17

Multiple Integration

17.1 Volume and Average Height

Consider a surface \( f(x, y) \); you might temporarily think of this as representing physical topography— a hilly landscape, perhaps. What is the average height of the surface (or average altitude of the landscape) over some region? As with most such problems, we start by thinking about how we might approximate the answer. Suppose the region is a rectangle, \([a, b] \times [c, d]\). We can divide the rectangle into a grid, \( m \) subdivisions in one direction and \( n \) in the other, as indicated in Figure 17.1.1. We pick \( x \) values \( x_0, x_1, \ldots, x_m \) in each subdivision in the \( x \) direction, and similarly in the \( y \) direction. At each of the points \((x_i, y_j)\) in one of the smaller rectangles in the grid, we compute the height of the surface: \( f(x_i, y_j) \). Now the average of these heights should be (depending on the fineness of the grid) close to the average height of the surface:

\[
\frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_i, y_j)
\]

As both \( m \) and \( n \) go to infinity, we expect this approximation to converge to a fixed value, the actual average height of the surface. For reasonably nice functions this does indeed happen.

Going back to the double sum, we can rewrite it to emphasize a particular order in which we want to add the terms:

\[
\sum_{i=0}^{m-1} \left( \sum_{j=0}^{n-1} f(x_i, y_j) \Delta x \right) \Delta y
\]

In the sum in parentheses, only the value of \( x \) is changing; \( y \) is temporarily constant. As \( n \) goes to infinity, this sum has the right form to turn into an integral:

\[
\lim_{n \to \infty} \sum_{i=0}^{m-1} \left( \sum_{j=0}^{n-1} f(x_i, y_j) \Delta x \right) \Delta y
\]

So after we take the limit as \( m \) goes to infinity, the sum is

\[
\int_a^b \int_c^d f(x, y) \, dx \, dy
\]

The two parts of this product have useful meaning: \((b - a)(d - c)\) is of course the area of the rectangle, and the double sum adds up \((y \text{-values})(x \text{-values})\) of the form \( f(x_i, y_j) \Delta x \Delta y \), which is the height of the surface at a point times the area of one of the small rectangles into which we have divided the large rectangle. In short, each term \( f(x_i, y_j) \Delta x \Delta y \) is the volume of a tall, thin, rectangular box, and is approximately the volume under the surface and above one of the small rectangles; see Figure 17.1.2. When we add all of these up, we get an approximation to the volume under the surface and above the rectangle \([a, b] \times [c, d]\).

We take when the limit as \( m \) and \( n \) go to infinity, the double sum becomes the actual volume under the surface, which we divide by \((y \text{-values})(x \text{-values})\) to get the average height.

Double sums like this come up in many applications, so in a way it is the most important part of this example; dividing by \((y \text{-values})(x \text{-values})\) is a simple extra step that allows the computation of an average. As we did in the single variable case, we introduce a special notation for the limit of such a double sum:

\[
\lim_{m \to \infty, \, n \to \infty} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_i, y_j) \Delta x \Delta y = \int_a^b \int_c^d f(x, y) \, dx \, dy
\]

the double integral of \( f \) over the region \( R \). The notation \( dA \) indicates a small bit of area, without specifying any particular order for the variables \( x \) and \( y \); it is shorter and

\[
\int_a^b \int_c^d f(x, y) \, dx \, dy
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the double integral of \( f \) over the region \( R \). The notation \( dA \) indicates a small bit of area, without specifying any particular order for the variables \( x \) and \( y \); it is shorter and

\[
\int_a^b \int_c^d f(x, y) \, dx \, dy
\]
Doing this gives a volume of approximately 8.84, so the average height is approximately \(8.84/6 \approx 1.47\).

Because addition and multiplication are commutative and associative, we can rewrite the original double sum:

\[
\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_i, y_j) \Delta x \Delta y = \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} f(x_i, y_j) \Delta x \Delta y.
\]

Now if we repeat the development above, the inner sum turns into an integral:

\[
\lim_{m \to \infty} \sum_{i=0}^{m-1} f(x_i, y_i) \Delta y = \int f(x, y) \, dy.
\]

and then the outer sum turns into an integral:

\[
\lim_{n \to \infty} \sum_{j=0}^{n-1} \left( \int f(x_i, y_j) \, dy \right) \Delta x = \int \int f(x, y) \, dy \, dx.
\]

In other words, we can compute the integrals in either order, first with respect to \(x\) then \(y\), or vice versa. Thinking of the loaf of bread, this corresponds to slicing the loaf in a direction perpendicular to the first.

We haven’t really proved that the value of a double integral is equal to the value of the corresponding two single integrals in either order of integration, but provided the function is reasonably nice, this is true; the result is called Fubini’s Theorem.
Exercises 17.1.

1. Compute \( \int_0^1 \int_0^1 (1 + z) \, dy \, dx \).

2. Compute \( \int_0^1 \int_0^1 (x + y) \, dy \, dx \).

3. Compute \( \int_0^1 \int_0^1 x^2 \, dy \, dx \).

4. Compute \( \int_0^1 \int_0^1 (x + y) \, dx \, dy \).

5. Compute \( \int_0^1 \int_0^1 y \, dx \, dy \).

6. Compute \( \int_0^1 \int_0^1 \frac{1}{2} \, dy \, dx \).

7. Compute \( \int_0^1 \int_0^1 x \cos y \, dy \, dx \).

8. Compute \( \int_0^1 \int_0^1 (\cos \theta - x) \, r \, dr \, d\theta \).

9. Compute \( \int_0^1 \int_0^1 \sqrt{r^2 + 1} \, dy \, dx \).

10. Compute \( \int_0^1 \int_0^1 y \sin (x^2) \, dy \, dx \).

11. Compute \( \int_0^1 \int_0^1 x \sqrt{1 + y^2} \, dy \, dx \).

12. Compute \( \int_0^1 \int_0^1 \frac{1}{x} \, dy \, dx \).

13. Compute \( \int_0^1 \int_0^1 x^2 \, dy \, dx \).

14. Compute \( \int_0^1 \int_0^1 x^2 - y \, dy \, dx \).

15. Compute \( \int_0^1 \int_0^1 \frac{1}{r} \, dy \, dx \).

16. Evaluate \( \int_0^1 \int_0^1 \frac{1}{\sqrt{r^2 + 1}} \, dy \, dx \).

17. Find the volume below \( z = 1 - y \) above the region \(-1 \leq x \leq 1, \, 0 \leq y \leq 1 - x^2 \).

18. Find the volume bounded by \( z = x^2 + y^2 \) and \( z = 4 \).

19. Find the volume in the first octant bounded by \( y^2 = 4 - 2x \) and \( y = 2x \).

20. Find the volume in the first octant bounded by \( y^2 = 4x, \, 2x + y = 4, \, z = y, \) and \( y = 0 \).

21. Find the volume in the first octant bounded by \( x + y + z = 9, \, 2x + 3y = 18, \) and \( x + 3y = 9 \).

22. Find the volume in the first octant bounded by \( x^2 + z = y^2 = a^2 \) and \( z = x + y \).

23. Find the volume bounded by \( 4z = x^2 + y^2 \) and \( z = 2 \).

24. Find the volume bounded by \( z = x^2 + y^2 \) and \( z = 9 \).

25. Find the volume under the surface \( z = xy \) above the triangle with vertices \((1,1,0), \,(4,1,0), \,(1,2,0)\).

26. Find the volume enclosed by \( y = x^2, \, y = 4, \, z = x^2, \) and \( z = 0 \).

27. A swimming pool is circular with a 40 meter diameter. The depth is constant along east-west lines and increases linearly from 2 meters at the south end to 7 meters at the north end. Find the volume of the pool.

28. Find the average value of \( f(x,y) = e^{-\frac{(x^2 + y^2)}{2}} \) on the rectangle with vertices \((0,0), \,(4,0), \,(4,1)\) and \((0,1)\).

29. Figure 17.1.5 is a temperature map of Colorado. Use the data to estimate the average temperature in the state using 4, 16, and 25 subregions. Give both an upper and lower estimate. Why do we like Colorado for this problem? What other state(s) might we like?

30. Three cylinders of radius 1 intersect at right angles at the origin, as shown in figure 17.3.6. Find the volume contained inside all three cylinders.

31. Prove that if \( f(x,y) \) is integrable and if \( y(x, y) = \int_0^x f(x, u) \, du \) then \( y_{x} = y_{xx} = f(x, y) \).

32. Reverse the order of integration on each of the following integrals
   a. \( \int_0^2 \int_0^1 f(x, y) \, dx \, dy \)
   b. \( \int_0^2 \int_0^1 f(x, y) \, dy \, dx \)

33. What are the parallels between Fubini’s Theorem and Clairaut’s Theorem?

17.2 Double Integrals in Cylindrical Coordinates

Suppose we have a surface given in cylindrical coordinates as \( z = f(r, \theta) \) and we wish to find the integral over some region. We could attempt to translate into rectangular coordinates and do the integration there, but it is often easier to stay in cylindrical coordinates.

How might we approximate the volume under such a surface in a way that uses cylindrical coordinates directly? The basic idea is the same as before: we divide the region into many small regions, multiply the area of each small region by the height of the surface somewhere in that little region, and add them up. What changes is the shape of the small regions; in order to have a nice representation in terms of \( r \), \( \theta \), we use small pieces of ring-shaped areas, as shown in figure 17.2.1. Each small region is roughly rectangular, except that two sides are segments of a circle and the other two sides are not quite parallel. Near a point \(( r, \theta) \), the length of either circular arc is about \( r \Delta \theta \) and the length of each straight side is simply \( \Delta r \). When \( \Delta r \) and \( \Delta \theta \) are very small, the region is nearly a rectangle with area \( r \Delta r \Delta \theta \), and the volume under the surface is approximately

\[
\sum \sum f(r, \theta) r \Delta r \Delta \theta.
\]
EXAMPLE 17.2.2 Find the volume under \( z = \sqrt{4 - r^2} \) above the region enclosed by the curve \( r = \cos \theta \), \(-\pi/2 \leq \theta \leq \pi/2\); see figure 17.2.3. The region is described in polar coordinates by the inequalities \(-\pi/2 \leq \theta \leq \pi/2\) and \(0 \leq r \leq 2 \cos \theta\), so the double integral is

\[
4 \int_{\pi/2}^{\pi/2} \int_{0}^{2 \cos \theta} r \, dr \, d\theta = 2 \int_{\pi/2}^{\pi/2} \int_{0}^{2 \cos \theta} \sqrt{4 - r^2} \, r \, dr \, d\theta.
\]

We can rewrite the integral as shown because of the symmetry of the volume; this avoids a complication during the evaluation. Proceeding:

\[
2 \int_{\pi/2}^{\pi/2} \int_{0}^{2 \cos \theta} \sqrt{4 - r^2} \, r \, dr \, d\theta = 2 \int_{\pi/2}^{\pi/2} \left( \frac{1}{2} (4 - r^2)^{1/2} \right) \bigg|_0^{2 \cos \theta} \, d\theta.
\]

\[
= 2 \int_{\pi/2}^{\pi/2} \left( \frac{8 \cos^3 \theta}{3} \right) \bigg|_0^{2 \cos \theta} \, d\theta = \frac{8 \sqrt{3}}{3}.
\]

\[\int_{\pi/2}^{\pi/2} \left( \frac{1}{2} \right) \bigg|_0^{2 \cos \theta} \, d\theta = \frac{8 \sqrt{3}}{3}.\]

Figure 17.2.3 Finding area by computing volume.

Exercises 17.2.
1. Find the volume above the xy-plane, under the surface \( z = r \), and inside \( r = 2 \).
2. Find the volume inside both \( r = 1 \) and \( r = 2 \) and \( z = 4 \).
3. Find the volume below \( z = r^2 \) and above the top half of the cone \( z = r \).
4. Find the volume below \( z = r \), above the xy-plane, and inside \( r = \cos \theta \).
5. Find the volume below \( z = r \), above the xy-plane, and inside \( r = 1 + \cos \theta \).
6. Find the volume between \( x^2 + y^2 = z^2 \) and \( x^2 + y^2 = z \).
7. Find the area inside \( r = 1 + \sin \theta \) and outside \( r = 3 \cos \theta \).
8. Find the area inside both \( r = 2 \sin \theta \) and \( r = 2 \cos \theta \).
9. Find the area inside the four-leaf rose \( r = \cos(2 \theta) \) and outside \( r = 1/2 \).
10. Find the area inside the cardioid \( r = 2(1 + \cos \theta) \) and outside \( r = 2 \).
11. Find the area of one loop of the three-leaf rose \( r = \cos(3 \theta) \).
12. Compute \( \int_{0}^{\pi/2} \int_{0}^{2 \cos \theta} \sin \theta \sin r \, dr \, d\theta \) by converting to cylindrical coordinates.
13. Compute \( \int_{0}^{\pi/2} \int_{0}^{2 \cos \theta} r^2 \sin \theta \, dr \, d\theta \) by converting to cylindrical coordinates.

17.3 Moment and Center of Mass

Using a single integral we were able to compute the center of mass for a one-dimensional object with variable density, and a two-dimensional object with constant density. With a double integral we can handle two dimensions and variable density.

14. Find the volume under \( z = y^2 + x + 2 \) above the region \( x^2 + y^2 \leq 4 \).
15. Find the volume between \( z = x^2 + y^2 \) and \( z = 1 \) above the region \( x^2 + y^2 \leq 1 \).
16. Find the volume inside \( x^2 + y^2 = 1 \) and \( x^2 + z^2 = 1 \).
17. Find the volume under \( z = r \) above \( r \leq z < 1 + \cos \theta \).
18. Figure 17.2.4 shows the plot of \( r = 1 + 4 \sin(5 \theta) \).

![Figure 17.2.4](image_url)

**a.** Describe the behavior of the graph in terms of the given equation. Specifically, explain maximum and minimum values, number of leaves, and the 'leaves within leaves'.

**b.** Give an integral or integrals to determine the area outside a smaller leaf but inside a larger leaf.

**c.** How would changing the value of \( a \) in the equation \( r = 1 + a \cos(5 \theta) \) change the relative sizes of the inner and outer leaves? Focus on values \( a \geq 1 \). (Hint: How would we change the maximum and minimum values?)

**d.** Consider the integral \( \int_{0}^{\pi} \int_{0}^{1} \frac{1}{\sqrt{1 - y^2}} \, dy \, dx \), where \( D \) is the unit disk centered at the origin. (See the graph here.)

**a.** Why might this integral be considered improper?

**b.** Calculate the value of the integral of the same function \( 1/\sqrt{1 - y^2} \) over the annulus with outer radius \( 1 \) and inner radius \( \lambda \).

**c.** Obtain a formula for the integral on the whole disk by letting \( \lambda \) approach 0. \( \Rightarrow \)

**d.** For which values \( \lambda \) can we replace the denominator with \((x^2 + y^2)^{1/2}\) in the original integral and still get a finite value for the improper integral?

17.3 Moment and Center of Mass

Just as before, the coordinates of the center of mass are

\[
\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M},
\]

where \( M \) is the total mass, \( M_y \) is the moment around the \( y \)-axis, and \( M_x \) is the moment around the \( x \)-axis. (You may want to review the concepts in section 11.1.)

The key to the computation, just as before, is the approximation of mass. In the two-dimensional case, we treat density \( \sigma \) as mass per square area, so when density is constant, mass is (density)(area). If we have a two-dimensional region with varying density given by \( \sigma(x, y) \), and we divide the region into small subregions with area \( \Delta A \), then the mass of one subregion is approximately \( \sigma(x, y) \Delta A \), the total mass is approximately the sum of many of these, and as usual the sum turns into an integral in the limit:

\[
M = \int_{A} \int_{A} \sigma(x, y) \, dx \, dy,
\]

and similarly for computations in cylindrical coordinates. Then as before:

\[
M_x = \int_{A} \int_{A} x \sigma(x, y) \, dx \, dy,
\]

\[
M_y = \int_{A} \int_{A} y \sigma(x, y) \, dx \, dy
\]

EXAMPLE 17.3.1 Find the center of mass of a thin, uniform plate whose shape is the region between \( y = \cos x \) and the \( x \)-axis between \( x = -\pi/2 \) and \( x = \pi/2 \). Since the density is constant, we must take \( \sigma(x, y) = 1 \).

It is clear that \( \bar{x} = 0 \), but for practice let’s compute it anyway. First we compute the mass:

\[
M = \int_{-\pi/2}^{\pi/2} \int_{0}^{\cos x} 1 \, dy \, dx = \int_{-\pi/2}^{\pi/2} \cos x \, dx = \sin x|_{-\pi/2}^{\pi/2} = 2.
\]

Next,

\[
M_x = \int_{-\pi/2}^{\pi/2} \int_{0}^{\cos x} x \, dy \, dx = \int_{-\pi/2}^{\pi/2} \cos x \, dx = \pi,
\]

Finally,

\[
M_y = \int_{-\pi/2}^{\pi/2} \int_{0}^{\cos x} y \, dy \, dx = \frac{\pi}{2} \int_{-\pi/2}^{\pi/2} \cos x \, dx = 0.
\]

So \( \bar{x} = 0 \) as expected, and \( \bar{y} = 0/4\pi = \pi/8 \). This is the same problem as in example 11.1.4, it may be helpful to compare the two solutions.
EXAMPLE 17.3.2 Find the center of mass of a two-dimensional plate that occupies the quarter circle \(x^2 + y^2 \leq 1\) in the first quadrant and has density \(k(x^2 + y^2)\). It seems clear that because of the symmetry of both the region and the density function (both are important!), \(\bar{\varphi} = \bar{\vartheta} = 0\). We'll do both to check our work.

Jumping right in, we have
\[
M = \int_{0}^{\pi/2} \int_{0}^{1} k(x^2 + y^2) \, dx \, dy = k \int_{0}^{\pi/2} \int_{0}^{1} \sqrt{1 - x^2} + \frac{(1 - x^2)^{3/2}}{3} \, dx. 
\]
This integral is something we can do, but it's a bit unpleasant. Since everything in sight is related to a circle, let's back up and try polar coordinates. Then \(x^2 + y^2 = r^2\) and
\[
M = \int_{0}^{\pi/2} \int_{0}^{1} k(r^2) \, dr \, d\theta = k \int_{0}^{\pi/2} \int_{0}^{1} \frac{r^2}{2} \, dr = k \frac{\pi}{4}.
\]

Alternatively, we could use the formula for the area of a surface of revolution:
\[
M = \int_{0}^{\pi/2} \int_{0}^{1} k(r^2) \, dr \, d\theta = k \int_{0}^{\pi/2} \frac{r^2}{2} \, dr = k \frac{\pi}{4}.
\]

EXAMPLE 17.4.1

Find the center of mass of a two-dimensional plate that occupies the region enclosed by the parabolas \(x = y^2\), \(y = x^2\) and has density function \(\sqrt{y}\). ⇒

Find the centroid of the area in the first quadrant bounded by \(x^2 - 8y + 4 = 0\), \(x^2 = 4y\), and \(x = 0\). (Recall that the centroid is the center of mass when the density is \(1\) everywhere.) ⇒

Find the centroid of one loop of the three-leaf rose \(r = \cos(3\vartheta)\). (Recall that the centroid is the center of mass when the density is \(1\) everywhere, and that the mass in this case is the same as the area, which was the subject of exercise 11 in section 17.2.) The computations of the integrals for the moments \(M_x\) and \(M_y\) are elementary but quite long; Sage can help. ⇒

Find the center of mass of a two-dimensional object that occupies the region \(0 \leq x \leq r\), \(0 \leq y \leq \sin r\), with density \(\varrho = r\) ⇒

A two-dimensional object has shape given by \(r = 1 + \cos \varphi\) and density \(\varrho = 2 + \cos \vartheta\). Set up the three integrals required to compute the center of mass. ⇒

A two-dimensional object has shape given by \(r = \cos \varphi\) and density \(\varrho(r, \vartheta) = r + 1\). Set up the three integrals required to compute the center of mass. ⇒

A two-dimensional object sits inside \(r = 1 + \cos \varphi\) and outside \(r = \cos \vartheta\), and has density \(1\) everywhere. Set up the integrals required to compute the center of mass. ⇒

### 17.4 Surface Area

We next seek to compute the area of a surface above (or below) a region in the \(x\)-\(y\) plane. How might we approximate this? We start, as usual, by dividing the region into a grid of small rectangles. We want to approximate the area of the surface above one of these small rectangles. The area is very close to the area of the tangent plane above the small rectangle. If the tangent plane just happened to be horizontal, of course the area would simply be the area of the rectangle. For a typical plane, however, the area is the area of a parallelogram, as indicated in figure 17.4.1. Note that the area of the parallelogram is obviously larger the more “tilted” the tangent plane. In the interactive figure you can see that viewed from above the four parallelograms exactly cover a rectangular region in the \(x\)-\(y\) plane.

Now recall a curious fact: the area of a parallelogram can be computed as the cross product of two vectors (page 327). We simply need to acquire two vectors, parallel to the sides of the parallelogram and with lengths to match. But this is easy: in the \(x\) direction we use the tangent vector we already know, namely \((1, 0, f_x)\) and multiply by \(\Delta x\) to shrink it to the right size: \(\langle \Delta x, 0, f_x \Delta x \rangle\). In the \(y\) direction we do the same thing and get \((0, \Delta y, f_y \Delta y)\). The cross product of these two vectors is \((f_y \Delta x, -f_x \Delta y, \Delta x \Delta y)\), the area of the parallelogram. Now we add these up and take the limit, to produce the integral:
\[
\int_{x_0}^{x_1} \int_{y_0}^{y_1} f_x \, dA_0 \, dA_1
\]
As before, the limits need not be constant.

### Exercises 17.4.

1. Find the area of the surface of a right circular cone of height \(h\) and base radius \(a\). ⇒

2. Find the area of the portion of the plane \(x + y + z = a^2\) inside the cylinder \(x^2 + y^2 = a^2\) ⇒

3. Find the area of the portion of the plane \(x + y + z = 1\) in the first octant. ⇒

4. Find the area of the upper half of the cone \(x^2 + y^2 = z^2\) inside the cylinder \(x^2 + y^2 = 2a^2\). ⇒

5. Find the area of the upper hemisphere of \(x^2 + y^2 + z^2 = a^2\) inside \(x^2 + y^2 = 2a^2\). ⇒

7. The plane \(ax + by + cz = d\) cuts a triangle in the first octant provided that \(a, b, c, d\) and are all positive. Find the area of this triangle. ⇒

8. Find the area of the portion of the cone \(x^2 + y^2 = z^2\) lying above the \(xy\) plane and inside the cylinder \(x^2 + y^2 = 4a^2\). ⇒

### 17.5 Triple Integrals

It will come as no surprise that we can also do triple integrals—integrals over a three-dimensional region. The simplest application allows us to compute volumes in an alternate way.

To approximate a volume in three dimensions, we can divide the three-dimensional region into small rectangular boxes, each \(\Delta x \times \Delta y \times \Delta z\) with volume \(\Delta x \Delta y \Delta z\). We add them all up and take the limit, to get an integral:
\[
\int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} f(x, y, z) \, dx \, dy \, dz
\]
If the limits are constant, we are simply computing the volume of a rectangular box.

### Exercises 17.5.1

We use an integral to compute the volume of the box with opposite corners at \((0, 0, 0)\) and \((1, 2, 3)\):
\[
\int_{0}^{1} \int_{0}^{2} \int_{0}^{3} f(x, y, z) \, dx \, dy \, dz
\]
Of course, this is more interesting and useful when the limits are not constant.

### EXAMPLE 17.5.2

Find the volume of the tetrahedron with corners at \((0, 0, 0)\), \((0, 3, 0)\), \((2, 3, 0)\), and \((2, 3, 5)\).⇒
The whole problem comes down to correctly describing the region by inequalities: 
\[0 \leq z \leq 2, \quad 3z/2 \leq y \leq 3, \quad 0 \leq z \leq 5/2\]. The lower \( y \) limit comes from the equation of the line \( y = 3z/2 \) that forms one edge of the tetrahedron in the \( x-y \) plane; the upper \( z \) limit comes from the equation of the plane \( z = 5z/2 \) that forms the “upper” side of the tetrahedron; see figure 17.5.1. Now the volume is:

\[
\int_0^{3z/2} \int_0^3 \int_{5z/2}^2 dz \, dy \, dx = \int_0^{3z/2} \int_0^3 x^{5z/2} dy \, dx
\]

\[
= \int_0^{3z/2} \frac{3z}{2} dy \, dx
\]

\[
= \int_0^{3z/2} \frac{3z}{2} x^{5z/2} \frac{dz}{2}
\]

\[
= \frac{15z^2}{2} - \frac{15z^3}{2}
\]

\[
= \frac{15 \times 2^2}{2} - \frac{15 \times 2^3}{2}
\]

\[
= 15 - 10 = 5.
\]

Pretty much just the way we did for two dimensions we can use triple integration to compute mass, center of mass, and various average quantities.

**EXAMPLE 17.5.3** Suppose the temperature at a point is given by \( T = x + y + z \). Find the average temperature in the cube with opposite corners at \((0,0,0)\) and \((2,2,2)\).

In two dimensions we add up the temperature at “each” point and divide by the area; here we add up the temperatures and divide by the volume, at 8:

\[
\frac{1}{8} \int_0^2 \int_0^2 \int_0^2 x + y + z \, dx \, dy \, dz
\]

\[
= \frac{1}{8} \int_0^2 \int_0^2 \left[ \frac{x^2}{2} + xy + xz \right]_0^2 dy \, dz
\]

\[
= \frac{1}{8} \int_0^2 \int_0^2 (2^2 + 2y + 2z) dy \, dz
\]

\[
= \frac{1}{8} \int_0^2 \left[ 2y + \frac{y^2}{2} + 2yz \right]_0^2 dz
\]

\[
= \frac{1}{8} \int_0^2 (2 \times 2 + \frac{2^2}{2} + 2 \times 2 \times 2) dz
\]

\[
= \frac{1}{8} \int_0^2 (4 + 2 + 8) dz
\]

\[
= \frac{1}{8} \int_0^2 14 dz
\]

\[
= \frac{14}{8} \int_0^2 dz
\]

\[
= \frac{7}{4} \times 2
\]

\[
= \frac{14}{4}
\]

\[
= \frac{7}{2}.
\]

Finally, the coordinates of the center of mass are \( \bar{x} = M_y/M = 1/\pi \), \( \bar{y} = M_z/M = 5/6 \), and \( \bar{z} = M_{xz}/M = 1/3 \).

**EXERCISES 17.5.**

1. Evaluate \( \int_0^1 \int_0^x y^2 \, dx \, dy \).
2. Evaluate \( \int_0^1 \int_0^x y^2 \, dx \, dy \).
3. Evaluate \( \int_0^1 \int_0^x e^{x+y-z} \, dx \, dy \).
4. Evaluate \( \int_0^1 \int_0^x \cos \theta \, dz \, dr \).
5. Evaluate \( \int_0^1 \int_0^x \sin^2 \theta \, dz \, dr \).
6. Evaluate \( \int_0^2 \int_0^{\sqrt{4-y^2}} x \, dy \, dx \).
7. Evaluate \( \int_0^2 \int_0^{\sqrt{4-y^2}} e^x \, dx \, dy \).
8. Compute \( \int_0^2 \int_0^{\sqrt{4-y^2}} z \, dx \, dy \).
9. For each of the integrals in the previous exercise, give a description of the volume (both algebraic and geometric) that is the domain of integration.
10. Compute \( \int_0^2 \int_0^x x + y + z \, dx \, dy \) over the region \( x^2 + y^2 + z^2 \leq 4 \) in the first quadrant.
11. Find the mass of a cube with edge length 2 and density 1 equal to the square of the distance from one corner.
12. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one edge.
13. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one side.
14. An object occupies the volume of the pyramid with corners at \((1,1,0), (1,-1,0), (-1,1,0), \) and \((0,0,2)\), and has density \( x^2 + y^2 + z^2 \) at \((x,y,z)\). Find the center of mass.
15. Verify the moments \(M_{xy}, M_{yz}, \) and \(M_{xz}\) of example 17.5.4 by evaluating the integrals.
16. Find the region \( E \) for which \( \iiint (1 - x^2 - y^2 - z^2) \, dx \, dy \, dz \) is a maximum.

**17.6 Cylindrical and Spherical Coordinates**

We have seen that sometimes double integrals are simplified by doing them in polar coordinates; not surprisingly, triple integrals are sometimes simpler in cylindrical coordinates or spherical coordinates. To set up integrals in polar coordinates, we had to understand the shape and area of a typical small region into which the region of integration was divided. We need to do the same thing here, for three dimensional regions.

As usual, the mass is the integral of density over the region:

\[
M = \iiint E \rho \, dx \, dy \, dz
\]

We compute moments as before, except now there is a third moment:

\[
M_{x} = \iiint E \rho x \, dx \, dy \, dz
\]

\[
M_{y} = \iiint E \rho y \, dx \, dy \, dz
\]

\[
M_{z} = \iiint E \rho z \, dx \, dy \, dz
\]

Finally, the coordinates of the center of mass are \( \bar{x} = M_{y}/M = 1/\pi \), \( \bar{y} = M_{z}/M = 5/6 \), and \( \bar{z} = M_{xz}/M = 1/3 \).
other arc is governed by \( \theta \), we need to imagine looking straight down the \( z \)-axis, so that the apparent angle we see is \( \Delta \theta \). In this view, the axes really are the \( x \)- and \( y \)-axes. In this graph, the apparent distance from the origin is not \( \rho \) but \( \rho \sin \phi \), as indicated in the left graph.

The upshot is that the volume of the little box is approximately \( \Delta \rho (\rho \Delta \phi)(\rho \sin \Delta \theta) = \rho^2 \sin \phi \Delta \rho \Delta \phi \Delta \theta \) or in the limit \( \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \).

**EXAMPLE 17.6.3** Suppose the temperature at \((x, y, z)\) is \( T = 1/(1 + x^2 + y^2 + z^2) \). Find the average temperature in the unit sphere centered at the origin.

### 17.7 Change of Variables

One of the most useful techniques for evaluating integrals is substitution, both “u-substitution” and trigonometric substitution, in which we change the variable to something more convenient. As we have seen, sometimes changing from rectangular coordinates to another coordinate system is helpful, and this too changes the variables. This is certainly a more complicated change, since instead of changing one variable for another we change an entire suite of variables, but as it turns out it is really very similar to the kinds of change of variables we already know as substitution.

![Figure 17.7.1](https://example.com/image17.7.1.png)  
**Figure 17.7.1** Single change of variable.

Let’s examine the single variable case again, from a slightly different perspective than we have previously used. Suppose we start with the problem

\[
\int_1^2 \sqrt{x - 1} - x^2 \, dx,
\]

this computes the area in the left graph of figure 17.7.1. We use the substitution \( x = \sin u \) to transform the function from \( x^2 \sqrt{1 - x^2} \) to \( \sin^2 u \sqrt{1 - \sin^2 u} \), and we also convert \( dx \) to \( \cos u \, du \). Finally, we convert the limits 0 and 1 to \( 0 \) and \( \pi/2 \). This transforms the integral:

\[
\int_0^{\pi/2} \sqrt{1 - \sin^2 u} - \sin^2 u \, \cos u \, du.
\]

We want to notice that there are three different conversions: the main function, the differential \( dx \), and the interval of integration. The function is converted to \( \sin^2 u \sqrt{1 - \sin^2 u} \),
Everywhere and gent vector \( \langle x, y \rangle \) is usually denoted \( f \) and \( g \theta \) is the absolute value of the two by two determinant
\[
\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix},
\]
which may be easier to remember. (Confusingly, the matrix, the determinant of the matrix, and the absolute value of the determinant are all called the Jacobian by various authors.) Because there are two things to worry about, namely, the form of the function and the region of integration, transformations in two (or more) variables are quite tricky to discover.

**Example 17.7.1** Integrate \( x^2 - xy + y^2 \) over the region \( x^2 - xy + y^2 \leq 2 \).

The equation \( x^2 - xy + y^2 = 2 \) describes an ellipse as in figure 17.7.5; the region of integration is the interior of the ellipse. We will use the transformation \( u = \sqrt{2a} - \sqrt{2b} \cos \theta \) and \( y = \sqrt{2a} + \sqrt{2b} \sin \theta \). Substituting into the function itself we get
\[
x^2 - xy + y^2 = 2 + 2u + 2v.
\]
The boundary of the ellipse is \( x^2 - xy + y^2 = 2 \), so the boundary of the corresponding region in the \( u-v \) plane is \( 2u = 2v = 1 \) or \( u = v = 1/2 \), the unit circle, so this substitution makes the region of integration simpler.

Next, we compute the Jacobian, using \( f = \sqrt{2a} - \sqrt{2b} \cos \theta \) and \( g = \sqrt{2a} + \sqrt{2b} \sin \theta \):
\[
\frac{f_u g_v - f_v g_u}{f_u^2 + g_v^2} = \frac{4\sqrt{2a} \sqrt{2b}}{\sqrt{2a} + \sqrt{2b}}.
\]
Hence the new integral is
\[
\int_R \frac{(2u + 2v)^2}{4u} \, du \, dv,
\]
where \( R \) is the interior of the unit circle. This is still not an easy integral, but it is easily transformed to polar coordinates, and then easily integrated.

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and called the Jacobian. If this is the absolute value of the two by two determinant
\[
\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix},
\]
where \( f = f(x, y) \) and \( g = g(x, y) \). Again, it will be straightforward to convert the function being integrated. Converting the limits will require, as above, an understanding of just how the functions \( f \) and \( g \) transform the \( u \) and \( v \) plane into the \( x \)-plane. Finally, the small vectors we need to approximate an area will be \( (f_u, f_v, 0) \) \( du \) and \( (g_u, g_v, 0) \) \( dv \). The cross product of these is \( (0, 0, f_u g_v - f_v g_u) \) \( dv \) with length \( |f_u g_v - f_v g_u| \) \( dv \). The quantity \( |f_u g_v - f_v g_u| \) is usually denoted
\[
\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}.
\]
There is a similar change of variables formula for triple integrals, though it is a bit more difficult to derive. Suppose we use three substitution functions, \( x = f(u, v, w), \) \( y = g(u, v, w), \) and \( z = h(u, v, w). \) The Jacobian determinant is now

\[
\left| \begin{array}{ccc}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}
\end{array} \right|
\]

Then the integral is transformed in a similar fashion:

\[
\iiint_{E} F(x, y, z) \, dV = \iiint_{S} F(f(u, v, w), g(u, v, w), h(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, du \, dv \, dw,
\]

where of course the region \( S \) in \( uvw \) space corresponds to the region \( R \) in \( xyz \) space.

**Exercises 17.7.**

1. Complete example 17.7.1 by converting to polar coordinates and evaluating the integral.
2. Evaluate \( \int \int xy \, dx \, dy \) over the square with corners \((0, 0), (1, 1), (1, -1), (0, -1)\) in two ways: directly, and using \( x = (u + v)/2, \, y = (u - v)/2. \)
3. Evaluate \( \int \int x^2 + y^2 \, dx \, dy \) over the square with corners \((-1, 0), (0, 1), (0, -1)\) in two ways: directly, and using \( x = (u + v)/2, \, y = (u - v)/2. \)
4. Evaluate \( \int \int (x + y)e^{-x} \, dy \, dx \) over the triangle with corners \((0, 0), (-1, 1), (1, 1)\) in two ways: directly, and using \( x = (u + v)/2, \, y = (u - v)/2. \)
5. Evaluate \( \int \int y(x - y) \, dy \, dx \) over the parallelogram with corners \((0, 0), (3, 3), (7, 3), (4, 0)\) in two ways: directly, and using \( x = u + v, \, y = u. \)
6. Evaluate \( \int \int \sqrt{x^2 + y^2} \, dx \, dy \) over the triangle with corners \((0, 0), (4, 4), (4, 0)\) using \( x = u, \, y = uv. \)
7. Evaluate \( \int \int y \sin(xy) \, dy \, dx \) over the region bounded by \( xy = 1, \, xy = 4, \, y = 1, \) and \( y = 4 \) using \( x = u/v, \, y = v. \)
8. Evaluate \( \int \int \sin(9x^2 + 4y^2) \, dA \) over the region in the first quadrant bounded by the ellipse \( 9x^2 + 4y^2 = 1. \)
9. Compute the Jacobian for the substitutions \( x = \rho \sin \theta \cos \phi, \, y = \rho \sin \theta \sin \phi, \, z = \rho \cos \theta. \)