15

Vector Functions

15.1 Space Curves

We have already seen that a convenient way to describe a line in three dimensions is to provide a vector that "points to" every point on the line as a parameter t varies, like

$$\langle 1,2,3\rangle + t\langle 1,-2,2\rangle = \langle 1+t, 2-2t, 3+2t\rangle.$$

Except that this gives a particularly simple geometric object, there is nothing special about the individual functions of t that make up the coordinates of this vector—any vector with a parameter, like $\langle f(t), g(t), h(t)\rangle$, will describe some curve in three dimensions as t varies through all possible values.

EXAMPLE 15.1.1 Describe the curves $\langle \cos t, \sin t, 0\rangle$, $\langle \cos t, \sin t, t\rangle$, and $\langle \cos t, \sin t, 2t\rangle$.

As t varies, the first two coordinates in all three functions trace out the points on the unit circle, starting with $(1,0)$ when $t=0$ and proceeding counter-clockwise around the circle as t increases. In the first case, the z coordinate is always 0, so this describes precisely the unit circle in the x-y plane. In the second case, the x and y coordinates still describe a circle, but now the z coordinate varies, so that the height of the curve matches the value of t. When $t=\pi$, for example, the resulting vector is $\langle -1,0,\pi\rangle$. A bit of thought should convince you that the result is a helix. In the third vector, the z coordinate varies twice as fast as the parameter t, so we get a stretched out helix. Both are shown in figure 15.1.1. On the left is the first helix, shown for t between 0 and 4π; on the right is the second helix, shown for t between 0 and 2π. Both start and end at the same point, but the first helix takes two full "turns" to get there, because its z coordinate grows more slowly. □

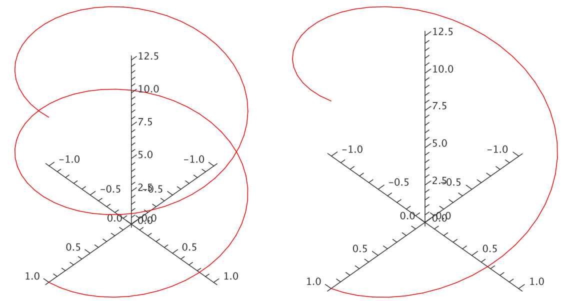


Figure 15.1.1 Two helixes. (AP)

A vector expression of the form $\langle f(t), g(t), h(t)\rangle$ is called a **vector function**; it is a function from the real numbers $\mathbb{R}$ to the set of all three-dimensional vectors. We can alternately think of it as three separate functions, $x=f(t)$, $y=g(t)$, and $z=h(t)$, that describe points in space. In this case we usually refer to the set of equations as **parametric equations** for the curve, just as for a line. While the parameter t in a vector function might represent any one of a number of physical quantities, or be simply a "pure number", it is often convenient and useful to think of t as representing time. The vector function then tells you where in space a particular object is at any time.

Vector functions can be difficult to understand, that is, difficult to picture. When available, computer software can be very helpful. When working by hand, one useful approach is to consider the "projections" of the curve onto the three standard coordinate planes. We have already done this in part: in example 15.1.1 we noted that all three curves project to a circle in the x-y plane, since $\langle \cos t, \sin t\rangle$ is a two dimensional vector function for the unit circle.

EXAMPLE 15.1.2 Graph the projections of $\langle \cos t, \sin t, 2t\rangle$ onto the x-z plane and the y-z plane. The two dimensional vector function for the projection onto the x-z plane is $\langle \cos t, 2t\rangle$, or in parametric form, $x=\cos t$, $z=2t$. By eliminating t we get the equation $x=\cos(z/2)$, the familiar curve shown on the left in figure 15.1.2. For the projection onto the y-z plane, we start with the vector function $\langle \sin t, 2t\rangle$, which is the same as $y=\sin t$, $z=2t$. Eliminating t gives $y=\sin(z/2)$, as shown on the right in figure 15.1.2. □

Exercises 15.1.

1. Describe the curve $\mathbf{r} = \langle \sin t, \cos t, \cos 8t\rangle$.
2. Describe the curve $\mathbf{r} = \langle t\cos t, t\sin t, t\rangle$.

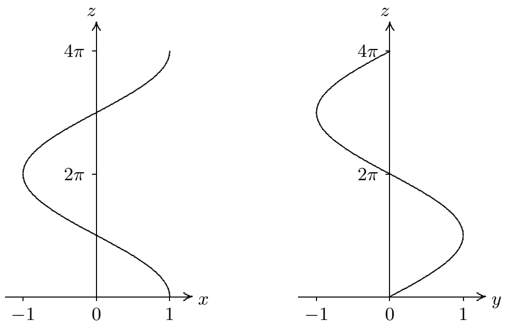


Figure 15.1.2 The projections of $\langle \cos t, \sin t, 2t\rangle$ onto the x-z and y-z planes.

3. Describe the curve $\mathbf{r} = \langle t, t^2, \cos t\rangle$.

4. Describe the curve $\mathbf{r} = \langle \cos(20t)\sqrt{1-t^2}, \sin(20t)\sqrt{1-t^2}, t\rangle$

5. Find a vector function for the curve of intersection of $x^2 + y^2 = 9$ and $y + z = 2$. ⇒

6. A bug is crawling outward along the spoke of a wheel that lies along a radius of the wheel. The bug is crawling at 1 unit per second and the wheel is rotating at 1 radian per second. Suppose the wheel lies in the y-z plane with center at the origin, and at time $t = 0$ the spoke lies along the positive y axis and the bug is at the origin. Find a vector function $\mathbf{r}(t)$ for the position of the bug at time t. ⇒

7. What is the difference between the parametric curves $f(t) = \langle t, t, t^2\rangle$, $g(t) = \langle t^2, t^2, t^4\rangle$, and $h(t) = \langle \sin(t), \sin(t), \sin^2(t)\rangle$as t runs over all real numbers?

8. Plot each of the curves below in 2 dimensions, projected onto each of the three standard planes (the x-y, x-z, and y-z planes).

a. $f(t) = \langle t, t^3, t^2\rangle$, t ranges over all real numbers

b. $f(t) = \langle t^2, t-1, t^2+5\rangle$ for $0 \le t \le 3$

9. Given points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, give parametric equations for the line *segment* connecting A and B. Be sure to give appropriate t values.

10. With a parametric plot and a set of t values, we can associate a 'direction'. For example, the curve $\langle \cos t, \sin t\rangle$ is the unit circle traced counterclockwise. How can we amend a set of given parametric equations and t values to get the same curve, only traced backwards?

15.2 Calculus with vector functions

A vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is a function of one variable—that is, there is only one "input" value. What makes vector functions more complicated than the functions $y = f(x)$ that we studied in the first part of this book is of course that the "output" values are now three-dimensional vectors instead of simply numbers. It is natural to wonder if there is a corresponding notion of derivative for vector functions. In the simpler case of

a function $y = s(t)$, in which t represents time and $s(t)$ is position on a line, we have seen that the derivative $s'(t)$ represents velocity; we might hope that in a similar way the derivative of a vector function would tell us something about the velocity of an object moving in three dimensions.

One way to approach the question of the derivative for vector functions is to write down an expression that is analogous to the derivative we already understand, and see if we can make sense of it. This gives us

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t\to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \\
&= \lim_{\Delta t\to 0} \frac{\langle f(t+\Delta t) - f(t), g(t+\Delta t) - g(t), h(t+\Delta t) - h(t)\rangle}{\Delta t} \\
&= \lim_{\Delta t\to 0} \langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t}, \frac{h(t+\Delta t) - h(t)}{\Delta t}\rangle \\
&= \langle f'(t), g'(t), h'(t)\rangle,
\end{aligned}$$

if we say that what we mean by the limit of a vector is the vector of the individual coordinate limits. So starting with a familiar expression for what appears to be a derivative, we find that we can make good computational sense out of it—but what does it actually mean?

We know how to interpret $\mathbf{r}(t+\Delta t)$ and $\mathbf{r}(t)$—they are vectors that point to locations in space; if t is time, we can think of these points as positions of a moving object at times that are Δt apart. We also know what $\Delta\mathbf{r} = \mathbf{r}(t+\Delta t) - \mathbf{r}(t)$ means—it is a vector that points from the head of $\mathbf{r}(t)$ to the head of $\mathbf{r}(t+\Delta t)$, assuming both have their tails at the origin. So when Δt is small, $\Delta\mathbf{r}$ is a tiny vector pointing from one point on the path of the object to a nearby point. As Δt gets close to 0, this vector points in a direction that is closer and closer to the direction in which the object is moving; geometrically, it approaches a vector tangent to the path of the object at a particular point.

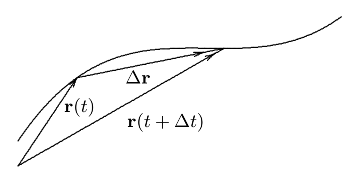


Figure 15.2.1 Approximating the derivative.

Unfortunately, the vector $\Delta\mathbf{r}$ approaches 0 in length; the vector $\langle 0, 0, 0\rangle$ is not very informative. By dividing by Δt, when it is small, we effectively keep magnifying the length of $\Delta\mathbf{r}$ so that in the limit it doesn't disappear. Thus the limiting vector $\langle f'(t), g'(t), h'(t)\rangle$ will (usually) be a good, non-zero vector that is tangent to the curve.

What about the length of this vector? It's nice that we've kept it away from zero, but what does it measure, if anything? Consider the length of one of the vectors that approaches the tangent vector:

$$\left|\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}\right| = \frac{|\mathbf{r}(t+\Delta t)-\mathbf{r}(t)|}{|\Delta t|}$$

The numerator is the length of the vector that points from one position of the object to a "nearby" position; this length is approximately the distance traveled by the object between times t and $t+\Delta t$. Dividing this distance by the length of time it takes to travel that distance gives the average speed. As Δt approaches zero, this average speed approaches the actual, instantaneous speed of the object at time t.

So by performing an "obvious" calculation to get something that looks like the derivative of $\mathbf{r}(t)$, we get precisely what we would want from such a derivative: the vector $\mathbf{r}'(t)$ points in the direction of travel of the object and its length tells us the speed of travel. In the case that t is time, then, we call $\mathbf{v}(t) = \mathbf{r}'(t)$ the velocity vector. Even if t is not time, $\mathbf{r}'(t)$ is useful—it is a vector tangent to the curve.

EXAMPLE 15.2.1 We have seen that $\mathbf{r} = \langle \cos t, \sin t, t\rangle$ is a helix. We compute $\mathbf{r}' = \langle -\sin t, \cos t, 1\rangle$, and $|\mathbf{r}'| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$. So thinking of this as a description of a moving object, its speed is always $\sqrt{2}$; see figure 15.2.2. □

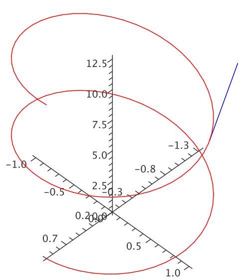


Figure 15.2.2 A tangent vector on the helix. (AP)

EXAMPLE 15.2.2 The velocity vector for $\langle \cos t, \sin t, \cos t\rangle$ is $\langle -\sin t, \cos t, -\sin t\rangle$. As before, the first two coordinates mean that from above this curve looks like a circle. The

z coordinate is now also periodic, so that as the object moves around the curve its height oscillates up and down. In fact it turns out that the curve is a tilted ellipse, as shown in figure 15.2.3. □

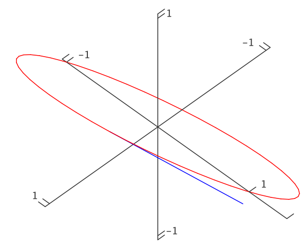


Figure 15.2.3 The ellipse $\mathbf{r} = \langle \cos t, \sin t, \cos t\rangle$. (AP)

EXAMPLE 15.2.3 The velocity vector for $\langle \cos t, \sin t, \cos 2t\rangle$ is $\langle -\sin t, \cos t, -2\sin 2t\rangle$. The z coordinate is now oscillating twice as fast as in the previous example, so the graph is not surprising; see figure 15.2.4. □

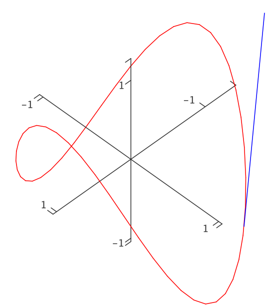


Figure 15.2.4 $\langle \cos t, \sin t, \cos 2t\rangle$. (AP)

EXAMPLE 15.2.4 Find the angle between the curves $\langle t, 1-t, 3+t^2\rangle$ and $\langle 3-t, t-2, t^2\rangle$ where they meet.

The angle between two curves at a point is the angle between their tangent vectors—any tangent vectors will do, so we can use the derivatives. We need to find the point of intersection, evaluate the two derivatives there, and finally find the angle between them.

To find the point of intersection, we need to solve the equations

$$\begin{aligned} t &= 3 - u \\ 1 - t &= u - 2 \\ 3 + t^2 &= u^2 \end{aligned}$$

Solving either of the first two equations for u and substituting in the third gives $3 + t^2 = (3 - t)^2$, which means $t = 1$. This together with $u = 2$ satisfies all three equations. Thus the two curves meet at $(1, 0, 4)$, the first when $t = 1$ and the second when $t = 2$.

The derivatives are $\langle 1, -1, 2t\rangle$ and $\langle -1, 1, 2t\rangle$; at the intersection point these are $\langle 1, -1, 2\rangle$ and $\langle -1, 1, 4\rangle$. The cosine of the angle between them is then

$$\cos\theta = \frac{-1 - 1 + 8}{\sqrt{6}\sqrt{18}} = \frac{1}{\sqrt{3}},$$

so $\theta = \arccos(1/\sqrt{3}) \approx 0.96$. □

The derivatives of vector functions obey some familiar looking rules, which we will occasionally need.

THEOREM 15.2.5 Suppose $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are differentiable functions, $f(t)$ is a differentiable function, and a is a real number.

a. $\displaystyle \frac{d}{dt}a\mathbf{r}(t) = a\mathbf{r}'(t)$

b. $\displaystyle \frac{d}{dt}(\mathbf{r}(t) + \mathbf{s}(t)) = \mathbf{r}'(t) + \mathbf{s}'(t)$

c. $\displaystyle \frac{d}{dt}f(t)\mathbf{r}(t) = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$

d. $\displaystyle \frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{s}(t)) = \mathbf{r}'(t) \cdot \mathbf{s}(t) + \mathbf{r}(t) \cdot \mathbf{s}'(t)$

e. $\displaystyle \frac{d}{dt}(\mathbf{r}(t) \times \mathbf{s}(t)) = \mathbf{r}'(t) \times \mathbf{s}(t) + \mathbf{r}(t) \times \mathbf{s}'(t)$

f. $\displaystyle \frac{d}{dt}\mathbf{r}(f(t)) = \mathbf{r}'(f(t))f'(t)$

■

Note that because the cross product is not commutative you must remember to do the three cross products in formula (e) in the correct order.

When the derivative of a function $f(t)$ is zero, we know that the function has a horizontal tangent line, and may have a local maximum or minimum point. If $\mathbf{r}'(t) = \mathbf{0}$, the geometric interpretation is quite different, though the interpretation in terms of motion is similar. Certainly we know that the object has speed zero at such a point, and it may thus be abruptly changing direction. In three dimensions there are many ways to change direction; geometrically this often means the curve has a cusp or a point, as in the path of a ball that bounces off the floor or a wall.

EXAMPLE 15.2.6 Suppose that $\mathbf{r}(t) = \langle 1 + t^3, t^2, 1\rangle$, so $\mathbf{r}'(t) = \langle 3t^2, 2t, 0\rangle$. This is $\mathbf{0}$ at $t = 0$, and there is indeed a cusp at the point $(1, 0, 1)$, as shown in figure 15.2.5. □

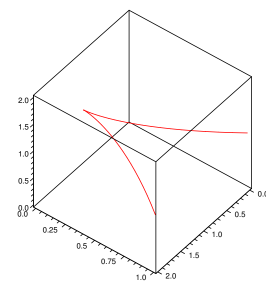


Figure 15.2.5 $\langle 1 + t^3, t^2, 1\rangle$ has a cusp at $(1, 0, 1)$. (AP)

Sometimes we will be interested in the direction of $\mathbf{r}'$ but not its length. In some cases, we can still work with $\mathbf{r}'$, as when we find the angle between two curves. On other occasions it will be useful to work with a unit vector in the same direction as $\mathbf{r}'$; of course, we can compute such a vector by dividing $\mathbf{r}'$ by its own length. This standard unit tangent vector is usually denoted by $\mathbf{T}$:

$$\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}.$$

In a sense, when we computed the angle between two tangent vectors we have already made use of the unit tangent, since

$$\cos\theta = \frac{\mathbf{r}' \cdot \mathbf{s}'}{|\mathbf{r}'||\mathbf{s}'|} = \frac{\mathbf{r}'}{|\mathbf{r}'|} \cdot \frac{\mathbf{s}'}{|\mathbf{s}'|}$$

Now that we know how to make sense of $\mathbf{r}'$, we immediately know what an antiderivative must be, namely

$$\int \mathbf{r}(t)\,dt = \langle \int f(t)\,dt, \int g(t)\,dt, \int h(t)\,dt \rangle,$$

if $\mathbf{r} = \langle f(t), g(t), h(t)\rangle$. What about definite integrals? Suppose that $\mathbf{v}(t)$ gives the velocity of an object at time t. Then $\mathbf{v}(t)\Delta t$ is a vector that approximates the displacement of the object over the time Δt: $\mathbf{v}(t)\Delta t$ points in the direction of travel, and $|\mathbf{v}(t)\Delta t| = |\mathbf{v}(t)||\Delta t|$ is the speed of the object times Δt, which is approximately the distance traveled. Thus, if we sum many such tiny vectors:

$$\sum_{i=0}^{n-1} \mathbf{v}(t_i)\Delta t$$

we get an approximation to the displacement vector over the time interval $[t_0, t_n]$. If we take the limit we get the exact value of the displacement vector:

$$\lim \sum_{i=0}^{n-1} \mathbf{v}(t_i)\Delta t = \int_{t_0}^{t_n} \mathbf{v}(t)\,dt = \mathbf{r}(t_n) - \mathbf{r}(t_0).$$

Denote $\mathbf{r}(t_0)$ by $\mathbf{r}_0$. Then given the velocity vector we can compute the vector function $\mathbf{r}$ giving the location of the object:

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_{t_0}^{t} \mathbf{v}(u)\,du.$$

EXAMPLE 15.2.7 An object moves with velocity vector $\langle \cos t, \sin t, \cos t\rangle$, starting at $(1, 1, 1)$ at time 0. Find the function $\mathbf{r}$ giving its location.

$$\begin{aligned}
\mathbf{r}(t) &= \langle 1,1,1\rangle + \int_0^t \langle \cos u, \sin u, \cos u\rangle\,du \\
&= \langle 1,1,1\rangle + \langle \sin u, -\cos u, \sin u\rangle\big|_0^t \\
&= \langle 1,1,1\rangle + \langle \sin t, -\cos t, \sin t\rangle - \langle 0,-1,0\rangle \\
&= \langle 1+\sin t, 2-\cos t, 1+\sin t\rangle
\end{aligned}$$

See figure 15.2.6. □

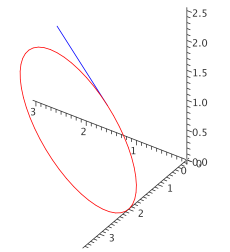


Figure 15.2.6 Path of the object with its initial velocity vector. (AP)

Exercises 15.2.

1. Find $\mathbf{r}'$ and $\mathbf{T}$ for $\mathbf{r} = \langle t^2, 1, t\rangle$. ⇒
2. Find $\mathbf{r}'$ and $\mathbf{T}$ for $\mathbf{r} = \langle \cos t, \sin 2t, t^2\rangle$. ⇒
3. Find $\mathbf{r}'$ and $\mathbf{T}$ for $\mathbf{r} = \langle \cos(e^t), \sin(e^t), \sin t\rangle$. ⇒
4. Find a vector function for the line tangent to the helix $\langle \cos t, \sin t, t\rangle$ when $t = \pi/4$. ⇒
5. Find a vector function for the line tangent to $\langle \cos t, \sin t, \cos 4t\rangle$ when $t = \pi/3$. ⇒
6. Find the cosine of the angle between the curves $\langle 0, t^2, t\rangle$ and $\langle \cos(\pi t/2), \sin(\pi t/2), t\rangle$ where they intersect. ⇒
7. Find the cosine of the angle between the curves $\langle \cos t, -\sin(t)/4, \sin t\rangle$ and $\langle \cos t, \sin t, \sin(2t)\rangle$ where they intersect. ⇒
8. Suppose that $|\mathbf{r}(t)| = k$, for some constant k. This means that $\mathbf{r}$ describes some path on the sphere of radius k with center at the origin. Show that $\mathbf{r}$ is perpendicular to $\mathbf{r}'$ at every point. Hint: Use Theorem 15.2.5, part (d).
9. A bug is crawling along the spoke of a wheel that lies along a radius of the wheel. The bug is crawling at 1 unit per second and the wheel is rotating at 1 radian per second. Suppose the wheel lies in the y-z plane with center at the origin, and at time $t = 0$ the spoke lies along the positive y axis and the bug is at the origin. Find a vector function $\mathbf{r}(t)$ for the position of the bug at time t, the velocity vector $\mathbf{r}'(t)$, the unit tangent $\mathbf{T}(t)$, and the speed of the bug $|\mathbf{r}'(t)|$. ⇒
10. An object moves with velocity vector $\langle \cos t, \sin t, t\rangle$, starting at $(0, 0, 0)$ when $t = 0$. Find the function $\mathbf{r}$ giving its location. ⇒
11. An object moves with velocity vector $\langle t, t^2, -t\rangle$, starting at $(1, 2, 3)$ when $t = 0$. Find the function $\mathbf{r}$ giving its location. ⇒
12. An object moves with velocity vector $\langle t^2, \sin t, \cos t\rangle$, starting at $(-1, 1, 2)$ when $t = 1$. Find the function $\mathbf{r}$ giving its location. ⇒
13. The position function of a particle is given by $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t\rangle$, $t \geq 0$. When is the speed of the particle a minimum? ⇒

14. A particle moves so that its position is given by $\langle \cos t, \sin t, \cos(6t)\rangle$. Find the maximum and minimum speeds of the particle. ⇒

15. An object moves with velocity vector $\langle t, t^2, \cos t\rangle$, starting at $(0,0,0)$ when $t = 0$. Find the function $\mathbf{r}$ giving its location. ⇒

16. What is the physical interpretation of the dot product of two vector valued functions? What is the physical interpretation of the cross product of two vector valued functions?

17. Show, using the rules of cross products and differentiation, that

$$\frac{d}{dt}(\mathbf{r}(t) \times \mathbf{r}'(t)) = \mathbf{r}(t) \times \mathbf{r}''(t).$$

18. Determine the point at which $\mathbf{f}(t) = \langle t, t^2, t^3\rangle$ and $\mathbf{g}(t) = \langle \cos(t), \cos(2t), t+1\rangle$ intersect, and find the angle between the curves at that point. (Hint: You'll need to set this one up like a line intersection problem, writing one in s and one in t.) If these two functions were the trajectories of two airplanes on the same scale of time, would the planes collide at their point of intersection? Explain. ⇒

19. Find the equation of the plane perpendicular to the curve $\mathbf{r}(t) = \langle 2\sin(3t), t, 2\cos(3t)\rangle$ at the point $(0, \pi, -2)$. ⇒

20. Find the equation of the plane perpendicular to $\langle \cos t, \sin t, \cos(6t)\rangle$ when $t = \pi/4$. ⇒

21. At what point on the curve $\mathbf{r}(t) = \langle t^3, 3t, t^4\rangle$ is the plane perpendicular to the curve also parallel to the plane $6x + 6y - 8z = 1$? ⇒

22. Find the equation of the line tangent to $\langle \cos t, \sin t, \cos(6t)\rangle$ when $t = \pi/4$. ⇒

15.3 Arc length and curvature

Sometimes it is useful to compute the length of a curve in space; for example, if the curve represents the path of a moving object, the length of the curve between two points may be the distance traveled by the object between two times.

Recall that if the curve is given by the vector function $\mathbf{r}$ then the vector $\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$ points from one position on the curve to another, as depicted in figure 15.2.1. If the points are close together, the length of $\Delta\mathbf{r}$ is close to the length of the curve between the two points. If we add up the lengths of many such tiny vectors, placed head to tail along a segment of the curve, we get an approximation to the length of the curve over that segment. In the limit, as usual, this sum turns into an integral that computes precisely the length of the curve. First, note that

$$|\Delta\mathbf{r}| = \frac{|\Delta\mathbf{r}|}{\Delta t}\,\Delta t \approx |\mathbf{r}'(t)|\,\Delta t,$$

when Δt is small. Then the length of the curve between $\mathbf{r}(a)$ and $\mathbf{r}(b)$ is

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}|\Delta\mathbf{r}| = \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{|\Delta\mathbf{r}|}{\Delta t}\,\Delta t = \lim_{n\to\infty}\sum_{i=0}^{n-1}|\mathbf{r}'(t)|\,\Delta t = \int_a^b |\mathbf{r}'(t)|\,dt.$$

(Well, sometimes. This works if between a and b the segment of curve is traced out exactly once.)

EXAMPLE 15.3.1 Let's find the length of one turn of the helix $\mathbf{r} = \langle \cos t, \sin t, t\rangle$ (see figure 15.1.1). We compute $\mathbf{r}' = \langle -\sin t, \cos t, 1\rangle$ and $|\mathbf{r}'| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$, so the length is

$$\int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\pi.$$

□

EXAMPLE 15.3.2 Suppose $y = \ln x$; what is the length of this curve between $x = 1$ and $x = \sqrt{3}$?

Although this problem does not appear to involve vectors or three dimensions, we can interpret it in those terms: let $\mathbf{r}(t) = \langle t, \ln t, 0\rangle$. This vector function traces out precisely $y = \ln x$ in the x-y plane. Then $\mathbf{r}'(t) = \langle 1, 1/t, 0\rangle$ and $|\mathbf{r}'(t)| = \sqrt{1 + 1/t^2}$ and the desired length is

$$\int_1^{\sqrt{3}} \sqrt{1 + \frac{1}{t^2}}\,dt = 2 - \sqrt{2} + \ln(\sqrt{2} + 1) - \frac{1}{2}\ln 3.$$

(This integral is a bit tricky, but requires only methods we have learned.) □

Notice that there is nothing special about $y = \ln x$, except that the resulting integral can be computed. In general, given any $y = f(x)$, we can think of this as the vector function $\mathbf{r}(t) = \langle t, f(t), 0\rangle$. Then $\mathbf{r}'(t) = \langle 1, f'(t), 0\rangle$ and $|\mathbf{r}'(t)| = \sqrt{1 + (f')^2}$. The length of the curve $y = f(x)$ between a and b is thus

$$\int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

Unfortunately, such integrals are often impossible to do exactly and must be approximated.

One useful application of arc length is the **arc length parameterization**. A vector function $\mathbf{r}(t)$ gives the position of a point in terms of the parameter t, which is often time, but need not be. Suppose s is the distance along the curve from some fixed starting point; if we use s for the variable, we get $\mathbf{r}(s)$, the position in space in terms of distance along the curve. We might still imagine that the curve represents the position of a moving object; now we get the position of the object as a function of how far the object has traveled.

EXAMPLE 15.3.3 Suppose $\mathbf{r}(t) = \langle \cos t, \sin t, 0\rangle$. We know that this curve is a circle of radius 1. While t might represent time, it can also in this case represent the usual angle between the positive x-axis and $\mathbf{r}(t)$. The distance along the circle from $(1,0,0)$ to

$(\cos t, \sin t, 0)$ is also t—this is the definition of radian measure. Thus, in this case $s = t$ and $\mathbf{r}(s) = \langle \cos s, \sin s, 0\rangle$. □

EXAMPLE 15.3.4 Suppose $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$. We know that this curve is a helix. The distance along the helix from $(1, 0, 0)$ to $(\cos t, \sin t, t)$ is

$$s = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{\cos^2 u + \sin^2 u + 1}\,du = \int_0^t \sqrt{2}\,du = \sqrt{2}t.$$

Thus, the value of t that gets us distance s along the helix is $t = s/\sqrt{2}$, and so the same curve is given by $\hat{\mathbf{r}}(s) = \langle \cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2}\rangle$. □

In general, if we have a vector function $\mathbf{r}(t)$, to convert it to a vector function in terms of arc length we compute

$$s = \int_a^t |\mathbf{r}'(u)|\,du = f(t),$$

solve $s = f(t)$ for t, getting $t = g(s)$, and substitute this back into $\mathbf{r}(t)$ to get $\hat{\mathbf{r}}(s) = \mathbf{r}(g(s))$.

Suppose that t is time. By the Fundamental Theorem of Calculus, if we start with arc length

$$s(t) = \int_a^t |\mathbf{r}'(u)|\,du$$

and take the derivative, we get

$$s'(t) = |\mathbf{r}'(t)|.$$

Here $s'(t)$ is the rate at which the arc length is changing, and we have seen that $|\mathbf{r}'(t)|$ is the speed of a moving object; these are of course the same.

Suppose that $\mathbf{r}(s)$ is given in terms of arc length; what is $|\mathbf{r}'(s)|$? It is the rate at which arc length is changing *relative to arc length*; it must be 1! In the case of the helix, for example, the arc length parameterization is $\langle \cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2}\rangle$, the derivative is $\langle -\sin(s/\sqrt{2})/\sqrt{2}, \cos(s/\sqrt{2})/\sqrt{2}, 1/\sqrt{2}\rangle$, and the length of this is

$$\sqrt{\frac{\sin^2(s/\sqrt{2})}{2} + \frac{\cos^2(s/\sqrt{2})}{2} + \frac{1}{2}} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1.$$

So in general, $\mathbf{r}'$ is a unit tangent vector.

Given a curve $\mathbf{r}(t)$, we would like to be able to measure, at various points, how sharply curved it is. Clearly this is related to how "fast" a tangent vector is changing direction, so a first guess might be that we can measure curvature with $|\mathbf{r}''(t)|$. A little thought shows that this is flawed; if we think of t as time, for example, we could be tracing out the curve

more or less quickly as time passes. The second derivative $|\mathbf{r}''(t)|$ incorporates this notion of time, so it depends not simply on the geometric properties of the curve but on how quickly we move along the curve.

EXAMPLE 15.3.5 Consider $\mathbf{r}(t) = \langle \cos t, \sin t, 0\rangle$ and $\mathbf{s}(t) = \langle \cos 2t, \sin 2t, 0\rangle$. Both of these vector functions represent the unit circle in the x-y plane, but if t is interpreted as time, the second describes an object moving twice as fast as the first. Computing the second derivatives, we find $|\mathbf{r}''(t)| = 1$, $|\mathbf{s}''(t)| = 4$. □

To remove the dependence on time, we use the arc length parameterization. If a curve is given by $\mathbf{r}(s)$, then the first derivative $\mathbf{r}'(s)$ is a unit vector, that is, $\mathbf{r}'(s) = \mathbf{T}(s)$. We now compute the second derivative $\mathbf{r}''(s) = \mathbf{T}'(s)$ and use $|\mathbf{T}'(s)|$ as the "official" measure of **curvature**, usually denoted κ.

EXAMPLE 15.3.6 We have seen that the arc length parameterization of a particular helix is $\mathbf{r}(s) = \langle \cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2}\rangle$. Computing the second derivative gives $\mathbf{r}''(s) = \langle -\cos(s/\sqrt{2})/2, -\sin(s/\sqrt{2})/2, 0\rangle$ with length $1/2$. □

What if we are given a curve as a vector function $\mathbf{r}(t)$, where t is not arc length? We have seen that arc length can be difficult to compute; fortunately, we do not need to convert to the arc length parameterization to compute curvature. Instead, let us imagine that we have done this, so we have found $t = g(s)$ and then formed $\hat{\mathbf{r}}(s) = \mathbf{r}(g(s))$. The first derivative $\hat{\mathbf{r}}'(s)$ is a unit tangent vector, so it is the same as the unit tangent vector $\mathbf{T}(t) = \mathbf{T}(g(s))$. Taking the derivative of this we get

$$\frac{d}{ds}\mathbf{T}(g(s)) = \mathbf{T}'(g(s))g'(s) = \mathbf{T}'(t)\frac{dt}{ds}.$$

The curvature is the length of this vector:

$$\kappa = |\mathbf{T}'(t)||\frac{dt}{ds}| = \frac{|\mathbf{T}'(t)|}{|ds/dt|} = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

(Recall that we have seen that $ds/dt = |\mathbf{r}'(t)|$.) Thus we can compute the curvature by computing only derivatives with respect to t; we do not need to do the conversion to arc length.

EXAMPLE 15.3.7 Returning to the helix, suppose we start with the parameterization $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$. Then $\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle$, $|\mathbf{r}'(t)| = \sqrt{2}$, and $\mathbf{T}(t) = \langle -\sin t, \cos t, 1\rangle/\sqrt{2}$. Then $\mathbf{T}'(t) = \langle -\cos t, -\sin t, 0\rangle/\sqrt{2}$ and $|\mathbf{T}'(t)| = 1/\sqrt{2}$. Finally, $\kappa = 1/\sqrt{2}/\sqrt{2} = 1/2$, as before. □

EXAMPLE 15.3.8 Consider this circle of radius a: $\mathbf{r}(t) = \langle a\cos t, a\sin t, 1\rangle$. Then $\mathbf{r}'(t) = \langle -a\sin t, a\cos t, 0\rangle$, $|\mathbf{r}'(t)| = a$, and $\mathbf{T}(t) = \langle -a\sin t, a\cos t, 0\rangle/a$. Now $\mathbf{T}'(t) = \langle -a\cos t, -a\sin t, 0\rangle/a$ and $|\mathbf{T}'(t)| = 1$. Finally, $\kappa = 1/a$: the curvature of a circle is everywhere the reciprocal of the radius. It is sometimes useful to think of curvature as describing what circle a curve most resembles at a point. The curvature of the helix in the previous example is $1/2$; this means that a small piece of the helix looks very much like a circle of radius 2, as shown in figure 15.3.1. □

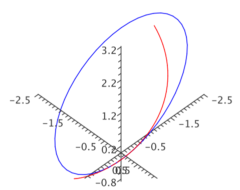


Figure 15.3.1 A circle with the same curvature as the helix. (AP)

EXAMPLE 15.3.9 Consider $\mathbf{r}(t) = \langle \cos t, \sin t, \cos 2t\rangle$, as shown in figure 15.2.4. $\mathbf{r}'(t) = \langle -\sin t, \cos t, -2\sin(2t)\rangle$ and $|\mathbf{r}'(t)| = \sqrt{1+4\sin^2(2t)}$, so

$$\mathbf{T}(t) = \left\langle \frac{-\sin t}{\sqrt{1+4\sin^2(2t)}}, \frac{\cos t}{\sqrt{1+4\sin^2(2t)}}, \frac{-2\sin 2t}{\sqrt{1+4\sin^2(2t)}} \right\rangle.$$

Computing the derivative of this and then the length of the resulting vector is possible but unpleasant. □

Fortunately, there is an alternate formula for the curvature that is often simpler than the one we have:

$$\kappa = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

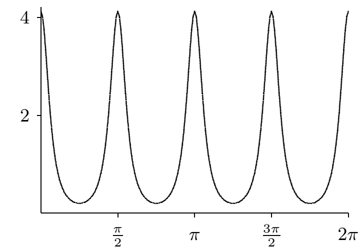
EXAMPLE 15.3.10 Returning to the previous example, we compute the second derivative $\mathbf{r}''(t) = \langle -\cos t, -\sin t, -4\cos(2t)\rangle$. Then the cross product $\mathbf{r}'(t)\times\mathbf{r}''(t)$ is

$$\langle -4\cos t\cos 2t - 2\sin t\sin 2t, 2\cos t\sin 2t - 4\sin t\cos 2t, 1\rangle.$$

Computing the length of this vector and dividing by $|\mathbf{r}'(t)|^3$ is still a bit tedious. With the aid of a computer we get

$$\kappa = \frac{\sqrt{48\cos^4 t - 48\cos^2 t + 17}}{(-16\cos^4 t + 16\cos^2 t + 1)^{3/2}}.$$

Graphing this we get



Compare this to figure 15.2.4—you may want to load the Java applet there so that you can see it from different angles. The highest curvature occurs where the curve has its highest and lowest points, and indeed in the picture these appear to be the most sharply curved portions of the curve, while the curve is almost a straight line midway between those points. □

Let's see why this alternate formula is correct. Starting with the definition of $\mathbf{T}$, $\mathbf{r}' = |\mathbf{r}'|\mathbf{T}$ so by the product rule $\mathbf{r}'' = |\mathbf{r}'|'\mathbf{T} + |\mathbf{r}'|\mathbf{T}'$. Then by Theorem 14.4.2 the cross product is

$$\begin{aligned}
\mathbf{r}'\times\mathbf{r}'' &= |\mathbf{r}'|\mathbf{T}\times(|\mathbf{r}'|'\mathbf{T} + |\mathbf{r}'|\mathbf{T}')\\
&= |\mathbf{r}'|\mathbf{T}\times|\mathbf{r}'|'\mathbf{T} + |\mathbf{r}'|\mathbf{T}\times|\mathbf{r}'|\mathbf{T}'\\
&= |\mathbf{r}'||\mathbf{r}'|'(\mathbf{T}\times\mathbf{T}) + |\mathbf{r}'|^2(\mathbf{T}\times\mathbf{T}')\\
&= |\mathbf{r}'|^2(\mathbf{T}\times\mathbf{T}')
\end{aligned}$$

because $\mathbf{T}\times\mathbf{T} = \mathbf{0}$, since $\mathbf{T}$ is parallel to itself. Then

$$\begin{aligned}
|\mathbf{r}'\times\mathbf{r}''| &= |\mathbf{r}'|^2|\mathbf{T}\times\mathbf{T}'|\\
&= |\mathbf{r}'|^2|\mathbf{T}||\mathbf{T}'|\sin\theta\\
&= |\mathbf{r}'|^2|\mathbf{T}'|
\end{aligned}$$

using exercise 8 in section 15.2 to see that $\theta = \pi/2$. Dividing both sides by $|\mathbf{r}'|^3$ then gives the desired formula.

We used the fact here that $\mathbf{T}'$ is perpendicular to $\mathbf{T}$; the vector $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$ is thus a unit vector perpendicular to $\mathbf{T}$, called the **unit normal** to the curve. Occasionally of use is the **unit binormal** $\mathbf{B} = \mathbf{T}\times\mathbf{N}$, a unit vector perpendicular to both $\mathbf{T}$ and $\mathbf{N}$.

Exercises 15.3.

1. Find the length of $\langle 3\cos t, 2t, 3\sin t\rangle$, $t \in [0, 2\pi]$. ⇒
2. Find the length of $\langle t^2, 2, t^3\rangle$, $t \in [0, 1]$. ⇒
3. Find the length of $\langle t^2, \sin t, \cos t\rangle$, $t \in [0, 1]$. ⇒
4. Find the length of the curve $y = x^{3/2}$, $x \in [1, 9]$. ⇒
5. Set up an integral to compute the length of $\langle \cos t, \sin t, e^t\rangle$, $t \in [0, 5]$. (It is tedious but not too difficult to compute this integral.) ⇒
6. Find the curvature of $\langle t, t^2, t\rangle$. ⇒
7. Find the curvature of $\langle t, t^2, t^2\rangle$. ⇒
8. Find the curvature of $\langle t, t^2, t^3\rangle$. ⇒
9. Find the curvature of $y = x^4$ at $(1, 1)$. ⇒

15.4 Motion along a curve

We have already seen that if t is time and an object's location is given by $\mathbf{r}(t)$, then the derivative $\mathbf{r}'(t)$ is the velocity vector $\mathbf{v}(t)$. Just as $\mathbf{v}(t)$ is a vector describing how $\mathbf{r}(t)$ changes, so is $\mathbf{v}'(t)$ a vector describing how $\mathbf{v}(t)$ changes, namely, $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$ is the **acceleration vector**.

EXAMPLE 15.4.1 Suppose $\mathbf{r}(t) = \langle \cos t, \sin t, 1\rangle$. Then $\mathbf{v}(t) = \langle -\sin t, \cos t, 0\rangle$ and $\mathbf{a}(t) = \langle -\cos t, -\sin t, 0\rangle$. This describes the motion of an object traveling on a circle of radius 1, with constant z coordinate 1. The velocity vector is of course tangent to the curve; note that $\mathbf{a}\cdot\mathbf{v} = 0$, so $\mathbf{v}$ and $\mathbf{a}$ are perpendicular. In fact, it is not hard to see that $\mathbf{a}$ points from the location of the object to the center of the circular path at $(0, 0, 1)$. □

Recall that the unit tangent vector is given by $\mathbf{T}(t) = \mathbf{v}(t)/|\mathbf{v}(t)|$, so $\mathbf{v} = |\mathbf{v}|\mathbf{T}$. If we take the derivative of both sides of this equation we get

$$\mathbf{a} = |\mathbf{v}|'\mathbf{T} + |\mathbf{v}|\mathbf{T}'. \tag{15.4.1}$$

Also recall the definition of the curvature, $\kappa = |\mathbf{T}'|/|\mathbf{v}|$, or $|\mathbf{T}'| = \kappa|\mathbf{v}|$. Finally, recall that we defined the unit normal vector as $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$, so $\mathbf{T}' = |\mathbf{T}'|\mathbf{N} = \kappa|\mathbf{v}|\mathbf{N}$. Substituting into equation 15.4.1 we get

$$\mathbf{a} = |\mathbf{v}|'\mathbf{T} + \kappa|\mathbf{v}|^2\mathbf{N}. \tag{15.4.2}$$

The quantity $|\mathbf{v}(t)|$ is the speed of the object, often written as $v(t)$; $|\mathbf{v}(t)|'$ is the rate at which the speed is changing, or the scalar acceleration of the object, $a(t)$. Rewriting equation 15.4.2 with these gives us

$$\mathbf{a} = a\mathbf{T} + \kappa v^2\mathbf{N} = a_T\mathbf{T} + a_N\mathbf{N};$$

a_T is the **tangential component of acceleration** and a_N is the **normal component of acceleration**. We have already seen that a_T measures how the speed is changing; if you are riding in a vehicle with large a_T you will feel a force pulling you into your seat. The other component, a_N, measures how sharply your direction is changing *with respect to time*. So it naturally is related to how sharply the path is curved, measured by κ, and also to how fast you are going. Because a_N includes v^2, note that the effect of speed is magnified; doubling your speed around a curve quadruples the value of a_N. You feel the effect of this as a force pushing you toward the outside of the curve, the "centrifugal force."

In practice, if want a_N we would use the formula for κ:

$$a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{r}'\times\mathbf{r}''|}{|\mathbf{r}'|^3}|\mathbf{r}'|^2 = \frac{|\mathbf{r}'\times\mathbf{r}''|}{|\mathbf{r}'|}.$$

To compute a_T we can project $\mathbf{a}$ onto $\mathbf{v}$:

$$a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|} = \frac{\mathbf{r}'\cdot\mathbf{r}''}{|\mathbf{r}'|}.$$

EXAMPLE 15.4.2 Suppose $\mathbf{r} = \langle t, t^2, t^3\rangle$. Compute $\mathbf{v}$, $\mathbf{a}$, a_T, and a_N.

Taking derivatives we get $\mathbf{v} = \langle 1, 2t, 3t^2\rangle$ and $\mathbf{a} = \langle 0, 2, 6t\rangle$. Then

$$a_T = \frac{4t + 18t^3}{\sqrt{1 + 4t^2 + 9t^4}} \quad\text{and}\quad a_N = \frac{\sqrt{4 + 36t^2 + 36t^4}}{\sqrt{1 + 4t^2 + 9t^4}}.$$

□

Exercises 15.4.

1. Let $\mathbf{r} = \langle \cos t, \sin t, t\rangle$. Compute $\mathbf{v}$, $\mathbf{a}$, a_T, and a_N. ⇒
2. Let $\mathbf{r} = \langle \cos t, \sin t, t^2\rangle$. Compute $\mathbf{v}$, $\mathbf{a}$, a_T, and a_N. ⇒
3. Let $\mathbf{r} = \langle \cos t, \sin t, e^t\rangle$. Compute $\mathbf{v}$, $\mathbf{a}$, a_T, and a_N. ⇒
4. Let $\mathbf{r} = \langle t^2, 2t - 3, 3t^2 - 3t\rangle$. Compute $\mathbf{v}$, $\mathbf{a}$, a_T, and a_N. ⇒
5. Let $\mathbf{r} = \langle e^t, \sin t, e^t\rangle$. Compute $\mathbf{v}$, $\mathbf{a}$, a_T, and a_N. ⇒
6. Suppose an object moves so that its acceleration is given by $\mathbf{a} = \langle -3\cos t, -2\sin t, 0\rangle$. At time $t = 0$ the object is at $(3, 0, 0)$ and its velocity vector is $\langle 0, 2, 0\rangle$. Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$ for the object. ⇒
7. Suppose an object moves so that its acceleration is given by $\mathbf{a} = \langle -3\cos t, -2\sin t, 0\rangle$. At time $t = 0$ the object is at $(3, 0, 0)$ and its velocity vector is $\langle 0, 2.1, 0\rangle$. Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$ for the object. ⇒
8. Suppose an object moves so that its acceleration is given by $\mathbf{a} = \langle -3\cos t, -2\sin t, 0\rangle$. At time $t = 0$ the object is at $(3, 0, 0)$ and its velocity vector is $\langle 0, 2, 1\rangle$. Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$ for the object. ⇒
9. Suppose an object moves so that its acceleration is given by $\mathbf{a} = \langle -3\cos t, -2\sin t, 0\rangle$. At time $t = 0$ the object is at $(3, 0, 0)$ and its velocity vector is $\langle 0, 2.1, 1\rangle$. Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$ for the object. ⇒

10. Describe a situation in which the normal component of acceleration is 0 and the tangential component of acceleration is non-zero. Is it possible for the tangential component of acceleration to be 0 while the normal component of acceleration is non-zero? Explain. Finally, is it possible for an object to move (not be stationary) so that both the tangential and normal components of acceleration are 0? Explain.