19

Differential Equations

Many physical phenomena can be modeled using the language of calculus. For example, observational evidence suggests that the temperature of a cup of tea (or some other liquid) in a room of constant temperature will cool over time at a rate proportional to the difference between the room temperature and the temperature of the tea.

In symbols, if t is the time, M is the room temperature, and $f(t)$ is the temperature of the tea at time t then $f'(t) = k(M - f(t))$ where $k > 0$ is a constant which will depend on the kind of tea (or more generally the kind of liquid) but not on the room temperature or the temperature of the tea. This is **Newton's law of cooling** and the equation that we just wrote down is an example of a **differential equation**. Ideally we would like to solve this equation, namely, find the function $f(t)$ that describes the temperature over time, though this often turns out to be impossible, in which case various approximation techniques must be used. The use and solution of differential equations is an important field of mathematics; here we see how to solve some simple but useful types of differential equation.

Informally, a differential equation is an equation in which one or more of the derivatives of some function appear. Typically, a scientific theory will produce a differential equation (or a system of differential equations) that describes or governs some physical process, but the theory will not produce the desired function or functions directly.

Recall from section 6.2 that when the variable is time the derivative of a function $y(t)$ is sometimes written as $\dot{y}$ instead of y'; this is quite common in the study of differential equations. Extending this idea, we also write $y''(t) = \ddot{y}$.

19.1 First Order Differential Equations

We start by considering equations in which only the first derivative of the function appears.

DEFINITION 19.1.1 A **first order differential equation** is an equation of the form $F(t, y, \dot{y}) = 0$. A solution of a first order differential equation is a function $f(t)$ that makes $F(t, f(t), f'(t)) = 0$ for every value of t. □

Here, F is a function of three variables which we label t, y, and $\dot{y}$. It is understood that $\dot{y}$ will explicitly appear in the equation although t and y need not. The term "first order" means that the first derivative of y appears, but no higher order derivatives do.

EXAMPLE 19.1.2 The equation from Newton's law of cooling, $\dot{y} = k(M - y)$, is a first order differential equation; $F(t, y, \dot{y}) = k(M - y) - \dot{y}$. □

EXAMPLE 19.1.3 $\dot{y} = t^2 + 1$ is a first order differential equation; $F(t, y, \dot{y}) = \dot{y} - t^2 - 1$. All solutions to this equation are of the form $t^3/3 + t + C$. □

DEFINITION 19.1.4 A **first order initial value problem** is a system of equations of the form $F(t, y, \dot{y}) = 0$, $y(t_0) = y_0$. Here t_0 is a fixed time and y_0 is a number. A solution of an initial value problem is a solution $f(t)$ of the differential equation that also satisfies the **initial condition** $f(t_0) = y_0$. □

EXAMPLE 19.1.5 The initial value problem $\dot{y} = t^2 + 1$, $y(1) = 4$ has solution $f(t) = t^3/3 + t + 8/3$. □

The general first order equation is rather too general, that is, we can't describe methods that will work on them all, or even a large portion of them. We can make progress with specific kinds of first order differential equations. For example, much can be said about equations of the form $\dot{y} = \phi(t, y)$ where ϕ is a function of the two variables t and y. Under reasonable conditions on ϕ, such an equation has a solution and the corresponding initial value problem has a unique solution. However, in general, these equations can be very difficult or impossible to solve explicitly.

EXAMPLE 19.1.6 Consider this specific example of an initial value problem for Newton's law of cooling: $\dot{y} = 2(25 - y)$, $y(0) = 40$. We first note that if $y(t_0) = 25$, the right hand side of the differential equation is zero, and so the constant function $y(t) = 25$ is a solution to the differential equation. It is not a solution to the initial value problem, since $y(0) \neq 40$. (The physical interpretation of this constant solution is that if a liquid is at the same temperature as its surroundings, then the liquid will stay at that temperature.)

So long as y is not 25, we can rewrite the differential equation as

$$\begin{aligned} \frac{dy}{dt}\frac{1}{25-y} &= 2 \\ \frac{1}{25-y}\,dy &= 2\,dt, \end{aligned}$$

so

$$\int \frac{1}{25-y}\,dy = \int 2\,dt,$$

that is, the two anti-derivatives must be the same except for a constant difference. We can calculate these anti-derivatives and rearrange the results:

$$\begin{aligned} \int \frac{1}{25-y}\,dy &= \int 2\,dt \\ (-1)\ln|25-y| &= 2t + C_0 \\ \ln|25-y| &= -2t - C_0 = -2t + C \\ |25-y| &= e^{-2t+C} = e^{-2t}e^C \\ y - 25 &= \pm e^C e^{-2t} \\ y &= 25 \pm e^C e^{-2t} = 25 + Ae^{-2t}. \end{aligned}$$

Here $A = \pm e^C = \pm e^{-C_0}$ is some non-zero constant. Since we want $y(0) = 40$, we substitute and solve for A:

$$\begin{aligned} 40 &= 25 + Ae^0 \\ 15 &= A, \end{aligned}$$

and so $y = 25 + 15e^{-2t}$ is a solution to the initial value problem. Note that y is never 25, so this makes sense for all values of t. However, if we allow $A = 0$ we get the solution $y = 25$ to the differential equation, which would be the solution to the initial value problem if we were to require $y(0) = 25$. Thus, $y = 25 + Ae^{-2t}$ describes all solutions to the differential equation $\dot{y} = 2(25 - y)$, and all solutions to the associated initial value problems. □

Why could we solve this problem? Our solution depended on rewriting the equation so that all instances of y were on one side of the equation and all instances of t were on the other; of course, in this case the only t was originally hidden, since we didn't write dy/dt in the original equation. This is not required, however.

EXAMPLE 19.1.7 Solve the differential equation $\dot{y} = 2t(25 - y)$. This is almost identical to the previous example. As before, $y(t) = 25$ is a solution. If $y \neq 25$,

$$\begin{aligned} \int \frac{1}{25-y}\,dy &= \int 2t\,dt \\ (-1)\ln|25-y| &= t^2 + C_0 \\ \ln|25-y| &= -t^2 - C_0 = -t^2 + C \\ |25-y| &= e^{-t^2+C} = e^{-t^2}e^C \\ y - 25 &= \pm e^C e^{-t^2} \\ y &= 25 \pm e^C e^{-t^2} = 25 + Ae^{-t^2}. \end{aligned}$$

As before, all solutions are represented by $y = 25 + Ae^{-t^2}$, allowing A to be zero. □

DEFINITION 19.1.8 A first order differential equation is **separable** if it can be written in the form $\dot{y} = f(t)g(y)$. □

As in the examples, we can attempt to solve a separable equation by converting to the form

$$\int \frac{1}{g(y)}\,dy = \int f(t)\,dt.$$

This technique is called **separation of variables**. The simplest (in principle) sort of separable equation is one in which $g(y) = 1$, in which case we attempt to solve

$$\int 1\,dy = \int f(t)\,dt.$$

We can do this if we can find an anti-derivative of $f(t)$.

Also as we have seen so far, a differential equation typically has an infinite number of solutions. Ideally, but certainly not always, a corresponding initial value problem will have just one solution. A solution in which there are no unknown constants remaining is called a **particular solution**.

The general approach to separable equations is this: Suppose we wish to solve $\dot{y} = f(t)g(y)$ where f and g are continuous functions. If $g(a) = 0$ for some a then $y(t) = a$ is a constant solution of the equation, since in this case $\dot{y} = 0 = f(t)g(a)$. For example, $\dot{y} = y^2 - 1$ has constant solutions $y(t) = 1$ and $y(t) = -1$.

To find the nonconstant solutions, we note that the function $1/g(y)$ is continuous where $g \neq 0$, so $1/g$ has an antiderivative G. Let F be an antiderivative of f. Now we write

$$G(y) = \int \frac{1}{g(y)}\,dy = \int f(t)\,dt = F(t) + C,$$

so $G(y) = F(t) + C$. Now we solve this equation for y.

Of course, there are a few places this ideal description could go wrong: we need to be able to find the antiderivatives G and F, and we need to solve the final equation for y. The upshot is that the solutions to the original differential equation are the constant solutions, if any, and all functions y that satisfy $G(y) = F(t) + C$.

EXAMPLE 19.1.9 Consider the differential equation $\dot{y} = ky$. When $k > 0$, this describes certain simple cases of population growth: it says that the change in the population y is proportional to the population. The underlying assumption is that each organism in the current population reproduces at a fixed rate, so the larger the population the more new organisms are produced. While this is too simple to model most real populations, it is useful in some cases over a limited time. When $k < 0$, the differential equation describes a quantity that decreases in proportion to the current value; this can be used to model radioactive decay.

The constant solution is $y(t) = 0$; of course this will not be the solution to any interesting initial value problem. For the non-constant solutions, we proceed much as before:

$$\begin{aligned}\int \frac{1}{y}\,dy &= \int k\,dt \\ \ln|y| &= kt + C \\ |y| &= e^{kt}e^{C} \\ y &= \pm\, e^{C}e^{kt} \\ y &= Ae^{kt}.\end{aligned}$$

Again, if we allow $A = 0$ this includes the constant solution, and we can simply say that $y = Ae^{kt}$ is the general solution. With an initial value we can easily solve for A to get the solution of the initial value problem. In particular, if the initial value is given for time $t = 0$, $y(0) = y_0$, then $A = y_0$ and the solution is $y = y_0e^{kt}$. □

Exercises 19.1.

1. Which of the following equations are separable?
   - **a.** $\dot{y} = \sin(ty)$
   - **b.** $\dot{y} = e^te^y$
   - **c.** $y\dot{y} = t$
   - **d.** $\dot{y} = (t^3 - t)\arcsin(y)$
   - **e.** $\dot{y} = t^2\ln y + 4t^3\ln y$
2. Solve $\dot{y} = 1/(1 + t^2)$. ⇒
3. Solve the initial value problem $\dot{y} = t^n$ with $y(0) = 1$ and $n \ge 0$. ⇒
4. Solve $\dot{y} = \ln t$. ⇒

5. Identify the constant solutions (if any) of $\dot{y} = t\sin y$. ⇒
6. Identify the constant solutions (if any) of $\dot{y} = te^y$. ⇒
7. Solve $\dot{y} = t/y$. ⇒
8. Solve $\dot{y} = y^2 - 1$. ⇒
9. Solve $\dot{y} = t/(y^3 - 5)$. You may leave your solution in implicit form: that is, you may stop once you have done the integration, without solving for y. ⇒
10. Find a non-constant solution of the initial value problem $\dot{y} = y^{1/3}$, $y(0) = 0$, using separation of variables. Note that the constant function $y(t) = 0$ also solves the initial value problem. This shows that an initial value problem can have more than one solution. ⇒
11. Solve the equation for Newton's law of cooling leaving M and k unknown. ⇒
12. After 10 minutes in Jean-Luc's room, his tea has cooled to 40° Celsius from 100° Celsius. The room temperature is 25° Celsius. How much longer will it take to cool to 35°? ⇒
13. Solve the **logistic equation** $\dot{y} = ky(M-y)$. (This is a somewhat more reasonable population model in most cases than the simpler $\dot{y} = ky$.) Sketch the graph of the solution to this equation when $M = 1000$, $k = 0.002$, $y(0) = 1$. ⇒
14. Suppose that $\dot{y} = ky$, $y(0) = 2$, and $\dot{y}(0) = 3$. What is y? ⇒
15. Radioactive substances "decay", that is, the amount (the mass) of the substance decreases over time. The mass of a radioactive substance obeys the equation $\dot{y} = ky$ where $k < 0$ and y is the mass of the substance at time t. Suppose that initially, the mass of the substance is $y(0) = M > 0$. At what time does half of the mass remain? (This is known as the half life. Note that the half life depends on k but not on M.) ⇒
16. Bismuth-210 has a half life of five days. If there is initially 600 milligrams, how much is left after 6 days? When will there be only 2 milligrams left? ⇒
17. The half life of carbon-14 is 5730 years. If one starts with 100 milligrams of carbon-14, how much is left after 6000 years? How long do we have to wait before there is less than 2 milligrams? ⇒
18. A certain species of bacteria doubles its population (or its mass) every hour in the lab. The differential equation that models this phenomenon is $\dot{y} = ky$, where $k > 0$ and y is the population of bacteria at time t. What is y? ⇒
19. If a certain microbe doubles its population every 4 hours and after 5 hours the total population has mass 500 grams, what was the initial mass? ⇒

19.2 First Order Homogeneous Linear Equations

A simple, but important and useful, type of separable equation is the **first order homogeneous linear equation**:

DEFINITION 19.2.1 A first order homogeneous linear differential equation is one of the form $\dot{y} + p(t)y = 0$ or equivalently $\dot{y} = -p(t)y$. □

"Linear" in this definition indicates that both $\dot{y}$ and y occur to the first power; "homogeneous" refers to the zero on the right hand side of the first form of the equation.

EXAMPLE 19.2.2 The equation $\dot{y} = 2t(25 - y)$ can be written $\dot{y} + 2ty = 50t$. This is linear, but not homogeneous. The equation $\dot{y} = ky$, or $\dot{y} - ky = 0$ is linear and homogeneous, with a particularly simple $p(t) = -k$. □

Because first order homogeneous linear equations are separable, we can solve them in the usual way:

$$\begin{aligned} \dot{y} &= -p(t)y \\ \int \frac{1}{y}\,dy &= \int -p(t)\,dt \\ \ln|y| &= P(t) + C \\ y &= \pm\, e^{P(t)+C} \\ y &= Ae^{P(t)}, \end{aligned}$$

where $P(t)$ is an anti-derivative of $-p(t)$. As in previous examples, if we allow $A = 0$ we get the constant solution $y = 0$.

EXAMPLE 19.2.3 Solve the initial value problems $\dot{y} + y\cos t = 0$, $y(0) = 1/2$ and $y(2) = 1/2$. We start with

$$P(t) = \int -\cos t\,dt = -\sin t,$$

so the general solution to the differential equation is

$$y = Ae^{-\sin t}.$$

To compute A we substitute:

$$\frac{1}{2} = Ae^{-\sin 0} = A,$$

so the solutions is

$$y = \frac{1}{2}e^{-\sin t}.$$

For the second problem,

$$\begin{aligned} \frac{1}{2} &= Ae^{-\sin 2} \\ A &= \frac{1}{2}e^{\sin 2} \end{aligned}$$

so the solution is

$$y = \frac{1}{2}e^{\sin 2}e^{-\sin t}.$$

□

EXAMPLE 19.2.4 Solve the initial value problem $t\dot{y} + 3y = 0$, $y(1) = 2$, assuming $t > 0$. We write the equation in standard form: $\dot{y} + 3y/t = 0$. Then

$$P(t) = \int -\frac{3}{t}\,dt = -3\ln t$$

and

$$y = Ae^{-3\ln t} = At^{-3}.$$

Substituting to find A: $2 = A(1)^{-3} = A$, so the solution is $y = 2t^{-3}$. □

Exercises 19.2.

Find the general solution of each equation in 1–4.

1. $\dot{y} + 5y = 0 \Rightarrow$

2. $\dot{y} - 2y = 0 \Rightarrow$

3. $\dot{y} + \dfrac{y}{1+t^2} = 0 \Rightarrow$

4. $\dot{y} + t^2 y = 0 \Rightarrow$

In 5–14, solve the initial value problem.

5. $\dot{y} + y = 0$, $y(0) = 4 \Rightarrow$

6. $\dot{y} - 3y = 0$, $y(1) = -2 \Rightarrow$

7. $\dot{y} + y\sin t = 0$, $y(\pi) = 1 \Rightarrow$

8. $\dot{y} + ye^t = 0$, $y(0) = e \Rightarrow$

9. $\dot{y} + y\sqrt{1+t^4} = 0$, $y(0) = 0 \Rightarrow$

10. $\dot{y} + y\cos(e^t) = 0$, $y(0) = 0 \Rightarrow$

11. $t\dot{y} - 2y = 0$, $y(1) = 4 \Rightarrow$

12. $t^2\dot{y} + y = 0$, $y(1) = -2$, $t > 0 \Rightarrow$

13. $t^3\dot{y} = 2y$, $y(1) = 1$, $t > 0 \Rightarrow$

14. $t^3\dot{y} = 2y$, $y(1) = 0$, $t > 0 \Rightarrow$

15. A function $y(t)$ is a solution of $\dot{y} + ky = 0$. Suppose that $y(0) = 100$ and $y(2) = 4$. Find k and find $y(t)$. $\Rightarrow$

16. A function $y(t)$ is a solution of $\dot{y} + t^k y = 0$. Suppose that $y(0) = 1$ and $y(1) = e^{-13}$. Find k and find $y(t)$. $\Rightarrow$

17. A bacterial culture grows at a rate proportional to its population. If the population is one million at $t = 0$ and 1.5 million at $t = 1$ hour, find the population as a function of time. $\Rightarrow$

18. A radioactive element decays with a half-life of 6 years. If a block of the element has mass 10 kilograms at $t = 0$, find the amount of the element at time t. $\Rightarrow$

19.3 First Order Linear Equations

As you might guess, a first order linear differential equation has the form $\dot{y}+p(t)y=f(t)$. Not only is this closely related in form to the first order homogeneous linear equation, we can use what we know about solving homogeneous equations to solve the general linear equation.

Suppose that $y_1(t)$ and $y_2(t)$ are solutions to $\dot{y}+p(t)y=f(t)$. Let $g(t)=y_1-y_2$. Then

$$\begin{aligned}
g'(t)+p(t)g(t) &= y_1'-y_2'+p(t)(y_1-y_2)\\
&= (y_1'+p(t)y_1)-(y_2'+p(t)y_2)\\
&= f(t)-f(t)=0.
\end{aligned}$$

In other words, $g(t)=y_1-y_2$ is a solution to the homogeneous equation $\dot{y}+p(t)y=0$. Turning this around, any solution to the linear equation $\dot{y}+p(t)y=f(t)$, call it y_1, can be written as $y_2+g(t)$, for some particular y_2 and some solution $g(t)$ of the homogeneous equation $\dot{y}+p(t)y=0$. Since we already know how to find all solutions of the homogeneous equation, finding just one solution to the equation $\dot{y}+p(t)y=f(t)$ will give us all of them.

How might we find that one particular solution to $\dot{y}+p(t)y=f(t)$? Again, it turns out that what we already know helps. We know that the general solution to the homogeneous equation $\dot{y}+p(t)y=0$ looks like $Ae^{P(t)}$. We now make an inspired guess: consider the function $v(t)e^{P(t)}$, in which we have replaced the constant parameter A with the function $v(t)$. This technique is called **variation of parameters**. For convenience write this as $s(t)=v(t)h(t)$ where $h(t)=e^{P(t)}$ is a solution to the homogeneous equation. Now let's compute a bit with $s(t)$:

$$\begin{aligned}
s'(t)+p(t)s(t) &= v(t)h'(t)+v'(t)h(t)+p(t)v(t)h(t)\\
&= v(t)(h'(t)+p(t)h(t))+v'(t)h(t)\\
&= v'(t)h(t).
\end{aligned}$$

The last equality is true because $h'(t)+p(t)h(t)=0$, since $h(t)$ is a solution to the homogeneous equation. We are hoping to find a function $s(t)$ so that $s'(t)+p(t)s(t)=f(t)$; we will have such a function if we can arrange to have $v'(t)h(t)=f(t)$, that is, $v'(t)=f(t)/h(t)$. But this is as easy (or hard) as finding an anti-derivative of $f(t)/h(t)$. Putting this all together, the general solution to $\dot{y}+p(t)y=f(t)$ is

$$v(t)h(t)+Ae^{P(t)}=v(t)e^{P(t)}+Ae^{P(t)}.$$

EXAMPLE 19.3.1 Find the solution of the initial value problem $\dot{y}+3y/t=t^2$, $y(1)=1/2$. First we find the general solution; since we are interested in a solution with a given condition at $t=1$, we may assume $t>0$. We start by solving the homogeneous equation as usual; call the solution g:

$$g=Ae^{-\int(3/t)\,dt}=Ae^{-3\ln t}=At^{-3}.$$

Then as in the discussion, $h(t)=t^{-3}$ and $v'(t)=t^2/t^{-3}=t^5$, so $v(t)=t^6/6$. We know that every solution to the equation looks like

$$v(t)t^{-3}+At^{-3}=\frac{t^6}{6}t^{-3}+At^{-3}=\frac{t^3}{6}+At^{-3}.$$

Finally we substitute to find A:

$$\begin{aligned}
\frac{1}{2} &= \frac{(1)^3}{6}+A(1)^{-3}=\frac{1}{6}+A\\
A &= \frac{1}{2}-\frac{1}{6}=\frac{1}{3}.
\end{aligned}$$

The solution is then

$$y=\frac{t^3}{6}+\frac{1}{3}t^{-3}.$$

□

Here is an alternate method for finding a particular solution to the differential equation, using an **integrating factor**. In the differential equation $\dot{y}+p(t)y=f(t)$, we note that if we multiply through by a function $I(t)$ to get $I(t)\dot{y}+I(t)p(t)y=I(t)f(t)$, the left hand side looks like it could be a derivative computed by the product rule:

$$\frac{d}{dt}(I(t)y)=I(t)\dot{y}+I'(t)y.$$

Now if we could choose $I(t)$ so that $I'(t)=I(t)p(t)$, this would be exactly the left hand side of the differential equation. But this is just a first order homogeneous linear equation, and we know a solution is $I(t)=e^{Q(t)}$, where $Q(t)=\int p\,dt$; note that $Q(t)=-P(t)$, where $P(t)$ appears in the variation of parameters method and $P'(t)=-p$. Now the modified differential equation is

$$\begin{aligned}
e^{-P(t)}\dot{y}+e^{-P(t)}p(t)y &= e^{-P(t)}f(t)\\
\frac{d}{dt}(e^{-P(t)}y) &= e^{-P(t)}f(t).
\end{aligned}$$

Integrating both sides gives

$$e^{-P(t)}y = \int e^{-P(t)} f(t)\,dt$$
$$y = e^{P(t)} \int e^{-P(t)} f(t)\,dt.$$

If you look carefully, you will see that this is exactly the same solution we found by variation of parameters, because $e^{-P(t)}f(t) = f(t)/h(t)$.

Some people find it easier to remember how to use the integrating factor method than variation of parameters. Since ultimately they require the same calculation, you should use whichever of the two you find easier to recall. Using this method, the solution of the previous example would look just a bit different: Starting with $\dot{y}+3y/t = t^2$, we recall that the integrating factor is $e^{\int 3/t} = e^{3\ln t} = t^3$. Then we multiply through by the integrating factor and solve:

$$\begin{aligned} t^3\dot{y} + t^3 3y/t &= t^3t^2 \\ t^3\dot{y} + t^2 3y &= t^5 \\ \frac{d}{dt}(t^3y) &= t^5 \\ t^3y &= t^6/6 \\ y &= t^3/6. \end{aligned}$$

This is the same answer, of course, and the problem is then finished just as before.

Exercises 19.3.

In problems 1–10, find the general solution of the equation.

1. $\dot{y} + 4y = 8 \Rightarrow$
2. $\dot{y} - 2y = 6 \Rightarrow$
3. $\dot{y} + ty = 5t \Rightarrow$
4. $\dot{y} + e^t y = -2e^t \Rightarrow$
5. $\dot{y} - y = t^2 \Rightarrow$
6. $2\dot{y} + y = t \Rightarrow$
7. $t\dot{y} - 2y = 1/t,\ t > 0 \Rightarrow$
8. $t\dot{y} + y = \sqrt{t},\ t > 0 \Rightarrow$
9. $\dot{y}\cos t + y \sin t = 1,\ -\pi/2 < t < \pi/2 \Rightarrow$
10. $\dot{y} + y\sec t = \tan t,\ -\pi/2 < t < \pi/2 \Rightarrow$

19.4 Approximation

We have seen how to solve a restricted collection of differential equations, or more accurately, how to attempt to solve them—we may not be able to find the required antiderivatives. Not surprisingly, non-linear equations can be even more difficult to solve. Yet much is known about solutions to some more general equations.

Suppose $\phi(t,y)$ is a function of two variables. A more general class of first order differential equations has the form $\dot{y} = \phi(t,y)$. This is not necessarily a linear first order equation, since ϕ may depend on y in some complicated way; note however that $\dot{y}$ appears in a very simple form. Under suitable conditions on the function ϕ, it can be shown that every such differential equation has a solution, and moreover that for each initial condition the associated initial value problem has exactly one solution. In practical applications this is obviously a very desirable property.

EXAMPLE 19.4.1 The equation $\dot{y} = t - y^2$ is a first order non-linear equation, because y appears to the second power. We will not be able to solve this equation. □

EXAMPLE 19.4.2 The equation $\dot{y} = y^2$ is also non-linear, but it is separable and can be solved by separation of variables. □

Not all differential equations that are important in practice can be solved exactly, so techniques have been developed to approximate solutions. We describe one such technique, **Euler's Method**, which is simple though not particularly useful compared to some more sophisticated techniques.

Suppose we wish to approximate a solution to the initial value problem $\dot{y} = \phi(t,y)$, $y(t_0) = y_0$, for $t \ge t_0$. Under reasonable conditions on ϕ, we know the solution exists, represented by a curve in the t-y plane; call this solution $f(t)$. The point (t_0, y_0) is of course on this curve. We also know the slope of the curve at this point, namely $\phi(t_0,y_0)$. If we follow the tangent line for a brief distance, we arrive at a point that should be almost on the graph of $f(t)$, namely $(t_0 + \Delta t, y_0 + \phi(t_0,y_0)\Delta t)$; call this point (t_1,y_1). Now we pretend, in effect, that this point really is on the graph of $f(t)$, in which case we again know the slope of the curve through (t_1,y_1), namely $\phi(t_1,y_1)$. So we can compute a new point, $(t_2,y_2) = (t_1 + \Delta t, y_1 + \phi(t_1,y_1)\Delta t)$ that is a little farther along, still close to the graph of $f(t)$ but probably not quite so close as (t_1,y_1). We can continue in this way, doing a sequence of straightforward calculations, until we have an approximation (t_n,y_n) for whatever time t_n we need. At each step we do essentially the same calculation, namely

$$(t_{i+1}, y_{i+1}) = (t_i + \Delta t, y_i + \phi(t_i,y_i)\Delta t).$$

We expect that smaller time steps Δt will give better approximations, but of course it will require more work to compute to a specified time. It is possible to compute a guaranteed upper bound on how far off the approximation might be, that is, how far y_n is from $f(t_n)$. Suffice it to say that the bound is not particularly good and that there are other more complicated approximation techniques that do better.

EXAMPLE 19.4.3 Let us compute an approximation to the solution for $\dot y = t - y^2$, $y(0) = 0$, when $t = 1$. We will use $\Delta t = 0.2$, which is easy to do even by hand, though we should not expect the resulting approximation to be very good. We get

$$(t_1, y_1) = (0 + 0.2, 0 + (0 - 0^2)0.2) = (0.2, 0)$$
$$(t_2, y_2) = (0.2 + 0.2, 0 + (0.2 - 0^2)0.2) = (0.4, 0.04)$$
$$(t_3, y_3) = (0.6, 0.04 + (0.4 - 0.04^2)0.2) = (0.6, 0.11968)$$
$$(t_4, y_4) = (0.8, 0.11968 + (0.6 - 0.11968^2)0.2) = (0.8, 0.23681533952)$$
$$(t_5, y_5) = (1.0, 0.23681533952 + (0.6 - 0.23681533952^2)0.2) = (1.0, 0.385599038513605)$$

So $y(1) \approx 0.3856$. As it turns out, this is not accurate to even one decimal place. Figure 19.4.1 shows these points connected by line segments (the lower curve) compared to a solution obtained by a much better approximation technique. Note that the shape is approximately correct even though the end points are quite far apart.

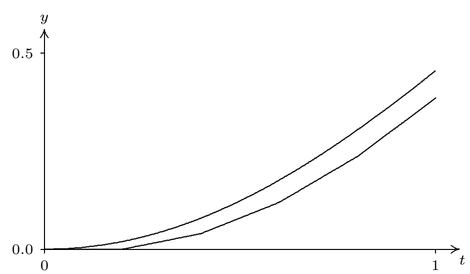

Figure 19.4.1 Approximating a solution to $\dot y = t - y^2$, $y(0) = 0$.

If you need to do Euler's method by hand, it is useful to construct a table to keep track of the work, as shown in figure 19.4.2. Each row holds the computation for a single step: the starting point (t_i, y_i); the stepsize Δt; the computed slope $\phi(t_i, y_i)$; the change in y, $\Delta y = \phi(t_i, y_i)\Delta t$; and the new point, $(t_{i+1}, y_{i+1}) = (t_i + \Delta t, y_i + \Delta y)$. The starting point in each row is the newly computed point from the end of the previous row.

It is easy to write a short function in Sage to do Euler's method; see this Sage worksheet. □

(t, y)	Δt	$\phi(t, y)$	$\Delta y = \phi(t, y)\Delta t$	$(t + \Delta t, y + \Delta y)$
$(0, 0)$	0.2	0	0	$(0.2, 0)$
$(0.2, 0)$	0.2	0.2	0.04	$(0.4, 0.04)$
$(0.4, 0.04)$	0.2	0.3984	0.07968	$(0.6, 0.11968)$
$(0.6, 0.11968)$	0.2	$0.58\ldots$	$0.117\ldots$	$(0.8, 0.236\ldots)$
$(0.8, 0.236\ldots)$	0.2	$0.743\ldots$	$0.148\ldots$	$(1.0, 0.385\ldots)$

Figure 19.4.2 Computing with Euler's Method.

Euler's method is related to another technique that can help in understanding a differential equation in a qualitative way. Euler's method is based on the ability to compute the slope of a solution curve at any point in the plane, simply by computing $\phi(t, y)$. If we compute $\phi(t, y)$ at many points, say in a grid, and plot a small line segment with that slope at the point, we can get an idea of how solution curves must look. Such a plot is called a **slope field**. A slope field for $\phi = t - y^2$ is shown in figure 19.4.3; compare this to figure 19.4.1. With a little practice, one can sketch reasonably accurate solution curves based on the slope field, in essence doing Euler's method visually.

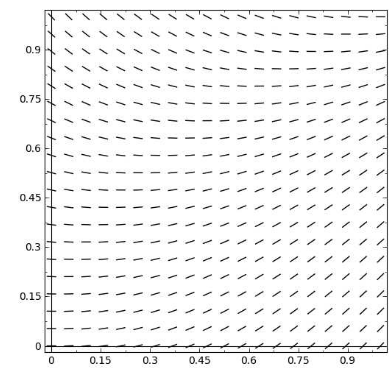

Figure 19.4.3 A slope field for $\dot y = t - y^2$.

Even when a differential equation can be solved explicitly, the slope field can help in understanding what the solutions look like with various initial conditions. Recall the logistic equation from exercise 13 in section 19.1, $\dot y = ky(M - y)$: y is a population at time t, M is a measure of how large a population the environment can support, and k measures the reproduction rate of the population. Figure 19.4.4 shows a slope field for this equation that is quite informative. It is apparent that if the initial population is smaller than M it rises to M over the long term, while if the initial population is greater than M it decreases to M. It is quite easy to generate slope fields with Sage; follow the AP link in the figure caption.

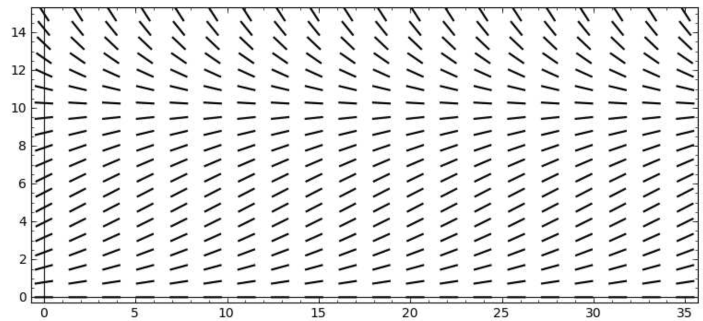

Figure 19.4.4 A slope field for $\dot{y} = 0.2y(10 - y)$. (AP)

Exercises 19.4.

In problems 1–4, compute the Euler approximations for the initial value problem for $0 \le t \le 1$ and $\Delta t = 0.2$. Using Sage, generate the slope field first and attempt to sketch the solution curve. Then use Sage to compute better approximations with smaller values of Δt.

1. $\dot{y} = t/y$, $y(0) = 1 \Rightarrow$

2. $\dot{y} = t + y^3$, $y(0) = 1 \Rightarrow$

3. $\dot{y} = \cos(t + y)$, $y(0) = 1 \Rightarrow$

4. $\dot{y} = t \ln y$, $y(0) = 2 \Rightarrow$

19.5 Second Order Homogeneous Equations

A second order differential equation is one containing the second derivative. These are in general quite complicated, but one fairly simple type is useful: the second order linear equation with constant coefficients.

EXAMPLE 19.5.1 Consider the initial value problem $\ddot{y} - \dot{y} - 2y = 0$, $y(0) = 5$, $\dot{y}(0) = 0$. We make an inspired guess: might there be a solution of the form e^{rt}? This seems at least plausible, since in this case $\ddot{y}$, $\dot{y}$, and y all involve e^{rt}.

If such a function is a solution then

$$\begin{aligned} r^2e^{rt} - re^{rt} - 2e^{rt} &= 0 \\ e^{rt}(r^2 - r - 2) &= 0 \\ (r^2 - r - 2) &= 0 \\ (r - 2)(r + 1) &= 0, \end{aligned}$$

so r is 2 or -1. Not only are $f = e^{2t}$ and $g = e^{-t}$ solutions, but notice that $y = Af + Bg$ is also, for any constants A and B:

$$\begin{aligned} (Af + Bg)'' - (Af + Bg)' - 2(Af + Bg) &= Af'' + Bg'' - Af' - Bg' - 2Af - 2Bg \\ &= A(f'' - f' - 2f) + B(g'' - g' - 2g) \\ &= A(0) + B(0) = 0. \end{aligned}$$

Can we find A and B so that this is a solution to the initial value problem? Let's substitute:

$$5 = y(0) = Af(0) + Bg(0) = Ae^0 + Be^0 = A + B$$

and

$$0 = \dot{y}(0) = Af'(0) + Bg'(0) = A2e^0 + B(-1)e^0 = 2A - B.$$

So we need to find A and B that make both $5 = A + B$ and $0 = 2A - B$ true. This is a simple set of simultaneous equations: solve $B = 2A$, substitute to get $5 = A + 2A = 3A$. Then $A = 5/3$ and $B = 10/3$, and the desired solution is $(5/3)e^{2t} + (10/3)e^{-t}$. You now see why the initial condition in this case included both $y(0)$ and $\dot{y}(0)$: we needed two equations in the two unknowns A and B □

You should of course wonder whether there might be other solutions; the answer is no. We will not prove this, but here is the theorem that tells us what we need to know:

THEOREM 19.5.2 Given the differential equation $a\ddot{y} + b\dot{y} + cy = 0$, where a, b, and c are constants, and $a \ne 0$, consider the quadratic polynomial $ax^2 + bx + c$, called the **characteristic polynomial**. Using the quadratic formula, this polynomial always has one or two roots, call them r and s. The general solution of the differential equation is:

(a) $y = Ae^{rt} + Be^{st}$, if the roots r and s are real numbers and $r \ne s$.

(b) $y = Ae^{rt} + Bte^{rt}$, if $r = s$ is real.

(c) $y = A\cos(\beta t)e^{\alpha t} + B\sin(\beta t)e^{\alpha t}$, if the roots r and s are complex numbers $\alpha + \beta i$ and $\alpha - \beta i$.

■

EXAMPLE 19.5.3 Suppose a mass m is hung on a spring with spring constant k. If the spring is compressed or stretched and then released, the mass will oscillate up and down. Because of friction, the oscillation will be damped: eventually the motion will cease. The damping will depend on the amount of friction; for example, if the system is suspended in oil the motion will cease sooner than if the system is in air. Using some simple physics, it is not hard to see that the position of the mass is described by this differential equation: $m\ddot{y} + b\dot{y} + ky = 0$. Using $m = 1$, $b = 4$, and $k = 5$ we find the motion of the mass. The characteristic polynomial is $x^2 + 4x + 5$ with roots $(-4 \pm \sqrt{16-20})/2 = -2 \pm i$. Thus the general solution is $y = A\cos(t)e^{-2t} + B\sin(t)e^{-2t}$. Suppose we know that $y(0) = 1$ and $\dot{y}(0) = 2$. Then as before we form two simultaneous equations: from $y(0) = 1$ we get $1 = A\cos(0)e^0 + B\sin(0)e^0 = A$. For the second we compute

$$\dot{y} = -2Ae^{-2t}\cos(t) + Ae^{-2t}(-\sin(t)) - 2Be^{-2t}\sin(t) + Be^{-2t}\cos(t),$$

and then

$$2 = -2Ae^0\cos(0) - Ae^0\sin(0) - 2Be^0\sin(0) + Be^0\cos(0) = -2A + B.$$

So we get $A = 1$, $B = 4$, and $y = \cos(t)e^{-2t} + 4\sin(t)e^{-2t}$.

Here is a useful trick that makes this easier to understand: We have $y = (\cos t + 4\sin t)e^{-2t}$. The expression $\cos t + 4\sin t$ is a bit reminiscent of the trigonometric formula $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ with $\alpha = t$. Let's rewrite it a bit as

$$\sqrt{17}\left(\frac{1}{\sqrt{17}}\cos t + \frac{4}{\sqrt{17}}\sin t\right).$$

Note that $(1/\sqrt{17})^2 + (4/\sqrt{17})^2 = 1$, which means that there is an angle β with $\cos\beta = 1/\sqrt{17}$ and $\sin\beta = 4/\sqrt{17}$ (of course, β may not be a "nice" angle). Then

$$\cos t + 4\sin t = \sqrt{17}(\cos t\cos\beta + \sin\beta\sin t) = \sqrt{17}\cos(t - \beta).$$

Thus, the solution may also be written $y = \sqrt{17}e^{-2t}\cos(t - \beta)$. This is a cosine curve that has been shifted β to the right; the $\sqrt{17}e^{-2t}$ has the effect of diminishing the amplitude of the cosine as t increases; see figure 19.5.1. The oscillation is damped very quickly, so in the first graph it is not clear that this is an oscillation. The second graph shows a restricted range for t. □

Other physical systems that oscillate can also be described by such differential equations. Some electric circuits, for example, generate oscillating current.

EXAMPLE 19.5.4 Find the solution to the initial value problem $\ddot{y} - 4\dot{y} + 4y = 0$, $y(0) = -3$, $\dot{y}(0) = 1$. The characteristic polynomial is $x^2 - 4x + 4 = (x-2)^2$, so there

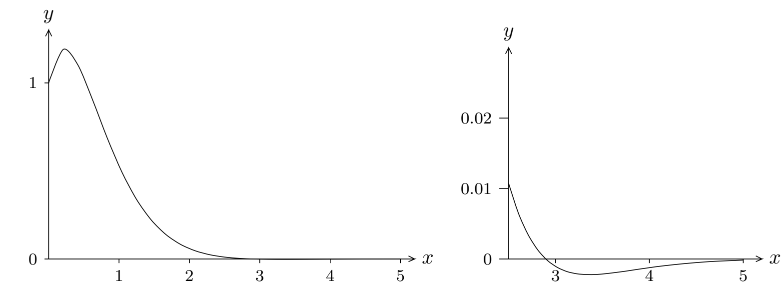

Figure 19.5.1 Graph of a damped oscillation.

is one root, $r = 2$, and the general solution is $Ae^{2t} + Bte^{2t}$. Substituting $t = 0$ we get $-3 = A + 0 = A$. The first derivative is $2Ae^{2t} + 2Bte^{2t} + Be^{2t}$; substituting $t = 0$ gives $1 = 2A + 0 + B = 2A + B = 2(-3) + B = -6 + B$, so $B = 7$. The solution is $-3e^{2t} + 7te^{2t}$. □

Exercises 19.5.

1. Verify that the function in part (a) of theorem 19.5.2 is a solution to the differential equation $a\ddot{y} + b\dot{y} + cy = 0$.

2. Verify that the function in part (b) of theorem 19.5.2 is a solution to the differential equation $a\ddot{y} + b\dot{y} + cy = 0$.

3. Verify that the function in part (c) of theorem 19.5.2 is a solution to the differential equation $a\ddot{y} + b\dot{y} + cy = 0$.

4. Solve the initial value problem $\ddot{y} - \omega^2 y = 0$, $y(0) = 1$, $\dot{y}(0) = 1$, assuming $\omega \neq 0$. ⇒

5. Solve the initial value problem $2\ddot{y} + 18y = 0$, $y(0) = 2$, $\dot{y}(0) = 15$. ⇒

6. Solve the initial value problem $\ddot{y} + 6\dot{y} + 5y = 0$, $y(0) = 1$, $\dot{y}(0) = 0$. ⇒

7. Solve the initial value problem $\ddot{y} - \dot{y} - 12y = 0$, $y(0) = 0$, $\dot{y}(0) = 14$. ⇒

8. Solve the initial value problem $\ddot{y} + 12\dot{y} + 36y = 0$, $y(0) = 5$, $\dot{y}(0) = -10$. ⇒

9. Solve the initial value problem $\ddot{y} - 8\dot{y} + 16y = 0$, $y(0) = -3$, $\dot{y}(0) = 4$. ⇒

10. Solve the initial value problem $\ddot{y} + 5y = 0$, $y(0) = -2$, $\dot{y}(0) = 5$. ⇒

11. Solve the initial value problem $\ddot{y} + y = 0$, $y(\pi/4) = 0$, $\dot{y}(\pi/4) = 2$. ⇒

12. Solve the initial value problem $\ddot{y} + 12\dot{y} + 37y = 0$, $y(0) = 4$, $\dot{y}(0) = 0$. ⇒

13. Solve the initial value problem $\ddot{y} + 6\dot{y} + 18y = 0$, $y(0) = 0$, $\dot{y}(0) = 6$. ⇒

14. Solve the initial value problem $\ddot{y} + 4y = 0$, $y(0) = \sqrt{3}$, $\dot{y}(0) = 2$. Put your answer in the form developed at the end of example 19.5.3. ⇒

15. Solve the initial value problem $\ddot{y} + 100y = 0$, $y(0) = 5$, $\dot{y}(0) = 50$. Put your answer in the form developed at the end of example 19.5.3. ⇒

16. Solve the initial value problem $\ddot{y}+4\dot{y}+13y=0$, $y(0)=1$, $\dot{y}(0)=1$. Put your answer in the form developed at the end of example 19.5.3. ⇒

17. Solve the initial value problem $\ddot{y}-8\dot{y}+25y=0$, $y(0)=3$, $\dot{y}(0)=0$. Put your answer in the form developed at the end of example 19.5.3. ⇒

18. A mass-spring system $m\ddot{y}+b\dot{y}+ky$ has $k=29$, $b=4$, and $m=1$. At time $t=0$ the position is $y(0)=2$ and the velocity is $\dot{y}(0)=1$. Find $y(t)$. ⇒

19. A mass-spring system $m\ddot{y}+b\dot{y}+ky$ has $k=24$, $b=12$, and $m=3$. At time $t=0$ the position is $y(0)=0$ and the velocity is $\dot{y}(0)=-1$. Find $y(t)$. ⇒

20. Consider the differential equation $a\ddot{y}+b\dot{y}=0$, with a and b both non-zero. Find the general solution by the method of this section. Now let $g=\dot{y}$; the equation may be written as $a\dot{g}+bg=0$, a first order linear homogeneous equation. Solve this for g, then use the relationship $g=\dot{y}$ to find y.

21. Suppose that $y(t)$ is a solution to $a\ddot{y}+b\dot{y}+cy=0$, $y(t_0)=0$, $\dot{y}(t_0)=0$. Show that $y(t)=0$.

19.6 Second Order Linear Equations

Now we consider second order equations of the form $a\ddot{y}+b\dot{y}+cy=f(t)$, with a, b, and c constant. Of course, if $a=0$ this is really a first order equation, so we assume $a\neq 0$. Also, much as in exercise 20 of section 19.5, if $c=0$ we can solve the related first order equation $a\dot{h}+bh=f(t)$, and then solve $h=\dot{y}$ for y. So we will only examine examples in which $c\neq 0$.

Suppose that $y_1(t)$ and $y_2(t)$ are solutions to $a\ddot{y}+b\dot{y}+cy=f(t)$, and consider the function $h=y_1-y_2$. We substitute this function into the left hand side of the differential equation and simplify:

$$a(y_1-y_2)''+b(y_1-y_2)'+c(y_1-y_2)=ay_1''+by_1'+cy_1-(ay_2''+by_2'+cy_2)=f(t)-f(t)=0.$$

So h is a solution to the homogeneous equation $a\ddot{y}+b\dot{y}+cy=0$. Since we know how to find all such h, then with just one particular solution y_2 we can express all possible solutions y_1, namely, $y_1=h+y_2$, where now h is the general solution to the homogeneous equation. Of course, this is exactly how we approached the first order linear equation.

To make use of this observation we need a method to find a single solution y_2. This turns out to be somewhat more difficult than the first order case, but if $f(t)$ is of a certain simple form, we can find a solution using the **method of undetermined coefficients**, sometimes more whimsically called the **method of judicious guessing**.

EXAMPLE 19.6.1 Solve the differential equation $\ddot{y}-\dot{y}-6y=18t^2+5$. The general solution of the homogeneous equation is $Ae^{3t}+Be^{-2t}$. We guess that a solution to the non-homogeneous equation might look like $f(t)$ itself, namely, a quadratic $y=at^2+bt+c$.

Substituting this guess into the differential equation we get

$$\ddot{y}-\dot{y}-6y=2a-(2at+b)-6(at^2+bt+c)=-6at^2+(-2a-6b)t+(2a-b-6c).$$

We want this to equal $18t^2+5$, so we need

$$\begin{aligned}-6a&=18\\-2a-6b&=0\\2a-b-6c&=5\end{aligned}$$

This is a system of three equations in three unknowns and is not hard to solve: $a=-3$, $b=1$, $c=-2$. Thus the general solution to the differential equation is $Ae^{3t}+Be^{-2t}-3t^2+t-2$. □

So the "judicious guess" is a function with the same form as $f(t)$ but with undetermined (or better, yet to be determined) coefficients. This works whenever $f(t)$ is a polynomial.

EXAMPLE 19.6.2 Consider the initial value problem $m\ddot{y}+ky=-mg$, $y(0)=2$, $\dot{y}(0)=50$. The left hand side represents a mass-spring system with no damping, i.e., $b=0$. Unlike the homogeneous case, we now consider the force due to gravity, $-mg$, assuming the spring is vertical at the surface of the earth, so that $g=980$. To be specific, let us take $m=1$ and $k=100$. The general solution to the homogeneous equation is $A\cos(10t)+B\sin(10t)$. For the solution to the non-homogeneous equation we guess simply a constant $y=a$, since $-mg=-980$ is a constant. Then $\ddot{y}+100y=100a$ so $a=-980/100=-9.8$. The desired general solution is then $A\cos(10t)+B\sin(10t)-9.8$. Substituting the initial conditions we get

$$\begin{aligned}2&=A-9.8\\50&=10B\end{aligned}$$

so $A=11.8$ and $B=5$ and the solution is $11.8\cos(10t)+5\sin(10t)-9.8$. □

More generally, this method can be used when a function similar to $f(t)$ has derivatives that are also similar to $f(t)$; in the examples so far, since $f(t)$ was a polynomial, so were its derivatives. The method will work if $f(t)$ has the form $p(t)e^{\alpha t}\cos(\beta t)+q(t)e^{\alpha t}\sin(\beta t)$, where $p(t)$ and $q(t)$ are polynomials; when $\alpha=\beta=0$ this is simply $p(t)$, a polynomial. In the most general form it is not simple to describe the appropriate judicious guess; we content ourselves with some examples to illustrate the process.

EXAMPLE 19.6.3 Find the general solution to $\ddot{y}+7\dot{y}+10y=e^{3t}$. The characteristic equation is $r^2+7r+10=(r+5)(r+2)$, so the solution to the homogeneous equation is

$Ae^{-5t} + Be^{-2t}$. For a particular solution to the inhomogeneous equation we guess Ce^{3t}. Substituting we get

$$9Ce^{3t} + 21Ce^{3t} + 10Ce^{3t} = e^{3t}40C.$$

When $C = 1/40$ this is equal to $f(t) = e^{3t}$, so the solution is $Ae^{-5t} + Be^{-2t} + (1/40)e^{3t}$. □

EXAMPLE 19.6.4 Find the general solution to $\ddot{y} + 7\dot{y} + 10y = e^{-2t}$. Following the last example we might guess Ce^{-2t}, but since this is a solution to the homogeneous equation it cannot work. Instead we guess Cte^{-2t}. Then

$$(-2Ce^{-2t} - 2Ce^{-2t} + 4Cte^{-2t}) + 7(Ce^{-2t} - 2Cte^{-2t}) + 10Cte^{-2t} = e^{-2t}(3C).$$

Then $C = 1/3$ and the solution is $Ae^{-5t} + Be^{-2t} + (1/3)te^{-2t}$. □

In general, if $f(t) = e^{kt}$ and k is one of the roots of the characteristic equation, then we guess Cte^{kt} instead of Ce^{kt}. If k is the only root of the characteristic equation, then Cte^{kt} will not work, and we must guess Ct^2e^{kt}.

EXAMPLE 19.6.5 Find the general solution to $\ddot{y} - 6\dot{y} + 9y = e^{3t}$. The characteristic equation is $r^2 - 6r + 9 = (r-3)^2$, so the general solution to the homogeneous equation is $Ae^{3t} + Bte^{3t}$. Guessing Ct^2e^{3t} for the particular solution, we get

$$(9Ct^2e^{3t} + 6Cte^{3t} + 6Cte^{3t} + 2Ce^{3t}) - 6(3Ct^2e^{3t} + 2Cte^{3t}) + 9Ct^2e^{3t} = e^{3t}2C.$$

The solution is thus $Ae^{3t} + Bte^{3t} + (1/2)t^2e^{3t}$. □

It is common in various physical systems to encounter an $f(t)$ of the form $a\cos(\omega t) + b\sin(\omega t)$.

EXAMPLE 19.6.6 Find the general solution to $\ddot{y} + 6\dot{y} + 25y = \cos(4t)$. The roots of the characteristic equation are $-3 \pm 4i$, so the solution to the homogeneous equation is $e^{-3t}(A\cos(4t) + B\sin(4t))$. For a particular solution, we guess $C\cos(4t) + D\sin(4t)$. Substituting as usual:

$$\begin{aligned}(-16C\cos(4t) + -16D\sin(4t)) + 6(-4C\sin(4t) + 4D\cos(4t)) + 25(C\cos(4t) + D\sin(4t))\\ = (24D + 9C)\cos(4t) + (-24C + 9D)\sin(4t).\end{aligned}$$

To make this equal to $\cos(4t)$ we need

$$\begin{aligned}24D + 9C &= 1\\ 9D - 24C &= 0\end{aligned}$$

which gives $C = 1/73$ and $D = 8/219$. The full solution is then $e^{-3t}(A\cos(4t) + B\sin(4t)) + (1/73)\cos(4t) + (8/219)\sin(4t)$.

The function $e^{-3t}(A\cos(4t) + B\sin(4t))$ is a damped oscillation as in example 19.5.3, while $(1/73)\cos(4t) + (8/219)\sin(4t)$ is a simple undamped oscillation. As t increases, the sum $e^{-3t}(A\cos(4t) + B\sin(4t))$ approaches zero, so the solution

$$e^{-3t}(A\cos(4t) + B\sin(4t)) + (1/73)\cos(4t) + (8/219)\sin(4t)$$

becomes more and more like the simple oscillation $(1/73)\cos(4t) + (8/219)\sin(4t)$—notice that the initial conditions don't matter to this long term behavior. The damped portion is called the **transient** part of the solution, and the simple oscillation is called the **steady state** part of the solution. A physical example is a mass-spring system. If the only force on the mass is due to the spring, then the behavior of the system is a damped oscillation. If in addition an external force is applied to the mass, and if the force varies according to a function of the form $a\cos(\omega t) + b\sin(\omega t)$, then the long term behavior will be a simple oscillation determined by the steady state part of the general solution; the initial position of the mass will not matter. □

As with the exponential form, such a simple guess may not work.

EXAMPLE 19.6.7 Find the general solution to $\ddot{y} + 16y = -\sin(4t)$. The roots of the characteristic equation are $\pm 4i$, so the solution to the homogeneous equation is $A\cos(4t) + B\sin(4t)$. Since both $\cos(4t)$ and $\sin(4t)$ are solutions to the homogeneous equation, $C\cos(4t) + D\sin(4t)$ is also, so it cannot be a solution to the non-homogeneous equation. Instead, we guess $Ct\cos(4t) + Dt\sin(4t)$. Then substituting:

$$\begin{aligned}(-16Ct\cos(4t) - 16Dt\sin(4t) + 8D\cos(4t) - 8C\sin(4t))) + 16(Ct\cos(4t) + Dt\sin(4t))\\ = 8D\cos(4t) - 8C\sin(4t).\end{aligned}$$

Thus $C = 1/8$, $D = 0$, and the solution is $A\cos(4t) + B\sin(4t) + (1/8)t\cos(4t)$. □

In general, if $f(t) = a\cos(\omega t) + b\sin(\omega t)$, and $\pm\omega i$ are the roots of the characteristic equation, then instead of $C\cos(\omega t) + D\sin(\omega t)$ we guess $Ct\cos(\omega t) + Dt\sin(\omega t)$.

Exercises 19.6.

Find the general solution to the differential equation.

1. $\ddot{y} - 10\dot{y} + 25y = \cos t \Rightarrow$
2. $\ddot{y} + 2\sqrt{2}\dot{y} + 2y = 10 \Rightarrow$
3. $\ddot{y} + 16y = 8t^2 + 3t - 4 \Rightarrow$
4. $\ddot{y} + 2y = \cos(5t) + \sin(5t) \Rightarrow$
5. $\ddot{y} - 2\dot{y} + 2y = e^{2t} \Rightarrow$
6. $\ddot{y} - 6y + 13 = 1 + 2t + e^{-t} \Rightarrow$

7. $\ddot{y} + \dot{y} - 6y = e^{-3t} \Rightarrow$

8. $\ddot{y} - 4\dot{y} + 3y = e^{3t} \Rightarrow$

9. $\ddot{y} + 16y = \cos(4t) \Rightarrow$

10. $\ddot{y} + 9y = 3\sin(3t) \Rightarrow$

11. $\ddot{y} + 12\dot{y} + 36y = 6e^{-6t} \Rightarrow$

12. $\ddot{y} - 8\dot{y} + 16y = -2e^{4t} \Rightarrow$

13. $\ddot{y} + 6\dot{y} + 5y = 4 \Rightarrow$

14. $\ddot{y} - \dot{y} - 12y = t \Rightarrow$

15. $\ddot{y} + 5y = 8\sin(2t) \Rightarrow$

16. $\ddot{y} - 4y = 4e^{2t} \Rightarrow$

Solve the initial value problem.

17. $\ddot{y} - y = 3t + 5$, $y(0) = 0$, $\dot{y}(0) = 0 \Rightarrow$

18. $\ddot{y} + 9y = 4t$, $y(0) = 0$, $\dot{y}(0) = 0 \Rightarrow$

19. $\ddot{y} + 12\dot{y} + 37y = 10e^{-4t}$, $y(0) = 4$, $\dot{y}(0) = 0 \Rightarrow$

20. $\ddot{y} + 6\dot{y} + 18y = \cos t - \sin t$, $y(0) = 0$, $\dot{y}(0) = 2 \Rightarrow$

21. Find the solution for the mass-spring equation $\ddot{y} + 4\dot{y} + 29y = 689\cos(2t)$. $\Rightarrow$

22. Find the solution for the mass-spring equation $3\ddot{y} + 12\dot{y} + 24y = 2\sin t$. $\Rightarrow$

23. Consider the differential equation $m\ddot{y} + b\dot{y} + ky = \cos(\omega t)$, with m, b, and k all positive and $b^2 < 2mk$; this equation is a model for a damped mass-spring system with external driving force $\cos(\omega t)$. Show that the steady state part of the solution has amplitude

$$\frac{1}{\sqrt{(k - m\omega^2)^2 + \omega^2 b^2}}.$$

Show that this amplitude is largest when $\omega = \dfrac{\sqrt{4mk - 2b^2}}{2m}$. This is the **resonant frequency** of the system.

19.7 Second Order Linear Equations, take two

The method of the last section works only when the function $f(t)$ in $a\ddot{y} + b\dot{y} + cy = f(t)$ has a particularly nice form, namely, when the derivatives of f look much like f itself. In other cases we can try variation of parameters as we did in the first order case.

Since as before $a \neq 0$, we can always divide by a to make the coefficient of $\ddot{y}$ equal to 1. Thus, to simplify the discussion, we assume $a = 1$. We know that the differential equation $\ddot{y} + b\dot{y} + cy = 0$ has a general solution $Ay_1 + By_2$. As before, we guess a particular solution to $\ddot{y} + b\dot{y} + cy = f(t)$; this time we use the guess $y = u(t)y_1 + v(t)y_2$. Compute the derivatives:

$$\begin{aligned}
\dot{y} &= \dot{u}y_1 + u\dot{y}_1 + \dot{v}y_2 + v\dot{y}_2 \\
\ddot{y} &= \ddot{u}y_1 + \dot{u}\dot{y}_1 + \dot{u}\dot{y}_1 + u\ddot{y}_1 + \ddot{v}y_2 + \dot{v}\dot{y}_2 + \dot{v}\dot{y}_2 + v\ddot{y}_2.
\end{aligned}$$

Now substituting:

$$\begin{aligned}
\ddot{y} + b\dot{y} + cy &= \ddot{u}y_1 + \dot{u}\dot{y}_1 + \dot{u}\dot{y}_1 + u\ddot{y}_1 + \ddot{v}y_2 + \dot{v}\dot{y}_2 + \dot{v}\dot{y}_2 + v\ddot{y}_2 \\
&\quad + b\dot{u}y_1 + bu\dot{y}_1 + b\dot{v}y_2 + bv\dot{y}_2 + cuy_1 + cvy_2 \\
&= (u\ddot{y}_1 + bu\dot{y}_1 + cuy_1) + (v\ddot{y}_2 + bv\dot{y}_2 + cvy_2) \\
&\quad + b(\dot{u}y_1 + \dot{v}y_2) + (\ddot{u}y_1 + \dot{u}\dot{y}_1 + \ddot{v}y_2 + \dot{v}\dot{y}_2) + (\dot{u}\dot{y}_1 + \dot{v}\dot{y}_2) \\
&= 0 + 0 + b(\dot{u}y_1 + \dot{v}y_2) + (\ddot{u}y_1 + \dot{u}\dot{y}_1 + \ddot{v}y_2 + \dot{v}\dot{y}_2) + (\dot{u}\dot{y}_1 + \dot{v}\dot{y}_2).
\end{aligned}$$

The first two terms in parentheses are zero because y_1 and y_2 are solutions to the associated homogeneous equation. Now we engage in some wishful thinking. If $\dot{u}y_1 + \dot{v}y_2 = 0$ then also $\ddot{u}y_1 + \dot{u}\dot{y}_1 + \ddot{v}y_2 + \dot{v}\dot{y}_2 = 0$, by taking derivatives of both sides. This reduces the entire expression to $\dot{u}\dot{y}_1 + \dot{v}\dot{y}_2$. We want this to be $f(t)$, that is, we need $\dot{u}\dot{y}_1 + \dot{v}\dot{y}_2 = f(t)$. So we would very much like these equations to be true:

$$\begin{aligned}
\dot{u}y_1 + \dot{v}y_2 &= 0 \\
\dot{u}\dot{y}_1 + \dot{v}\dot{y}_2 &= f(t).
\end{aligned}$$

This is a system of two equations in the two unknowns $\dot{u}$ and $\dot{v}$, so we can solve as usual to get $\dot{u} = g(t)$ and $\dot{v} = h(t)$. Then we can find u and v by computing antiderivatives. This is of course the sticking point in the whole plan, since the antiderivatives may be impossible to find. Nevertheless, this sometimes works out and is worth a try.

EXAMPLE 19.7.1 Consider the equation $\ddot{y} - 5\dot{y} + 6y = \sin t$. We can solve this by the method of undetermined coefficients, but we will use variation of parameters. The solution to the homogeneous equation is $Ae^{2t} + Be^{3t}$, so the simultaneous equations to be solved are

$$\begin{aligned}
\dot{u}e^{2t} + \dot{v}e^{3t} &= 0 \\
2\dot{u}e^{2t} + 3\dot{v}e^{3t} &= \sin t.
\end{aligned}$$

If we multiply the first equation by 2 and subtract it from the second equation we get

$$\begin{aligned}
\dot{v}e^{3t} &= \sin t \\
\dot{v} &= e^{-3t}\sin t \\
v &= -\frac{1}{10}(3\sin t + \cos t)e^{-3t},
\end{aligned}$$

using integration by parts. Then from the first equation:

$$\begin{aligned}
\dot{u} &= -e^{-2t}\dot{v}e^{3t} = -e^{-2t}e^{-3t}\sin(t)e^{3t} = -e^{-2t}\sin t \\
u &= \frac{1}{5}(2\sin t + \cos t)e^{-2t}.
\end{aligned}$$

Now the particular solution we seek is

$$\begin{aligned} ue^{2t}+ve^{3t} &= \frac{1}{5}(2\sin t+\cos t)e^{-2t}e^{2t}-\frac{1}{10}(3\sin t+\cos t)e^{-3t}e^{3t} \\ &= \frac{1}{5}(2\sin t+\cos t)-\frac{1}{10}(3\sin t+\cos t) \\ &= \frac{1}{10}(\sin t+\cos t), \end{aligned}$$

and the solution to the differential equation is $Ae^{2t}+Be^{3t}+(\sin t+\cos t)/10$. For comparison (and practice) you might want to solve this using the method of undetermined coefficients. □

EXAMPLE 19.7.2 The differential equation $\ddot{y}-5\dot{y}+6y=e^t\sin t$ can be solved using the method of undetermined coefficients, though we have not seen any examples of such a solution. Again, we will solve it by variation of parameters. The equations to be solved are

$$\begin{aligned} \dot{u}e^{2t}+\dot{v}e^{3t} &= 0 \\ 2\dot{u}e^{2t}+3\dot{v}e^{3t} &= e^t\sin t. \end{aligned}$$

If we multiply the first equation by 2 and subtract it from the second equation we get

$$\begin{aligned} \dot{v}e^{3t} &= e^t\sin t \\ \dot{v} &= e^{-3t}e^t\sin t=e^{-2t}\sin t \\ v &= -\frac{1}{5}(2\sin t+\cos t)e^{-2t}. \end{aligned}$$

Then substituting we get

$$\begin{aligned} \dot{u} &= -e^{-2t}\dot{v}e^{3t}=-e^{-2t}e^{-2t}\sin(t)e^{3t}=-e^{-t}\sin t \\ u &= \frac{1}{2}(\sin t+\cos t)e^{-t}. \end{aligned}$$

The particular solution is

$$\begin{aligned} ue^{2t}+ve^{3t} &= \frac{1}{2}(\sin t+\cos t)e^{-t}e^{2t}-\frac{1}{5}(2\sin t+\cos t)e^{-2t}e^{3t} \\ &= \frac{1}{2}(\sin t+\cos t)e^{t}-\frac{1}{5}(2\sin t+\cos t)e^{t} \\ &= \frac{1}{10}(\sin t+3\cos t)e^t, \end{aligned}$$

and the solution to the differential equation is $Ae^{2t}+Be^{3t}+e^t(\sin t+3\cos t)/10$. □

EXAMPLE 19.7.3 The differential equation $\ddot{y}-2\dot{y}+y=e^t/t^2$ is not of the form amenable to the method of undetermined coefficients. The solution to the homogeneous equation is Ae^t+Bte^t and so the simultaneous equations are

$$\begin{aligned} \dot{u}e^t+\dot{v}te^t &= 0 \\ \dot{u}e^t+\dot{v}te^t+\dot{v}e^t &= \frac{e^t}{t^2}. \end{aligned}$$

Subtracting the equations gives

$$\begin{aligned} \dot{v}e^t &= \frac{e^t}{t^2} \\ \dot{v} &= \frac{1}{t^2} \\ v &= -\frac{1}{t}. \end{aligned}$$

Then substituting we get

$$\begin{aligned} \dot{u}e^t &= -\dot{v}te^t=-\frac{1}{t^2}te^t \\ \dot{u} &= -\frac{1}{t} \\ u &= -\ln t. \end{aligned}$$

The solution is $Ae^t+Bte^t-e^t\ln t-e^t$. □

Exercises 19.7.

Find the general solution to the differential equation using variation of parameters.

1. $\ddot{y}+y=\tan t \Rightarrow$

2. $\ddot{y}+y=e^{2t} \Rightarrow$

3. $\ddot{y}+4y=\sec t \Rightarrow$

4. $\ddot{y}+4y=\tan t > \Rightarrow$

5. $\ddot{y}+\dot{y}-6y=t^2e^{2t} \Rightarrow$

6. $\ddot{y}-2\dot{y}+2y=e^t\tan(t) \Rightarrow$

7. $\ddot{y}-2\dot{y}+2y=\sin(t)\cos(t)$ (This is rather messy when done by variation of parameters; compare to undetermined coefficients.) $\Rightarrow$