16 Partial Differentiation

16.1 Functions of Several Variables

In single-variable calculus we were concerned with functions of the form $f: \mathbb{R} \rightarrow \mathbb{R}$, that is, for each input value we get a single output value. However, in many situations a single input may determine more than one output and hence we need to use functions of the form $f: \mathbb{R}^n \rightarrow \mathbb{R}$. We may think of $f$ as a function that associates an $n$-tuple $(x_1, x_2, \ldots, x_n)$ with a real number $f(x_1, x_2, \ldots, x_n)$. For example, the function $f(x, y) = x^2 + y^2$ takes ordered pairs of real numbers $(x, y)$ and assigns to each pair the sum of the squares of the components. The graph of such a function is a subset of $\mathbb{R}^3$, that is, a subset of the three-dimensional space with coordinates $(x, y, f(x, y))$. The graph of a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a generalization of the graph of a function $f: \mathbb{R} \rightarrow \mathbb{R}$.

EXAMPLE 16.1.1 Consider the function $f(x, y) = x^2 + y^2$. This function represents a surface in $\mathbb{R}^3$, called a paraboloid. The graph of this function is obtained by rotating the parabola $y = x^2$ around the $y$-axis.

EXAMPLE 16.1.2 Consider the function $f(x, y) = 1/(x^2 + y^2)$. This function represents a surface in $\mathbb{R}^3$, called the hyperboloid of one sheet. The graph of this function is obtained by rotating the hyperbola $x^2 - y^2 = 1$ around the $x$-axis.

EXAMPLE 16.1.3 Consider the function $f = \sqrt{x^2 + y^2}$. This function is defined only when both $x$ and $y$ are non-negative. When $y = 0$ we get $f(x, y) = x$, the familiar square root function in the $x$-$y$ plane, and when $x = 0$ we get the same curve in the $y$-$z$ plane. Generally speaking, we see that starting from $f(0, 0) = 0$ this function gets larger in every direction roughly the same way that the square root function gets larger. For example, if we restrict attention to the line $x = y$, we get $f(x, y) = 2\sqrt{x}$ and along the line $y = 2x$ we have $f(x, y) = \sqrt{x^2 + \sqrt{x^2}}$. In the previous example, it is often a good idea to examine the function on restricted subsets of the plane, especially lines. It can also be useful to identify those points $(x, y)$ that share a common $z$-value.

EXAMPLE 16.1.4 Consider the function $f(x, y) = x^2 + y^2$. This function is defined for all $x$ and $y$, and its graph is a cone. When $x = 0$ we get $f(0, y) = y^2$, a parabola in the $y$-$z$ plane, and when $y = 0$ we get the same “same” parabola $f(x) = x^2$ in the $x$-$z$ plane.

In order to pretend that this line is the horizontal axis, we need to write the function in terms of the distance from the origin, which is $\sqrt{x^2 + y^2} = \sqrt{x^2 + k^2}$. Now $f(x, y) = x^2 + k^2 = (\sqrt{x^2 + k^2})^2$. The level curves of this function are circles centered at the origin.

Finally, picking a value $z = k$, at what points does $f(x, y) = k$? This means $x^2 + y^2 = k$, which we recognize as the equation of a circle of radius $\sqrt{k}$. So the graph of $f(x, y) = k$ is a paraboloid cross-sections, and the same height everywhere on concentric circles with center at the origin. This fits with what we have already discovered.

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In this example, the points $(x, y)$ such that $f(x, y) = k$ usually form a curve, called a level curve of the function. A graph of some level curves can give a good idea of the shape of the surface; it looks much like a topographic map of the surface. In figure 16.1.2 both the surface and its associated level curves are shown. Note that, as with a topographic map, the heights corresponding to the level curves are evenly spaced, so that where curves are closer together the surface is steeper.

Functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ behave much like functions of two variables; we will on occasion discuss functions of several variables. The principal difficulty with such functions is visualizing them, as they do not “fit” in the three dimensions we are familiar with. For three variables there are various ways to interpret functions that make them easier to understand. For example, $f(x, y, z)$ could represent the temperature at the point $(x, y, z)$, or the pressure, or the strength of a magnetic field. It remains useful to consider those points at which $f(x, y, z) = k$, where $k$ is some constant value. If $f(x, y, z)$ is temperature, the set of points $(x, y, z)$ such that $f(x, y, z) = k$ is the collection of points in space with temperature equal to $k$; in general this is called a level set; for three variables, a level set is typically a surface, called a level surface.

EXAMPLE 16.1.5 Suppose the temperature at $(x, y, z)$ is $T(x, y, z) = e^{-[x^2+y^2+z^2]}$. This function has a maximum value of 1 at the origin, and tends to 0 in all directions. If $k$ is positive and at most 1, the set of points for which $T(x, y, z) = k$ is those points satisfying $x^2 + y^2 + z^2 = -\ln k$, a sphere centered at the origin. The level surfaces are the concentric spheres centered at the origin.

Exercises 16.1

1. Let $f(x, y) = (x - y)^2$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, $x = y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface.

2. Let $f(x, y) = |x| + |y|$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, $x = y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface.

3. Let $f(x, y) = e^{-(x^2+y^2)}$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, $x = y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface.

4. Let $f(x, y) = \sin(x - y)$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, $x = y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface.

5. Let $f(x, y) = (x^2 - y^2)^2$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, $x = y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface.

6. Find the domain of each of the following functions of two variables:
   a. $\sqrt{3-x^2} + \sqrt{y^2-4}$
   b. $\arcsin(x^2 + y^2 - 2)$
   c. $e^{-\ln(x^2+y^2)}$

7. Below are two sets of level curves. One is for a cone, one is for a paraboloid. Which is which? Explain.
16.2 Limits and Continuity

To develop calculus for functions of one variable, we needed to make sense of the concept of a limit, which we needed to understand continuous functions and to define the derivative. Limits involving functions of two variables can be considerably more difficult to deal with; fortunately, most of the functions we encounter are fairly easy to understand.

The potential difficulty is largely due to the fact that there are many ways to "approach" a point in the x-y plane. If we want to say that \( \lim_{(x,y) \to (a,b)} f(x,y) = L \), we need to capture the idea that as \((x,y)\) gets close to \((a,b)\) then \(f(x,y)\) gets close to \(L\). For functions of one variable, \(f(x)\), there are only two ways that \(x\) can approach \(a\) from the left or right. But there are an infinite number of ways to approach \((a,b)\): along any one of an infinite number of lines, or an infinite number of parabolas, or an infinite number of sine curves, and so on. We might hope that it’s really not so bad—suppose, for example, that along every possible line through \((a,b)\) the value of \(f(x,y)\) gets close to \(L\); surely this means that "\(f(x,y)\) approaches \(L\) as \((x,y)\) approaches \((a,b)\)". Sadly, no.

**EXAMPLE 16.2.1** Consider \(f(x,y) = x^2y/(x^2+y^2)\). When \(x = 0\) or \(y = 0\), \(f(x,y)\) is 0, so the limit of \(f(x,y)\) approaching the origin along either the x or y axis is 0. Moreover, along the line \(y = mx, f(x,y) = m^2x^2/(x^2 + m^2x^2)\). As \(x\) approaches 0 this expression approaches 0 as well. So along every line through the origin \(f(x,y)\) approaches 0. Now suppose we approach the origin along \(x = y^2\). Then

\[
f(x,y) = \frac{x^2y}{y^2 + y^2} = \frac{x^2}{2y} \to \frac{1}{2}
\]

so the limit is 1/2. Looking at figure 16.2.1, it is apparent that there is a ridge above \(x = y^2\). Approaching the origin along a straight line, we go over the ridge and then drop down toward 0, but approaching along the ridge the height is a constant 1/2. Thus, there is no limit at \((0,0)\).

Fortunately, we can define the concept of limit without needing to specify how a particular point is approached—indeed, in definition 2.3.2, we didn’t need the concept of "approach." Roughly, that definition says that when \(x\) is close to \(a\) then \(f(x)\) is close to \(L\); there is no mention of "how" we get close to \(a\). We can adapt that definition to two variables quite easily:

**DEFINITION 16.2.2 Limit** Suppose \(f(x,y)\) is a function. We say that \(\lim_{(x,y) \to (a,b)} f(x,y) = L\) if for every \(\epsilon > 0\) there is a \(\delta > 0\) so that whenever \(0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta\), \(|f(x,y) - L| < \epsilon\).

We want to force this to be less than \(\epsilon\) by picking \(\delta\) "small enough." If we choose \(\delta = \epsilon/3\) then

\[
|\frac{3x^2y}{x^2 + y^2} - \frac{1}{3}| < 1 \cdot \frac{x^2}{3} + \frac{1}{3} < \epsilon.
\]

Recall that a function \(f(x)\) is continuous at \(x = a\) if \(\lim_{x \to a} f(x) = f(a)\); roughly this says that there is no "hole" or "jump" at \(x = a\). We can say exactly the same thing about a function of two variables.

**DEFINITION 16.2.4** \(f(x,y)\) is continuous at \((a,b)\) if \(\lim_{(x,y) \to (a,b)} f(x,y) = f(a,b)\).

**EXAMPLE 16.2.5** The function \(f(x,y) = 3x^2y/(x^2+y^2)\) is not continuous at \((0,0)\), because \(f(0,0)\) is not defined. However, we know that \(\lim_{(x,y) \to (0,0)} f(x,y) = 0\), so we can easily "fix" the problem, by extending the definition of \(f\) so that \(f(0,0) = 0\). This surface is shown in figure 16.2.2.

![Figure 16.2.2](image)

\(f(x,y) = \frac{3x^2y}{x^2+y^2}\) (AP)

This says that there is no "hole" or "jump" at \((a,b)\), so the limit of \(f(x,y)\) approaching the origin along either the x or y axis is 0. Moreover, along the line \(y = mx, f(x,y) = m^2x^2/(x^2 + m^2x^2)\). As \(x\) approaches 0 this expression approaches 0 as well. So along every line through the origin \(f(x,y)\) approaches 0. Now suppose we approach the origin along \(x = y^2\). Then

\[
f(x,y) = \frac{x^2y}{y^2 + y^2} = \frac{x^2}{2y} \to \frac{1}{2}
\]

so the limit is 1/2. Looking at figure 16.2.1, it is apparent that there is a ridge above \(x = y^2\). Approaching the origin along a straight line, we go over the ridge and then drop down toward 0, but approaching along the ridge the height is a constant 1/2. Thus, there is no limit at \((0,0)\).

Note that in contrast to this example we cannot fix example 16.2.1 at \((0,0)\) because the limit does not exist. No matter what value we try to assign to \(f(0,0)\) the surface will have a "jump" there.

Fortunately, the functions we will examine will typically be continuous almost everywhere. Usually this follows easily from the fact that closely related functions of one variable are continuous. As with single variable functions, two classes of common functions are particularly useful and easy to describe. A polynomial in two variables is a sum of terms of the form \(a_nx^n y^m\), where \(a_n\) is a real number and \(m\) and \(n\) are non-negative integers. A rational function is a quotient of polynomials.

**THEOREM 16.2.6** Polynomials are continuous everywhere. Rational functions are continuous everywhere they are defined.

**Exercises 16.2.**

Determine whether each limit exists. If it does, find the limit and prove that it is the limit; if it does not, explain why you know:

1. \(\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{x^2 + y^2}\)
2. \(\lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2}\)
3. \(\lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2}\)
4. \(\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{x^2 + y^2}\)
5. \(\lim_{(x,y) \to (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}\)
6. \(\lim_{(x,y) \to (0,0)} \frac{\sqrt{x^2 + y^2}}{x^2 + y^2}\)
7. \(\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{x^2 + y^2}\)
8. \(\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{x^2 + y^2}\)
9. \(\lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{x^2 + y^2}\)
10. \(\lim_{(x,y) \to (0,0)} \frac{(x-1)^2(x+1)}{x^2 + y^2}\)
11. \(\lim_{(x,y) \to (0,0)} x^2 + 4y\)
12. \(\lim_{(x,y) \to (0,0)} x^2 + y^2\)
13. Does the function \( f(x, y) = \frac{x - y}{\sqrt{1 + x^2 + y^2}} \) have any discontinuities? What about \( f(x, y) = x^2 + y^2 \)?

16.3 Partial Differentiation

When we first considered what the derivative of a vector function might mean, there was really not much difficulty in understanding either how such a thing might be computed or what it might measure. In the case of functions of two variables, things are a bit harder to understand. If we think of a function of two variables in terms of its graph, a surface, there is a more-or-less obvious derivative-like question we might ask: namely, how “steep” is the surface. But it’s not clear that this has a simple answer, nor how we might proceed.

We will start with what seems to be very small steps toward the goal; surprisingly, it turns out that these simple ideas hold the keys to a more general understanding.

Imagine a particular point on a surface; what might we be able to say about how steep it is? We can limit the question to make it more familiar: how steep is the surface in a particular direction? What does this even mean? Here’s one way to think of it: Suppose we’re interested in the point \((a, b, c)\). Pick a straight line in the \(xy\)-plane through the point \((a, b, 0)\), then extend the line vertically into a plane. Look at the intersection of the plane with the surface. If we pay attention to just the plane, we see the chosen straight line where the \(z\)-axis would normally be, and the intersection with the surface shows up as a curve in the plane. Figure 16.3.1 shows the parabolic surface from figure 16.1.2, exposing its cross-section above the line \(x + y = 1\).

In principle, this is a problem we know how to solve: find the slope of a curve in a plane. Let’s start by looking at some particularly easy lines: those parallel to the \(x\) or \(y\) axis. Suppose we are interested in the cross-section of \(f(x, y)\) above the line \(y = b\). If we substitute \(b\) for \(y\) in \(f(x, y)\), we get a function in one variable, describing the height of the cross-section as a function of \(x\). Because \(y = b\) is parallel to the \(x\)-axis, if we view it from a vantage point on the negative \(y\)-axis, we will see what appears to be simply an ordinary curve in the \(x\)-\(z\) plane.

Consider again the parabolic surface \(f(x, y) = x^2 + y^2\). The cross-section above the line \(y = 1\) consists of all points \((x, 2, x^2 + 4)\). Looking at this cross-section from somewhere on the negative \(y\)-axis, we see what appears to be just the curve \(f(x) = x^2 + 4\). At any point on the cross-section, \((a, 2, a^2 + 4)\), the steepness of the surface in the direction of the line \(y = 2\) is simply the slope of the curve \(f(x) = x^2 + 4\), namely \(2x\). Figure 16.3.2 shows the same parabolic surface as before, but now cut by the plane \(y = 2\). The left graph shows the cut-off surface, the right shows just the cross-section, looking up from the negative \(y\)-axis toward the origin.

Imagine a particular point on a surface; what might we be able to say about how steep it is? We can limit the question to make it more familiar: how steep is the surface in a particular direction? What does this even mean? Here’s one way to think of it: Suppose we’re interested in the point \((a, b, c)\). Pick a straight line in the \(xy\)-plane through the point \((a, b, 0)\), then extend the line vertically into a plane. Look at the intersection of the plane with the surface. If we pay attention to just the plane, we see the chosen straight line where the \(z\)-axis would normally be, and the intersection with the surface shows up as a curve in the plane. Figure 16.3.1 shows the parabolic surface from figure 16.1.2, exposing its cross-section above the line \(x + y = 1\).

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EXAMPLE 16.3.3 Find the tangent plane to the surface $z = x^2 + y^2 - 4$ at $(1, 1, 2)$. This plane is in the upper hemisphere, so we have $f(z, y) = \sqrt{4 - x^2 - y^2}$. Then $f_x(x, y) = -4x/(4 - x^2 - y^2)$ and $f_y(x, y) = -2y/(4 - x^2 - y^2)$ so $(f_1(1), f_2(1)) = -3/\sqrt{2}$ and the equation of the plane is

$$z = -\frac{3}{\sqrt{2}}(1 - 1) - \frac{1}{\sqrt{2}}(y - 1) + \sqrt{2}.$$ 

The hemisphere and this tangent plane are pictured in figure 16.3.3.

It seems that to find a tangent plane, we need only find two quite simple ordinary derivatives, namely $f_x$ and $f_y$. This is true if the tangent plane exists. It is, unfortunately, not always the case that if $f_x$ and $f_y$ exist there is a tangent plane. Consider the function $x^2/(x^2 + y^2)$ pictured in figure 16.2.1. This function has value 0 when $x = 0$ or $y = 0$, and we can "plug the hole" by agreeing that $f(0,0) = 0$, because in the $x$ and $y$ directions the surface is simply a horizontal line. But it's also clear from the picture that this surface does not have anything that deserves to be called a "tangent plane" at the origin, certainly not the $x$-$y$ plane containing those two tangent lines.

16.3 Partial Differentiation

right side does the same, because as $(x, y)$ approaches $(x_0, y_0)$, $f$ approaches 0. Essentially the same calculation works for $f_y$.

Almost all of the functions we will encounter are differentiable at points we will be interested in, and often at all points. This is usually because the functions satisfy the hypotheses of this theorem.

**THEOREM 16.3.5** If $f(x, y)$ and its partial derivatives are continuous at a point $(x_0, y_0)$, then $f$ is differentiable there.

**Exercises 16.3.**

1. Find $f_x$ and $f_y$ where $f(x, y) = \cos(xy) + y^3$. →

2. Find $f_x$ and $f_y$ where $f(x, y) = \frac{xy}{y^2 + 1}$. →

3. Find $f_x$ and $f_y$ where $f(x, y) = xe^{xy}$. →

4. Find $f_x$ and $f_y$ where $f(x, y) = xy(x^2 + y^2)$. →

5. Find $f_x$ and $f_y$ where $f(x, y) = \sqrt{x^2 + y^2}$. →

6. Find $f_x$ and $f_y$ where $f(x, y) = \tan(xy)$. →

7. Find $f_x$ and $f_y$ where $f(x, y) = \frac{1}{x+y}$. →

8. Find an equation for the plane tangent to $2x^2 + 3y^2 - z^2 = 4$ at $(1, 1, -1)$. →

9. Find an equation for the plane tangent to $x^2 + (y+1)^2$ at $(1, 2, 1)$. →

10. Find an equation for the plane tangent to $f(x, y) = x^2 + y^2$ at $(3, 1, 3)$. →

11. Find an equation for the plane tangent to $f(x, y) = x^2y^2$ at $(2/2, 0)$. →

12. Find an equation for the line normal to $x^2 + 4y^2 = 2z$ at $(2, 1, 4)$. →

13. Explain in your own words why, when taking a partial derivative of a function of multiple variables, we can treat the variables not being differentiated as constants.

14. Consider a differentiable function, $f(x, y)$. Give physical interpretations of the meanings of $f_x(x_0, b)$ and $f_y(a_0, b)$ as they relate to the graph of $f$.

15. In much the same way that we used the tangent line to approximate the value of a function from a single variable calculus, we can use the tangent plane to approximate a function from multivariable calculus. Consider the tangent plane found in Exercise 11. Use this plane to approximate $(1.98, 0.4)$.

16. Suppose that one of your colleagues has calculated the partial derivatives of a given function, and reported to you that $f(x, y) = 2x + 3y$ and that $f_x(x, y) = 2 + 6y$. Do you believe them? Why or why not? If not, what answer might you have accepted for $f_x$?

17. Suppose $f(t)$ and $g(t)$ are single variable differentiable functions. Find $\partial f/\partial t$ and $\partial g/\partial t$ for each of the following two variable functions.

   a. $z = f(t^2)$
   b. $z = f(xy)$
   c. $z = f(x^2)$

### 16.4 Chain Rule

16.4.1 Chain Rule

Consider the surface $z = x^2 + xy$, and suppose that $z = x + t$ and $y = 1 - t$. We can think of the latter two equations as describing how $x$ and $y$ change relative to, say, time. Then $z = x^2 + xy = (2 + t)^2(1 - t^2) + (2 + t)(1 - t)^2$ tells us explicitly how $z$ changes, and the coordinate on the surface depends on $x$ and $y$ as

$$\frac{dz}{dt} = 2x(2 + t) + y(2 + t) \frac{dx}{dt} + x(2 + t) \frac{dy}{dt}.$$

If we look carefully at the middle step, $dx/dt = (2 + t)^2y^2 + x^2 + 2y$, we notice that $2x + y = \partial f/\partial x$ and $2x + 2y$ is $\partial f/\partial y$. This turns out to be true in general, and gives us a new chain rule:

**THEOREM 16.4.1** Suppose that $z = f(x, y)$, $f$ is differentiable, $x = g(t)$, and $y = h(t)$. Assuming that the relevant derivatives exist,

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}.$$

**Proof.** If $f$ is differentiable, then

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + c_1\Delta x + c_2\Delta y,$$

where $c_1$ and $c_2$ approach 0 as $(x, y)$ approaches $(x_0, y_0)$. Then

$$\lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = f_x(x_0, y_0) + c_1 \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = f_x(x_0, y_0)$$

and so taking the limit of (16.4.1) as $\Delta x$ goes to 0 gives

$$\frac{dz}{dt} = f_x(x_0, y_0) \frac{dx}{dt} + f_y(x_0, y_0) \frac{dy}{dt}$$

as desired.
We can write the chain rule in way that is somewhat closer to the single variable chain rule:
\[
\frac{df}{dx} = \left( \frac{\partial f}{\partial x_1} \right) \cdot \left( \frac{\partial x_1}{\partial y} \right)
\]
or (roughly) the derivatives of the outside function “times” the derivatives of the inside functions. Not surprisingly, essentially the same chain rule works for functions of more than two variables, for example, given a function of three variables \(f(x,y,z)\), where each of \(x, y, \text{ and } z\) is a function of \(t\):
\[
\frac{df}{dt} = \left( \frac{\partial f}{\partial x_1} \right) \cdot \left( \frac{\partial x_1}{\partial t} \right) \cdot x'(t)
\]
We can even extend the idea further. Suppose that \(f(x,y)\) is a function and \(z = g(x,y)\) are functions of two variables \(s\) and \(t\). Then \(f\) is “really” a function of \(s\) and \(t\) as well, and
\[
\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}
\]
EXAMPLE 16.4.2 \(x^2 + y^2 + z^2 = 4\) defines a sphere, which is not a function of \(x\) and \(y\). However, it can be thought of as two functions, the top and bottom hemispheres. We can think of \(x, y\) as one of these two functions, so \(z = z(x,y)\), and we can think of \(x\) and \(y\) as particularly simple functions of \(z\) and \(y\), and let \(f(x,y,z) = x^2 + y^2 + z^2\). Since \(f(x,y,z) = 4\), \(\partial f/\partial x = 0\), but using the chain rule:
\[
0 = \frac{\partial f}{\partial x} = 2x + \frac{\partial x}{\partial s} \frac{\partial x}{\partial z} + \frac{\partial y}{\partial s} \frac{\partial y}{\partial z} = (2x)(1) + (2y)(0) + (2z) \frac{\partial z}{\partial s},
\]
noting that since \(y\) is temporarily held constant its derivative \(\partial y/\partial s = 0\). Now we can solve for \(\partial z/\partial s\):
\[
\frac{\partial z}{\partial s} = \frac{2z}{2x} = \frac{-x}{2z}
\]
EXAMPLE 16.5.2 immediately provides us with some additional information. We know that
\[
D_u f = \nabla \cdot \mathbf{u}
\]
and \(\mathbf{u} \cdot \nabla f = \mathbf{u} \cdot (\nabla f)\). This tells us immediately that the largest value of \(D_u f\) occurs when \(\mathbf{u} \cdot \nabla f = 1\), namely, when \(\theta = 0\), so \(\nabla f\) is parallel to \(\mathbf{u}\). In other words, the gradient \(\nabla f\) points in the direction of steepest ascent of the surface, and \(-\nabla f\) is the slope in that direction. Likewise, the smallest value of \(D_u f\) occurs when \(\cos \theta = -1\), namely, when \(\theta = \pi\), so \(\nabla f\) is anti-parallel to \(\mathbf{u}\). In other words, \(-\nabla f\) points in the direction of steepest descent of the surface, and \(-\nabla f\) is the slope in that direction.

EXAMPLE 16.5.3 Investigate the direction of steepest ascent and descent for \(z = x^2 + y^2\).

The gradient is \((2x, 2y)\), which is a vector parallel to the vector \((x, y)\), so the direction of steepest ascent is directly away from the origin, starting at the point \((x, y)\). The direction of steepest descent is directly toward the origin from \((x, y)\). Note that at \((0,0)\) the gradient vector is \((0,0)\), which has no direction, and it is clear from the plot of this surface that there is a minimum point at the origin, and tangent vectors in all directions are parallel toward the \(x-y\) plane.

\[
\nabla f = (2x, 2y) = (2z, 2z) = z(2x, 2y) = z(\mathbf{e}_x, \mathbf{e}_y)
\]
and \(-\nabla f\) points in the direction of steepest descent of the surface, and \(-\nabla f\) is the slope in that direction. Note that when \(\mathbf{u} \cdot \nabla f = 1\), namely, when \(\theta = 0\), so \(\nabla f\) is parallel to \(\mathbf{u}\). In other words, \(-\nabla f\) points in the direction of steepest descent of the surface, and \(-\nabla f\) is the slope in that direction.
16.5 Directional Derivatives

The gradient points directly at the origin from the point \((x, y, z)\) by moving directly toward the heat source, we increase the temperature as quickly as possible.

EXAMPLE 16.5.5

Find the points on the surface defined by \(x^2 + 2y^2 + 3z^2 = 1\) where the tangent plane is parallel to the plane defined by \(3z - y + 3z = 1\).

Two planes are parallel if their normals are parallel or anti-parallel, so we want to find the points on the surface with normal parallel or anti-parallel to \((-3, -1, 3)\). Let \(f = x^2 + 2y^2 + 3z^2\), the gradient of \(f\) is normal to the level surface at every point, so we are looking for a gradient parallel or anti-parallel to \((-3, -1, 3)\). The gradient is \((2x, 4y, 6z)\); if it is parallel or anti-parallel to \((-3, -1, 3)\), then

\[
(2x, 4y, 6z) = k(3, -1, 3) \quad \text{for some \(k\).}
\]

This means we need a solution to the equations

\[
2x = 3k, \quad 4y = -k, \quad 6z = 3k.
\]

but this is three equations in four unknowns—we need another equation. What we haven’t used so far is that the points we seek are on the surface \(x^2 + 2y^2 + 3z^2 = 1\); this is the fourth equation. If we solve the first three equations for \(y, z\), and substitute into the fourth equation we get

\[
1 = \left(\frac{3k}{2}\right)^2 + \left(\frac{k}{2}\right)^2 + \left(\frac{3k}{2}\right)^2 = \frac{1}{2}k^2.
\]

so \(k = \pm \frac{1}{\sqrt{2}}\). The desired points are \((\frac{3\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{3\sqrt{2}}{4})\) and \((\frac{3\sqrt{2}}{4}, \frac{-\sqrt{2}}{4}, \frac{-3\sqrt{2}}{4})\). The ellipsoid and the three planes are shown in figure 16.5.1.

16.6 Higher order derivatives

10. Suppose the temperature at \((x, y, z)\) is given by \(T = x^2y^2z\). In what direction can you go from the point \((1, 1, 1)\) to maintain the same temperature? ⇒

11. Find an equation for the plane tangent to \(z = x^2 + y^2 - 1\) at \((1, 1, 0)\). ⇒

12. Find an equation for the plane tangent to \(z = x^2 + y^2 - 1\) at \((1, 2, 3)\). ⇒

13. Find a vector function for the line normal to \(x^2 + y^2 = 9\) at \((3, 2, -1)\). ⇒

14. Find a vector function for the line normal to \(x^2 + y^2 = 9\) at \((3, 2, -1)\). ⇒

15. Find a vector function for the line normal to \(x^2 + y^2 = 9\) at \((3, 2, -1)\). ⇒

16. Find the directions in which the directional derivative of \(f(x, y) = x^2 + \sin(x+y)\) at the point \((1, 0)\) has the value 1. ⇒

17. Show that the curve \(r(t) = (\ln(t), \ln(t), 0)\) is tangent to the surface \(x^2 + y^2 + z = 9\) at the point \((0, 0, 1)\). ⇒

18. A bug is crawling on the surface of a hot plate, the temperature of which at the point \(x\) units to the right of the lower left corner and \(x\) units up from the lower left corner is given by \(T(x, y) = 100 - x^2 - 3y\).

a. If the bug is at the point \((2, 1)\), in which direction should it move to cool off the fastest?

b. If the bug is at the point \((1, 3)\), in which direction should it move in order to maintain its temperature?

19. The elevation on a portion of a hill is given by \(f(x, y) = 100 - 4x^2 - 2y^2\). From the location above \((2, 1)\), in which direction will water run off? ⇒

20. Suppose that \(g(x, y) = y - x^2\). Find the gradient at the point \((1, -1, 3)\). Sketch the level curve of the graph of \(g(x, y) = 2\), and plot both the tangent line and the gradient vector at the point \((1, -1, 3)\). (Make your sketch large. What do you notice, geometrically?) ⇒

21. The gradient \(\nabla f\) is a vector valued function of two variables. Prove the following gradient rules. Assume \(f(x, y)\) and \(g(x, y)\) are differentiable functions.

a. \(\nabla (f+g) = \nabla f + \nabla g\)

b. \(\nabla (cf) = c\nabla f\)

c. \(\nabla (f \cdot g) = f \nabla g + g \nabla f\)

d. \(\nabla (f/g) = [g \nabla f - f \nabla g]/g^2\)

16.6 Higher order derivatives

In single variable calculus we saw that the second derivative is often useful: in appropriate circumstances it measures acceleration; it can be used to identify maximum and minimum points; it tells us something about how sharply curved a graph is. Not surprisingly, second derivatives are also useful in the multi-variable case, but again not surprisingly, things are a bit more complicated.

It’s easy to see where some complication is going to come from: with two variables there are four possible second derivatives. To take a “derivative,” we must take a partial derivative with respect to \(x\) or \(y\), and there are four ways to do it: \(x\) then \(x\), \(x\) then \(y\), \(y\) then \(x\), \(y\) then \(y\).

EXAMPLE 16.6.1

Compute all four second derivatives of \(f(x, y) = x^2y^2\).

Using an obvious notation, we get:

\[
f_{xx} = 2y^2, \quad f_{xy} = 4xy, \quad f_{yx} = 4xy, \quad f_{yy} = 2x^2.
\]

You will have noticed that two of these are the same, the “mixed partials” computed by taking partial derivatives with respect to both variables in the two possible orders. This is not an accident—as long as the function is reasonably nice, this will always be true.

THEOREM 16.6.2 Clairaut’s Theorem

If the mixed partial derivatives are continuous, they are equal.

EXAMPLE 16.6.3

Compute the mixed partials of \(f(x, y) = xy(x^2 + y^2)\).

\[
f_x = y(x^2 + y^2) + 2x^2y, \quad f_y = x(x^2 + y^2) + 2xy^2
\]

We leave \(f_{xx}\) as an exercise.

16.6 Higher order derivatives

1. Find all first and second partial derivatives of \(f = xy/(x^2 + y^2)\). ⇒

2. Find all first and second partial derivatives of \(e^{x+y} + y^2\). ⇒

3. Find all first and second partial derivatives of \(x^2 + xy + x\). ⇒

4. Find all first and second partial derivatives of \(xy - y\). ⇒

5. Find all first and second partial derivatives of \(\sin(x+y)\). ⇒

6. Find all first and second partial derivatives of \(\sin(x^2y^2)\). ⇒

7. Find all first and second partial derivatives of \(\ln(x^2 + y^2)\). ⇒

8. Find all first and second partial derivatives of \(e^{x+y^2} + 16x^2 - 64\). ⇒

9. Find all first and second partial derivatives of \(x+y\) with respect to \(x\) and \(y\) if \(xy + yz + xz = 1\). ⇒

10. Let \(a\) and \(k\) be constants. Prove that the function \(f(x, t) = e^{-kx^2/2}\sin(at)\) is a solution to the heat equation \(u_t = a^{2}u_{xx}\).

11. Let \(a\) be a constant. Prove that \(u = \sin(x-ct) + \sin(x+ct)\) is a solution to the wave equation \(u_{tt} = a^{2}u_{xx}\).

12. How many third-order derivatives does a function of 2 variables have? How many of these are distinct?
13. How many real order derivatives does a function of 2 variables have? How many of these are distinct?

16.7 Maxima and minima

Suppose a surface given by \( f(x, y) \) has a local maximum at \((x_0, y_0, z_0)\); geometrically, this point on the surface looks like the top of a hill. If we look at the cross-section in the plane \( y = y_0 \), we will see a local maximum on the curve at \((x_0, y_0)\), and we know from single-variable calculus that \( \frac{dz}{dx} = 0 \) at this point. Likewise, in the plane \( x = x_0 \) we know from single-variable calculus that \( \frac{dz}{dy} = 0 \) at this point. So if there is a local maximum at \((x_0, y_0, z_0)\), both partial derivatives at the point must be zero, and likewise for a local minimum. Thus, to find local maximum and minimum points, we need only consider those points at which both partial derivatives are zero. As in the single-variable case, it is possible for the derivatives to be zero at a point that is neither a maximum or a minimum, so we need to test these points further.

You will recall that in the single variable case, we examined three methods to identify maximum and minimum points; the most useful is the second derivative test, though it doesn’t always work. For functions of two variables there is also a second derivative test; again it is by far the most useful test, though it doesn’t always work.

**THEOREM 16.7.1** Suppose that the second partial derivatives of \( f(x, y) \) are continuous near \((x_0, y_0, z_0)\); and \( f(x_0, y_0) = z_0 \). We denote by \( D \) the discriminant
\[
D(x_0, y_0) = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2.
\]
If \( D > 0 \) and \( f_{xx}(x_0, y_0) > 0 \) there is a local maximum at \((x_0, y_0)\); if \( D > 0 \) and \( f_{xx}(x_0, y_0) < 0 \) there is a local minimum at \((x_0, y_0)\); if \( D < 0 \) there is neither a maximum nor a minimum at \((x_0, y_0)\); if \( D = 0 \), the test fails.

**EXAMPLE 16.7.2** Verify that \( f(x, y) = x^2 + y^2 \) has a minimum at \((0, 0)\).

First, we compute all the needed derivatives:
\[
f_x = 2x \quad f_y = 2y \quad f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = 0.
\]
The derivatives \( f_x \) and \( f_y \) are zero only at \((0, 0)\). Applying the second derivative test there:
\[
D(0, 0) = (2)(2) - (0)^2 = 4 > 0
\]
and
\[
f_{xx}(0, 0) = 2 > 0,
\]
so there is a local minimum at \((0, 0)\), and there are no other possibilities.

**EXAMPLE 16.7.3** Find all local maxima and minima for \( f(x, y) = x^4 - y^2 \).

The derivatives:
\[
f_x = 4x^3 \quad f_y = -2y \quad f_{xx} = 12x^2 \quad f_{yy} = 2 \quad f_{xy} = 0.
\]
Again there is a single critical point, at \((0, 0)\), and
\[
D(0, 0) = 2 - 0 = 2 > 0
\]
so we get no information. However, in this case it is easy to see that there is a minimum at \((0, 0)\), because \( f(0, 0) = 0 \) and at all other points \( f(x, y) > 0 \).

**EXAMPLE 16.7.4** Find all local maxima and minima for \( f(x, y) = x^2 + y^4 \).

The derivatives:
\[
f_x = 2x \quad f_y = 4y^3 \quad f_{xx} = 2 \quad f_{yy} = 12y^2 \quad f_{xy} = 0.
\]
Again there is a single critical point, at \((0, 0)\), and
\[
D(0, 0) = 2 - 0 = 2 > 0
\]
so we get no information. However, in this case it is easy to see that there is a minimum at \((0, 0)\), because \( f(0, 0) = 0 \) and at all other points \( f(x, y) > 0 \).

**EXAMPLE 16.7.5** Find all local maxima and minima for \( f(x, y) = x^3 + y^2 \).

The derivatives:
\[
f_x = 3x^2 \quad f_y = 2y \quad f_{xx} = 6x \quad f_{yy} = 0 \quad f_{xy} = 0.
\]
Again there is a single critical point, at \((0, 0)\), and
\[
D(0, 0) = (0)(0) - (0)^2 = 0
\]
so we get no information. In this case, a little thought shows there is neither a maximum nor a minimum at \((0, 0)\), when \( x \) and \( y \) are both positive, \( f(x, y) > 0 \), and when \( x \) and \( y \) are both negative, \( f(x, y) > 0 \), and there are points of both kinds arbitrarily close to \((0, 0)\). Alternately, if we look at the cross-section when \( y = 0 \), we get \( f(x, 0) = x^3 \) which does not have either a maximum or minimum at \( x = 0 \).

**EXAMPLE 16.7.6** Suppose a box with no top is to hold a certain volume \( V \). Find the dimensions for the box that result in the minimum surface area.

The area of the box is \( A = 2lh + 2wh + 4w \), and the volume is \( V = lwh \), so we can write the area as a function of two variables,
\[
A(l, w) = 2V \frac{l}{w} + 2w + 4w.
\]
Then
\[
a_l = \frac{2V}{w} + \frac{l}{w} \quad a_w = \frac{2l}{w} + 4.
\]
If we set these equal to zero and solve, we find \( l = (2V)^{1/3} \) and \( w = (2V)^{1/3} \), and the corresponding height is \( h = V/(2V)^{2/3} \).

The second derivatives are
\[
a_{ll} = \frac{2V}{W} \quad a_{lw} = \frac{4V}{W} \quad a_{ww} = 4
\]
so the discriminant is
\[
D = \frac{4V}{W} - \frac{4V}{W} = \frac{4V}{W} > 0.
\]
Since \( A_{ll} \) is 2, there is a local minimum at the critical point. Is this a global minimum? It is, but it is difficult to see this analytically; physically and graphically it is clear that there is a minimum, in which case it must be at the single critical point. This applet shows an example of such a graph. Note that we must choose a value for \( V \) in order to graph it.

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Recall that when we did single variable global maximum and minimum problems, the easiest cases were those for which the variable could be limited to a finite closed interval, for then we simply had to check all critical values and the endpoints. The previous example is difficult because there is no finite boundary to the domain of the problem—both \( w \) and \( l \) can be in \([0, \infty)\). As in the single variable case, the problem is often simpler when there is a finite boundary.

**THEOREM 16.7.7** If \( f(x, y) \) is continuous on a closed and bounded subset of \( \mathbb{R}^2 \), then it has both a maximum and minimum value.

As in the case of single variable functions, this means that the maximum and minimum values must occur at a critical point or on the boundary; in the two variable case, however, the boundary is a curve, not merely two endpoints.

**EXAMPLE 16.7.8** The length of the diagonal of a box is to be 1 meter; find the maximum possible volume.

If the box is placed with one corner at the origin, and sides along the axes, the length of the diagonal is \( \sqrt{x^2 + y^2 + z^2} \), and the volume is
\[
V = xyz = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.
\]
Clearly, \( x^2 + y^2 \leq 1 \), so the domain we are interested in is the quarter of the unit disk in the first quadrant. Computing derivatives:
\[
V_x = \frac{y - 2xy^2 - y^3}{\sqrt{x^2 + y^2 + z^2}}
\]
and
\[
V_y = \frac{-2xy - y^3}{\sqrt{x^2 + y^2 + z^2}}.
\]
If these are both 0, then \( x = 0 \) or \( y = 0 \), or \( x = y = 0 \). The boundary of the domain is composed of three curves: \( x = 0 \) for \( y \in [0, 1] \); \( y = 0 \) for \( x \in [0, 1] \); and \( x^2 + y^2 = 1 \), where \( x \geq 0 \) and \( y \geq 0 \). In all three cases, the volume \( \frac{y}{\sqrt{1 - x^2}} \) is 0, so the maximum occurs at the only critical point \((1/\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2}) \). See figure 16.7.1.

**Exercises 16.7**

1. Find all local maximum and minimum points of \( f = x^2 + 4y^2 - 2x + 8y - 1 \).
2. Find all local maximum and minimum points of \( f = x^2 - y^2 + 6x - 10y + 2 \).
3. Find all local maximum and minimum points of \( f = xy \).
4. Find all local maximum and minimum points of \( f = 9 + 4x - y - 2x^3 - 3y^3 \).
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5. Find all local maximum and minimum points of \( f = x^2 + 4y + y^2 - 6y + 1 \).

6. Find all local maximum and minimum points of \( f = x^2 - 9y + 2y^2 - 5x + 6y - 5 \).

7. Find the absolute maximum and minimum points of \( f = x^3 + 3y - 3xy \) over the region bounded by \( y = x \), \( y = 0 \), and \( x = 2 \).

8. A two-sided rectangular box is to hold 1/2 cubic meter; what shape should the box be to minimize surface area?

9. The post office will accept packages whose combined length \( L \) and girth \( G \) is at most 130 inches. The volume of a box with fixed length diagonal.

10. The bottom of a rectangular box costs twice as much per unit area as the sides and top.

11. The gradient of \( f(x, y, z) = x^2 + y^2 + z^2 \).

12. Using the methods of this section, find the shortest distance from the point \((x_0, y_0, z_0)\) to the plane \( ax + by + cz = d \).

13. A trough is to be formed by bending up two sides of a long metal rectangle so that the cross-section of the trough is an isosceles trapezoid, as in figure 6.2.6. If the width of the metal sheet is 2 meters, how should it be bent to maximize the volume of the trough?

16.8 Lagrange Multipliers 391

- Many applied max/min problems take the form of the last two examples: we want to find an extreme value of a function, like \( V = xyz \), subject to a constraint, like \( 1 = \sqrt{x^2 + y^2 + z^2} \).
- Often this can be done, as we have, by explicitly combining the equations and then finding critical points.
- There is another approach that is often convenient, the method of Lagrange multipliers.

- Lagrange multipliers is somewhat easier to understand two variable problems, so we begin with one as an example. Suppose the perimeter of a rectangle is to be 100 units. Find the rectangle with largest area.
- This is a fairly straightforward problem from single variable calculus. We write down the two equations: \( A = xy \), \( P = 2x + 2y \), solve the second of these for \( y \), substitute into the first, and end up with a one-variable maximization problem. Let’s now think of it differently: the equation \( A = xy \) defines a surface, and the equation \( P = 2x + 2y \) defines a curve (a line, in this case) in the \( x,y \)-plane. If we graph both of these in the three-dimensional coordinate system, we can phrase the problem like this: what is the highest point on the surface above the line? The solution we already understand effectively produces the equation of the cross-section of the surface above the line and then treats it as a single-variable problem.

- Instead, imagine that we draw the level curves of the function, like \( z = x^2 + y^2 \), and the gradient of \( z = xy \) is \( (1, 1) \). They are parallel when \( (2, 2) = \lambda (y, x) \), that is, when \( 2x = 2y \) and \( 2 = 2\lambda \). We have two equations in three unknowns, which typically results in many solutions (as we expected). A third equation will reduce the number of solutions; the third equation is the original constraint, \( 100 = 2x + 2y \).

- So we have the following system to solve:
  
  \[
  2 = 2y \\
  2 = 2x \\
  100 = 2x + 2y.
  \]

- In the first two equations, \( \lambda \) can’t be 0, so we may divide by it to get \( x = y = 2 \). Substituting into the third equation we get

\[
\frac{2^2}{\lambda} + \frac{2^2}{\lambda} + 2 = 100 \Rightarrow \lambda = 25.
\]

- Note that we are not really interested in the value of \( \lambda \)—it is a clever tool, the Lagrange multiplier, introduced to solve the problem. In many cases, as here, it is easier to find \( \lambda \) than to find everything else without using \( \lambda \).

- The same method works for functions of three variables, except of course everything is one dimension higher: the function to be optimized is a function of three variables and the constraint represents a surface—for example, the function may represent temperature, and we may be interested in the maximum temperature on some surface, like a sphere. The points we seek are those at which the constraint surface is tangent to a level surface of the function. Once again, we consider the constraint surface to be a level surface of some function, and we look for points at which the two gradients are parallel, giving us three equations in four unknowns. The constraint provides a fourth equation.

**Example 16.8.1** Recall example 16.7.8: the diagonal of a box is 1, we seek to maximize the volume. The constraint is \( 1 = \sqrt{x^2 + y^2 + z^2} \), which is the same as \( 1 = \sqrt{x^2 + y^2 + z^2} \)
As before, this gives us three equations, one for each component of the vectors, but now the plane total of five equations in five unknowns, and so can usually find the solutions we need.

If \( \lambda = 0 \) then at least two of \( x, y, z \) must be 0, giving a volume of 0, which will not be the maximum. If we multiply the first two equations by \( x \) and \( y \) respectively, we get

\[
\begin{align*}
yz &= 2x\lambda \\
xz &= 2y\lambda \\
yx &= 2z\lambda \\
x^2 + y^2 + z^2 &= 1
\end{align*}
\]

so \( 2x\lambda = 2y\lambda \) or \( x^2 = y^2 \), in the same way we can show \( x^2 = z^2 \). Hence the fourth equation becomes \( 1 = x^2 + y^2 + z^2 \) or \( x = 1/\sqrt{2} \), and so \( x = y = z = 1/\sqrt{2} \) gives the maximum volume. This is of course the same answer we obtained previously.

Another possibility is that we have a function of three variables, and we want to find a maximum or minimum value not on a surface but on a curve; often the curve is the intersection of two surfaces, so that we really have two constraint equations, say \( g(x, y, z) = c_1 \) and \( h(x, y, z) = c_2 \). It turns out that at points on the intersection of the surfaces where \( f \) has a maximum or minimum value,

\[
\nabla f = \lambda \nabla g + \mu \nabla h.
\]

As before, this gives us three equations, one for each component of the vectors, but now in five unknowns, \( x, y, z, \lambda, \mu \). Since there are two constraint functions, we have a total of five equations in five unknowns, and so can usually find the solutions we need.

**Example 16.8.2** The plane \( x + y + z = 1 \) intersects the cylinder \( x^2 + y^2 = 1 \) in an ellipse. Find the points on the ellipse closest to and farthest from the origin.

We want the extreme values of \( f = \sqrt{x^2 + y^2 + z^2} \) subject to the constraints \( g = x^2 + y^2 = 1 \) and \( h = x + y + z = 1 \). To simplify the algebra, we may use instead \( f = x^2 + y^2 + z^2 \), since this has a maximum or minimum value at exactly the points at which \( \sqrt{x^2 + y^2 + z^2} \) does. The gradients are

\[
\nabla f = (2x, 2y, 2z) \quad \nabla g = (2x, 2y, 0) \quad \nabla h = (1, 1, -1),
\]

so the equations we need to solve are

\[
\begin{align*}
yz &= 2x\lambda \\
xz &= 2y\lambda \\
yx &= 2z\lambda \\
x^2 + y^2 + z^2 &= 1
\end{align*}
\]

Subtracting the first two we get \( 2y - 2z = \lambda(2x - 2z) \), so either \( \lambda = 1 \) or \( x = y \). If \( \lambda = 1 \) then \( y = 0 \), so \( z = 0 \) and the last two equations are

\[
\begin{align*}
x^2 + y^2 &= 1 \\
1 &= x + y.
\end{align*}
\]

Solving these gives \( x = 1, y = 0, \) or \( x = 0, y = 1 \), so the points of interest are \((1,0,0)\) and \((0,1,0)\), which are both distance 1 from the origin. If \( x = y \), the fourth equation is \( 2x^2 = 1 \), giving \( x = y = \pm 1/\sqrt{2} \), and from the fifth equation we get \( z = -1 \pm \sqrt{2} \). The distance from the origin to \((1/\sqrt{2}, 1/\sqrt{2}, -1 + \sqrt{2})\) is \(1.08\) and the distance from the origin to \((-1/\sqrt{2}, -1/\sqrt{2}, -1 - \sqrt{2})\) is \(2.24\). Thus, the points \((1,0,0)\) and \((0,1,0)\) are closest to the origin and \((-1/\sqrt{2}, -1/\sqrt{2}, -1 - \sqrt{2})\) is farthest from the origin.

This applet shows the cylinder, the plane, the four points of interest, and the origin.

**Exercises 16.8.**

1. A tin-sided rectangular box is to hold 1/2 cubic meter; what shape should the box be to minimize surface area? \( \Rightarrow \)
2. The post office will accept packages whose combined length and girth are at most 130 inches (girth is the maximum distance around the package perpendicular to the length). What is the largest volume that can be sent in a rectangular box? \( \Rightarrow \)
3. The bottom of a rectangular box costs twice as much per unit area as the sides and top. Find the shape of the cheapest aquarium that holds a given volume. \( \Rightarrow \)
4. Using Lagrange multipliers, find the shortest distance from the point \((x_0, y_0, z_0)\) to the plane \( ax + by + cz = d = d \) is \( \Rightarrow \)
5. Find all points on the surface \( xyz = 1 \) that are closest to the origin. \( \Rightarrow \)
6. The material for the bottom of an aquarium costs half as much as the high strength glass for the four sides. Find the shape of the cheapest aquarium that holds a given volume. \( \Rightarrow \)
7. The plane \( x - y + z = 2 \) intersects the cylinder \( x^2 + y^2 &= 4 \) in an ellipse. Find the points on the ellipse closest to and farthest from the origin. \( \Rightarrow \)
8. Find all points on the plane \( x + y + z = 5 \) in the first octant at which \( f(x, y, z) = xy^2z^3 \) has a maximum value. \( \Rightarrow \)

10. Find the points on the surface \( x^2 - yz = 5 \) that are closest to the origin. \( \Rightarrow \)
11. A manufacturer makes two models of an item, standard and deluxe. It costs 840 to manufacture the standard model and 860 for the deluxe. A market research firm estimates that if the standard model is priced at \( x \) dollars and the deluxe at \( y \) dollars, then the manufacturer will sell 500(x - y) of the standard items and 45,000 + 500(x - 2y) of the deluxe each year. How should the items be priced to maximize profit? \( \Rightarrow \)
12. A length of sheet metal is to be made into a water trough by bending up two sides as shown in figure 16.8.3. Find \( x \) and \( \phi \) so that the trapezoid-shaped cross section has maximum area, when the width of the metal sheet is 27 inches (that is, \( 2e + y + z = 27 \)). \( \Rightarrow \)

**Figure 16.8.3** Cross-section of a trough.

13. Find the maximum and minimum values of \( f(x, y, z) = 6x + 3y + 2z \) subject to the constraint \( g(x, y, z) = x^2 + y^2 + z^2 = 70 = 70 \). \( \Rightarrow \)
14. Find the maximum and minimum values of \( f(x, y) = e^{xy} \) subject to the constraint \( g(x, y) = x^2 + y^2 = 16 = 16 \). \( \Rightarrow \)
15. Find the maximum and minimum values of \( f(x, y) = xy + \sqrt{y^2 - x^2 - y^2} \) when \( x^2 + y^2 \leq 9 \). \( \Rightarrow \)
16. Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible. \( \Rightarrow \)
17. Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere. \( \Rightarrow \)