14 Three Dimensions

14.1 The Coordinate System

So far, we have been investigating functions of the form $y = f(x)$, with one independent and one dependent variable. Such functions can be represented in two dimensions, using two numerical axes that allow us to identify every point in the plane with two numbers. We now want to talk about three-dimensional space, to identify every point in three dimensions. We require three numerical values. The obvious way to make this association is to add one new axis, perpendicular to the $x$ and $y$ axes we already understand. We could, for example, add a third axis, the $z$ axis, with the positive $z$ axis coming straight out of the page, and the negative $z$ axis going out the back of the page. This is difficult to work with on a printed page, so more often we draw a view of the three axes from an angle:

![Figure 14.1.1: The plane $z = 1$. (AP)](image)

Use just $P_1$. Distance between two points in either two or three dimensions is sometimes denoted by $d$, so for example the formula for the distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ might be expressed as:

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

![Figure 14.1.2: Distance in three dimensions.](image)

In two dimensions, the distance formula immediately gives us the equation of a circle: the circle of radius $r$ and center at $(h, k)$ consists of all points $(x, y)$ at distance $r$ from $(h, k)$.

Note that if you imagine looking down from above, along the $z$ axis, the positive $z$ axis will come straight toward you, the positive $x$ axis will point up right, as usual. Any point in space is identified by providing the three coordinates of the point, as shown; naturally, we list the coordinates in the order $(x, y, z)$.

One useful way to think of this is to use the $x$ and $y$ coordinates to identify a point in the $x$-$y$ plane, then move straight up (or down) a distance given by the $z$ coordinate.

It is now fairly simple to understand some “shapes” in three dimensions that correspond to simple conditions on the coordinates. In two dimensions, the equation $x = 1$ describes the vertical line through $(1, 0)$. In three dimensions, it still describes all points with $x$-coordinate 1, but this is now a plane, as in figure 14.1.1.

Recall the very useful distance formula in two dimensions: the distance between points $(x_1, y_1)$ and $(x_2, y_2)$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$; this comes directly from the Pythagorean theorem. What is the distance between two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ in three dimensions? Geometrically, we want the length of the long diagonal labeled $c$ in the “box” in figure 14.1.2. Since $a$, $b$, $c$ form a right triangle, $a^2 + b^2 = c^2$. $b$ is the vertical distance between $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$, so $b = |z_1 - z_2|$. The length $a$ runs parallel to the $x$-$y$ plane, so it is simply the distance between $(x_1, y_1)$ and $(x_2, y_2)$, that is, $a^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$. Now we see that $c^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$ and $v = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

It is sometimes useful to give names to points, for example, we might let $P_1 = (x_1, y_1, z_1)$, or more concisely we might refer to the point $(x_1, y_1, z_1)$, and subsequently $(x_2, y_2, z_2)$.

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$h$, $k$, so the equation is $v = \sqrt{(x - h)^2 + (y - k)^2}$ or $v = \sqrt{(x - h)^2 + (y - k)^2}$. Now we can get the similar equation $v = \sqrt{(x - h)^2 + (y - k)^2} + (z - l)^2$, which describes all points $(x, y, z)$ at distance $r$ from $(h, k, l)$, namely, the sphere with radius $r$ and center $(h, k, l)$.

**Exercises 14.1.**

1. Sketch the location of the points $(1, 0, 0)$, $(2, 3, -1)$, and $(-1, 2, 3)$ on a single set of axes.
2. Describe geometrically the set of points $(x, y, z)$ that satisfy $z = 3$.
3. Describe geometrically the set of points $(x, y, z)$ that satisfy $y = -3$.
4. Describe geometrically the set of points $(x, y, z)$ that satisfy $x + y = 2$.
5. The equation $x + y + z = 1$ describes some collection of points in $\mathbb{R}^3$. Describe and sketch the points that satisfy $x + y + z = 1$ and are in the $x$-$y$ plane, in the $x$-$z$ plane, and in the $y$-$z$ plane.
6. Find the length of the sides of the triangle with vertices $(1, 0, 0), (2, 2, 1)$, and $(-3, 2, 2)$.
7. Find the length of the sides of the triangle with vertices $(2, 2, 3), (8, 6, 5)$, and $(-1, 0, 2)$.
8. Why do the results tell you that this isn’t really a triangle?
9. Find an equation of the sphere with center at $(1, 1, 1)$ and radius $2$.
10. Find an equation of the sphere with center at $(2, -1, 3)$ and radius $5$.
11. Find an equation of the sphere with center at $(2, 1, -1)$ and that goes through the point $(4, 2, 5)$.
12. Find an equation of the sphere with center at $(2, 1, -1)$ and radius $4$. Find an equation for the intersection of this sphere with the $y$-$z$ plane; describe this intersection geometrically.
13. Consider the sphere of radius $5$ centered at $(2, 3, 4)$. What is the intersection of this sphere with each of the coordinate planes?
14. Show that for all values of $\theta$ and $\phi$, the point $(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ lies on the sphere given by $x^2 + y^2 + z^2 = a^2$.
15. Prove that the midpoint of the line segment connecting $(x_1, y_1, z_1)$ to $(x_2, y_2, z_2)$ is at $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.
16. Any three points $P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2), P_3(x_3, y_3, z_3)$ lie in a plane and form a triangle. The triangle inequality says that $d(P_1, P_2) \leq d(P_1, P_3) + d(P_3, P_2)$. Prove the triangle inequality using either algebra (Jason) or the law of cosines (Jason).
17. Is it possible for a plane to intersect a sphere in exactly two points? Explain.

14.2 Vectors

A vector is a quantity consisting of a non-negative magnitude and a direction. We could represent a vector in two dimensions as $(m, \theta)$, where $m$ is the magnitude and $\theta$ is the direction, measured as an angle from some agreed upon direction. For example, we might think of the vector $(5, 45^\circ)$ as representing “5 km toward the northeast”; that is, this...
vector might be a displacement vector, indicating, say, that your grandmother walked 5 kilometers toward the northeast to school in the snow. On the other hand, the same vector could represent a velocity, indicating that your grandmother walked at 5 km/hr toward the northeast. What the vector does not indicate is where this walk occurred: a vector represents a magnitude and a direction, but not a location. Pictorially it is useful to represent a vector as an arrow; the direction of the vector, naturally, is the direction in which the arrow points; the magnitude of the vector is reflected in the length of the arrow.

It turns out that many, many quantities behave as vectors, e.g., displacement, velocity, acceleration, force. Already we can get some idea of their usefulness using displacement vectors. Suppose that your grandmother walked 5 km NE and then 2 km SSE; if the terrain allows, and perhaps armed with a compass, how could your grandmother have walked directly to her destination? We can use vectors (and a bit of geometry) to answer this question. We begin by noting that since vectors do not include a specification of position, we can "place" them anywhere that is convenient. So we can picture your grandmother’s journey as two displacement vectors drawn head to tail:

The displacement vector for the shortcut route is the vector drawn with a dashed line, from the tail of the first to the head of the second. With a little trigonometry, we can compute that the third vector has magnitude approximately 4.62 and direction 21.43°, so walking 4.62 km in the direction 21.43° north of east (approximately ENE) would get your grandmother to school.

This sort of calculation is so common, we dignify it with a name: we say that the third vector is the sum of the other two vectors. There is another common way to picture the sum of two vectors. Put the vectors tail to tail and then complete the parallelogram they indicate; the sum of the two vectors is the diagonal of the parallelogram:

This is a more natural representation in some circumstances. For example, if the two original vectors represent forces acting on an object, the sum of the two vectors is the net or effective force on the object, and it is nice to draw all three with their tails at the location of the object.

We also define scalar multiplication for vectors: if \( A \) is a vector (\( m, \theta \)) and \( a \geq 0 \) is a real number, the vector \( aA \) is \((am, \theta)\); namely, it points in the same direction but has \( a \) times the magnitude. If \( a < 0 \), \( aA \) is \((-|a|m, \theta + \pi)\); with \( |a| \) times the magnitude and pointing in the opposite direction (unless we specify otherwise; angles are measured in radians).

Now we can understand subtraction of vectors: \( A - B = A + (-1)B \).

Not surprisingly, we can also use vectors to represent a walk in ways other than (\( m, \theta \)), and in fact (\( m, \theta \)) is not generally used at all. How else could we describe a particular vector? Consider again the vector \((5, 45°)\). Let’s draw it again, but impose a coordinate system. If we put the tail of the arrow at the origin, the head of the arrow ends up at the point \((5\sqrt{2}/2, 5\sqrt{2}/2) \approx (3.54, 3.54)\).

In this picture the coordinates \((3.54, 3.54)\) identify the head of the arrow, provided we know that the tail of the arrow has been placed at \((0,0)\). Then in fact the vector can always be identified as \((3.54, 3.54)\), no matter where it is placed; we just have to remember that the numbers 3.54 must be interpreted as a change from the position of the tail, not as the actual coordinates of the arrow head; to emphasize this we will write \((3.54, 3.54)\) to mean the vector and \((3.54, 3.54)\) to mean the point. Then if the vector \((3.54, 3.54)\) is drawn with its tail at \((1,2)\) it looks like this:

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Consider again the two part trip: 5 km NE and then 2 km SSE. The vector representing the first part of the trip is \((5\sqrt{2}/2, 5\sqrt{2}/2)\); and the second part of the trip is represented by \((2\cos(-3\pi/8), 2\sin(-3\pi/8)) \approx (0.77, -1.85)\). We can represent the sum of these with the usual head to tail picture:

It is clear from the picture that the coordinates of the destination point are \((5\sqrt{2}/2 + 2\cos(-3\pi/8), 5\sqrt{2}/2 + 2\sin(-3\pi/8)) \approx (4.3, 1.69)\), so the sum of the two vectors is \((5\sqrt{2}/2 + 2\cos(-3\pi/8), 5\sqrt{2}/2 + 2\sin(-3\pi/8)) \approx (4.3, 1.69)\). Adding the two vectors is easier in this form than in the \((m, \theta)\) form, provided we’re willing to have the answer in this form as well. In general: \((v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)\).

It is easy to see that scalar multiplication and vector subtraction are also easy to compute in this form: \(av, w) = (av, aw)\) and \((v_1, w_1) - (v_2, w_2) = (v_1 - v_2, w_1 - w_2)\).

What about the magnitude? The magnitude of the vector \((v, w)\) is still the length of the corresponding arrow representation. This is the distance from the origin to the point \((v, w)\); namely, the distance from the tail to the head of the arrow. We know how to compute distances, so the magnitude of the vector is simply \(\sqrt{v^2 + w^2}\), which we also denote with absolute value bars: \(|(v, w)| = \sqrt{v^2 + w^2}\).

In three dimensions, vectors are still quantities consisting of a magnitude and a direction, but of course there are many more possible directions. It’s not clear how we might represent the direction explicitly, but the coordinate version of vectors makes just as much sense in three dimensions as in two. By \((1, 2, 3)\) we mean the vector whose head is at \((1,2,3)\) if its tail is at the origin. As before, we can place the vector anywhere we want; if it has its tail at \((4,5,6)\) then its head is at \((5,7,9)\). It remains true that arithmetic is easy to do with vectors in this form:

\[ (v_1, w_1, z_1) + (v_2, w_2, z_2) = (v_1 + v_2, w_1 + w_2, z_1 + z_2) \]

\[ (v_1, w_1, z_1) - (v_2, w_2, z_2) = (v_1 - v_2, w_1 - w_2, z_1 - z_2) \]

The magnitude of the vector is again the distance from the origin to the head of the arrow, or \(|(v_1, w_1, z_1)| = \sqrt{v_1^2 + w_1^2 + z_1^2}\).

Three particularly simple vectors turn out to be quite useful: \(i = (1,0,0), j = (0,1,0), \) and \(k = (0,0,1)\). These play much the same role for vectors that the axes play for points. In particular, notice that:

\[ (v_1, v_2, v_3) = (v_1, 0, 0) + (0, v_2, 0) + (0, 0, v_3) = v_1i + v_2j + v_3k \]

We will frequently want to produce a vector that points from one point to another. That is, if \(P\) and \(Q\) are points, we seek the vector \(x\) such that when the tail of \(x\) is placed at \(P\), its head is at \(Q\); we refer to this vector as \(\overrightarrow{PQ}\). If we know the coordinates of \(P\) and \(Q\), the coordinates of the vector are easy to find.

EXAMPLE 14.2.1 Suppose \(P = (1, -2, 4)\) and \(Q = (2, 3, -1)\). The vector \(\overrightarrow{PQ}\) is \((-2 - 1, -2 - 3, 4 - (-1)) = (-3, -5, 5)\). Note that this is the same as subtracting the vectors with tails at the origin and heads at \(P\) and \(Q\): \((-2 - 1, -2 - 3, 4 - (-1)) = (-3, -5, 5)\).

Arithmetic with vectors has some familiar properties, listed in the next theorem. These are all quite easy to prove, by simply representing the vectors in standard form.

THEOREM 14.2.2 If \(u, v, \) and \(w\) are vectors and \(a\) and \(b\) are real numbers, then:

1. \(u + v = v + u\)
2. \(au = ua\)
3. \((u + v) = au + ov\)
4. \((a + b) \mathbf{u} = a \mathbf{u} + b \mathbf{u}\)
5. \((a + \mathbf{v}) + \mathbf{w} = a + (\mathbf{v} + \mathbf{w})\)
6. \([\mathbf{a}] \mathbf{u} = [a] \mathbf{u}\)

**Proof.** We do one of these as an example, part 3. Write \(\mathbf{u} = (x_1, y_1, z_1), \mathbf{v} = (x_2, y_2, z_2)\). Then

\[
\mathbf{a}(\mathbf{u} + \mathbf{v}) = \mathbf{a}(x_1 + y_1, y_1 + z_1 + z_2) = \mathbf{a}(x_1, y_1, z_1) + \mathbf{a}(x_2, y_2, z_2) = \mathbf{a}(x_1, y_1, z_1) + \mathbf{a}(x_2, y_2, z_2)
\]

Consider the 12 vectors that have their tails at each of the 12 digits. What is the sum of these vectors? What if we remove the vector corresponding to 4 o'clock? What if, instead, all vectors have their tails at 12 o'clock, and their heads on the remaining digits? 

14. Let \(a\) and \(b\) be nonzero vectors in two dimensions that are not parallel or anti-parallel.
2. Show, algebraically, that if \(a\) is any two-dimensional vector, there are scalars \(s\) and \(t\) such that \(c = sa + tb\).
3. Does the statement in the previous exercise hold if the vectors \(a, b,\) and \(c\) are three-dimensional vectors? Explain.
4. Prove the remaining parts of Theorem 14.2.2.

### 14.3 The Dot Product

Here's a question whose answer turns out to be very useful: Given two vectors, what is the angle between them?

It may not be immediately clear that the question makes sense, but it's not hard to turn it into a question that does. Since vectors have no place to vector wherever we like. If the two vectors are placed tail-to-tail, there is now a reasonable interpretation of the question: we seek the measure of the smallest angle between the two vectors, in the plane in which they lie. Figure 14.3.1 illustrates the situation.

![Figure 14.3.1](image)

**Exercises 14.3.**

1. A. B is the vector with tail at the origin and head at (1, 2).
2. Let \(A\) be the vector with tail at the origin and head at (1, 2). Draw \(A\) and \(B\), a vector \(C\) with tail at (1, 2) and head at (3, 1). Draw \(C\) with its tail at the origin.
3. Let \(A\) be the vector with tail at the origin and head at (1, 2). Let \(B\) be the vector with tail at the origin and head at (3, 1). Draw \(A\) and \(B\) and a vector \(C\) with tail at (1, 2) and head at (3, 1).
4. Find \(P = (4, 5, 6)\), a point. Find \(PQ\), a vector with tail at \((4, 5, 6)\) and head at \((1, 2, 3)\).

**Solution.**

1. The angle \(\theta\) lies in a triangle, we can compute it using a bit of trigonometry, namely, the law of cosines. The lengths of the sides of the triangle in figure 14.4.1 are \(|\mathbf{A}|, |\mathbf{B}|,\) and \(|\mathbf{A} - \mathbf{B}|\). Let \(A = (a_1, a_2, a_3)\) and \(B = (b_1, b_2, b_3)\), then

\[
|\mathbf{A} - \mathbf{B}| = |\mathbf{A}| + |\mathbf{B}| - 2|\mathbf{A}| \cos \theta
\]

2. Find \((\mathbf{A} \cdot \mathbf{B})^2 = |\mathbf{A}|^2 |\mathbf{B}|^2 - 2|\mathbf{A}| \mathbf{B} \cdot \mathbf{A} + |\mathbf{B}|^2 |\mathbf{A}| \cdot \mathbf{A} + |\mathbf{B}|^2|\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}| \cos \theta
\]

Consider the 12 vectors that have their tails at each of a clock and their respective heads at each of the 12 digits. What is the sum of these vectors? What if we remove the vector corresponding to 4 o'clock? What if, instead, all vectors have their tails at 12 o'clock, and their heads on the remaining digits?

Generalizing the examples, note the following useful facts:

1. If \(A\) is parallel or anti-parallel to \(B\) then \(A \cdot |\mathbf{B}| = \pm |\mathbf{A}| = -|\mathbf{B}| \cdot |\mathbf{A}|\), and conversely, if \(A \cdot |\mathbf{B}| = 1\), \(A\) and \(B\) are parallel, while if \(A \cdot |\mathbf{B}| = -1\), \(A\) and \(B\) are anti-parallel. (Vectors are parallel if they point in the same direction, anti-parallel if they point in opposite directions.)

2. If \(A\) is perpendicular to \(B\) then \(A \cdot |\mathbf{B}| = 0\), and conversely if \(A \cdot |\mathbf{B}| = 0\) then \(A\) and \(B\) are perpendicular.

Given two vectors, it is often useful to find the projection of one vector onto the other, because this turns out to have important meaning in many circumstances. More precisely, given \(A\) and \(B\), we seek a vector parallel to \(B\) but with length determined by \(A\) in a natural way, as shown in figure 14.4.2. We choose so that the triangle formed by \(A, V,\) and \(A + V\) is a right triangle.

![Figure 14.4.2](image)

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Using a little trigonometry, we see that

\[
|\mathbf{V}| = |\mathbf{A}| \cos \theta = |\mathbf{A}| \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}| |\mathbf{B}|} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|}
\]

this is sometimes called the scalar projection of \(A\) onto \(B\). To get \(V\) itself, we multiply this length by a vector of length one parallel to \(B\):

\[
\mathbf{V} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|} \frac{\mathbf{B}}{|\mathbf{B}|} = \mathbf{A} \cdot \mathbf{B}
\]

Be sure that you understand why \(\mathbf{B} / |\mathbf{B}|\) is a vector of length one (also called a unit vector) parallel to \(B\).

The discussion so far implicitly assumed that \(0 \leq \theta < \pi/2\). If \(\pi/2 < \theta \leq \pi\), the picture is like figure 14.3.3. In this case \(A\) is negative, so the vector

\[
\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|}
\]
is anti-parallel to $B$, and its length is $|A| \cdot \frac{B}{|B|}$.

So in general, the scalar projection of $A$ onto $B$ may be positive or negative. If it is negative, it means that the projection vector is anti-parallel to $B$ and that the length of the projection vector is the absolute value of the scalar projection. Of course, you can also compute the length of the projection vector as usual, by applying the distance formula to the vector.

![Figure 14.3.3](image1)

$V$ is the projection of $A$ onto $B$.

Note that the phrase "projection onto $B$" is a bit misleading if taken literally, all that $B$ provides is a direction; the length of $B$ has no impact on the final vector. In figure 14.3.4, for example, $B$ is shorter than the projection vector, but this is perfectly acceptable.

![Figure 14.3.4](image2)

$V$ is the projection of $A$ onto $B$.

### EXAMPLE 14.3.4

Physical force is a vector quantity. It is often necessary to compute the "component" of a force acting in a different direction than the force is being applied. For example, suppose a ten pound weight is resting on an inclined plane—a pitched roof, for example. Gravity exerts a force of ten pounds on the object, directed straight down. It is useful to think of the component of this force directed down and parallel to the roof, and the component down and directly into the roof. These forces are the projections of the force vector onto vectors parallel and perpendicular to the roof. Suppose the roof is tilted at a $30^\circ$ angle, as in figure 14.3.5. A vector parallel to the roof is $(-\sqrt{3}, -1)$, and a vector perpendicular to the roof is $(1, -\sqrt{3})$. The force vector is $F = (0, -10)$. The component of the force directed down the roof is then

$$ F_1 = \frac{F \cdot (-\sqrt{3}, -1)}{|(-\sqrt{3}, -1)|^2} (-\sqrt{3}, -1) = -\frac{10(-\sqrt{3} - 1)}{2} = (-\sqrt{3}5/2, -5/2) $$

with length $5\sqrt{3}$. Thus, a force of 5 pounds is pulling the object down the roof, while a force of $5\sqrt{3}$ pounds is pulling the object into the roof.

![Figure 14.3.5](image3)

Components of a force.

The dot product has some familiar-looking properties that will be useful later, so we list them here. These may be proved by writing the vectors in coordinate form and then performing the indicated calculations; subsequently it can be easier to use the properties instead of calculating with coordinates.

### THEOREM 14.3.5

If $u$, $v$, and $w$ are vectors and $a$ is a real number, then

1. $u \cdot v = |u|^2$
2. $u \cdot v = v \cdot u$
3. $u \cdot (v + w) = u \cdot v + u \cdot w$
4. $(au) \cdot v = a(u \cdot v) = u \cdot (av)$

### Exercises 14.3

1. Find $(1, 1, 1) \cdot (2, -3, 4)$. ⇒
2. Find $(1, 2, 0) \cdot (0, 0, 5)$. ⇒
3. Find $(1, 2, 1) \cdot (0, 1, 0)$. ⇒
4. Find $(-1, -2, 5) \cdot (0, 1, -4)$. ⇒
5. Find $(3, 6, 0) \cdot (2, -3, 4)$. ⇒
6. Find the cosine of the angle between $(1, 2, 3)$ and $(1, 1, 1)$; use a calculator if necessary to find the angle. ⇒
7. Find the cosine of the angle between $(-1, -2, 3)$ and $(5, 0, 2)$; use a calculator if necessary to find the angle. ⇒
8. Find the cosine of the angle between $(7, 100, 0)$ and $(0, 0, 5)$; use a calculator if necessary to find the angle. ⇒
9. Find the cosine of the angle between $(1, 0, 1)$ and $(0, 1, 1)$; use a calculator if necessary to find the angle. ⇒
10. Find the cosine of the angle between $(2, 0, 0)$ and $(-1, -1, -1)$; use a calculator if necessary to find the angle. ⇒
11. Find the angle between the diagonal of a cube and one of the edges adjacent to the diagonal. ⇒
12. Find the scalar and vector projections of $(1, 2, 3)$ onto $(1, 2, 0)$. ⇒
13. Find the scalar and vector projections of $(1, 1, 1)$ onto $(2, 1, 3)$. ⇒
14. A 20 pound object sits on a ramp at an angle of $30^\circ$ from the horizontal, as in figure 14.3.5. Find the force pulling the object down the ramp and the force pulling the object directly into the ramp. ⇒
15. A 20 pound object sits on a ramp at an angle of $45^\circ$ from the horizontal. Find the force pulling the object down the ramp and the force pulling the object directly into the ramp. ⇒
16. A force of 10 pounds is applied to a wagon, directed at an angle of $30^\circ$ from the horizontal as shown. Find the component of this force pulling the wagon straight up, and the component pulling it horizontally along the ground. ⇒
17. A force of 15 pounds is applied to a wagon, directed at an angle of $45^\circ$ from the horizontal. Find the component of this force pulling the wagon straight up, and the component pulling it horizontally along the ground. ⇒
18. A force $F$ is to be applied to a wagon, directed at an angle of $30^\circ$ from the horizontal. The resulting force pulling the wagon horizontally along the ground is to be 10 pounds. What is the magnitude of the required force $F$? ⇒
19. Use the dot product to find a non-zero vector $w$ perpendicular to both $u = (1, 2, -3)$ and $v = (2, 0, 1)$. ⇒
20. Let $x = (1, 1, 0)$ and $y = (2, 4, 2)$. Find a unit vector that is perpendicular to both $x$ and $y$. ⇒
21. Do the three points $(1, 2, 0), (-2, 1, 1),$ and $(0, 3, -1)$ form a right triangle? ⇒
22. Do the three points $(1, 1, 1), (2, 3, 2),$ and $(0, 1, -1)$ form a right triangle? ⇒
23. Show that $|A \cdot B| \leq |A||B|$. Let $x$ and $y$ be non-zero vectors. Use Theorem 14.3.5 to prove that $|x|^2 + |y|^2 = |x + y|^2$. What is this result better known as? ⇒
24. Prove that the diagonals of a rhombus intersect at right angles. ⇒
25. Suppose that $x = |x|y$ where $x, y$, and $z$ are all nonzero vectors. Prove that $y$ bisects the angle between $x$ and $z$. ⇒
26. Prove Theorem 14.3.5.

### 14.4 The Cross Product

Another useful operation: Given two vectors, find a third (non-zero) vector perpendicular to the first two. There are of course an infinite number of such vectors of different lengths. Nevertheless, let us find one. Suppose $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$. We want to find a vector $v = (v_1, v_2, v_3)$ with $v \cdot A = v \cdot B = 0$, or $a_1v_1 + a_2v_2 + a_3v_3 = 0$, $b_1v_1 + b_2v_2 + b_3v_3 = 0$. Multiply the first equation by $b_2$ and the second by $a_2$ and subtract to get $b_1v_1a_2 + b_2v_2a_2 + b_3v_3a_2 = 0$, $a_1v_1b_2 + a_2v_2b_2 + a_3v_3b_2 = 0$, $(a_1b_2 - b_1a_2)v_1 + (b_2a_3 - a_2b_3)v_2 = 0$. Of course, this equation in two variables has many solutions; a particularly easy one to see is $v_1 = a_2b_3 - b_2a_3$, $v_2 = b_2a_1 - a_2b_1$, $v_3 = a_1b_2 - b_1a_2$. Substituting back into either of the original equations and solving for $v_3$ gives $v_3 = a_1b_3 - b_1a_3$. This particular answer to the problem turns out to have some nice properties, and it is dignified with a name: the cross product:

$$ A \times B = (a_2b_3 - b_2a_3, a_3b_1 - b_3a_1, a_1b_2 - b_1a_2). $$

While there is a nice pattern to this vector, it can be a bit difficult to memorize; here is a convenient mnemonic. The determinant of a two by two matrix is

$$ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. $$
This is extended to the determinant of a three by three matrix:

\[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i
\end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)
\]

Each of the two by two matrices is formed by deleting the top row and one column of the matrix directly from the 3 by 3 matrix. This is extended to the determinant of a three by three matrix:

\[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i
\end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)
\]

This is known as the dot product, which is a way to multiply two vectors. The dot product of two vectors \( \mathbf{u} = (u_1, u_2, u_3) \) and \( \mathbf{v} = (v_1, v_2, v_3) \) is defined as:

\[
\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3
\]

The cross product of two vectors \( \mathbf{u} \) and \( \mathbf{v} \) is defined as:

\[
\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)
\]

This is known as the cross product, which is a way to multiply two vectors. The cross product of two vectors \( \mathbf{u} = (u_1, u_2, u_3) \) and \( \mathbf{v} = (v_1, v_2, v_3) \) is defined as:

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\[
\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)
\]
That is, suppose we know that \((a, b, c)\) is normal to a plane containing the point \(v_1, v_2, v_3\). Then \((x, y, z)\) is in the plane if and only if \((a, b, c)\) is perpendicular to \((x - v_1, y - v_2, z - v_3)\). In turn, we know that this is true precisely when \((a, b, c) \cdot (x - v_1, y - v_2, z - v_3) = 0\). Thus, \((x, y, z)\) is in the plane if and only if

\[
(a, b, c) \cdot (x - v_1, y - v_2, z - v_3) = 0
\]

or equivalently

\[
a(x - v_1) + b(y - v_2) + c(z - v_3) = 0
\]

or

\[
a x + b y + c z = a v_1 + b v_2 + c v_3.
\]

Working backwards, note that if \((x, y, z)\) is a point satisfying \(a x + b y + c z = d\) then

\[
(x - v_1) a + (y - v_2) b + (z - v_3) c = 0
\]

or equivalently

\[
(x - v_1, y - v_2, z - v_3) \cdot (a, b, c) = 0
\]

where \((x_1, y_1, z_1)\) is any point on the plane. It is easy to find a point on the plane, say \((1, 0, 0)\) and head at \((1, 2, 3)\). Therefore an equation of the plane is

\[
\langle a, b, c \rangle \cdot \langle x - 1, y, z - 3 \rangle = 0
\]

or equivalently

\[
a(x - 1) + b y + c(z - 3) = 0
\]

Note that had we used \(a(x - 1) + b(y - 2) + c(z - 3) = 0\) in place of \(a(x - 1) + b y + c(z - 3) = 0\), then we might well have noticed that we could divide both sides by \(a\) to get the equivalent

\[
(x - 1, y, z - 3) \cdot (a, b, c) = 0
\]


EXAMPLE 14.5.1

Find an equation for the plane perpendicular to \((1, 2, 3)\) and contain- ing the point \((5, 0, 7)\).

Using the derivation above, the plane is \(1x + 2y + 3z = 5 + 2 \cdot 0 + 3 \cdot 7 = 26\).

Alternatively, we know that the plane is \(x + 2y + 3z = d\) and to find \(d\) we may substitute the known point on the plane to get \(5 + 2 \cdot 0 + 3 \cdot 7 = 26\). We could also write this simply as \((x - 5) + 2(y - 0) + 3(z - 0) = 0\), which is for many purposes a fine representation; it can be always be multiplied out to get \(x + 2y + 3z = 26\).

EXAMPLE 14.5.2

Find a vector normal to the plane \(2x - 3y + z = 15\).

One example is \((2, -3, 1)\). Any vector parallel or anti-parallel to this works as well, so for example \(-2(-2, 3, -1) = (-4, 6, -2)\) is also normal to the plane.

We will frequently need to find an equation for a plane given certain information about the plane. While there may occasionally be slightly shorter ways to get to the desired result,


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perfectly well, any vector parallel or anti-parallel to it will work as well, so for example we might choose \((1, 1, 1)\) which is anti-parallel to it.

Now we know that \((1, 1, 1)\) is normal to the desired plane and \((2, 1, 1)\) is a point on the plane. Therefore an equation of the plane is \(x + y + z = 4\). As a quick check, since \((1, 1, 0)\) is also on the line, it should be on the plane; since \(1 + 1 + 0 = 4\), we see that this is indeed the case.

Note that had we used \((-3, -3, -3)\) as the normal, we would have discovered the equation \(-3x - 3y - 3z = -12\), then we might well have noticed that we could divide both sides by \(-3\) to get the equivalent \(x + y + z = 4\).

So we now understand equations of planes; let us turn to lines. Unfortunately, it turns out to be quite inconvenient to represent a typical line with a single equation; we need to approach lines in a different way.

Unlike a plane, a line in three dimensions does have an obvious direction, namely, the direction of any vector parallel to it. In fact a line can be defined and uniquely identified by providing one point on the line and a vector parallel to the line (in one of two possible directions). That is, the line consists of exactly those points we can reach by starting at the point and going for some distance in the direction of the vector. Let’s see how we can translate this into more mathematical language.

Suppose a line contains the point \((v_1, v_2, v_3)\) and is parallel to the vector \((a, b, c)\); we call \((a, b, c)\) a direction vector for the line. If we place the vector \((v_1, v_2, v_3)\) with its tail at the origin and its head at \((v_1, v_2, v_3)\), and if we place the vector \((a, b, c)\) with its tail at \((v_1, v_2, v_3)\), then the head of \((a, b, c)\) is at a point on the line. We can get any point on the line by doing the same thing, except using \(tv_1, b, c\) in place of \((a, b, c)\), where \(t\) is some real number. Because of the way vector addition works, the point at the head of the vector \(v_1, v_2, v_3 + t(a, b, c)\) is the point at the head of the vector \(v_1, v_2, v_3 + t(a, b, c)\) points to every point on the line when its tail is placed at the origin. Another common way to write this is as a set of parametric equations:

\[
x = v_1 + ta \\
y = v_2 + tb \\
z = v_3 + tc.
\]

It is occasionally useful to use this form of a line even in two dimensions; a vector form for a line in the \(x\)-\(y\) plane is \((v_1, v_2) + t(a, b)\), which is the same as \((v_1, v_2 + 0) + t(a, b)\).

EXAMPLE 14.5.4

Find a vector expression for the line through \((6, -1, -3)\) and \((2, 4, 5)\). To get a vector parallel to the line we subtract \((6, -1, -3) - (2, 4, 5) = (-4, -3, -8)\). The line is then given by \((2, 4, 5) + t(-4, -3, -8)\); there are of course many other possibilities, such as \((6, -1, -3) + t(-4, -3, 8)\).

EXAMPLE 14.5.5

Determine whether the lines \((1, 1, -1) + t(-1, 2, -1)\) and \((3, 2, 1) + t(-1, -5, 3)\) are parallel, intersect, or neither.

In two dimensions, two lines either intersect or are parallel; in three dimensions, lines that do not intersect might not be parallel. In this case, since the direction vectors for the lines are not parallel or anti-parallel we know the lines are not parallel. If they intersect, there must be two values \(u\) and \(v\) so that \((1, 1, -1) + u(-1, 2, -1) = (3, 2, 1) + v(-1, -5, 3)\), that is,

\[
\begin{align*}
1 + u & = 3 + v - b \\
2u & = 2 - 5v \\
3u & = 1 + 3v - 3b
\end{align*}
\]

This gives three equations in two unknowns, so there may or may not be a solution in general. In this case, it is easy to discover that \(u = -3\) and \(v = 1\) satisfies all three equations, so the lines do intersect at the point \((4, 7, -2)\).

EXAMPLE 14.5.6

Find the distance from the point \((1, 2, 3)\) to the plane \(2x - 9y + 3z = 5\).

The distance from a point \(P\) to a plane is the shortest distance from \(P\) to any point on the plane; this is the distance measured from \(P\) perpendicular to the plane; see figure 14.5.3.

This distance is the absolute value of the scalar projection of \(\overrightarrow{QP}\) onto a normal vector \(n\), where \(Q\) is any point on the plane. It is easy to find a point on the plane, say \((1, 0, 1)\).

Thus the distance is

\[
\frac{|\overrightarrow{QP}|}{|n|} = \frac{|[0, 2, 2] \cdot [2, -1, 3]|}{|(-1, 5, 3)|} = \frac{1}{771.4}
\]

EXAMPLE 14.5.7

Find the distance from the point \((-1, 2, 1)\) to the line \((1, 1, 1) + t(2, 3, -1)\). Again we want the distance measured perpendicular to the line, as indicated.
6. Find an equation of the plane containing the line of intersection of \(x + y + z = 1\) and \(x - y + 2z = 2\), and perpendicular to the plane \(2x + 3y - z = 4\). ⇒

7. Find an equation of the plane containing the line of intersection of \(x + 2y - z = 3\) and \(3x - y + 4z = 7\), and perpendicular to the plane \(6x + y + 3z = 16\). ⇒

8. Find an equation of the plane containing the line of intersection of \(x + 3y - z = 6\) and \(2x + 2y - 3z = 11\), and perpendicular to the plane \(3x + y + z = 11\). ⇒

9. Find an equation of the line through \((1, 0, 3)\) and \((1, 2, 4)\). ⇒

10. Find an equation of the line through \((0, 3, 3)\) and \((1, 0, 3)\) and perpendicular to the plane \(x + 2y - z = 1\). ⇒

11. Find an equation of the line through the origin and perpendicular to the plane \(x + y - z = 2\). ⇒

12. Find \(n\) and \(s\) so that \(na + n\) is the line through \((0, 2, 3)\) and \((2, 7, 5)\). ⇒

13. Explain how to discover the solution in example 14.5.5.

14. Determine whether the lines \((3, 1, -1) + t(1, 1, 0)\) and \((0, 6, 0) + t(1, 4, 5)\) are parallel, intersect, or neither. ⇒

15. Determine whether the lines \((1, 0, 2) + t(-1, -1, 2)\) and \((4, 4, 2) + t(2, 2, -4)\) are parallel, intersect, or neither. ⇒

16. Determine whether the lines \((3, 1, 2) + t(1, 2, -3)\) and \((1, 0, 1) + t(2, 3, 4)\) are parallel, intersect, or neither. ⇒

17. Determine whether the lines \((1, 1, 2) + t(1, 2, -3)\) and \((2, 3, -1) + t(2, 4, -6)\) are parallel, intersect, or neither. ⇒

18. Find a unit normal vector to each of the coordinate planes.⇒

19. Show that \((2, 1, 3) + t(1, 1, 2)\) and \((3, 2, 5) + t(2, 4, 6)\) are the same line.

20. Give a prose description for each of the following processes:
   a. Given two distinct points, find the line that goes through them.
   b. Given three points (not all on the same line), find the plane that goes through them.
   c. Given a line and a point not on the line, find the plane that contains them both.
   d. Given a plane and a point not on the plane, find the line that is perpendicular to the plane through the given point.
   e. Find the distance from \((2, 3, 4)\) to \(x + y + z = -1\). ⇒
   f. Find the distance from \((2, -1, -1)\) to \(2x - 3y + z = 2\). ⇒
   g. Find the distance from \((2, 1, -1)\) to \((2, 0, 1) + t(1, 2, 3)\). ⇒
   h. Find the distance from \((1, 0, 1)\) to \((1, -1, 1) + t(2, 1, -2)\). ⇒
   i. Find the distance between the lines \((5, 3, 1) + t(2, 4, 3)\) and \((6, 1, 0) + t(3, 5, 7)\). ⇒
   j. Find the distance between the lines \((2, 1, 3) + t(-1, -2, -3)\) and \((1, -3, 4) + t(-4, -4, 3)\). ⇒
   k. Find the distance between the lines \((1, 2, 3) + t(-1, -2, -3)\) and \((4, 5, 6) + t(-4, 2, -6)\). ⇒
   l. Find the distance between the lines \((3, 2, 1) + t(1, 4, -1)\) and \((3, 1, 3) + t(2, 8, -2)\). ⇒
   m. Find the cosine of the angle between the planes \(x + y - z = 2\) and \(x + 2y + 3z = 8\). ⇒
   n. Find the cosine of the angle between the planes \(x + y - 2z = 2\) and \(2x - 2y + z = 5\). ⇒

14.6 Other Coordinate Systems 337

Coordinate systems are tools that let us use algebraic methods to understand geometry. While the rectangular (also called Cartesian) coordinates that we have been discussing are the most common, some problems are easier to analyze in alternate coordinate systems.

A coordinate system is a scheme that allows us to identify any point in the plane or in three-dimensional space by a set of numbers. In rectangular coordinates these numbers are interpreted, roughly speaking, as the lengths of the sides of a rectangular "box."

In two dimensions you may already be familiar with an alternative, called polar coordinates. In this system, each point in the plane is identified by a pair of numbers \((r, \theta)\).

The number \(r\) measures the distance from the origin to the point. Either of these may be negative; a negative \(r\) indicates the angle is measured clockwise from the positive \(x\)-axis instead of counterclockwise, and a negative \(\theta\) indicates the point at distance \(r\) in the opposite of the direction given by \(\theta\).

Figure 14.6.1 also shows the point with rectangular coordinates \((1, \sqrt{3})\) and polar coordinates \((2, \pi/3)\), 2 units from the origin and \(\pi/3\) radians from the positive \(x\)-axis.

The cylinder \(r = 2\).

Given a point \((r, \theta)\) in polar coordinates, it is easy to see (as in figure 14.6.1) that the rectangular coordinates of the same point are \((r \cos \theta, r \sin \theta)\), and so the point \((r, \theta, z)\) in cylindrical coordinates is \((r \cos \theta, r \sin \theta, z)\) in rectangular coordinates. This means it is usually easy to convert any equation from rectangular to cylindrical coordinates: simply substitute

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]
and leave \( z \) alone. For example, starting with \( x^2 + y^2 = 4 \) and substituting \( x = r \cos \theta \), \( y = r \sin \theta \) gives

\[
\begin{align*}
r^2 \cos^2 \theta + r^2 \sin^2 \theta &= 4 \\
r^2 (\cos^2 \theta + \sin^2 \theta) &= 4 \\
r^2 &= 4 \\
r &= 2.
\end{align*}
\]

Of course, it’s easy to see directly that this defines a cylinder as mentioned above.

Cylindrical coordinates are an obvious extension of polar coordinates to three dimensions, but the use of the \( z \) coordinate means they are not as closely analogous to polar coordinates as another standard coordinate system. In polar coordinates, we identify a point by a direction and distance from the origin; in three dimensions we can do the same thing, in a variety of ways. The question is: how do we represent a direction? One way is to specify a line through the point \((r, \theta, \phi)\), where \( r \) is the distance from the origin, and \( \theta \) and \( \phi \) are longitude and latitude. (Earth longitude is measured as a positive or negative angle from the prime meridian, and is always between 0 and 180 degrees, east or west, so can be any positive or negative angle, and we use radians except in informal circumstances. Earth latitude is measured north or south from the equator; \( \phi \) is measured from the north pole down.) This system is called spherical coordinates; the coordinates are listed in the order \((r, \theta, \phi)\), where \( r \) is the distance from the origin, and \( \theta \) and \( \phi \) in cylindrical coordinates may be negative. The general case and an example are pictured in figure 14.6.4: the length marked \( r \) is the distance from the origin in cylindrical coordinates.

![Cylindrical coordinates](image)

Figure 14.6.4 Cylindrical coordinates: the general case and the point with rectangular coordinates \((1, \sqrt{3}, 3)\).

As with cylindrical coordinates, we can easily convert equations in rectangular coordinates to the equivalent in spherical coordinates, though it is a bit more difficult to discover the proper substitutions. Figure 14.6.5 shows the typical point in spherical coordinates.

![Spherical coordinates](image)

### Exercises 14.6

1. Convert the following points in rectangular coordinates to cylindrical and spherical coordinates:
   a. \((1, 1, 1)\)
   b. \((-7, -5, 0)\)
   c. \((\cos(1), \sin(1), 1)\)
   d. \((0, 0, -r)\)

2. Find an equation for the sphere \( x^2 + y^2 + z^2 = 4 \) in cylindrical coordinates.

3. Find an equation for the \( z \) plane in cylindrical coordinates.

4. Find an equation equivalent to \( x^2 + y^2 + z^2 = 5 \) in cylindrical coordinates.

5. Suppose the curve \( z = e^{-t} \) in the \( x-z \) plane is rotated around the \( z \) axis. Find an equation for the resulting surface in cylindrical coordinates.

6. Suppose the curve \( z = x \) in the \( x-z \) plane is rotated around the \( z \) axis. Find an equation for the resulting surface in cylindrical coordinates.

7. Find an equation for the plane \( y = 0 \) in spherical coordinates.

8. Find an equation for the plane \( z = 1 \) in spherical coordinates.

9. Find an equation for the sphere with radius 1 and center at \((0, 0, 0)\) in spherical coordinates.

10. Find an equation for the cylinder \( x^2 + y^2 = 9 \) in spherical coordinates.

11. Suppose the curve \( z = x \) in the \( x-z \) plane is rotated around the \( z \) axis. Find an equation for the resulting surface in spherical coordinates.

12. Plot the polar equations \( r = \sin \theta \) and \( r = \cos \theta \) and comment on their similarities. (If you get stuck on how to plot these, you can multiply both sides of each equation by \( r \) and convert back to rectangular coordinates.)

13. Extend exercises 6 and 11 by rotating the curve \( z = mx \) around the \( z \) axis and converting to both cylindrical and spherical coordinates.