17

17.1 Volume and Average Height

Consider a surface \( f(x, y) \); you might temporarily think of this as representing physical topography—a hilly landscape, perhaps. What is the average height of the surface (or average altitude of the landscape) over some region?

As with most such problems, we start by thinking about how we might approximate the answer. Suppose the region is a rectangle, \([a, b] \times [c, d]\). We can divide the rectangle into a grid, \(m\) subdivisions in one direction and \(n\) in the other, as indicated in figure 17.1.1. We pick \(x_i\) values \(x_0, x_1, \ldots, x_m\) in each subdivision in the \(x\) direction, and similarly in the \(y\) direction. At each of the points \((x_i, y_j)\) in one of the smaller rectangles in the grid, we compute the height of the surface: \(f(x_i, y_j)\). Now the average of these heights should be (depending on the fineness of the grid) close to the average height of the surface:

\[
\frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_i, y_j) \Delta y.
\]

As both \(m\) and \(n\) go to infinity, we expect this approximation to converge to a fixed value, the actual average height of the surface. For reasonably nice functions this does indeed happen.

The next question, of course, is: How do we compute these double integrals? You might think that we will need some two-dimensional version of the Fundamental Theorem of Calculus, but as it turns out we can get away with just the single-variable version, applied twice.

Going back to the double sum, we can rewrite it to emphasize a particular order in which we want to add the terms:

\[
\frac{1}{mn} \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} f(x_i, y_j) \Delta y.
\]

In the sum in parentheses, only the value of \(x_i\) is changing; \(y_j\) is temporarily constant. As \(m\) goes to infinity, this sum has the right form to turn into an integral.

\[
\lim_{m \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f(x_i, y_j) \Delta y = \int_a^b f(x, y) \, dx.
\]

So after we take the limit as \(m\) goes to infinity, the sum is

\[
\frac{1}{n} \sum_{j=0}^{n-1} \int_a^b f(x, y_j) \, dx \Delta y.
\]

Using sigma notation, we can rewrite the approximation:

\[
\frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_i, y_j) \Delta x \Delta y = \frac{1}{n} \sum_{i=0}^{n-1} \left( \int_a^b f(x, y_i) \, dx \right) \Delta y.
\]

The two parts of this product have useful meaning: \((b-a)\Delta y = 1/n\) is of course the area of the rectangle, and the double sum adds up our terms of the form \(f(x_i, y_j)\Delta x\Delta y\), which is the height of the surface at a point times the area of one of the small rectangles into which we have divided the large rectangle. In short, each term \(f(x_i, y_j)\Delta x\Delta y\) is the volume of a tall, thin, rectangular box, and is approximately the volume under the surface and above one of the small rectangles; see figure 17.1.2. When we add all of those up, we get an approximation to the volume under the surface and above the rectangle \([a, b] \times [c, d]\).

When we take the limit as \(m\) and \(n\) go to infinity, the double sum becomes the actual volume under the surface, which we divide by \((b-a)(d-c)\) to get the average height.

Double sums like this come up in many applications, so in a way it is the most important part of this example; dividing by \((b-a)(d-c)\) is a simple extra step that allows the computation of an average. As we did in the single variable case, we introduce a special notation for the limit of such a double sum:

\[
\lim_{m, n \to \infty} \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_i, y_j) \Delta x \Delta y = \int_a^b \int_c^d f(x, y) \, dx \, dy = \int_a^b \int_c^d f(x, y) \, dy \, dx,
\]

the double integral of \(f\) over the region \(R\). The notation \(dA\) indicates a small bit of area, without specifying any particular order for the variables \(x\) and \(y\); it is shorter and more "generic" than writing \(dx \, dy\). The average height of the surface in this notation is

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dx \, dy.
\]

Of course, for different values of \(y\), this integral has different values; in other words, it is really a function applied to \(y\):

\[
G(y) = \int_a^b f(x, y) \, dx.
\]

If we substitute back into the sum we get

\[
\sum_{y=0}^{y_m} G(y) \Delta y.
\]

This sum has a nice interpretation. The value \(G(y)\) is the area of a cross-section of the region under the surface \(f(x, y)\), namely, when \(y = y_j\). The quantity \(G(y)\Delta y\) can be interpreted as the volume of a solid with face area \(G(y)\) and thickness \(\Delta y\). Think of the surface \(f(x, y)\) as the top of a loaf of sliced bread. Each slice has a cross-sectional area and a thickness. \(G(y)\Delta y\) corresponds to the volume of a single slice of bread. Adding these up approximates the total volume of the loaf. (This is very similar to the technique we used to compute volumes in section 8.3, except that there we need the cross-sections to be in some way "the same"). Figure 17.1.3 shows this "sliced loaf" approximation using the same surface as shown in figure 17.1.2. Nicely enough, this sum looks just like the sort of sum that turns into an integral, namely:

\[
\lim_{m, n \to \infty} \sum_{i=0}^{m-1} G(y_i) \Delta y = \int_c^d \left( \int_a^b f(x, y) \, dx \right) \, dy.
\]

Let’s be clear about what this means: we first will compute the inner integral, temporarily treating \(y\) as a constant. We will do this by finding an anti-derivative with respect to \(x\), then substituting \(x = a\) and \(x = b\) and subtracting, as usual. The result will be an expression with no \(x\) variable but some occurrences of \(y\). Then the outer integral will be an ordinary one-variable problem, with \(y\) as the variable.

EXAMPLE 17.1.1 Figure 17.1.2 shows the function \(\sin(xy) + 6/5\) on \([0, 5] \times [0, 5, 2.5]\). The volume under this surface is

\[
\int_0^5 \int_0^{2.5} \sin(xy) + \frac{6}{5} \, dx \, dy.
\]

The inner integral is

\[
\int_0^5 \sin(xy) + \frac{6}{5} \, dx = -\cos(xy) \bigg|_0^5 + \frac{6}{5} \int_0^5 \cos(3.5y) + \cos(0.5y) = \frac{18}{5}.
\]

Unfortunately, this gives a function for which we can’t find a simple anti-derivative. To complete the problem we could use Sigma or similar software to approximate the integral.
Figure 17.1.3  Approximating the volume under a surface with slices. (AP)

Doing this gives us a volume of approximately 8.84, so the average height is approximately 8.84/6 ≈ 1.47.

Because addition and multiplication are commutative and associative, we can rewrite the original double sum:

\[ \sum_{i=1}^{m} \sum_{j=0}^{n-1} f(x_i, y_j) \Delta x \Delta y = \sum_{j=0}^{n-1} \sum_{i=1}^{m} f(x_i, y_j) \Delta x \Delta y. \]

Now if we repeat the development above, the inner sum turns into an integral:

\[ \lim_{m,n \to \infty} \sum_{i=1}^{m} \int_{y_j}^{y_{j+1}} f(x_i, y) \, dy = \int_{y_0}^{y_n} \left( \lim_{m \to \infty} \sum_{i=1}^{m} f(x_i, y) \right) \, dy. \]

and then the outer sum turns into an integral:

\[ \lim_{m \to \infty} \sum_{j=0}^{n-1} \left( \int_{y_j}^{y_{j+1}} f(x_i, y) \, dy \right) \Delta x = \int_{y_0}^{y_n} \left( \int_{x_0}^{x_m} f(x, y) \, dx \right) \, dy. \]

In other words, we can compute the integrals in either order, first with respect to \( x \) then \( y \) or vice versa. Thinking of the loaf of bread, this corresponds to slicing the loaf in a direction perpendicular to the first.

We haven’t really proved that the value of a double integral is equal to the value of the corresponding two single integrals in either order of integration, but provided the function is reasonably nice, this is true; the result is called Fubini’s Theorem.

EXAMPLE 17.1.2 We compute \( \int_0^1 \int_0^1 (x-y)^2 + 4y^2 \, dx \, dy \), where \( R = [0, 3] \times [0, 2] \), in two ways.

First,

\[
\int_0^1 \int_0^1 (x-y)^2 + 4y^2 \, dx \, dy = \int_0^1 \left[ \frac{(x-y)^3}{3} + 4y^2 x \right]_0^1 \, dy
\]

\[
= \int_0^1 \left( \frac{1}{3} + 4y^2 \right) \, dy
\]

\[ = \frac{1}{3} + \frac{8}{3} + \frac{32}{3}
\]

\[ = 4. \]

In the other order,

\[
\int_0^3 \int_0^1 (x-y)^2 + 4y^2 \, dx \, dy = \int_0^3 \left[ \frac{(x-y)^3}{3} + 4y^2 x \right]_0^1 \, dy
\]

\[ = \int_0^3 \frac{8}{3} + \frac{12y^2}{3} \, dy
\]

\[ = 6 + \frac{46}{3} + \frac{32}{3}
\]

\[ = 44. \]

In this example there is no particular reason to favor one direction over the other; in some cases, one direction might be much easier than the other, so it is usually worth considering the two different possibilities.

Frequently we will be interested in a region that is not simply a rectangle. Let’s compute the volume under the surface \( z = x^2 \) above the region described by \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq x^2 \), shown in figure 17.1.4.

In principle there is nothing more difficult about this problem. If we imagine the three-dimensional region under the surface and above the parabolic region as an oddly shaped loaf of bread, we can still slice it up, approximate the volume of each slice, and add these

EXAMPLE 17.1.3 Find the volume under the surface \( z = \sqrt{1-x^2} \) and above the triangle formed by \( y = x, x = 1 \), and the \( x \)-axis.

Let’s consider the two possible ways to set this up:

\[
\int_0^1 \int_0^{\sqrt{1-x^2}} \, dy \, dx \quad \text{or} \quad \int_0^1 \int_0^{\sqrt{1-x^2}} \, dx \, dy.
\]

Which appears easier? In the first, the inner integral is easy, because we need an anti-derivative with respect to \( y \) and the entire integral \( \sqrt{1-x^2} \) is constant with respect to \( y \). Of course, the outer integral may be more difficult. In the second, the inner integral is mildly unpleasant—a trig substitution. So let’s try the first one, since the first step is easy, and see where that leaves us.

\[
\int_0^1 \left[ y \sqrt{1-x^2} \right]_0^1 \, dx = \int_0^1 x \sqrt{1-x^2} \, dx.
\]

This is quite easy, since the substitution \( u = 1-x^2 \) works:

\[
\int \sqrt{1-x^2} \, dx = -\frac{1}{2} \sqrt{1-x^2} = -\frac{1}{3} (1-x^2)^{3/2}.
\]

Then

\[
\int_0^1 x \sqrt{1-x^2} \, dx = \frac{1}{3} (1-x^2)^{3/2} = \frac{1}{3}.
\]

This is a good example of how the order of integration can affect the complexity of the problem. In this case it is possible to do the other order, but it is a bit messier. In some cases one order may lead to a very difficult or impossible integral; it’s usually worth considering both possibilities before going very far.

0.1
0 1

Figure 17.1.4  A parabolic region of integration.
Exercises 17.1.
1. Compute \( \int_{y=1}^{2} \int_{x=0}^{2} x \, dx \, dy \).
2. Compute \( \int_{x=0}^{2} \int_{y=0}^{3} x + y \, dy \, dx \).
3. Compute \( \int_{x=0}^{2} \int_{y=0}^{3} xy \, dx \, dy \).
4. Compute \( \int_{x=0}^{1} \int_{y=0}^{2} x \, dx \, dy \).
5. Compute \( \int_{x=0}^{1} \int_{y=0}^{2} x^2 \, dx \, dy \).
6. Compute \( \int_{x=0}^{1} \int_{y=0}^{2} y \, dy \, dx \).
7. Compute \( \int_{x=0}^{1} \int_{y=0}^{2} x^4 \, dx \, dy \).
8. Compute \( \int_{x=0}^{1} \int_{y=0}^{2} x^2 y \, dx \, dy \).
9. Compute \( \int_{x=0}^{1} \int_{y=0}^{2} \sqrt{x^2 + 1} \, dx \, dy \).
10. Compute \( \int_{x=0}^{1} \int_{y=0}^{2} y \ln(x^2) \, dx \, dy \).
11. Compute \( \int_{x=0}^{1} \int_{y=0}^{2} x \sqrt{1 + y^2} \, dx \, dy \).
12. Compute \( \int_{x=0}^{1} \int_{y=0}^{2} \frac{1}{x} \, dx \, dy \).
13. Compute \( \int_{x=0}^{1} \int_{y=0}^{2} \frac{1}{y} \, dx \, dy \).
14. Compute \( \int_{x=0}^{1} \int_{y=0}^{2} x^2 - y \, dx \, dy \).
15. Compute \( \int_{x=0}^{1} \int_{y=0}^{2} x^2 \, dx \, dy \).
16. Evaluate \( \int_{x=0}^{1} \int_{y=0}^{2} x^2 \, dx \, dy \) over the region in the first quadrant bounded by the hyperbola \( xy = 16 \) and the lines \( y = x, y = 8 \), and \( x = 8 \).
17. Find the volume below \( z = 1 - y \) above the region \(-1 \leq x \leq 1, 0 \leq y \leq 1 - x^2\).
18. Find the volume bounded by \( z = x^2 + y^2 \) and \( z = 4 \).
19. Find the volume in the first octant bounded by \( z = 4 - x \) and \( y = 2z \).
20. Find the volume in the first octant bounded by \( z = 4 - x, 2x + y = 4, z = y, \) and \( y = 0 \).

17.2 Double Integrals in Cylindrical Coordinates
Suppose we have a surface given in cylindrical coordinates as \( z = f(r, \theta) \) and we wish to find the integral over some region. We could attempt to translate into rectangular coordinates and do the integration there, but it is often easier to stay in cylindrical coordinates.

How might we approximate the volume under such a surface in a way that uses cylindrical coordinates directly? The basic idea is the same as before: we divide the region into many small regions, multiply the area of each small region by the height of the surface somewhere in that little region, and add them up. What changes is the shape of the small regions; in order to have a nice representation in terms of \( r \) and \( \theta \), we use small pieces of ring-shaped areas, as shown in figure 17.2.1. Each small region is roughly rectangular, except that two sides are segments of a circle and the other two sides are not quite parallel. Near a point \((r, \theta)\), the length of either circular arc is about \(r \Delta \theta\) and the length of each straight side is simply \(\Delta r\). When \(\Delta r\) and \(\Delta \theta\) are very small, the region is nearly a rectangle with area \(r \Delta r \Delta \theta\), and the volume under the surface is approximately

\[
\sum \sum f(r, \theta) r \Delta r \Delta \theta.
\]

Figure 17.2.1 A cylindrical coordinates “grid”.

Example 17.2.1 Find the volume under \(z = \sqrt{4 - r^2}\) above the quarter circle bounded by the two axes and the circle \(x^2 + y^2 = 4\) in the first quadrant.

In terms of \( r \) and \( \theta \), this region is described by the restrictions \(0 \leq r \leq 2\) and \(0 \leq \theta \leq \pi/2\), so we have

\[
\int_{0}^{\pi/2} \int_{0}^{2} \sqrt{4 - r^2} \, r \, dr \, d\theta = \int_{0}^{\pi/2} \left(4r - \frac{1}{2}r^3\right)_{0}^{2} \, d\theta = \left(4\pi - 2\right)\theta \bigg|_{0}^{\pi/2} = \frac{4\pi}{3}.
\]

The surface is a portion of the sphere of radius 2 centered at the origin, in fact exactly one-eighth of the sphere. We know the formula for volume of a sphere is \(4/3)\pi r^3\), so the volume we have computed is \(1/8)(4/3)\pi r^3 = (1/3)\pi r^3\), in agreement with our answer.

This example is much like a simple one in rectangular coordinates: the region of interest may be described exactly by a constant range for each of the variables. As with rectangular coordinates, we can adapt the method to deal with more complicated regions.
17.2 Double Integrals in Cylindrical Coordinates

EXAMPLE 17.2.2 Find the volume under \( z = \sqrt{1-x^2} \) above the region enclosed by the curve \( r = 2 \cos \theta \), \(-\pi/2 \leq \theta \leq \pi/2\); see figure 17.2.2. The region is described in polar coordinates by the inequalities \(-\pi/2 \leq \theta \leq \pi/2\) and \(0 \leq r \leq 2 \cos \theta\); so the double integral is

\[
\int_{-\pi/2}^{\pi/2} \int_{0}^{2 \cos \theta} \sqrt{1-x^2} \, r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2} \left(1 - r^2\right)^{1/2} \int_{0}^{2 \cos \theta} r \, dr \, d\theta.
\]

We can rewrite the integral as shown because of the symmetry of the volume; this avoids a complication during the evaluation. Proceeding:

\[
2 \int_{\pi/2}^{\pi/2} \int_{0}^{2 \cos \theta} \sqrt{1-x^2} \, r \, dr \, d\theta = 2 \int_{0}^{\pi/2} \int_{0}^{2 \cos \theta} \left(1 - r^2\right)^{1/2} \frac{1}{2} r^2 \, dr \, d\theta = 2 \int_{0}^{\pi/2} \left[\frac{1}{2} r^2 \left(1 - r^2\right)^{1/2}\right]_{0}^{2 \cos \theta} \, d\theta.
\]

You might have learned a formula for computing areas in polar coordinates. It is

\[
\frac{1}{2} \int_{0}^{2 \pi} r^2 \, d\theta = \frac{1}{2} \int_{0}^{2 \pi} (2 \cos \theta)^2 \, d\theta = \frac{1}{2} \int_{0}^{2 \pi} 4 \cos^2 \theta \, d\theta = 2 \int_{0}^{\pi} \cos^2 \theta \, d\theta = \pi.
\]


17.3 Moment and Center of Mass

Using a single integral we were able to compute the center of mass for a one-dimensional object with variable density, and a two-dimensional object with constant density. With a double integral we can handle two dimensions and variable density.

14. Find the volume under \( z = y^2 + x + 2 \) above the region \( x^2 + y^2 \leq 4 \) ⇒

15. Find the volume between \( z = xy^2 \) and \( z = 1 \) above the region \( x^2 + y^2 \leq 1 \) ⇒

16. Find the volume inside \( x^2 + y^2 = 1 \) and \( x^2 + z^2 = 1 \) ⇒

17. Find the volume under \( z = r \) above \( r = 3 + \cos \theta \) ⇒

Figure 17.2.4 shows the plot of \( r = 1 + 4 \sin(\theta) \).

\[ \int_{0}^{\pi} \int_{0}^{2 \cos \theta} \sqrt{1-x^2} \, r \, dr \, d\theta \]

a. Describe the behavior of the graph in terms of the given equation. Specifically, explain maximum and minimum values, number of leaves, and the ‘leaves within leaves’.  
   b. Give an integral or integrals to determine the area outside a smaller leaf but inside a larger leaf.  
   c. How would changing the value of \( a \) in the equation \( r = 1 + a \cos(\theta) \) change the relative sizes of the inner and outer leaves? Focus on values \( a \geq 1 \). (Hint: How would we change the maximum and minimum values?)

19. Consider the integral \( \int_{0}^{\pi} \int_{0}^{2 \cos \theta} \sqrt{1-x^2} \, dA \), where \( D \) is the unit disk centered at the origin. (See the graph here.)  
   a. Why might this integral be considered improper?  
   b. Calculate the value of the integral of the same function \( \int_{0}^{2 \cos \theta} \frac{1}{\sqrt{1-x^2}} \, dA \) over the annulus with outer radius \( 2 \) and inner radius \( \lambda \).  
   c. Obtain a value for the integral on the whole disk by letting \( \lambda \) approach 0. ⇒
   d. For which values \( \lambda \) can we replace the denominator with \( (x^2 + y^2)^{1/2} \) in the original integral and still get a finite value for the improper integral?

17.3 Moment and Center of Mass

Just as before, the coordinates of the center of mass are

\[ \bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}, \]

where \( M \) is the total mass, \( M_y \) is the moment around the \( y \)-axis, and \( M_x \) is the moment around the \( x \)-axis. (You may want to review the concepts in section 11.1.)

The key to the computation, just as before, is the approximation of mass. In the two-dimensional case, we treat density \( \sigma \) as mass per square area, so when density is constant, mass is \( \sigma \cdot \text{area} \). If we have a two-dimensional region with varying density given by \( \sigma(x, y) \), and we divide the region into small subregions with area \( dA \), then the mass of one subregion is approximately \( \sigma(x, y) \, dA \), the total mass is approximately the sum of many of these, and as usual the sum turns into an integral in the limit:

\[ M = \int_{D} \int_{x}^{0} \sigma(x, y) \, dy \, dx, \]

and similarly for computations in cylindrical coordinates. Then as before

\[ M_x = \int_{D} \int_{x}^{0} \sigma(x, y) y \, dy \, dx, \quad M_y = \int_{D} \int_{x}^{0} \sigma(x, y) x \, dy \, dx, \]

\[ M_{xy} = \int_{D} \int_{x}^{0} \sigma(x, y) (x, y) \, dy \, dx. \]

EXAMPLE 17.3.1 Find the center of mass of a thin, uniform plate whose shape is the region between \( y = \cos x \) and the \( x \)-axis between \( x = -\pi/2 \) and \( x = \pi/2 \). Since the density is constant, we may take \( \sigma(x, y) = 1 \). It is clear that \( \bar{x} = 0 \), but for practice let’s compute it anyway. First we compute the mass:

\[ M = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} 1 \, dx \, dy = \int_{-\pi/2}^{\pi/2} \left[ \frac{1}{2} \sin x^2 \right]_{-\pi/2}^{\pi/2} = 2. \]

Next,

\[ M_y = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \cos x \, dy \, dx = \frac{\pi}{2} \int_{-\pi/2}^{\pi/2} \cos x \, dx = \frac{\pi}{2} \int_{-\pi/2}^{\pi/2} \sin x \, dx = 0. \]

Finally,

\[ M_{xy} = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} x \, dy \, dx = \frac{\pi}{2} \int_{-\pi/2}^{\pi/2} x \, dx = 0. \]

So \( \bar{x} = 0 \) as expected, and \( \bar{y} = \sigma/4\pi = \pi/8 \). This is the same problem as in example 11.1.4, it may be helpful to compare the two solutions.
EXAMPLE 17.3.2 Find the center of mass of a two-dimensional plate that occupies the quarter circle \( x^2 + y^2 \leq 1 \) in the first quadrant and has density \( k(x^2 + y^2) \). It seems clear that because of the symmetry of both the region and the density function (both are important!), \( \overline{x} = \overline{y} \). We'll do both to check our work.

Jumping right in:

\[
M = \int_0^{\pi/2} \int_0^1 \frac{x^2 + y^2}{2} \, r \, dr \, d\theta = \frac{k}{3} \int_0^{\pi/2} \left( \frac{1}{2} r^4 \right) \, d\theta.
\]

This integral is something we can do, but it's a bit unpleasant. Since everything in sight is related to a circle, let's back up and try polar coordinates. Then \( x^2 + y^2 = r^2 \) and

\[
M = \frac{k}{3} \int_0^{\pi/2} \int_0^1 r^2 \, dr \, d\theta = \frac{k}{3} \int_0^{\pi/2} \frac{1}{2} \, d\theta = \frac{k \pi}{6}.
\]

\[
\overline{y} = \frac{1}{M} \int_0^{\pi/2} \int_0^1 \frac{r^2 \sin \theta}{2} \, dr \, d\theta = \frac{k}{3} \int_0^{\pi/2} \frac{1}{3} \, d\theta = \frac{k \pi}{18}.
\]

Much better. Next, since \( y = r \sin \theta \)

\[
M_y = \frac{k}{3} \int_0^{\pi/2} \int_0^1 r \sin \theta \, dr \, d\theta = \frac{k}{3} \int_0^{\pi/2} \frac{1}{2} \, d\theta = \frac{k \pi}{6}.
\]

Similarly,

\[
M_x = \frac{k}{3} \int_0^{\pi/2} \int_0^1 r \cos \theta \, dr \, d\theta = \frac{k}{3} \int_0^{\pi/2} \frac{1}{2} \, d\theta = \frac{k \pi}{6}.
\]

Finally, \( \overline{x} = \overline{y} = \frac{8}{3\pi} \).

\[\square\]

Exercises 17.3.

1. Find the center of mass of a two-dimensional plate that occupies the square \([0,1] \times [0,1]\) and has density function \( y \).
2. Find the center of mass of a two-dimensional plate that occupies the triangle \( 0 \leq x \leq 1, 0 \leq y \leq x \), and has density function \( x \).
3. Find the center of mass of a two-dimensional plate that occupies the upper unit semicircle centered at \((0,0)\) and has density function \( x^2 \).
4. Find the center of mass of a two-dimensional plate that occupies the upper unit semicircle centered at \((0,0)\) and has density function \( x^3 \).
5. Find the center of mass of a two-dimensional plate that occupies the triangle formed by \( x = 2, y = x, \) and \( y = 2x \) and has density function \( 2x \).
6. Find the center of mass of a two-dimensional plate that occupies the triangle formed by \( x = 1, y = x, \) and \( 2x + y = 6 \) and has density function \( x^2 \).

### 17.4 Surface Area

We next seek to compute the area of a surface above (or below) a region in the \( x,y \)-plane. How might we approximate this? We start, as usual, by dividing the region into a grid of small rectangles. We want to approximate the surface area above one of these small rectangles. The area is very close to the area of the tangent plane above the small rectangle. If the tangent plane just happened to be horizontal, of course the area would simply be the area of the rectangle. For a typical plane, however, the area is the area of a parallelogram, as indicated in figure 17.4.1. Note that the area of the parallelogram is obviously larger the more “tilted” the tangent plane. In the interactive figure you can see that viewed from above the four parallelograms exactly cover a rectangular region in the \( x,y \)-plane.

Now recall a curious fact: the area of a parallelogram can be computed as the cross product of two vectors (page 327). We simply need to acquire two vectors, parallel to the sides of the parallelogram and with lengths to match. But this is easy: in the \( \vec{x} \)-direction we use the tangent vector we already know, namely \((1,0,f_x)\) and multiply by \( \Delta x \) to shrink it to the right size: \((\Delta x, f_x \Delta x)\). In the \( \vec{y} \)-direction we do the same thing and get \((0,\Delta y, f_y \Delta y)\). The cross product of these vectors is \((f_y \Delta y - f_x \Delta x, \Delta x \Delta y, \Delta x \Delta y)\). The area of the parallelogram. Now we add those up and take the limit, to produce the integral

\[
\int_a^b \int_c^d \sqrt{f_x'^2 + f_y'^2 + 1} \, dy \, dx.
\]

As before, the limits need not be constant.

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Exercises 17.4.

1. Find the area of the surface of a right circular cone of height \( h \) and base radius \( a \).
2. Find the area of the portion of the plane \( x+y+z=1 \) inside the cylinder \( x^2 + y^2 = a^2 \).
3. Find the area of the portion of the plane \( x+y+z=1 \) in the first octant.
4. Find the area of the upper half of the cone \( x^2 + y^2 = z^2 \) inside the cylinder \( x^2 + y^2 = 2z \).
5. Find the area of the upper half of the cone \( x^2 + y^2 = z^2 \) above the interior of one loop of \( r = \cos(\theta) \).
6. Find the area of the upper hemisphere of \( x^2 + y^2 + z^2 = 1 \) above the interior of one loop of \( r = \cos(\theta) \).
7. The plane \( ax+by+cz=d \) cuts a triangle in the first octant provided that \( a, b, c \) and \( d \) are all positive. Find the area of this triangle.
8. Find the area of the portion of the cone \( x^2 + y^2 = z^2 \) lying above the \( xy \)-plane and inside the cylinder \( x^2 + y^2 = 4 \).

### 17.5 Triple Integrals

It will come as no surprise that we can also do triple integrals—integrals over a three-dimensional region. The simplest application allows us to compute volumes in an alternate way.

To approximate a volume in three dimensions, we can divide the three-dimensional region into small rectangular boxes, each \( \Delta x \times \Delta y \times \Delta z \) with volume \( \Delta x \Delta y \Delta z \). We then add them all up and take the limit, to get an integral:

\[
\int_a^b \int_c^d \int_e^f \Delta x \Delta y \Delta z.
\]

If the limits are constant, we are simply computing the volume of a rectangular box.

EXAMPLE 17.5.1 We use an integral to compute the volume of the box with opposite corners at \((0,0,0)\) and \((1,2,3)\):

\[
\int_0^1 \int_0^2 \int_0^3 \Delta x \Delta y \Delta z = \int_0^1 \int_0^2 \int_0^3 1 \, dx \, dy \, dz = \int_0^1 \int_0^2 3 \, dy \, dz = \int_0^2 6 \, dy = 6.
\]

Of course, this is more interesting and useful when the limits are not constant.

EXAMPLE 17.5.2 Find the volume of the tetrahedron with corners at \((0,0,0)\), \((0,3,0)\), \((2,3,0)\), and \((2,3,5)\).
The whole problem comes down to correctly describing the region by inequalities: 
0 ≤ z ≤ 2, 3x/2 ≤ y ≤ 2, 0 ≤ x ≤ 5/2. The lower y limit comes from the equation of the line y = 3x/2 that forms one edge of the tetrahedron in the x-y plane; the upper z limit comes from the equation of the plane z = 5x/2 that forms the "upper" side of the tetrahedron; see figure 17.5.1. Now the volume is

\[
\int_0^1 \int_{3x/2}^2 \int_{9z/2}^y dy \, dx \, dz = \int_0^1 \int_{3x/2}^2 \frac{9z^2}{2} \, dx \, dz
\]

Pretty much just the way we did for two dimensions we can use triple integration to compute mass, center of mass, and various quantities averages.

**EXAMPLE 17.5.3** Suppose the temperature at a point is given by T = xyz. Find the average temperature in the cube with opposite corners at (0, 0, 0) and (2, 2, 2).

In two dimensions we add up the temperatures and divide by the volume, 8:

\[
\frac{1}{8} \int_0^2 \int_0^2 \int_0^2 T \, dx \, dy \, dz = \frac{1}{8} \int_0^2 \int_0^2 \int_0^2 xyz \, dx \, dy \, dz = \frac{1}{8} \int_0^2 \int_0^2 \int_0^2 \frac{8}{27} \, dx \, dy \, dz = 1.\]

**EXAMPLE 17.5.4** Suppose the density of an object is given by \( \rho = x \), and the object occupies the tetrahedron with corners at (1, 0, 0), (0, 1, 0), (1, 1, 0), and (0, 1, 1). Find the mass and center of mass of the object.

**Exercises 17.5.**

1. Evaluate \( \iiint 2x + y - 1 \, dx \, dy \, dz \).
2. Evaluate \( \iiint xy \, dx \, dy \, dz \).
3. Evaluate \( \iiint x^3 \, dx \, dy \, dz \).
4. Evaluate \( \iiint x \, dV \).
5. Evaluate \( \iiint x \cos \theta \, dz \, dr \, d\theta \).
6. Evaluate \( \iiint x^2 \, dx \, dy \).
7. Evaluate \( \iiint x^2 \, dx \, dy \).
8. Evaluate \( \iiint x^2 y \, dx \, dy \, dz \).
9. Evaluate \( \iiint x^2 + y^2 \, dV \) over the region \( x^2 + y^2 + z^2 \leq 1 \) in the first octant.
10. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one corner.
11. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one edge.
12. A solid occupies the volume of the upper hemisphere of \( x^2 + y^2 + z^2 = 4 \) and has density \( z \) at \((x, y, z)\). Find the center of mass.
13. An object occupies the volume of the pyramid of corners at \((1, 1, 0), (1, -1, 0), (-1, 1, 0), (0, 0, 2)\) and has density \( x + y \) at \((x, y, z)\). Find the center of mass.
14. Verify the moments \( M_{xy}, M_{xz}, \) and \( M_{yz} \) of example 17.5.4 by evaluating the integrals.
15. Find the region \( E \) for which \( \iiint (1 - x^2 - y^2 - z^2) \, dV \) is a maximum.

**17.6 Cylindrical and Spherical Coordinates**

We have seen that sometimes double integrals are simplified by doing them in polar coordinates; not surprisingly, triple integrals are sometimes simpler in cylindrical coordinates or spherical coordinates. To set up integrals in polar coordinates, we had to understand the shape and area of a typical small region into which the region of integration was divided. We need to do the same thing here, for three dimensional regions.

**Figure 17.5.1** A tetrahedron. (AP)

As usual, the mass is the integral of density over the region:

\[
M = \iiint_{\text{region}} \rho \, dV = \iiint_{\text{region}} x \, dy \, dx \, dz = \frac{4}{3} \iiint_{\text{region}} \frac{y(1-x^2)}{2} \, dz \, dx = \frac{1}{3} \iiint_{\text{region}} \frac{(x(1-x^2))}{2} \, dz \, dx
\]

We compute moments as before, except now there is a third moment:

\[
M_{x,y} = \iiint_{\text{region}} x^2 \, dy \, dx \, dz = \frac{1}{5}\pi \\
M_{y,z} = \iiint_{\text{region}} y^2 \, dx \, dy \, dz = \frac{2}{5}\pi \\
M_{z,x} = \iiint_{\text{region}} z^2 \, dx \, dy \, dz = \frac{1}{5}\pi.
\]

Finally, the coordinates of the center of mass are \( \bar{x} = M_{y,z}/M = 1/5, \bar{y} = M_{z,x}/M = 2/5, \) and \( \bar{z} = M_{x,y}/M = 1/5 \).

**Chapter 17 Multiple Integration**

The cylindrical coordinate system is the simplest, since it is just the polar coordinate system plus a z coordinate. A typical small unit of volume is the shape shown in figure 17.2.1 “fattened up” in the z direction, so its volume is \( r \, \Delta r \, \Delta \theta \, \Delta z \), or in the limit, \( r \, dr \, d\theta \, dz \).

**EXAMPLE 17.6.1** Find the volume under \( z = \sqrt{x^2 + y^2} \) above the quarter circle inside \( x^2 + y^2 = 4 \) in the first quadrant.

We could of course do this with a double integral, but we’ll use a triple integral:

\[
\iiint_{\text{region}} r^2 \, dz \, dr \, d\theta = \int_0^\pi \int_0^2 r^2 \, dr \, d\theta = \frac{4\pi}{3}
\]

Compare this to example 17.2.1.

**EXAMPLE 17.6.2** An object occupies the space inside both the cylinder \( x^2 + y^2 = 1 \) and the sphere \( x^2 + y^2 + z^2 = 4 \), and has density \( x^2 \) at \((x, y, z)\). Find the total mass.

We set this up in cylindrical coordinates, recalling that \( r = r \cos \theta \):

\[
\int_0^1 \int_0^\pi \int_0^{\sqrt{4-r^2}} r^2 \cos^2(\theta) \, dr \, d\theta = \int_0^2 \int_0^\pi 2\sqrt{4-r^2} r \cos^2(\theta) \, dr \, d\theta = \frac{2\pi}{3}
\]

Spherical coordinates are somewhat more difficult to understand. The small volume we want will be defined by \( \Delta \rho, \Delta \theta, \) and \( \Delta \phi \), as pictured in figure 17.6.1. To gain a better understanding, just look at the Javaview applet. The small volume is nearly box-shaped, with 4 flat sides and two sides formed from bits of concentric spheres. When \( \Delta \rho, \Delta \theta, \) and \( \Delta \phi \) are all very small, the volume of this little region will be nearly the volume we get by treating it as a box. One dimension of the box is simply \( \Delta \rho \), the change in distance from the origin. The other two dimensions are the lengths of small circular arcs, so they are \( r \Delta \rho \Delta \phi \) for suitable \( r \) and \( \phi \) just as in the polar coordinates case.

The easiest of these to understand is the arc corresponding to a change in \( \phi \), which is nearly identical to the derivative for polar coordinates, as shown in the left graph in figure 17.6.2. In that graph we are looking “face on” at the side of the box we are interested in, so the small angle pictured is so close to \( \Delta \theta \), the horizontal axis really is the z axis, but the horizontal axis is not a real axis—it is just some line in the x-y plane. Because the
10. An object occupies the region inside the unit sphere at the origin, and has density proportional to the distance from its axis of symmetry.

11. An object occupies the region inside the unit sphere at the origin, and has density equal to the distance from the x-axis. Find the mass.

12. An object occupies the region inside the unit sphere at the origin, and has density equal to the square of the distance from the origin. Find the mass.

13. An object occupies the region inside the unit sphere at the origin, and has density equal to the square of the distance from the origin. Find the mass.

14. An object occupies the region bounded by the plane x = 0, y = 0, and z = 0. Find the mass.

15. An object occupies the region bounded by the cone x = y^2 and the plane y = 0. Find the mass.

16. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

17. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

18. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

19. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

20. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

21. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

22. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

23. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

24. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

25. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

26. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

27. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

28. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

29. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

30. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

31. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

32. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

33. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

34. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

35. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

36. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

37. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

38. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

39. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

40. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

41. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

42. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

43. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

44. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

45. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

46. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

47. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

48. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

49. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.

50. An object occupies the region bounded by the paraboloid z = x^2 + y^2 and the plane z = 1. Find the mass.
coordinates:
\[
\int_{0}^{\pi} \int_{0}^{a} r \, dr \, d\theta = \frac{\pi a^2}{2}
\]
But let’s instead approach this as a substitution problem, starting with \( x = r \cos \theta, \ y = r \sin \theta \). This pair of equations describes a function from “\( r \theta \) space” to “\( x \ y \) space”, and because it involves familiar concepts, it is not too hard to understand what it does. In figure 17.7.2 we have indicated geometrically a bit about how this function behaves. The four dots labeled \( a \) in the \( r \theta \) plane correspond to the three dots in the \( x \ y \) plane; dots \( a \) and \( b \) both go to the origin, \( b \) at \( r = \pi \) everywhere and \( \theta \) ranges from 0 to \( \pi \), so the corresponding points \( x = r \cos \theta, \ y = r \sin \theta \) start at \((1,0)\) and follow the unit circle counter-clockwise. Finally, the vertical arrow has \( \theta = \pi/4 \) and \( r \) ranges from 0 to 1, so it maps to the straight arrow in the \( x \ y \) plane. Extrapolating from these few examples, it’s not hard to see that every vertical line in the \( r \theta \) plane is transformed to a line through the origin in the \( x \ y \) plane, and every horizontal line in the \( r \theta \) plane is transformed to a circle with center at the origin in the \( x \ y \) plane.

Since we are interested in integrating over the half-disk in the \( r \theta \) plane, we will integrate over the rectangle \([0, \pi] \times [0, 1]\) in the \( r \theta \) plane, because we now see that the points in this rectangle are sent precisely to the upper half disk by \( x = r \cos \theta \) and \( y = r \sin \theta \).

At this point we are two-thirds done with the task: we know the \( r \theta \) limits of integration, and we can easily convert the function to the new variables:
\[
\sqrt{x^2 + y^2} = \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta} = r \sqrt{\cos^2 \theta + \sin^2 \theta} = r.
\]
(17.7.1)
The final, and most difficult, task is to figure out what replaces \( dx \, dy \). (Of course, we actually know the answer, because we are in effect converting to polar coordinates. What we really want is a series of steps that gets us to that right answer but that will also work for other substitutions that are not so familiar.)

Let’s take a step back and remember how integration arises from approximation. When we approximate the integral in the \( x \ y \) plane, we are computing the volumes of tall thin boxes, in this case boxes that are \( dx \times dy \times \sqrt{x^2 + y^2} \). We are aiming to come up with an integral in the \( r \theta \) plane that looks like this:
\[
\int_{0}^{\pi} \int_{0}^{a} r \, dr \, d\theta.
\]
(17.7.2)
What we’re missing is exactly the right quantity to replace the “\( r \)” so that we get the correct answer. Of course, this integral is also the result of an approximation, in which we add up volumes of boxes that are \( \Delta r \times \Delta \theta \times \text{height} \); the problem is that the height that will give us the correct answer is not simply \( r \). Or put another way, we can think of the correct height as \( r \), but the area of the base \( \Delta r \Delta \theta \) as being wrong. The height \( r \) comes from equation 17.7.1, which is to say, it is precisely the same as the corresponding height in the \( x-y \) version of the integral. The problem is that the area of the base \( \Delta r \Delta \theta \) is not the same as the area of the base \( \Delta r \Delta \theta \). We can think of the “\( r \)” in the integral as a correction factor that is needed so that \( r \, dr \, d\theta = dx \, dy \).

So let’s think about what that little base \( \Delta r \Delta \theta \) corresponds to. We know that each bit of horizontal line in the \( r \theta \) plane corresponds to a bit of circular arc in the \( r \theta \) plane, and each bit of vertical line in the \( r \theta \) plane corresponds to a bit of “radial line” in the \( x \ y \) plane. In figure 17.7.3 we show a typical rectangle in the \( r \theta \) plane and its corresponding area in the \( x \ y \) plane.

For \( r \), a vector tangent to this path is given by the derivative \( v'(r) = (\cos \theta_0, \sin \theta_0, 0) \) and a small tangent vector, with length approximately equal to the side of the region, is \( (\cos \theta_0, \sin \theta_0, 0) \, dr \). Likewise, if we fix \( r = r_0 \approx 0.5 \), we get the vector function \( w(\theta) = (r_0 \cos \theta, r_0 \sin \theta, 0) \) with derivative \( w'(\theta) = (r_0 \cos \theta, r_0 \sin \theta, 0) \) and a small tangent vector \( (-r_0 \sin \theta, r_0 \cos \theta, 0) \, d\theta \) when \( \theta = \theta_0 \). These vectors are shown in figure 17.7.4, with the actual region outlined by a dotted boundary. Of course, since both \( \Delta r \) and \( \Delta \theta \) are quite large, the parallelogram is not a particularly good approximation to the true area.

The area of this parallelogram is the length of the cross product:
\[
\begin{vmatrix}
- r_0 \sin \theta_0 & r_0 \cos \theta_0 & 0 \\
0 & \cos \theta_0 & \sin \theta_0 \\
0 & -\sin \theta_0 & \cos \theta_0 \\
\end{vmatrix} \, dr \, d\theta = r_0 \, dr \, d\theta.
\]

The length of this vector is \( r_0 \, dr \, d\theta \) so in general, for any values of \( r \) and \( \theta \), the area in the \( x \ y \) plane corresponding to a small rectangle anchored at \((0, \theta)\) in the \( r \theta \) plane is approximately \( r \, dr \, d\theta \). In other words, “\( r \)” replaces the “\( r \)” in equation 17.7.2.

In general, a substitution will start with equations \( x = f(u, v) \) and \( y = g(u, v) \). Again, it will be straightforward to convert the function being integrated. Converting the limits will require, as above, an understanding of just how the functions \( f \) and \( g \) transform the \( u \ v \) plane into the \( x \ y \) plane. Finally, the small vectors we need to approximate an area will be \((f_\theta, g_\theta, 0) \) \( du \) and \((f_r, g_r, 0) \) \( dv \). The cross product of these is \((0, 0, f_g \times g_f) \) \( du \, dv \). Hence we have \( f_g \times g_f \) \( f_r \) \( g_r \) \( du \, dv \). The quantity \( f_g \times g_f \) \( du \, dv \) is usually denoted
\[
\frac{\partial(x, y)}{\partial(u, v)} = f_g \times g_f.
\]

and called the Jacobian. Note that this is the absolute value of the two by two determinant
\[
\begin{vmatrix}
f_r & g_r \\
g_r & f_r \\
\end{vmatrix},
\]
which may be easier to remember. (Confusingly, the matrix, the determinant of the matrix, and the absolute value of the determinant are all called the Jacobian by various authors.)

Because there are two things to worry about, namely, the form of the function and the region of integration, transformations in two (or more) variables are quite tricky to discover.

EXAMPLE 17.7.1 Integrate \( x^2 + y^2 \) over the region \( x^2 + y^2 \leq 2 \).

The equation \( x^2 + y^2 = 2 \) describes an ellipse as in figure 17.7.5; the region of integration is the interior of the ellipse. We will use the transformation \( x = \sqrt{2u - \sqrt{2} \theta} \) and \( y = \sqrt{2u + \sqrt{2} \theta} \). Substituting into the function itself we get
\[
x^2 + y^2 = 2u^2 + 2u \theta.
\]
The boundary of the ellipse is \( x^2 + y^2 = 2 \), so the boundary of the corresponding region in the \( u \, v \) plane is \( 2u^2 + 2u \theta = 2 \) or \( u^2 + \theta = 1 \), the unit circle, so this substitution makes the region of integration simpler.

Next, we compute the Jacobian, using \( f \omega = \sqrt{2u - \sqrt{2} \theta} \) and \( y = \sqrt{2u + \sqrt{2} \theta} \):
\[
\begin{vmatrix}
f_u & g_u \\
g_u & f_u \\
\end{vmatrix} = \frac{2u}{\sqrt{2u^2 - 2}}\frac{\sqrt{2u^2 - 2}}{\sqrt{2u^2 + 2}} = 4.
\]

Hence the new integral is
\[
\int\int_R (2u^2 + 2u \theta) \frac{4}{\sqrt{2u^2 - 2}} \, du \, dv,
\]
where \( R \) is the interior of the unit circle. This is still not an easy integral, but it is easily transformed to polar coordinates, and then easily integrated.
There is a similar change of variables formula for triple integrals, though it is a bit more difficult to derive. Suppose we use three substitution functions, \( x = f(u, v, w) \), \( y = g(u, v, w) \), and \( z = h(u, v, w) \). The Jacobian determinant is now
\[
\begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}
\end{vmatrix}
\]
Then the integral is transformed in a similar fashion:
\[
\iiint_R F(x, y, z) \, dV = \iiint_S F(f(u, v, w), g(u, v, w), h(u, v, w)) \left| \frac{\partial (x, y, z)}{\partial (u, v, w)} \right| \, du \, dv \, dw,
\]
where of course the region \( S \) in \( uvw \) space corresponds to the region \( R \) in \( xyz \) space.

**Exercises 17.7.**

1. Complete example 17.7.1 by converting to polar coordinates and evaluating the integral.

2. Evaluate \( \iiint_S x \, dx \, dy \) over the square with corners \((0, 0), (1, 1), (2, 0), \) and \((1, -1)\) in two ways: directly, and using \( x = \frac{u + v}{2}, y = \frac{u - v}{2} \).

3. Evaluate \( \iiint_S x^2 + y^2 \, dx \, dy \) over the square with corners \((-1, 0), (0, 1), \) and \((0, -1)\) in two ways: directly, and using \( x = \frac{u + v}{2}, y = \frac{u - v}{2} \).

4. Evaluate \( \iiint_S (x + y)e^{-x+y} \, dx \, dy \) over the triangle with corners \((0, 0), (-1, 1), \) and \((1, 1)\) in two ways: directly, and using \( x = \frac{u + v}{2}, y = \frac{u - v}{2} \).

5. Evaluate \( \iiint_S x(x - y) \, dx \, dy \) over the parallelogram with corners \((0, 0), (3, 3), (7, 3), \) and \((4, 0)\) in two ways: directly, and using \( x = u + v, y = w \).

6. Evaluate \( \iiint_S \sqrt{x^2 + y^2} \, dx \, dy \) over the triangle with corners \((0, 0), (1, 0), \) and \((0, -1)\) using \( x = u, y = v \).

7. Evaluate \( \iiint_S y \sin(x+y) \, dx \, dy \) over the region bounded by \( xy = 1, xy = 4, y = 1, \) and \( y = 4 \) using \( x = u/v, y = v \).

8. Evaluate \( \iiint_S \sin(9x^2 + 4y^2) \, dA \) over the region in the first quadrant bounded by the ellipse \( 9x^2 + 4y^2 = 1 \).

9. Compute the Jacobian for the substitutions \( x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi \).