individual terms get small very quickly, so that the sum of all of them stays finite, or, as in the case of $\sum(-1)^{n-1}/n$, the terms don't get small fast enough ($\sum 1/n$ diverges), but a mixture of positive and negative terms provides enough cancellation to keep the sum finite. You might guess from what we've seen that if the terms get small fast enough to do the job, then whether or not some terms are negative and some positive the series converges.

THEOREM 13.6.1 If $\displaystyle\sum_{n=0}^{\infty}|a_n|$ converges, then $\displaystyle\sum_{n=0}^{\infty}a_n$ converges.

Proof. Note that $0\le a_n+|a_n|\le 2|a_n|$ so by the comparison test $\displaystyle\sum_{n=0}^{\infty}(a_n+|a_n|)$ converges. Now

$$\sum_{n=0}^{\infty}(a_n+|a_n|)-\sum_{n=0}^{\infty}|a_n|=\sum_{n=0}^{\infty}a_n+|a_n|-|a_n|=\sum_{n=0}^{\infty}a_n$$

converges by theorem 13.2.2. ■

So given a series $\sum a_n$ with both positive and negative terms, you should first ask whether $\sum|a_n|$ converges. This may be an easier question to answer, because we have tests that apply specifically to terms with non-negative terms. If $\sum|a_n|$ converges then you know that $\sum a_n$ converges as well. If $\sum|a_n|$ diverges then it still may be true that $\sum a_n$ converges—you will have to do more work to decide the question. Another way to think of this result is: it is (potentially) easier for $\sum a_n$ to converge than for $\sum|a_n|$ to converge, because the latter series cannot take advantage of cancellation.

If $\sum|a_n|$ converges we say that $\sum a_n$ converges **absolutely**; to say that $\sum a_n$ converges absolutely is to say that any cancellation that happens to come along is not really needed, as the terms already get small so fast that convergence is guaranteed by that alone. If $\sum a_n$ converges but $\sum|a_n|$ does not, we say that $\sum a_n$ converges **conditionally**. For example $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ converges absolutely, while $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges conditionally.

EXAMPLE 13.6.2 Does $\displaystyle\sum_{n=2}^{\infty}\frac{\sin n}{n^2}$ converge?

In example 13.5.2 we saw that $\sum_{n=2}^{\infty} \frac{|\sin n|}{n^2}$ converges, so the given series converges absolutely. □

EXAMPLE 13.6.3 Does $\sum_{n=0}^{\infty}(-1)^n \frac{3n+4}{2n^2+3n+5}$ converge?

Taking the absolute value, $\sum_{n=0}^{\infty} \frac{3n+4}{2n^2+3n+5}$ diverges by comparison to $\sum_{n=1}^{\infty} \frac{3}{10n}$, so if the series converges it does so conditionally. It is true that $\lim_{n\to\infty}(3n+4)/(2n^2+3n+5) = 0$, so to apply the alternating series test we need to know whether the terms are decreasing. If we let $f(x) = (3x+4)/(2x^2+3x+5)$ then $f'(x) = -(6x^2+16x-3)/(2x^2+3x+5)^2$, and it is not hard to see that this is negative for $x \geq 1$, so the series is decreasing and by the alternating series test it converges. □

Exercises 13.6.

Determine whether each series converges absolutely, converges conditionally, or diverges.

1. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n^2+3n+5}$ ⇒

2. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3n^2+4}{2n^2+3n+5}$ ⇒

3. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln n}{n}$ ⇒

4. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\ln n}{n^3}$ ⇒

5. $\sum_{n=2}^{\infty}(-1)^{n}\frac{1}{\ln n}$ ⇒

6. $\sum_{n=0}^{\infty}(-1)^{n}\frac{3^n}{2^n+5^n}$ ⇒

7. $\sum_{n=0}^{\infty}(-1)^{n}\frac{3^n}{2^n+3^n}$ ⇒

8. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\arctan n}{n}$ ⇒

13.7 The Ratio and Root Tests

Does the series $\sum_{n=0}^{\infty} \frac{n^5}{5^n}$ converge? It is possible, but a bit unpleasant, to approach this with the integral test or the comparison test, but there is an easier way. Consider what happens as we move from one term to the next in this series:

$$\cdots + \frac{n^5}{5^n} + \frac{(n+1)^5}{5^{n+1}} + \cdots$$

The denominator goes up by a factor of 5, $5^{n+1} = 5 \cdot 5^n$, but the numerator goes up by much less: $(n+1)^5 = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1$, which is much less than $5n^5$ when n is large, because $5n^4$ is much less than n^5. So we might guess that in the long run it

begins to look as if each term is 1/5 of the previous term. We have seen series that behave like this:

$$\sum_{n=0}^{\infty} \frac{1}{5^n} = \frac{5}{4},$$

a geometric series. So we might try comparing the given series to some variation of this geometric series. This is possible, but a bit messy. We can in effect do the same thing, but bypass most of the unpleasant work.

The key is to notice that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^5}{5^{n+1}} \frac{5^n}{n^5} = \lim_{n\to\infty} \frac{(n+1)^5}{n^5} \frac{1}{5} = 1 \cdot \frac{1}{5} = \frac{1}{5}.$$

This is really just what we noticed above, done a bit more officially: in the long run, each term is one fifth of the previous term. Now pick some number between 1/5 and 1, say 1/2. Because

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{5},$$

then when n is big enough, say $n \geq N$ for some N,

$$\frac{a_{n+1}}{a_n} < \frac{1}{2} \quad \text{and} \quad a_{n+1} < \frac{a_n}{2}.$$

So $a_{N+1} < a_N/2$, $a_{N+2} < a_{N+1}/2 < a_N/4$, $a_{N+3} < a_{N+2}/2 < a_{N+1}/4 < a_N/8$, and so on. The general form is $a_{N+k} < a_N/2^k$. So if we look at the series

$$\sum_{k=0}^{\infty} a_{N+k} = a_N + a_{N+1} + a_{N+2} + a_{N+3} + \cdots + a_{N+k} + \cdots,$$

its terms are less than or equal to the terms of the sequence

$$a_N + \frac{a_N}{2} + \frac{a_N}{4} + \frac{a_N}{8} + \cdots + \frac{a_N}{2^k} + \cdots = \sum_{k=0}^{\infty} \frac{a_N}{2^k} = 2a_N.$$

So by the comparison test, $\sum_{k=0}^{\infty} a_{N+k}$ converges, and this means that $\sum_{n=0}^{\infty} a_n$ converges, since we've just added the fixed number $a_0 + a_1 + \cdots + a_{N-1}$.

Under what circumstances could we do this? What was crucial was that the limit of a_{n+1}/a_n, say L, was less than 1 so that we could pick a value r so that $L < r < 1$. The fact that $L < r$ ($1/5 < 1/2$ in our example) means that we can compare the series $\sum a_n$ to $\sum r^n$, and the fact that $r < 1$ guarantees that $\sum r^n$ converges. That's really all that is

required to make the argument work. We also made use of the fact that the terms of the series were positive; in general we simply consider the absolute values of the terms and we end up testing for absolute convergence.

THEOREM 13.7.1 The Ratio Test Suppose that $\lim_{n\to\infty} |a_{n+1}/a_n| = L$. If $L < 1$ the series $\sum a_n$ converges absolutely, if $L > 1$ the series diverges, and if $L = 1$ this test gives no information.

Proof. The example above essentially proves the first part of this, if we simply replace $1/5$ by L and $1/2$ by r. Suppose that $L > 1$, and pick r so that $1 < r < L$. Then for $n \geq N$, for some N,

$$\frac{|a_{n+1}|}{|a_n|} > r \quad \text{and} \quad |a_{n+1}| > r|a_n|.$$

This implies that $|a_{N+k}| > r^k|a_N|$, but since $r > 1$ this means that $\lim_{k\to\infty} |a_{N+k}| \neq 0$, which means also that $\lim_{n\to\infty} a_n \neq 0$. By the divergence test, the series diverges.

To see that we get no information when $L = 1$, we need to exhibit two series with $L = 1$, one that converges and one that diverges. It is easy to see that $\sum 1/n^2$ and $\sum 1/n$ do the job. ■

EXAMPLE 13.7.2 The ratio test is particularly useful for series involving the factorial function. Consider $\sum_{n=0}^{\infty} 5^n/n!$.

$$\lim_{n\to\infty} \frac{5^{n+1}}{(n+1)!}\frac{n!}{5^n} = \lim_{n\to\infty} \frac{5^{n+1}}{5^n}\frac{n!}{(n+1)!} = \lim_{n\to\infty} 5\frac{1}{(n+1)} = 0.$$

Since $0 < 1$, the series converges. □

A similar argument, which we will not do, justifies a similar test that is occasionally easier to apply.

THEOREM 13.7.3 The Root Test Suppose that $\lim_{n\to\infty} |a_n|^{1/n} = L$. If $L < 1$ the series $\sum a_n$ converges absolutely, if $L > 1$ the series diverges, and if $L = 1$ this test gives no information. ■

The proof of the root test is actually easier than that of the ratio test, and is a good exercise.

EXAMPLE 13.7.4 Analyze $\sum_{n=0}^{\infty} \frac{5^n}{n^n}$.

The ratio test turns out to be a bit difficult on this series (try it). Using the root test:

$$\lim_{n\to\infty} \left(\frac{5^n}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{(5^n)^{1/n}}{(n^n)^{1/n}} = \lim_{n\to\infty} \frac{5}{n} = 0.$$

Since $0 < 1$, the series converges. □

The root test is frequently useful when n appears as an exponent in the general term of the series.

Exercises 13.7.

1. Compute $\lim_{n\to\infty} |a_{n+1}/a_n|$ for the series $\sum 1/n^2$.

2. Compute $\lim_{n\to\infty} |a_{n+1}/a_n|$ for the series $\sum 1/n$.

3. Compute $\lim_{n\to\infty} |a_n|^{1/n}$ for the series $\sum 1/n^2$.

4. Compute $\lim_{n\to\infty} |a_n|^{1/n}$ for the series $\sum 1/n$.

Determine whether the series converge.

5. $\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{5^n}$ ⇒

6. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ ⇒

7. $\sum_{n=1}^{\infty} \frac{n^5}{n^n}$ ⇒

8. $\sum_{n=1}^{\infty} \frac{(n!)^2}{n^n}$ ⇒

9. Prove theorem 13.7.3, the root test.

13.8 Power Series

Recall that we were able to analyze all geometric series "simultaneously" to discover that

$$\sum_{n=0}^{\infty} kx^n = \frac{k}{1-x},$$

if $|x| < 1$, and that the series diverges when $|x| \geq 1$. At the time, we thought of x as an unspecified constant, but we could just as well think of it as a variable, in which case the

series

$$\sum_{n=0}^{\infty} kx^n$$

is a function, namely, the function $k/(1-x)$, as long as $|x|<1$. While $k/(1-x)$ is a reasonably easy function to deal with, the more complicated $\sum kx^n$ does have its attractions: it appears to be an infinite version of one of the simplest function types—a polynomial. This leads naturally to the questions: Do other functions have representations as series? Is there an advantage to viewing them in this way?

The geometric series has a special feature that makes it unlike a typical polynomial—the coefficients of the powers of x are the same, namely k. We will need to allow more general coefficients if we are to get anything other than the geometric series.

DEFINITION 13.8.1 A power series has the form

$$\sum_{n=0}^{\infty} a_n x^n,$$

with the understanding that a_n may depend on n but not on x. □

EXAMPLE 13.8.2 $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ is a power series. We can investigate convergence using the ratio test:

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{n+1}\frac{n}{|x|^n} = \lim_{n\to\infty} |x|\frac{n}{n+1} = |x|.$$

Thus when $|x|<1$ the series converges and when $|x|>1$ it diverges, leaving only two values in doubt. When $x=1$ the series is the harmonic series and diverges; when $x=-1$ it is the alternating harmonic series (actually the negative of the usual alternating harmonic series) and converges. Thus, we may think of $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$ as a function from the interval $[-1,1)$ to the real numbers. □

A bit of thought reveals that the ratio test applied to a power series will always have the same nice form. In general, we will compute

$$\lim_{n\to\infty} \frac{|a_{n+1}||x|^{n+1}}{|a_n||x|^n} = \lim_{n\to\infty} |x|\frac{|a_{n+1}|}{|a_n|} = |x| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L|x|,$$

assuming that $\lim |a_{n+1}|/|a_n|$ exists. Then the series converges if $L|x|<1$, that is, if $|x|<1/L$, and diverges if $|x|>1/L$. Only the two values $x=\pm 1/L$ require further

investigation. Thus the series will definitely define a function on the interval $(-1/L, 1/L)$, and perhaps will extend to one or both endpoints as well. Two special cases deserve mention: if $L=0$ the limit is 0 no matter what value x takes, so the series converges for all x and the function is defined for all real numbers. If $L=\infty$, then no matter what value x takes the limit is infinite and the series converges only when $x=0$. The value $1/L$ is called the **radius of convergence** of the series, and the interval on which the series converges is the **interval of convergence**.

Consider again the geometric series,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Whatever benefits there might be in using the series form of this function are only available to us when x is between -1 and 1. Frequently we can address this shortcoming by modifying the power series slightly. Consider this series:

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{x+2}{3}\right)^n = \frac{1}{1-\frac{x+2}{3}} = \frac{3}{1-x},$$

because this is just a geometric series with x replaced by $(x+2)/3$. Multiplying both sides by $1/3$ gives

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^{n+1}} = \frac{1}{1-x},$$

the same function as before. For what values of x does this series converge? Since it is a geometric series, we know that it converges when

$$\begin{aligned} |x+2|/3 &< 1 \\ |x+2| &< 3 \\ -3 < x+2 &< 3 \\ -5 < x &< 1. \end{aligned}$$

So we have a series representation for $1/(1-x)$ that works on a larger interval than before, at the expense of a somewhat more complicated series. The endpoints of the interval of convergence now are -5 and 1, but note that they can be more compactly described as -2 ± 3. We say that 3 is the radius of convergence, and we now say that the series is centered at -2.

DEFINITION 13.8.3 A power series centered at a has the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n,$$

with the understanding that a_n may depend on n but not on x. □

Exercises 13.8.

Find the radius and interval of convergence for each series. In exercises 3 and 4, do not attempt to determine whether the endpoints are in the interval of convergence.

1. $\displaystyle\sum_{n=0}^{\infty} nx^n \Rightarrow$

2. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}x^n \Rightarrow$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}(x-2)^n \Rightarrow$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{n^n}(x-2)^n \Rightarrow$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+5)^n}{n(n+1)} \Rightarrow$

13.9 Calculus with Power Series

Now we know that some functions can be expressed as power series, which look like infinite polynomials. Since calculus, that is, computation of derivatives and antiderivatives, is easy for polynomials, the obvious question is whether the same is true for infinite series. The answer is yes:

THEOREM 13.9.1 Suppose the power series $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of convergence R. Then

$$f'(x) = \sum_{n=0}^{\infty} na_n(x-a)^{n-1},$$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1},$$

and these two series have radius of convergence R as well. ■

EXAMPLE 13.9.2 Starting with the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\int \frac{1}{1-x}\,dx = -\ln|1-x| = \sum_{n=0}^{\infty} \frac{1}{n+1}x^{n+1}$$

$$\ln|1-x| = \sum_{n=0}^{\infty} -\frac{1}{n+1}x^{n+1}$$

when $|x| < 1$. The series does not converge when $x = 1$ but does converge when $x = -1$ or $1 - x = 2$. The interval of convergence is $[-1, 1)$, or $0 < 1 - x \leq 2$, so we can use the series to represent $\ln(x)$ when $0 < x \leq 2$. For example

$$\ln(3/2) = \ln(1 - -1/2) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1}\frac{1}{2^{n+1}}$$

and so

$$\ln(3/2) \approx \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \frac{1}{160} - \frac{1}{384} + \frac{1}{896} = \frac{909}{2240} \approx 0.406.$$

Because this is an alternating series with decreasing terms, we know that the true value is between $909/2240$ and $909/2240 - 1/2048 = 29053/71680 \approx .4053$, so correct to two decimal places the value is 0.41.

What about $\ln(9/4)$? Since $9/4$ is larger than 2 we cannot use the series directly, but

$$\ln(9/4) = \ln((3/2)^2) = 2\ln(3/2) \approx 0.82,$$

so in fact we get a lot more from this one calculation than first meets the eye. To estimate the true value accurately we actually need to be a bit more careful. When we multiply by two we know that the true value is between 0.8106 and 0.812, so rounded to two decimal places the true value is 0.81. □

Exercises 13.9.

1. Find a series representation for $\ln 2$. ⇒

2. Find a power series representation for $1/(1-x)^2$. ⇒

3. Find a power series representation for $2/(1-x)^3$. ⇒

4. Find a power series representation for $1/(1-x)^3$. What is the radius of convergence? ⇒

5. Find a power series representation for $\displaystyle\int \ln(1-x)\,dx$. ⇒

13.10 Taylor Series

We have seen that some functions can be represented as series, which may give valuable information about the function. So far, we have seen only those examples that result from manipulation of our one fundamental example, the geometric series. We would like to start with a given function and produce a series to represent it, if possible.

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on some interval of convergence. Then we know that we can compute derivatives of f by taking derivatives of the terms of the series. Let's look at the first few in general:

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots$$

$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3} = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4 x + \cdots$$

By examining these it's not hard to discern the general pattern. The kth derivative must be

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1) a_n x^{n-k} \\ &= k(k-1)(k-2)\cdots(2)(1)a_k + (k+1)(k)\cdots(2)a_{k+1}x + \\ &\quad + (k+2)(k+1)\cdots(3)a_{k+2}x^2 + \cdots \end{aligned}$$

We can shrink this quite a bit by using factorial notation:

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k} = k!a_k + (k+1)!a_{k+1}x + \frac{(k+2)!}{2!}a_{k+2}x^2 + \cdots$$

Now substitute $x = 0$:

$$f^{(k)}(0) = k!a_k + \sum_{n=k+1}^{\infty} \frac{n!}{(n-k)!} a_n 0^{n-k} = k!a_k,$$

and solve for a_k:

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

Note the special case, obtained from the series for f itself, that gives $f(0) = a_0$.

So if a function f can be represented by a series, we know just what series it is. Given a function f, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

is called the **Maclaurin series** for f.

EXAMPLE 13.10.1 Find the Maclaurin series for $f(x) = 1/(1-x)$. We need to compute the derivatives of f (and hope to spot a pattern).

$$\begin{aligned} f(x) &= (1-x)^{-1} \\ f'(x) &= (1-x)^{-2} \\ f''(x) &= 2(1-x)^{-3} \\ f'''(x) &= 6(1-x)^{-4} \\ f^{(4)}(x) &= 4!(1-x)^{-5} \\ &\vdots \\ f^{(n)}(x) &= n!(1-x)^{-n-1} \end{aligned}$$

So

$$\frac{f^{(n)}(0)}{n!} = \frac{n!(1-0)^{-n-1}}{n!} = 1$$

and the Maclaurin series is

$$\sum_{n=0}^{\infty} 1 \cdot x^n = \sum_{n=0}^{\infty} x^n,$$

the geometric series. □

A warning is in order here. Given a function f we may be able to compute the Maclaurin series, but that does not mean we have found a series representation for f. We still need to know where the series converges, and if, where it converges, it converges to $f(x)$. While for most commonly encountered functions the Maclaurin series does indeed converge to f on some interval, this is not true of all functions, so care is required.

As a practical matter, if we are interested in using a series to approximate a function, we will need some finite number of terms of the series. Even for functions with messy derivatives we can compute these using computer software like Sage. If we want to know the whole series, that is, a typical term in the series, we need a function whose derivatives fall into a pattern that we can discern. A few of the most important functions are fortunately very easy.

EXAMPLE 13.10.2 Find the Maclaurin series for $\sin x$.

The derivatives are quite easy: $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x) = \sin x$, and then the pattern repeats. We want to know the derivatives at zero: 1, 0, -1, 0, 1, 0, -1, 0,..., and so the Maclaurin series is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

We should always determine the radius of convergence:

$$\lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!} \frac{(2n+1)!}{|x|^{2n+1}} = \lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0,$$

so the series converges for every x. Since it turns out that this series does indeed converge to $\sin x$ everywhere, we have a series representation for $\sin x$ for every x. Here is an interactive plot of the sine and some of its series approximations. □

Sometimes the formula for the nth derivative of a function f is difficult to discover, but a combination of a known Maclaurin series and some algebraic manipulation leads easily to the Maclaurin series for f.

EXAMPLE 13.10.3 Find the Maclaurin series for $x\sin(-x)$.

To get from $\sin x$ to $x\sin(-x)$ we substitute $-x$ for x and then multiply by x. We can do the same thing to the series for $\sin x$:

$$x\sum_{n=0}^{\infty} (-1)^n \frac{(-x)^{2n+1}}{(2n+1)!} = x\sum_{n=0}^{\infty} (-1)^n (-1)^{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+2}}{(2n+1)!}.$$

□

As we have seen, a general power series can be centered at a point other than zero, and the method that produces the Maclaurin series can also produce such series.

EXAMPLE 13.10.4 Find a series centered at -2 for $1/(1-x)$.

If the series is $\sum_{n=0}^{\infty} a_n(x+2)^n$ then looking at the kth derivative:

$$k!(1-x)^{-k-1} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (x+2)^{n-k}$$

and substituting $x = -2$ we get $k!3^{-k-1} = k!a_k$ and $a_k = 3^{-k-1} = 1/3^{k+1}$, so the series is

$$\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^{n+1}}.$$

We've already seen this, on page 296. □

Such a series is called the **Taylor series** for the function, and the general term has the form

$$\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

A Maclaurin series is simply a Taylor series with $a = 0$.

Exercises 13.10.

For each function, find the Maclaurin series or Taylor series centered at a, and the radius of convergence.

1. $\cos x \Rightarrow$
2. $e^x \Rightarrow$
3. $1/x$, $a = 5 \Rightarrow$
4. $\ln x$, $a = 1 \Rightarrow$
5. $\ln x$, $a = 2 \Rightarrow$
6. $1/x^2$, $a = 1 \Rightarrow$
7. $1/\sqrt{1-x} \Rightarrow$
8. Find the first four terms of the Maclaurin series for $\tan x$ (up to and including the x^3 term). $\Rightarrow$
9. Use a combination of Maclaurin series and algebraic manipulation to find a series centered at zero for $x\cos(x^2)$. $\Rightarrow$
10. Use a combination of Maclaurin series and algebraic manipulation to find a series centered at zero for xe^{-x}. $\Rightarrow$

13.11 Taylor's Theorem

One of the most important uses of infinite series is the potential for using an initial portion of the series for f to approximate f. We have seen, for example, that when we add up the first n terms of an alternating series with decreasing terms that the difference between this and the true value is at most the size of the next term. A similar result is true of many Taylor series.

THEOREM 13.11.1 Suppose that f is defined on some open interval I around a and suppose $f^{(N+1)}(x)$ exists on this interval. Then for each $x \neq a$ in I there is a value z between x and a so that

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1}.$$

Proof. The proof requires some cleverness to set up, but then the details are quite elementary. We want to define a function $F(t)$. Start with the equation

$$F(t) = \sum_{n=0}^{N} \frac{f^{(n)}(t)}{n!}\,(x-t)^n + B(x-t)^{N+1}.$$

Here we have replaced a by t in the first $N+1$ terms of the Taylor series, and added a carefully chosen term on the end, with B to be determined. Note that we are temporarily keeping x fixed, so the only variable in this equation is t, and we will be interested only in t between a and x. Now substitute $t = a$:

$$F(a) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}\,(x-a)^n + B(x-a)^{N+1}.$$

Set this equal to $f(x)$:

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}\,(x-a)^n + B(x-a)^{N+1}.$$

Since $x \neq a$, we can solve this for B, which is a "constant"—it depends on x and a but those are temporarily fixed. Now we have defined a function $F(t)$ with the property that $F(a) = f(x)$. Consider also $F(x)$: all terms with a positive power of $(x-t)$ become zero when we substitute x for t, so we are left with $F(x) = f^{(0)}(x)/0! = f(x)$. So $F(t)$ is a function with the same value on the endpoints of the interval $[a, x]$. By Rolle's theorem (6.5.1), we know that there is a value $z \in (a, x)$ such that $F'(z) = 0$. Let's look at $F'(t)$. Each term in $F(t)$, except the first term and the extra term involving B, is a product, so to take the derivative we use the product rule on each of these terms. It will help to write out the first few terms of the definition:

$$\begin{aligned} F(t) = f(t) &+ \frac{f^{(1)}(t)}{1!}(x-t)^1 + \frac{f^{(2)}(t)}{2!}(x-t)^2 + \frac{f^{(3)}(t)}{3!}(x-t)^3 + \cdots \\ &+ \frac{f^{(N)}(t)}{N!}(x-t)^N + B(x-t)^{N+1}. \end{aligned}$$

Now take the derivative:

$$\begin{aligned} F'(t) = f'(t) &+ \left(\frac{f^{(1)}(t)}{1!}(x-t)^0(-1) + \frac{f^{(2)}(t)}{1!}(x-t)^1\right) \\ &+ \left(\frac{f^{(2)}(t)}{1!}(x-t)^1(-1) + \frac{f^{(3)}(t)}{2!}(x-t)^2\right) \\ &+ \left(\frac{f^{(3)}(t)}{2!}(x-t)^2(-1) + \frac{f^{(4)}(t)}{3!}(x-t)^3\right) + \ldots + \\ &+ \left(\frac{f^{(N)}(t)}{(N-1)!}(x-t)^{N-1}(-1) + \frac{f^{(N+1)}(t)}{N!}(x-t)^N\right) \\ &+ B(N+1)(x-t)^N(-1). \end{aligned}$$

Now most of the terms in this expression cancel out, leaving just

$$F'(t) = \frac{f^{(N+1)}(t)}{N!}(x-t)^N + B(N+1)(x-t)^N(-1).$$

At some z, $F'(z) = 0$ so

$$\begin{aligned} 0 &= \frac{f^{(N+1)}(z)}{N!}(x-z)^N + B(N+1)(x-z)^N(-1) \\ B(N+1)(x-z)^N &= \frac{f^{(N+1)}(z)}{N!}(x-z)^N \\ B &= \frac{f^{(N+1)}(z)}{(N+1)!}. \end{aligned}$$

Now we can write

$$F(t) = \sum_{n=0}^{N} \frac{f^{(n)}(t)}{n!}\,(x-t)^n + \frac{f^{(N+1)}(z)}{(N+1)!}(x-t)^{N+1}.$$

Recalling that $F(a) = f(x)$ we get

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}\,(x-a)^n + \frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1},$$

which is what we wanted to show. ∎

It may not be immediately obvious that this is particularly useful; let's look at some examples.

EXAMPLE 13.11.2 Find a polynomial approximation for $\sin x$ accurate to ± 0.005.

From Taylor's theorem:

$$\sin x = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1}.$$

What can we say about the size of the term

$$\frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1}?$$

Every derivative of $\sin x$ is $\pm\sin x$ or $\pm\cos x$, so $|f^{(N+1)}(z)| \le 1$. The factor $(x-a)^{N+1}$ is a bit more difficult, since $x-a$ could be quite large. Let's pick $a=0$ and $|x| \le \pi/2$; if we can compute $\sin x$ for $x \in [-\pi/2, \pi/2]$, we can of course compute $\sin x$ for all x.

We need to pick N so that

$$\left|\frac{x^{N+1}}{(N+1)!}\right| < 0.005.$$

Since we have limited x to $[-\pi/2, \pi/2]$,

$$\left|\frac{x^{N+1}}{(N+1)!}\right| < \frac{2^{N+1}}{(N+1)!}.$$

The quantity on the right decreases with increasing N, so all we need to do is find an N so that

$$\frac{2^{N+1}}{(N+1)!} < 0.005.$$

A little trial and error shows that $N=8$ works, and in fact $2^9/9! < 0.0015$, so

$$\begin{aligned}\sin x &= \sum_{n=0}^{8} \frac{f^{(n)}(0)}{n!}x^n \pm 0.0015 \\ &= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \pm 0.0015.\end{aligned}$$

Figure 13.11.1 shows the graphs of $\sin x$ and and the approximation on $[0, 3\pi/2]$. As x gets larger, the approximation heads to negative infinity very quickly, since it is essentially acting like $-x^7$. □

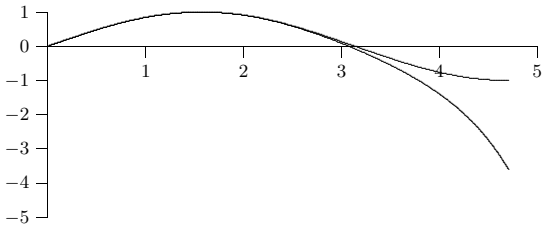

Figure 13.11.1 $\sin x$ and a polynomial approximation. (AP)

We can extract a bit more information from this example. If we do not limit the value of x, we still have

$$\left|\frac{f^{(N+1)}(z)}{(N+1)!}x^{N+1}\right| \le \left|\frac{x^{N+1}}{(N+1)!}\right|$$

so that $\sin x$ is represented by

$$\sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}x^n \pm \left|\frac{x^{N+1}}{(N+1)!}\right|.$$

If we can show that

$$\lim_{N\to\infty}\left|\frac{x^{N+1}}{(N+1)!}\right| = 0$$

for each x then

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!},$$

that is, the sine function is actually equal to its Maclaurin series for all x. How can we prove that the limit is zero? Suppose that N is larger than $|x|$, and let M be the largest integer less than $|x|$ (if $M=0$ the following is even easier). Then

$$\begin{aligned}\frac{|x^{N+1}|}{(N+1)!} &= \frac{|x|}{N+1}\frac{|x|}{N}\frac{|x|}{N-1}\cdots\frac{|x|}{M+1}\frac{|x|}{M}\frac{|x|}{M-1}\cdots\frac{|x|}{2}\frac{|x|}{1} \\ &\le \frac{|x|}{N+1}\cdot 1\cdot 1\cdots 1\cdot\frac{|x|}{M}\frac{|x|}{M-1}\cdots\frac{|x|}{2}\frac{|x|}{1} \\ &= \frac{|x|}{N+1}\frac{|x|^M}{M!}.\end{aligned}$$

The quantity $|x|^M/M!$ is a constant, so

$$\lim_{N\to\infty}\frac{|x|}{N+1}\frac{|x|^M}{M!} = 0$$

and by the Squeeze Theorem (13.1.3)

$$\lim_{N\to\infty}\left|\frac{x^{N+1}}{(N+1)!}\right| = 0$$

as desired. Essentially the same argument works for $\cos x$ and e^x; unfortunately, it is more difficult to show that most functions are equal to their Maclaurin series.

EXAMPLE 13.11.3 Find a polynomial approximation for e^x near $x = 2$ accurate to ± 0.005.

From Taylor's theorem:

$$e^x = \sum_{n=0}^{N}\frac{e^2}{n!}(x-2)^n + \frac{e^z}{(N+1)!}(x-2)^{N+1},$$

since $f^{(n)}(x) = e^x$ for all n. We are interested in x near 2, and we need to keep $|(x-2)^{N+1}|$ in check, so we may as well specify that $|x-2| \le 1$, so $x \in [1,3]$. Also

$$\left|\frac{e^z}{(N+1)!}\right| \le \frac{e^3}{(N+1)!},$$

so we need to find an N that makes $e^3/(N+1)! \le 0.005$. This time $N = 5$ makes $e^3/(N+1)! < 0.0015$, so the approximating polynomial is

$$e^x = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3 + \frac{e^2}{24}(x-2)^4 + \frac{e^2}{120}(x-2)^5 \pm 0.0015.$$

This presents an additional problem for approximation, since we also need to approximate e^2, and any approximation we use will increase the error, but we will not pursue this complication. □

Note well that in these examples we found polynomials of a certain accuracy only on a small interval, even though the series for $\sin x$ and e^x converge for all x; this is typical. To get the same accuracy on a larger interval would require more terms.

Exercises 13.11.

1. Find a polynomial approximation for $\cos x$ on $[0,\pi]$, accurate to $\pm 10^{-3}$ ⇒
2. How many terms of the series for $\ln x$ centered at 1 are required so that the guaranteed error on $[1/2, 3/2]$ is at most 10^{-3}? What if the interval is instead $[1, 3/2]$? ⇒
3. Find the first three nonzero terms in the Taylor series for $\tan x$ on $[-\pi/4, \pi/4]$, and compute the guaranteed error term as given by Taylor's theorem. (You may want to use Sage or a similar aid.) ⇒

4. Show that $\cos x$ is equal to its Taylor series for all x by showing that the limit of the error term is zero as N approaches infinity.
5. Show that e^x is equal to its Taylor series for all x by showing that the limit of the error term is zero as N approaches infinity.

13.12 Additional exercises

These problems require the techniques of this chapter, and are in no particular order. Some problems may be done in more than one way.

Determine whether the series converges.

1. $\displaystyle\sum_{n=0}^{\infty}\frac{n}{n^2+4}$ ⇒
2. $\displaystyle\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\frac{1}{7\cdot 8}+\cdots$ ⇒
3. $\displaystyle\sum_{n=0}^{\infty}\frac{n}{(n^2+4)^2}$ ⇒
4. $\displaystyle\sum_{n=0}^{\infty}\frac{n!}{8^n}$ ⇒
5. $\displaystyle 1-\frac{3}{4}+\frac{5}{8}-\frac{7}{12}+\frac{9}{16}+\cdots$ ⇒
6. $\displaystyle\sum_{n=0}^{\infty}\frac{1}{\sqrt{n^2+4}}$ ⇒
7. $\displaystyle\sum_{n=0}^{\infty}\frac{\sin^3(n)}{n^2}$ ⇒
8. $\displaystyle\sum_{n=0}^{\infty}\frac{n}{e^n}$ ⇒
9. $\displaystyle\sum_{n=0}^{\infty}\frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)}$ ⇒
10. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$ ⇒
11. $\displaystyle\frac{1}{2\cdot 3\cdot 4}+\frac{2}{3\cdot 4\cdot 5}+\frac{3}{4\cdot 5\cdot 6}+\frac{4}{5\cdot 6\cdot 7}+\cdots$ ⇒
12. $\displaystyle\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2n)!}$ ⇒
13. $\displaystyle\sum_{n=0}^{\infty}\frac{6^n}{n!}$ ⇒
14. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ ⇒

15. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n 3^{n-1}}{n!} \Rightarrow$

16. $\displaystyle 1 + \frac{5^2}{2^2} + \frac{5^4}{(2\cdot 4)^2} + \frac{5^6}{(2\cdot 4\cdot 6)^2} + \frac{5^8}{(2\cdot 4\cdot 6\cdot 8)^2} + \cdots \Rightarrow$

17. $\displaystyle\sum_{n=1}^{\infty} \sin(1/n) \Rightarrow$

Find the interval and radius of convergence; you need not check the endpoints of the intervals.

18. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n \Rightarrow$

19. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{1+3^n} \Rightarrow$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n3^n} \Rightarrow$

21. $\displaystyle x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots \Rightarrow$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^2} x^n \Rightarrow$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 3^n} x^{2n} \Rightarrow$

24. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!} \Rightarrow$

Find a series for each function, using the formula for Maclaurin series and algebraic manipulation as appropriate.

25. $2^x \Rightarrow$

26. $\ln(1+x) \Rightarrow$

27. $\displaystyle \ln\left(\frac{1+x}{1-x}\right) \Rightarrow$

28. $\sqrt{1+x} \Rightarrow$

29. $\displaystyle \frac{1}{1+x^2} \Rightarrow$

30. $\arctan(x) \Rightarrow$

31. Use the answer to the previous problem to discover a series for a well-known mathematical constant. $\Rightarrow$