17
Multipl e Integration

17.1 Volume and Average Height

Consider a surface \( f(x, y) \); you might temporarily think of this as representing physical topography—a hilly landscape, perhaps. What is the average height of the surface (or average altitude of the landscape) over some region?

As with most such problems, we start by thinking about how we might approximate the answer. Suppose the region is a rectangle, \([a, b] \times [c, d]\). We can divide the rectangle into a grid, \( m \times n \) subregions in one direction and \( n \times m \) in the other, as indicated in figure 17.1.1. We pick \( x \) values \( x_0, x_1, \ldots, x_m \) in each subdivision in the \( x \) direction, and similarly in the \( y \) direction. At each of the points \((x_i, y_j)\) in one of the smaller rectangles in the grid, we compute the height of the surface: \( f(x_i, y_j) \). Now the average of these heights should be (depending on the fineness of the grid) close to the average height of the surface:

\[
\frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_i, y_j) \Delta y \Delta x
\]

As both \( m \) and \( n \) go to infinity, we expect this approximation to converge to a fixed value, the actual average height of the surface. For reasonably nice functions this does indeed happen.

Figure 17.1.2 Approximating the volume under a surface.

more "generic" than writing \( \int_a^b \int_c^d f(x, y) \, dx \, dy \). The next question, of course, is: How do we compute these double integrals? You might think that we will need some two-dimensional version of the Fundamental Theorem of Calculus, but as it turns out we can get away with just the single variable version, applied twice.

Going back to the double sum, we can rewrite it to emphasize a particular order in which we want to add the terms:

\[
\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_i, y_j) \Delta y \Delta x
\]

In the sum in parentheses, only the value of \( x_j \) is changing; \( y_i \) is temporarily constant. As \( m \) goes to infinity, this sum has the right form to turn into an integral:

\[
\lim_{m \to \infty} \sum_{i=0}^{m-1} f(x_i, y_j) \Delta x = \int_a^b f(x, y) \, dx.
\]

So after we take the limit as \( m \) goes to infinity, the sum is

\[
\int_a^b \int_c^d f(x, y) \, dx \, dy.
\]

Using sigma notation, we can rewrite the approximation:

\[
\frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_i, y_j) \Delta y \Delta x
\]

The two parts of this product have useful meaning: \((b-a)(d-c)\) is of course the area of the rectangle, and the double sum adds up \( m \times n \) terms of the form \( f(x_i, y_j) \Delta y \Delta x \), which is the height of the surface at a point times the area of one of the small rectangles into which we have divided the large rectangle. In short, each term \( f(x_i, y_j) \Delta y \Delta x \) is the volume of a tall, thin, rectangular box, and is approximately the volume under the surface and above one of the small rectangles; see figure 17.1.2. When we add all of these up, we get an approximation to the volume under the surface and above the rectangle \([a, b] \times [c, d]\). When we take the limit as \( m \) and \( n \) go to infinity, the double sum becomes the actual volume under the surface, which we divide by \((b-a)(d-c)\) to get the average height.

Double sums like this come up in many applications, so in a way it is the most important part of this example; dividing by \((b-a)(d-c)\) is a simple extra step that allows the computation of an average. As we did in the single variable case, we introduce a special notation for the limit of such a double sum:

\[
\lim_{m,n \to \infty} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_i, y_j) \Delta y \Delta x
\]

the double integral of \( f \) over the region \( R \). The notation \( dA \) indicates a small bit of area, without specifying any particular order for the variables \( x \) and \( y \); it is shorter and

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Of course, for different values of \( y \), this integral has different values; in other words, it is really a function applied to \( y \):

\[
G(y) = \int_a^b f(x, y) \, dx.
\]

If we substitute back into the sum we get

\[
\sum_{i=0}^{m-1} G(y_i) \Delta y \Delta x
\]

This sum has a nice interpretation. The value \( G(y_i) \) is the area of a cross-section of the region under the surface \( f(x, y) \), namely, when \( y = y_i \). The quantity \( G(y) \Delta y \) can be interpreted as the volume of a solid with face area \( G(y) \) and thickness \( \Delta y \). Think of the surface \( f(x, y) \) as the top of a loaf of sliced bread. Each slice has a cross-sectional area and a thickness; \( G(y) \Delta y \) corresponds to the volume of a single slice of bread. Adding these up approximates the total volume of the loaf. (This is very similar to the technique we used to compute volumes in section 8.3, except that there we need the cross-sections to be in some way "the same".) Figure 17.1.3 shows this "sliced loaf" approximation using the same surface as shown in figure 17.1.2. Nicely enough, this sum looks just like the sort of sum that turns into an integral, namely,

\[
\lim_{m,n \to \infty} \sum_{i=0}^{m-1} G(y_i) \Delta y
\]

the double integral of \( f \) over the region \( R \). The notation \( dA \) indicates a small bit of area, without specifying any particular order for the variables \( x \) and \( y \); it is shorter and

\[
\int_a^b \int_c^d f(x, y) \, dx \, dy
\]

the double integral of \( f \) over the region \( R \). The notation \( dA \) indicates a small bit of area, without specifying any particular order for the variables \( x \) and \( y \); it is shorter and
17.1 Volume and Average Height

Doing this gives us a volume of approximately 8.84, so the average height is approximately 8.84/6 ≈ 1.47.

Because addition and multiplication are commutative and associative, we can rewrite the original double sum:

\[ \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_j, y_i) \Delta x \Delta y = \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} f(x_j, y_i) \Delta x \Delta y \]

Now if we repeat the development above, the inner sum turns into an integral:

\[ \lim_{m,n \to \infty} \sum_{i=0}^{n-1} f(x_j, y_i) \Delta y = \int_{y_0}^{y_1} f(x_j, y) \, dy \]

and then the outer sum turns into an integral:

\[ \lim_{m,n \to \infty} \sum_{j=0}^{m-1} \left( \int_{y_0}^{y_1} f(x_j, y) \, dy \right) \, \Delta x = \int_{x_0}^{x_1} \int_{y_0}^{y_1} f(x, y) \, dy \, dx \]

In other words, we can compute the integrals in either order, first with respect to \( x \) then \( y \), or vice versa. Thinking of the loaf of bread, this corresponds to slicing the loaf in a direction perpendicular to the first.

We haven’t really proved that the value of a double integral is equal to the value of the corresponding two single integrals in either order of integration, but provided the function is reasonably nice, this is true; the result is called Fubini’s Theorem.

\[ \int \int_R f(x, y) \, dA = \int_{x_0}^{x_1} \left( \int_{y_0}^{y_1} f(x, y) \, dy \right) \, dx \]

\[ \int \int_R f(x, y) \, dA = \int_{y_0}^{y_1} \left( \int_{x_0}^{x_1} f(x, y) \, dx \right) \, dy \]

EXAMPLE 17.1.2 We compute \( \int_0^1 \int_0^1 (x - 1)^2 + 4y^2 \, dx \, dy \), where \( R = [0, 3] \times [0, 2] \), in two ways.

First,

\[ \int_0^1 \int_0^1 (x - 1)^2 + 4y^2 \, dx \, dy = \int_0^1 \int_0^1 (x - 1)^2 + 4y^2 \, dx \, dy \]

\[ = \int_0^1 \left[ 2 + 2(x - 1)^2 + \frac{32}{3} y^2 \right]_0^1 \, dy \]

\[ = 4 + \frac{8}{3} \cdot 4 + \frac{32}{3} \cdot \frac{1}{3} \]

\[ = 44 \]

In the other order:

\[ \int_0^2 \int_0^3 (x - 1)^2 + 4y^2 \, dx \, dy = \int_0^3 \int_0^1 (x - 1)^2 + 4y^2 \, dx \, dy \]

\[ = \int_0^3 \left[ 3 + \frac{8}{3} + 12y^2 + \frac{1}{3} y^4 \right]_0^3 \, dy \]

\[ = 6 + \frac{16}{3} + 32 + \frac{2}{3} \]

\[ = 44 \]

In this example there is no particular reason to favor one direction over the other; in some cases, one direction might be much easier than the other, so it’s usually worth considering the two different possibilities.

Frequently we will be interested in a region that is not simply a rectangle. Let’s compute the volume under the surface \( z = x + 2y^3 \), or vice versa. Thinking of the loaf of bread, this corresponds to slicing the loaf in a direction perpendicular to the first.

We haven’t really proved that the value of a double integral is equal to the value of the corresponding two single integrals in either order of integration, but provided the function is reasonably nice, this is true; the result is called Fubini’s Theorem.

\[ \int \int_R f(x, y) \, dA = \int_{x_0}^{x_1} \left( \int_{y_0}^{y_1} f(x, y) \, dy \right) \, dx \]

\[ \int \int_R f(x, y) \, dA = \int_{y_0}^{y_1} \left( \int_{x_0}^{x_1} f(x, y) \, dx \right) \, dy \]

EXAMPLE 17.1.3 Find the volume of the region bounded above by the function \( z = x + 2y^3 \) and below by the plane \( z = 1 - x^2 \).

Let’s consider the two possible ways to set this up:

\[ \int_0^1 \int_0^1 \sqrt{1-x^2} \, dx \, dz \quad \text{or} \quad \int_0^1 \int_0^1 \sqrt{1-x^2} \, dz \, dx \]

Which appears easier? In the first, the first (inner) integral is easy, because we need an anti-derivative with respect to \( y \), and the entire integrand \( \sqrt{1-x^2} \) is constant with respect to \( y \). Of course, the second integral may be more difficult. In the second, the first integral is mildly unpleasant—a trig substitution. So let’s try the first one, since the first step is easy, and see where that leaves us.

\[ \int_0^1 \int_0^1 \sqrt{1-x^2} \, dx \, dz = \int_0^1 \left[ \frac{1}{2} \sqrt{1-x^2} \right]_0^1 \, dz \]

\[ = \int_0^1 \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot 1 \, dz \]

\[ = \frac{1}{12} \cdot 1 = \frac{1}{12} \]

\[ = \frac{1}{12} \]

This is quite easy, since the substitution \( u = 1 - x^2 \) works:

\[ \int_0^1 \int_0^1 \sqrt{1-x^2} \, dx \, dz = \int_0^1 \int_0^1 (1-x^2)^{1/2} \, dx \, dz \]

\[ = \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot 1 \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} \]

Then

\[ \int_0^1 (1-x^2)^{1/2} \, dx = \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \]

This is a good example of how the order of integration can affect the complexity of the problem. In this case it is possible to do the other order, but it is a bit messier. In some cases one order may lead to a very difficult or impossible integral; it’s usually worth considering both possibilities before going very far.
Suppose we have a surface given in cylindrical coordinates as $r = \theta$ and we wish to find the integral over some region. We could attempt to translate into rectangular coordinates and do the integration there, but it is often easier to stay in cylindrical coordinates.

How might we approximate the volume under such a surface in a way that uses cylindrical coordinates directly? The basic idea is the same as before: we divide the region into many small regions, multiply the area of each small region by the height of the surface somewhere in that little region, and add them up. What changes is the shape of the small regions; in order to have a nice representation in terms of $r$ and $\theta$, we use small pieces of ring-shaped areas, as shown in figure 17.2.1. Each small region is roughly rectangular, except that the two sides are segments of a circle and the other two sides are not quite parallel.

Near a point $(r, \theta)$ the length of either circular arc is about $r \Delta \theta$ and the length of each straight side is simply $\Delta r$. When $\Delta r$ and $\Delta \theta$ are very small, the region is nearly a rectangle with area $r \Delta r \Delta \theta$, and the volume under the surface is approximately

$$\sum \sum f(r, \theta) r \Delta r \Delta \theta.$$
EXAMPLE 17.2.2 Find the volume under \( z = \sqrt{r^2 - 4} \) above the region enclosed by the curve \( r = 2 \cos \theta \), \(-\pi/2 \leq \theta \leq \pi/2\); see figure 17.2.3. The region is described in polar coordinates by the inequalities \(-\pi/2 \leq \theta \leq \pi/2\) and \(0 \leq r \leq 2 \cos \theta\), so the double integral is

\[
\int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} \sqrt{r^2 - 4} \, r \, dr \, d\theta
\]

We can rewrite the integral as shown because of the symmetry of the volume; this avoids a complication during the evaluation. Proceeding:

\[
2 \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} \sqrt{r^2 - 4} \, r \, dr \, d\theta = 2 \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} (4 - r^2)^{1/2} \, dr \, d\theta
\]

\[
= 2 \int_{-\pi/2}^{\pi/2} \left[ \frac{8 \cos^3 \theta + 8 \cos \theta}{3} \right]^{2 \cos \theta} \, d\theta
\]

\[
= \frac{32}{9}
\]

Figure 17.2.2 Volume over a region with non-constant limits.

You might have learned a formula for computing areas in polar coordinates. It is possible to compute areas as volumes, so that you need only remember one technique. Consider the surface \( r = 1 \), a horizontal plane. The volume under this surface and above a region in the \( x-y \) plane is simply \( 1 \) (area of the region), so computing the volume really just computes the area of the region.

17.3 Moment and Center of Mass

Using a single integral we were able to compute the center of mass for a one-dimensional object with variable density, and a two-dimensional object with constant density. With a double integral we can handle two dimensions and variable density.

EXERCISES 17.2.

14. Find the volume under \( z = y^2 + x + 2 \) above the region \( x^2 + y^2 \leq 4 \) ⇒

15. Find the volume between \( z = y^2 + x \) and \( z = 0 \) above the region \( x^2 + y^2 \leq 1 \) ⇒

16. Find the volume inside \( x^2 + y^2 = 1 \) and \( x^2 + z^2 = 1 \) ⇒

17. Find the volume under \( z = 2 \) above \( r = 1 + 4 \sin \theta \) ⇒

18. Figure 17.2.4 shows the plot of \( r = 1 + 4 \sin \theta \).

a. Describe the behavior of the graph in terms of the given equation. Specifically, explain maximum and minimum values, number of leaves, and the ‘leaves within leaves’.

b. Give an integral or integrals to determine the area outside a smaller leaf but inside a larger leaf.

c. How would changing the value of \( a \) in the equation \( r = a + \cos \theta \) change the relative sizes of the inner and outer leaves? Focus on values \( a \geq 1 \). (Hint: How would we change the maximum and minimum values?)

d. Consider the integral \( \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a+\cos \theta} \frac{1}{\sqrt{r^2-a^2}} \, dr \, d\theta \), where \( D \) is the unit disk centered at the origin. (See the graph here.)

a. Why might this integral be considered improper?

b. Calculate the value of the integral of the same function \( 1/\sqrt{r^2-a^2} \) over the annulus with outer radius \( b \) and inner radius \( a \).

c. Obtain a formula for the integral on the whole disk by letting \( \delta \) approach 0. ⇒

d. For which values \( a \) can we replace the denominator with \((x^2+y^2)^{1/2}\) in the original integral and still get a finite value for the improper integral?

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EXAMPLE 17.2.3 Find the area outside the circle \( r = 2 \) and inside \( r = 4 \sin \theta \); see figure 17.2.3. The region is described by \( \pi/6 \leq \theta \leq \pi/2 \) and \( 2 \leq r \leq 4 \sin \theta \), so the integral is

\[
\int_{\pi/6}^{\pi/2} \int_2^{4 \sin \theta} r \, dr \, d\theta = \int_{\pi/6}^{\pi/2} \left[ \frac{r^2}{2} \right]_2^{4 \sin \theta} \, d\theta
\]

\[
= \int_{\pi/6}^{\pi/2} \frac{1}{2} (8 \sin^2 \theta - 2 \theta) \, d\theta
\]

\[
= \frac{4}{3} \theta^2 + 2 \sqrt{3} \sin \theta - \theta \cos \theta + \sqrt{3} \sin^2 \theta \bigg|_{\pi/6}^{\pi/2}
\]

\[
= \frac{4 \pi}{9}
\]

Figure 17.2.3 Finding area by computing volume.

EXERCISES 17.3.1

1. Find the center of mass of a thin, uniform plate whose shape is the region between \( y = \cos x \) and the \( y \)-axis between \( x = -\pi/2 \) and \( x = \pi/2 \).

2. Since the density is constant, we must take \( \sigma(x, y) = 1 \).

3. It is clear that \( \bar{x} = 0 \), but for practice let’s compute it anyway. First we compute the mass:

\[
M = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} 1 \, dy \, dx = \int_{-\pi/2}^{\pi/2} \cos x \, dx = \sin \theta^{\pi/2} = 2
\]

Next, the moment about the \( x \)-axis:

\[
M_y = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} y \, dy \, dx = \int_{-\pi/2}^{\pi/2} \frac{1}{2} \cos^2 x \, dx = \frac{\pi}{4}
\]

Finally, the moment about the \( y \)-axis:

\[
M_x = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} x \, dy \, dx = \int_{-\pi/2}^{\pi/2} \cos x \, dx = 0
\]

So \( \bar{x} = 0 \) as expected, and \( \bar{y} = \sqrt{\pi}/4 = \pi/8 \). This is the same problem as in example 11.1.4; it may be helpful to compare the two solutions.
EXAMPLE 17.3.2 Find the center of mass of a two-dimensional plate that occupies the quarter circle $x^2 + y^2 \leq 1$ in the first quadrant and has density $k(x^2 + y^2)$. It seems clear that because of the symmetry of both the region and the density function (both are important!), $\bar{r} = \bar{y}$. We'll do both to check our work.

Jumping right in:

$$M = \frac{1}{3} \int_0^{\pi/2} \int_0^1 (x^2 + y^2) \, r \, dr \, d\theta = k \int_0^{\pi/2} \frac{1}{3} (1 - r^2) \, d\theta.$$

This integral is something we can do, but it's a bit unpleasant. Since everything in sight is related to a circle, let's back up and try polar coordinates. Then $x^2 + y^2 = r^2$ and

$$M = k \int_0^{\pi/2} r^3/2 \, d\theta = k \int_0^{1} r^3/2 \, dr = k \frac{1}{5}.$$

Much better. Next, since $y = r \sin \theta$,

$$M_x = k \int_0^{\pi/2} \int_0^1 r^2 \, d\theta = k \int_0^{1} \frac{1}{2} \sin \theta \, dr = k \frac{1}{2} \sin \theta \biggr|_0^1 = 0.$$

Similarly,

$$M_y = k \int_0^{\pi/2} \int_0^1 r^2 \, d\theta = k \int_0^{1} \frac{1}{2} \cos \theta \, dr = k \frac{1}{2} \cos \theta \biggr|_0^1 = 0.$$

Finally, $\bar{x} = \bar{y} = \frac{8}{5\pi}$.

Exercises 17.3.

1. Find the center of mass of a two-dimensional plate that occupies the square $[0, 1] \times [0, 1]$ and has density function $y \Rightarrow$

2. Find the center of mass of a two-dimensional plate that occupies the triangle $0 \leq x \leq 1$, $0 \leq y \leq x$, and has density function $xy \Rightarrow$

3. Find the center of mass of a two-dimensional plate that occupies the upper unit semicircle centered at $(0, 0)$ and has density function $x^2 \Rightarrow$

4. Find the center of mass of a two-dimensional plate that occupies the upper unit semicircle centered at $(0, 1)$ and has density function $y^3 \Rightarrow$

5. Find the center of mass of a two-dimensional plate that occupies the triangle formed by $x = 2$, $y = x$, and $y = 2x$ and has density function $2x \Rightarrow$

6. Find the center of mass of a two-dimensional plate that occupies the triangle formed by $x = 1$, $y = x$, and $2x + y = 6$ and has density function $x^3 \Rightarrow$.

Figure 17.4.1 Small parallelograms at points of tangency. (AP)

EXAMPLE 17.4.1 We find the area of the hemisphere $z = \sqrt{1 - x^2 - y^2}$. We compute the derivatives

$$f_x = \frac{-x}{\sqrt{1 - x^2 - y^2}} \quad f_y = \frac{-y}{\sqrt{1 - x^2 - y^2}},$$

and then the area is

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 - x^2 - y^2} + \frac{x^2}{2} + \frac{y^2}{2} + 1 \, dy \, dx.$$

This is a bit on the messy side, but we can use polar coordinates:

$$\int_{0}^{2\pi} \int_{0}^{1} \sqrt{1 - r^2} \, r \, dr \, d\theta.$$

This integral is improper, since the function is undefined at the limit 1. We therefore compute

$$\lim_{a \to 1} \int_{0}^{a} \frac{1}{\sqrt{1 - r}} \, r \, dr = \lim_{a \to 1} -\sqrt{1 - x^2} + 1 = 1,$$

using the substitution $u = 1 - r^2$. Then the area is

$$\int_{0}^{2\pi} \frac{1}{2} \, d\theta = \pi.$$

You may recall that the area of a sphere of radius $r$ is $4\pi r^2$, so half the area of a unit sphere is $(1/2)(4\pi r^2) = 2\pi$, in agreement with our answer.

17.4 Surface Area

We next seek to compute the area of a surface above (or below) a region in the $x$-$y$ plane. How might we approximate this? We start, as usual, by dividing the region into a grid of small rectangles. We want to approximate the area of the surface above one of these small rectangles. The area is very close to the area of the tangent plane above the small rectangle. If the tangent plane just happened to be horizontal, of course the area would simply be the area of the rectangle. For a typical plane, however, the area is the area of a parallelogram, as indicated in figure 17.4.1. Note that the area of the parallelogram is obviously larger the more “tilted” the tangent plane.

The cross product of these vectors is $\mathbf{u} \times \mathbf{v} = (f_y, f_z - f_x, f_x - f_y)$.

As before, the limits need not be constant.

EXAMPLE 17.5.1 We use an integral to compute the volume of the box with opposite corners at $(0, 0, 0)$ and $(1, 2, 3)$.

$$\int_{0}^{1} \int_{0}^{2} \int_{0}^{3} 1 \, dx \, dy \, dz = \int_{0}^{1} \int_{0}^{2} \int_{0}^{3} 1 \, dy \, dx \, dz = \int_{0}^{3} \int_{0}^{1} \int_{0}^{2} 3 \, dz \, dx \, dy = \int_{0}^{3} 6 \, dx = 6.$$

Of course, this is more interesting and useful when the limits are not constant.

EXAMPLE 17.5.2 Find the volume of the tetrahedron with corners at $(0, 0, 0)$, $(2, 3, 0)$, and $(2, 3, 5)$.
The whole problem comes down to correctly describing the region by inequalities:
\[0 \leq x \leq 2, 3x/2 \leq y \leq 5, 0 \leq z \leq 1/2.\] The lower \( y \) limit comes from the equation of the line \( y = 3x/2 \) that forms one edge of the tetrahedron in the \( x-y \) plane; the upper \( z \) limit comes from the equation of the plane \( z = 5/2 \) that forms the “upper” side of the tetrahedron; see figure 17.5.1. Now the volume is
\[
\int_0^2 \int_{3x/2}^5 \frac{3x^2}{2} dy dx = \int_0^2 \left[ \frac{3x^2}{2} y \right]_{3x/2}^5 dx = \int_0^2 \frac{3x^2}{2} \left( \frac{5}{2} - \frac{3x}{2} \right) dx
\]
\[
= \frac{15}{4} - \frac{15x^3}{8} \bigg|_0^2 = 15 - 10 = 5.
\]
Pretty much just the way we did for two dimensions we can use triple integration to compute mass, center of mass, and various average quantities.

**EXAMPLE 17.5.3** Suppose the temperature at a point is given by \( T = xyz \). Find the average temperature in the cube with opposite corners at \((0,0,0)\) and \((2,2,2)\).

In two dimensions we add up the temperature at each point and divide by the area; here we add up the temperatures and divide by the volume, 8:
\[
\frac{1}{8} \int_0^2 \int_0^2 \int_0^2 xyz \, dx \, dy \, dz = \frac{1}{8} \int_0^2 \int_0^2 \left[ \frac{x^2y^2}{2} \right]_0^2 \, dy \, dx
\]
\[
= \frac{1}{8} \int_0^2 \int_0^2 2^3 \, dy \, dx = \frac{1}{8} \int_0^2 2 \cdot 2 \, dx = \frac{1}{2} \int_0^2 2 \, dx = \frac{3}{4}.
\]

**EXAMPLE 17.5.4** Suppose the density of an object is given by \( xz \); and the object occupies the tetrahedron with corners \((0,0,0)\), \((0,1,0)\), \((1,1,0)\), and \((0,1,1)\). Find the mass and center of mass of the object.

**Exercises 17.5.**

1. Evaluate \( \int_0^1 \int_0^1 \int_0^1 x^2 + y - 1 \, dx \, dy \, dz = \)

2. Evaluate \( \int_0^1 \int_0^1 \int_0^1 x^2 + yz + z \, dx \, dy \, dz = \)

3. Evaluate \( \int_0^1 \int_0^1 \int_0^1 x^2 e^{x+z} \, dx \, dy \, dz = \)

4. Evaluate \( \int_0^1 \int_0^1 \int_0^1 x^2 e^{x+z} \, dx \, dy \, dz = \)

5. Evaluate \( \int_0^1 \int_0^1 \int_0^1 x \cos \theta \, dx \, dy \, dz = \)

6. Evaluate \( \int_0^1 \int_0^1 \int_0^1 r \, dx \, dy \, dz = \)

7. Evaluate \( \int_0^1 \int_0^1 \int_0^1 e^{x+y} \, dx \, dy \, dz = \)

8. Compute \( \int_0^1 \int_0^1 \int_0^1 z \sin x + z \cos y \, dx \, dy \, dz = \)

9. For each of the integrals in the previous exercise, give a description of the volume (both algebraic and geometric) that is the domain of integration.

10. Compute \( \int_0^1 \int_0^1 \int_0^1 \frac{1}{x} \, dx \, dy \, dz \) over the region \( x^2 + y^2 + z^2 \leq 1 \) in the first octant.

11. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one corner.

12. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one edge.

13. An object occupies the volume of the upper hemisphere of \( x^2 + y^2 + z^2 = 4 \) and has density \( z \) at \((x,y,z)\). Find the center of mass.

14. An object occupies the volume of the pyramid with corners at \((1,1,0), (-1,-1,0), (-1,1,0), \) and \((0,0,2)\) and has density \( z^3 \) at \((x,y,z)\). Find the center of mass.

15. Verify the moments \( M_{xy}, M_{xz}, \) and \( M_{yz} \) of example 17.5.4 by evaluating the integrals.

16. Find the region \( R \) for which \( \int_0^1 \int_0^1 \int_0^1 (1 - x^2 - y^2 - z^2) \, dx \, dy \, dz \) is a maximum.

### 17.6 Cylindrical and Spherical Coordinates

We have seen that sometimes double integrals are simplified by doing them in polar coordinates; not surprisingly, triple integrals are sometimes simpler in cylindrical coordinates or spherical coordinates. To set up integrals in polar coordinates, we had to understand the shape and area of a typical small region into which the region of integration was divided. We need to do the same thing here, for three dimensional regions.

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The cylindrical coordinate system is the simplest, since it is just the polar coordinate system plus a \( z \) coordinate. A typical small unit of volume in the shape shown in figure 17.2.1 “fattened up” in the \( z \) direction, so its volume is \( r \, dr \, d\theta \, dz \), or in the limit, \( r \, dr \, d\theta \, dz \).

**EXAMPLE 17.6.1** Find the volume under \( z = \sqrt{x^2 + y^2} \) above the quarter circle inside \( x^2 + y^2 = 4 \) in the first quadrant.

We could of course do this with a double integral, but we’ll use a triple integral:
\[
\int_{\theta=0}^{\pi/4} d\theta \int_{r=0}^{2 \cos \theta} r^2 \, dr \, dz = \int_{\theta=0}^{\pi/4} d\theta \int_{r=0}^{2 \cos \theta} r^3 \, dr \, dz = \frac{16}{3}.
\]
Compare this to example 17.2.1.

**EXAMPLE 17.6.2** An object occupies the space inside both the cylinder \( x^2 + y^2 = 1 \) and the sphere \( x^2 + y^2 + z^2 = 4 \), and has density \( z^3 \) at \((x,y,z)\). Find the total mass. The small volume we want will be defined by \( \Delta \rho, \Delta \phi, \) and \( \Delta \theta \), as pictured in figure 17.6.1. To gain a better understanding, see the Java applet. The small volume is nearly box shaped, with 4 flat sides and two sides formed from bits of concentric spheres. When \( \Delta \rho, \Delta \phi, \) and \( \Delta \theta \) are all very small, the volume of this little region will be nearly the volume we got by treating it as a box. One dimension of the box is simply \( \Delta \rho \), the change in distance from the origin. The other two dimensions are the lengths of small circular arcs, so they are \( r \, \Delta \rho \) for some suitable \( r \) and \( \rho \), just as in the polar coordinates case.

The easiest of these to understand is the arc, corresponding to a change in \( \phi \), which is nearly identical to the derivation for polar coordinates, as shown in the left graph in figure 17.6.2. In that graph we are looking “face on” at the side of the box we are interested in, so the small angle pictured is precisely \( \Delta \phi \). The vertical axis really is the \( z \) axis, but the horizontal axis is not a real axis—it is just some line in the \( x-y \) plane. Because the
Find the average temperature in the unit sphere centered at the origin. Suppose the temperature at \((x, y, z)\) is \(T = 1 / (1 + x^2 + y^2 + z^2)\). Find the average temperature in the unit sphere centered at the origin.

**Example 17.6.3** Suppose the temperature at \((x, y, z)\) is \(T = 1 / (1 + x^2 + y^2 + z^2)\). Find the average temperature in the unit sphere centered at the origin.

**10.** An object occupies the region in the first octant bounded by the cones \(\phi = \pi / 4\) and \(\phi = \arctan(2)\), and the sphere \(\rho = \sqrt{x^2 + y^2 + z^2}\) has density proportional to the distance from the origin. Find the mass. \(\Rightarrow\)

**11.** Find the mass of a right circular cone of height \(h\) and base radius \(a\) if the density is proportional to the distance from the base. \(\Rightarrow\)

**12.** An object occupies the region inside the unit sphere at the origin, and has density equal to \(\frac{3}{4\pi} (x^2 + y^2)\). Find the mass. \(\Rightarrow\)

### 17.7 Change of Variables

One of the most useful techniques for evaluating integrals is substitution, both "u-substitution" and trigonometric substitution, in which we change the variable to something more convenient. As we have seen, sometimes changing from rectangular coordinates to another coordinate system is helpful, and this too changes the variables. This is certainly a more complicated change, since instead of changing one variable for another we change an entire suite of variables, but as it turns out it is really very similar to the kinds of change of variables we already know as substitution.

**13.** An object occupies the region inside the unit sphere at the origin, and has density equal to the square of the distance from the origin. Find the mass. \(\Rightarrow\)

**14.** An object occupies the region between the unit sphere at the origin and a sphere of radius \(a\) with center at the origin, and has density equal to the distance from the origin. Find the mass. \(\Rightarrow\)

**15.** An object occupies the region in the first octant bounded by the cones \(\phi = \pi / 4\) and \(\phi = \arctan(2)\), and the sphere \(\rho = \sqrt{x^2 + y^2 + z^2}\) has density proportional to the distance from the origin. Find the mass. \(\Rightarrow\)

In two dimensions we add up the temperature at "each" point and divide by the area; here we add up the temperatures and divide by the volume, \(\left(4/3\pi\right)\).

\[
\frac{3}{4\pi} \int_0^1 \int_{-\pi/2}^{\pi/2} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{1+x^2+y^2} \, dz \, dy \, dx
\]

This looks quite messy, since everything in the problem is closely related to a sphere, we'll convert to spherical coordinates.

\[
\frac{3}{4\pi} \int_0^1 \int_{-\pi/2}^{\pi/2} \int_{\pi/4}^{\pi / 2} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

**Exercises 17.6.**

1. Evaluate \(\int \int \int_{V} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \). \(\Rightarrow\)

2. Evaluate \(\int \int \int_{V} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \). \(\Rightarrow\)

3. Evaluate \(\int \int \int_{V} x^2 y^2 \, dx \, dy \, dz \) over the interior of the cylinder \(x^2 + y^2 = 1\) between \(z = 0\) and \(z = 5\). \(\Rightarrow\)

4. Evaluate \(\int \int \int_{V} x^2 y \, dx \, dy \, dz \) over the interior of the cylinder \(x^2 + y^2 = 1\) between \(z = 0\) and \(z = 5\). \(\Rightarrow\)

5. Evaluate \(\int \int \int_{V} x^2 y \, dz \, dy \, dx \) over the region above the x-y plane, inside \(x^2 + y^2 = 2\) and under \(x^2 + y^2 + z^2 = 4\). \(\Rightarrow\)

6. Evaluate \(\int \int \int_{V} z^2 y \, dx \, dy \, dz \) over the region in the first octant, inside \(x^2 + y^2 + z^2 = 2\) and under \(x^2 + y^2 + z^2 = 4\). \(\Rightarrow\)

7. Evaluate \(\int \int \int_{V} x^2 y^2 \, dx \, dy \, dz \) over the interior of \(x^2 + y^2 + z^2 = 4\). \(\Rightarrow\)

8. Evaluate \(\int \int \int_{V} x^2y^2 \, dz \, dy \, dx \) over the interior of \(x^2 + y^2 = 4\). \(\Rightarrow\)

9. Compute \(\int \int \int_{V} x \, dz \, dy \, dx \) over the region inside \(x^2 + y^2 + z^2 = 1\) in the first octant. \(\Rightarrow\)

10. Find the mass of a right circular cone of height \(h\) and base radius \(a\) if the density is proportional to the distance from the base. \(\Rightarrow\)

11. Find the mass of a right circular cone of height \(h\) and base radius \(a\) if the density is proportional to the distance from its axis of symmetry. \(\Rightarrow\)

**12.** An object occupies the region inside the unit sphere at the origin, and has density equal to \(\frac{3}{4\pi} (x^2 + y^2)\). Find the mass. \(\Rightarrow\)

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coordinates: \[
\int_0^1 \int_0^1 \sqrt{x^2 + y^2} \, dx \, dy = \frac{\pi}{3}
\]

But let's instead approach this as a substitution problem, starting with \[ x = r \cos \theta, \quad y = r \sin \theta. \] This pair of equations describes a function from “\(r\)-\(\theta\) space” to “\(x\)-\(y\) space”, and because it involves familiar concepts, it is not too hard to understand what it does. In figure 17.7.2 we have indicated geometrically a bit about how this function behaves. The four dots labeled \(a\) in the \(r\)-\(\theta\) plane correspond to the three dots in the \(x\)-\(y\) plane; dots \(a\) and \(b\) both go to the origin because \(r = 0\). The horizontal arrow in the \(r\)-\(\theta\) plane has \(r = 1\) everywhere and \(\theta\) ranges from \(0\) to \(\pi\), so the corresponding points \(x = r \cos \theta, \quad y = r \sin \theta\) start at \((1, 0)\) and follow the unit circle counter-clockwise. Finally, the vertical arrow has \(\theta = \pi/4\) and \(r\) ranges from \(0\) to \(1\), so it maps to the straight arrow in the \(x\)-\(y\) plane. Extrapolating from these few examples, it’s not hard to see that every vertical line in the \(r\)-\(\theta\) plane is transformed to a line through the origin in the \(x\)-\(y\) plane, and every horizontal line in the \(r\)-\(\theta\) plane is transformed to a circle centered at the origin in the \(x\)-\(y\) plane.

Since we are interested in integrating over the half-disk in the \(x\)-\(y\) plane, we will integrate over the rectangle \([0, \pi] \times [0, 1]\) in the \(r\)-\(\theta\) plane, because we now see that the points in this rectangle are sent precisely to the upper half disk by \(x = r \cos \theta\) and \(y = r \sin \theta\).

At this point we are two-thirds done with the task: we know the \(r\)-\(\theta\) limits of integration, and we can easily convert the function to the new variables:

\[
\sqrt{x^2 + y^2} = \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta} = r \cos \theta + r \sin \theta = r.
\]

The final, and most difficult, task is to figure out what replaces the “\(\pi\)” so that we get the correct answer. Of course, this integral is also the result of an approximation, in which we add up volumes of boxes that are \(\Delta r \times \Delta \theta \times \text{height}\); the problem is that the height that will give us the correct answer is not simply \(r\). Or put another way, we can think of the correct height as \(r\), but the area of the base \(\Delta r \Delta \theta\) as being wrong. The height \(r\) comes from equation 17.7.1, which is to say, it is precisely the same as the corresponding height in the \(x\)-\(y\) version of the integral. The problem is that the area of the base \(\Delta r \Delta \theta\) is not the same as the area of the base \(\Delta x \Delta y\). We can think of the “\(\pi\)” in the integral as a correction factor that is needed so that \(\Delta r \Delta \theta = \Delta x \Delta y\).

So let’s think about what that little base \(\Delta r \Delta \theta\) corresponds to. We know that each bit of horizontal line in the \(r\)-\(\theta\) plane corresponds to a bit of circular arc in the \(x\)-\(y\) plane, and each bit of vertical line in the \(r\)-\(\theta\) plane corresponds to a bit of “radial line” in the \(x\)-\(y\) plane. In figure 17.7.3 we show a typical rectangle in the \(r\)-\(\theta\) plane and its corresponding area in the \(x\)-\(y\) plane.

For a vector tangent to this path is given by the derivative \(\psi'(r) = (r \cos \theta, r \sin \theta, 0)\), and a small tangent vector, with length approximately equal to the side of the region, is \((\cos \theta, \sin \theta, 0) \, dr\).

Likewise, if we fix \(r = r_0 = 0.5\), we get the vector function \(\psi(\theta) = (r_0 \cos \theta, r_0 \sin \theta, 0)\) with derivative \(\psi'(\theta) = (-r_0 \sin \theta, r_0 \cos \theta, 0)\) and a small tangent vector \(-r_0 \sin \theta, r_0 \cos \theta, 0\) \(\, dr\) when \(\theta = \theta_0\) (at the corner we’re focusing on). These vectors are shown in figure 17.4, with the actual region outlined by a dotted boundary. Of course, since both \(\Delta r\) and \(\Delta \theta\) are quite large, the parallelogram is not a particularly good approximation to the true area.

The area of this parallelogram is the length of the cross product:

\[
\begin{vmatrix}
-\cos \theta & r_0 \cos \theta & 0 \\
\sin \theta & r_0 \sin \theta & 0 \\
0 & 0 & 1
\end{vmatrix} \, dr \, d\theta = \begin{vmatrix} 1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{vmatrix} \, dr \, d\theta.
\]

The length of this vector is \(r \, dr \, d\theta\). So in general, for any values of \(r\) and \(\theta\), the area in the \(x\)-\(y\) plane corresponding to a small rectangle anchored at \((0, r)\) in the \(r\)-\(\theta\) plane is approximately \(r \, dr \, d\theta\). In other words, “\(\pi\)” replaces the “\(r^2\)” in equation 17.7.2.

In general, a substitution will start with equations \(x = f(x, y)\) and \(y = g(x, y)\). Again, it will be straightforward to convert the function being integrated. Converting the limits will require, as above, an understanding of just how the functions \(f\) and \(g\) transform the \(x\)-\(y\) plane into the \(r\)-\(\theta\) plane. Finally, the small vectors we need to approximate an area will be \((f_\theta, g_\theta, 0) \, du \, dv\) and \((f_x, g_y, 0) \, du \, dv\). The cross product of these is \((0, 0, f_\theta g_y - f_y g_\theta) \, du \, dv\) with length \(|f_\theta g_y - f_y g_\theta| \, du \, dv\). The quantity \(|f_\theta g_y - f_y g_\theta|\) is usually denoted

\[
\left| \frac{\partial (x, y)}{\partial (u, v)} \right| = |f_\theta g_y - f_y g_\theta|
\]

and called the Jacobian. Note that this is the absolute value of the two by two determinant

\[
f_x \cdot g_y - f_y \cdot g_x.
\]

which may be easier to remember. (Confusingly, the matrix, the determinant of the matrix, and the absolute value of the determinant are all called the Jacobian by various authors.)

Because there are two things to worry about, namely, the form of the function and the region of integration, transformations in two (or more) variables are quite tricky to discover.

**EXAMPLE 17.7.1** Integrate \(x^2 + y^2\) over the region \(x^2 + y^2 \leq 2\).

The equation \(x^2 + y^2 = 2\) describes an ellipse as in figure 17.7.5; the region of integration is the interior of the ellipse. We will use the transformation \(x = 2u + \sqrt{2}v\) and \(y = 2u - \sqrt{2}v\). Substituting into the function itself we get

\[
x^2 + y^2 = 4u^2 + 2v^2 + 2u v^2.
\]

The boundary of the ellipse is \(x^2 + y^2 = 2\), so the boundary of the corresponding region in the \(u-v\) plane is \(2u^2 + 2v^2 = 2\) or \(u^2 + v^2 = 1\), the unit circle, so this substitution makes the region of integration simpler.

Next, we compute the Jacobian, using \(f = 2u + \sqrt{2}v\) and \(g = 2u - \sqrt{2}v\):

\[
f_u \cdot g_v - f_v \cdot g_u = -\sqrt{2}\sqrt{\frac{2}{3}} + \sqrt{2}\sqrt{\frac{2}{3}} = 4
\]

Hence the new integral is

\[
\int \int_R \left(2u^2 + 2v^2\right) \frac{1}{4} \, du \, dv,
\]

where \(R\) is the interior of the unit circle. This is still not an easy integral, but it is easily transformed to polar coordinates, and then easily integrated.
There is a similar change of variables formula for triple integrals, though it is a bit more difficult to derive. Suppose we use three substitution functions, \( x = f(u, v, w) \), \( y = g(u, v, w) \), and \( z = h(u, v, w) \). The Jacobian determinant is now
\[
\left| \frac{\partial (x, y, z)}{\partial (u, v, w)} \right| = \begin{vmatrix}
 f_u & g_u & h_u \\
 f_v & g_v & h_v \\
 f_w & g_w & h_w \\
\end{vmatrix}.
\]

Then the integral is transformed in a similar fashion:
\[
\iiint_R F(x, y, z) \, dV = \iiint_S F(f(u, v, w), g(u, v, w), h(u, v, w)) \left| \frac{\partial (x, y, z)}{\partial (u, v, w)} \right| \, du \, dv \, dw,
\]
where of course the region \( S \) in \( uvw \) space corresponds to the region \( R \) in \( xyz \) space.

Exercises 17.7.

1. Complete example 17.7.1 by converting to polar coordinates and evaluating the integral.

2. Evaluate \( \iint xy \, dx \, dy \) over the square with corners \((0,0), (1,1), (1,-1), (0,-1)\) in two ways: directly, and using \( x = (u + v)/2, y = (u - v)/2 \).

3. Evaluate \( \iiint x^2 + y^2 \, dx \, dy \) over the square with corners \((-1,0), (0,1), (0,-1)\) in two ways: directly, and using \( x = (u + v)/2, y = (u - v)/2 \).

4. Evaluate \( \iiint (x + y)e^{-x} \, dx \, dy \) over the triangle with corners \((0,0), (-1,1), (1,1)\) in two ways: directly, and using \( x = (u + v)/2, y = (u - v)/2 \).

5. Evaluate \( \iint y(x - y) \, dx \, dy \) over the parallelogram with corners \((0,0), (3,3), (7,3), (4,0)\) in two ways: directly, and using \( x = u + v, y = w \).

6. Evaluate \( \iint \sqrt{x^2 + y^2} \, dx \, dy \) over the triangle with corners \((0,0), (4,4), (4,0)\) using \( x = u, y = v \).

7. Evaluate \( \iint y \sin(xy) \, dx \, dy \) over the region bounded by \( xy = 1, x = 4, y = 1, \) and \( y = 4 \) using \( x = u/v, y = v \).

8. Evaluate \( \iiint \sin(9x^2 + 4y^2) \, dA \) over the region in the first quadrant bounded by the ellipse \( 9x^2 + 4y^2 = 1 \).

9. Compute the Jacobian for the substitutions \( x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi \).

10. Evaluate \( \iiint dV \) where \( E \) is the solid enclosed by the ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \), using the transformation \( x = au, y = bv, \) and \( z = cw \).