11
More Applications of Integration

11.1 Center of Mass

Suppose a beam is 10 meters long, and that there are three weights on the beam: a 10 kilogram weight 3 meters from the left end, a 5 kilogram weight 6 meters from the left end, and a 4 kilogram weight 8 meters from the left end. Where should a fulcrum be placed so that the beam balances? Let’s assign a scale to the beam, from 0 at the left end to 10 at the right, so that we can denote locations on the beam simply as $x$ coordinates; the weights are at $x = 3$, $x = 6$, and $x = 8$, as in figure 11.1.1.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$m$</th>
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<tbody>
<tr>
<td>3</td>
<td>10</td>
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<tr>
<td>6</td>
<td>5</td>
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<tr>
<td>8</td>
<td>4</td>
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Figure 11.1.1 A beam with three masses.

Suppose to begin with that the fulcrum is placed at $x = 5$. What will happen? Each weight applies a force to the beam that tends to rotate it around the fulcrum; this effect is measured by a quantity called torque, proportional to the mass times the distance from the fulcrum. Of course, weights on different sides of the fulcrum rotate the beam in opposite directions. We can distinguish this by using a signed distance in the formula for torque. So with the fulcrum at 5, the torques induced by the three weights will be proportional to $(3 - 5)10 = -20$, $(6 - 5)5 = 5$, and $(8 - 5)4 = 12$. For the beam to balance, the sum of the torques must be zero; since the sum is $-20 + 5 + 12 = -3$, the beam rotates counter-clockwise, and to get the beam to balance we need to move the fulcrum to the left. To calculate exactly where the fulcrum should be, we let $\bar{x}$ denote the location of the fulcrum when the beam is in balance. The total torque on the beam is then $(3 - 3)10 + (6 - 5)5 + (8 - 5)4 = 92 - 192$. Since the beam balances at $\bar{x}$ it must be that $92 - 192 = 0$ or $\bar{x} = 92/19 \approx 4.84$, that is, the fulcrum should be placed at $x = 92/19$ to balance the beam.

Now suppose that we have a beam with varying density—some portions of the beam contain more mass than other portions of the same size. We want to figure out where to put the fulcrum so that the beam balances.

EXAMPLE 11.1.1 Suppose the beam is 10 meters long and that the density is $1 + x$ kilograms per meter at location $x$ on the beam. To approximate the solution, we can think of the beam as a sequence of weights “on” a beam. For example, we can think of the portion of the beam between $x = 0$ and $x = 1$ as a weight sitting at $x = 0$, the portion between $x = 1$ and $x = 2$ as a weight sitting at $x = 1$, and so on, as indicated in figure 11.1.2. We then approximate the mass of the weights by assuming that each portion of the beam has constant density. So the mass of the first weight is approximately $m_1 = (1 + 0)1 = 1$ kilograms, namely, $(1 + 0)$ kilograms per meter times 1 meter. The second weight is $m_2 = (1 + 1)1 = 2$ kilograms, and so on to the tenth weight with $m_{10} = (1 + 9)1 = 10$ kilograms. So in this case the total torque is $\sum (0 - x)m_0 + (1 - x)m_1 + \cdots + (9 - x)m_9 = (0 - 0)1 + (1 - 1)2 + \cdots + (9 - 9)10$.

If we set this to zero and solve for $\bar{x}$ we get $\bar{x} = 6$. In general, if we divide the beam into $n$ portions, the mass of weight number $i$ will be $m_i = (1 + i)(x_{i+1} - x_i) = (1 + i)\Delta x$ and the torque induced by weight number $i$ will be $(x_i - x) m_i = (x_i - x)(1 + i)\Delta x$. The total torque is then

$$\sum_{i=1}^{n-1} (x_{i+1} - x_i)(1 + i)\Delta x = \sum_{i=0}^{n-1} (1 + x_i)\Delta x = \sum_{i=0}^{n-1} x_i(1 + x_{i+1})\Delta x - \frac{1}{2} \sum_{i=0}^{n-1} (1 + x_i)\Delta x.$$

The numerator of this fraction has a very familiar interpretation. Consider one term of the sum in the denominator: $(1 + x_i)\Delta x$. This is the density near $x_i$, times a short length, $\Delta x$, which in other words is approximately the mass of the beam between $x_i$ and $x_{i+1}$. When we add these up we get approximately the mass of the beam.

Now each of the sums in the fraction has the right form to turn into an integral, which in turn gives us the exact value of $\bar{x}$:

$$\bar{x} = \frac{\int_0^a x f(x) \, dx}{\int_0^a f(x) \, dx}.$$

The numerator of this fraction is called the moment of the system around zero:

$$\int_0^a x f(x) \, dx = \int_0^a x \cdot (1 + x) \, dx = \frac{1150}{1}.$$

and the denominator is the mass of the beam:

$$\int_0^a (1 + x) \, dx = 60,$$

and the balance point, officially called the center of mass, is

$$\bar{x} = \frac{\frac{1150}{1}}{\frac{60}{1}} \approx 18.3.$$
the “beam,” say between \( x_1 \) and \( x_{i+1} \), is the mass of a strip of the plate between \( x_i \) and \( x_{i+1} \). See figure 11.1.3 showing the plate from above and as it appears edge on. Since the plate has uniform density we may as well assume that \( \sigma = 1 \). Then the mass of the plate between \( x_i \) and \( x_{i+1} \) is approximately \( m_i = \sigma (1 - x_i^2) \Delta x_i = (1 - x_i^2) \Delta x_i \). Now we can compute the moment around the \( y \)-axis:

\[
M_y = \int_0^2 x(1 - x^2) \, dx = \frac{1}{4}
\]

and the total mass

\[
M = \int_0^2 (1 - x^2) \, dx = \frac{2}{3}
\]

and finally

\[
\bar{y} = 1 \frac{3}{2} - \frac{3}{8}
\]

Next we do the same thing to find \( \bar{x} \). The mass of the plate between \( y_1 \) and \( y_{i+1} \) is approximately \( m_i = \sqrt{\pi} \Delta y_i \), so

\[
M_x = \int_0^1 y_2 \sqrt{\pi} \, dy = \frac{2}{5}
\]

and

\[
\bar{y} = 2 \frac{1}{3} - \frac{3}{7}
\]

since the total mass \( M \) is the same. The center of mass is shown in figure 11.1.3.

\section{Example 11.1.4}

Find the center of mass of a thin, uniform plate whose shape is the region between \( y = \cos x \) and the \( x \)-axis between \( x = -\pi/2 \) and \( x = \pi/2 \). It is clear that \( \bar{x} = 0 \), but for practice let’s compute it anyway. We will need the total mass, so we compute it first:

\[
M = \int_{-\pi/2}^{\pi/2} \cos x \, dx = \sin \left( \frac{\pi}{2} \right) - \sin \left( \frac{-\pi}{2} \right) = 2
\]

The moment around the \( y \)-axis is

\[
M_y = \int_{-\pi/2}^{\pi/2} x \cos x \, dx = \cos x + x \sin x \bigg|_{-\pi/2}^{\pi/2} = 0
\]

and the moment around the \( x \)-axis is

\[
M_x = \int_{-\pi/2}^{\pi/2} y(2 \arccos y - y \sqrt{1 - y^2}) \, dy = \frac{\pi}{4}
\]

Thus

\[
\bar{x} = \frac{0}{2} = 0
\]

\section{Exercises 11.1.1}

1. A beam 10 meters long has density \( \sigma(x) = x^3 \) at distance \( x \) from the left end of the beam. Find the center of mass \( \bar{x} \).

2. A beam 10 meters long has density \( \sigma(x) = \sin(\pi x/10) \) at distance \( x \) from the left end of the beam. Find the center of mass \( \bar{x} \).

3. A beam 4 meters long has density \( \sigma(x) = x^3 \) at distance \( x \) from the left end of the beam. Find the center of mass \( \bar{x} \).

4. Verify that \( \int_0^L \sigma(x) \, dx = \sigma L \) for any function \( \sigma \) and any interval of length \( L \).

5. A thin plate lies in the region between \( y = x^2 \) and the \( x \)-axis between \( x = 0 \) and \( x = 2 \). Find the centroid.

6. A thin plate fills the upper half of the unit circle \( x^2 + y^2 = 1 \). Find the centroid.

7. A thin plate lies in the region contained by \( y = x \) and \( y = x^2 \). Find the centroid.

8. A thin plate lies in the region contained by \( y = 4 - x^2 \) and the \( x \)-axis. Find the centroid.

9. A thin plate lies in the region contained by \( y = x^3 \) and the \( x \)-axis between \( x = 0 \) and \( x = 1 \). Find the centroid.

10. A thin plate lies in the region contained by \( y = \sqrt{x} + 1 \) and the \( x \)-axis in the first quadrant. Find the centroid.

11. A thin plate lies in the region between the circle \( x^2 + y^2 = 4 \) and the circle \( x^2 + y^2 = 1 \) above the \( x \)-axis. Find the centroid.

12. A thin plate lies in the region between the circle \( x^2 + y^2 = 4 \) and the circle \( x^2 + y^2 = 1 \) in the first quadrant. Find the centroid.

13. A thin plate lies in the region between the circle \( x^2 + y^2 = 25 \) and the circle \( x^2 + y^2 = 16 \) above the \( x \)-axis. Find the centroid.

\section{11.2 Kinetic energy; improper integrals}

Recall example 8.5.3 in which we computed the work required to lift an object from the surface of the earth to some large distance \( D \). Since \( F = k/x^2 \) we computed

\[
\int_0^D \frac{k}{x^2} \, dx = \frac{k}{D} - \frac{k}{x}
\]

We noticed that as \( D \) increases, \( k/D \) decreases to zero so that the amount of work increases to \( k/\alpha \). More precisely,

\[
\lim_{D \to \infty} \int_0^D \frac{k}{x^2} \, dx = \lim_{D \to \infty} \left[ -\frac{k}{x} \right]_0^D = -\frac{k}{\alpha}
\]

We might reasonably describe this calculation as computing the amount of work required to lift the object “to infinity,” and abbreviate the limit as

\[
\lim_{x \to \infty} \int_0^D \frac{k}{x^2} \, dx = \int_{-\infty}^{\infty} \frac{k}{x^2} \, dx.
\]

Such an integral, with a limit of infinity, is called an improper integral. This is a bit unfortunate, since it’s not really “improper” to do this; it is really “too infinite” — it is an abbreviation for the limit of a particular sort of integral. Nevertheless, we’re stuck with the term, and the operation itself is perfectly legitimate. It may at first seem odd that a finite amount of work is sufficient to lift an object to “infinity,” but sometimes surprising things are nevertheless true, and this is such a case. If the value of an improper integral is a finite number, as in this example, we say that the integral converges, and if not we say that the integral diverges.

Here’s another way, perhaps even more surprising, to interpret this calculation. We know that one interpretation of

\[
\int_0^D \frac{1}{D} \, dx = \frac{1}{D} \cdot \frac{1}{x}
\]

is the area under \( y = 1/x \) from \( x = 1 \) to infinity is finite.
backward.” This makes the work W negative when it should be positive, so typically the work in this case is defined as

\[ W = - \int_a^b F \, dx. \]

Also, by Newton’s Law, \( F = ma(t) \). This means that

\[ W = - \int_a^b ma(t) \, dx. \]

Unfortunately this integral is a bit problematic: \( a(t) \) is in terms of \( t \), while the limits and the “d”s are in terms of \( x \). But \( x \) and \( t \) are certainly related here: \( x = \alpha(t) \) is the function that gives the position of the object at time \( t \), so \( x = v(t) \cdot dt = \alpha'(t) \) is its velocity and \( a(t) = \alpha''(t) \). We can use \( x = \alpha(t) \) as a substitution to convert the integral from “dx” to “d\( x \)” in the usual way, with a bit of cleverness along the way:

\[
\frac{dx}{dt} = a(t) \, dx = a(\alpha(t)) \, d\alpha(t) = a(\alpha(t)) \, d\alpha(t)
\]

Substituting in the integral:

\[ W = - \int_{\alpha(a)}^{\alpha(b)} ma(\alpha(t)) \, d\alpha(t). \]

You may recall seeing the expression \( m\alpha^2/2 \) in a physics course—it is called the kinetic energy of the object. We have shown here that the work done in moving the object from one place to another is the same as the change in kinetic energy.

We know that the work required to move an object from the surface of the earth to infinity is

\[ W = \int_{\alpha(b)}^{\alpha(b)} \frac{k}{r^2} \, dr \]

At the surface of the earth the acceleration due to gravity is approximately 9.8 meters per second squared, so the force on an object of mass \( m \) is \( F = 9.8 \, m \). The radius of the earth is approximately 6378.1 kilometers or 6378100 meters. Since the force due to gravity obeys an inverse square law, \( F = k/\alpha^2 \) and 9.8m = \( k/6378100^2 \), \( k = 39865564178006m \) and \( W = 62503880m \).

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12. Does \( \int_{-\infty}^{\infty} \sin x \, dx \) converge or diverge? If it converges, find the value. Also find the Cauchy Principal Value, if it exists. \( \Rightarrow \)

13. Suppose the curve \( y = 1/x \) is rotated around the x-axis generating a sort of funnel or horn shape, called Gabriel’s horn or Torricelli’s trumpet. Is this the volume of this funnel from \( x = 1 \) to infinity finite or infinite? If finite, compute the volume. \( \Rightarrow \)

14. An officially sanctioned baseball must be between 142 and 149 grams. How much work, in Newton-meters, does it take to throw a ball at 90 miles per hour? At 90 mph? At 100.9 mph? (According to the Guinness Book of World Records, at http://www.baseball-almanac.com/rebooks/rb_guin.shtml, “The greatest reliably recorded speed at which a baseball has been pitched is 106.7 mph by Nolan Ryan (California Angels) at Anaheim Stadium in California on August 20, 1974.”) \( \Rightarrow \)

11.3 Probability

You perhaps have at least a rudimentary understanding of discrete probability, which measures the likelihood of an “event” when there are a finite number of possibilities. For example, when an ordinary six-sided die is rolled, the probability of getting any particular number is 1/6. In general, the probability of an event is the number of ways the event can happen divided by the number of ways that “anything” can happen.

For a slightly more complicated example, consider the case of two six-sided dice. The dice are physically distinct, which means that rolling a 2-5 is different than rolling a 5-2; each is an equally likely way out of a total of 36 ways the dice can land, so each has a probability of 1/36.

Most interesting events are not so simple. More interesting is the probability of rolling a certain sum out of the possibilities 2 through 12. It is clearly not true that all sums are equally likely: the only way to roll a 2 is to roll 1–1, while there are many ways to roll a 7. Because the number of possibilities is quite small, and because a pattern quickly becomes evident, it is easy to see that the probabilities of the various sums are:

\[
P(2) = P(12) = \frac{1}{36},
\]

\[
P(3) = P(11) = \frac{2}{36},
\]

\[
P(4) = P(10) = \frac{3}{36},
\]

\[
P(5) = P(9) = \frac{4}{36},
\]

\[
P(6) = P(8) = \frac{5}{36},
\]

\[
P(7) = \frac{6}{36}.
\]

Here we use \( P(x) \) to mean “the probability of rolling an \( x \).” Since we have correctly accounted for all possibilities, the sum of all those probabilities is 36/36 = 1; the probability that the sum is one of 2 through 12 is 1, because there are no other possibilities.
DEFINITION 11.3.1 Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. If \( f(x) \geq 0 \) for every \( x \) and \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \) then \( f \) is a probability density function.

We associate a probability density function with a random variable \( X \) by stipulating that the probability that \( X \) is between \( a \) and \( b \) is \( \int_a^b f(x) \, dx \). Because of the requirement that the integral from \(-\infty\) to \( \infty \) be 1, all probabilities are less than or equal to 1, and the probability that \( X \) takes on some value between \(-\infty\) and \( \infty \) is 1, as it should be.

EXAMPLE 11.3.2 Consider again the two dice example; we can view it in a way that takes on only a finite portion of the beam, say between \(-\infty\) and \( \infty \). We have shown that \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \).

\( \sum_{c} \) is a positive constant.

EXAMPLE 11.3.3 Suppose that \( a < c \) and \( f(x) = \frac{1}{b - a} \) if \( a \leq x \leq b \) otherwise.

Then \( f(x) \) is the uniform probability density function on \([a, b]\) and the corresponding distribution is the uniform distribution on \([a, b]\).

EXAMPLE 11.3.4 Consider the function \( f(x) = e^{-x^2/2} \). What can we say about \( \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \)?

We cannot find an antiderivative of \( f \), but we can see that this integral is some finite number. Notice that \( 0 < f(x) = e^{-x^2/2} \leq x^{1/2} \) for \( |x| > 1 \). This implies that the area under \( e^{-x^2/2} \) is less than the area under \( e^{-x^2} \), over the interval \([1, \infty)\). It is easy to compute the latter area, namely

\[
\int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \frac{\sqrt{\pi}}{\sqrt{2}}
\]

so

\[
\int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \frac{\sqrt{\pi}}{\sqrt{2}}
\]

is some finite number smaller than \( 2/\sqrt{7} \). Because \( f \) is symmetric around the y-axis,

\[
\int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \int_{0}^{\infty} e^{-x^2/2} \, dx.
\]

This means that

\[
\int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \int_{-\infty}^{0} e^{-x^2/2} \, dx + \int_{0}^{\infty} e^{-x^2/2} \, dx + \int_{0}^{\infty} e^{-x^2/2} \, dx = A
\]

for some finite positive number \( A \). Note if we let \( g(x) = f(x)/A \),

\[
\int_{-\infty}^{\infty} g(x) \, dx = \frac{1}{A} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \frac{1}{A} A = 1,
\]

so \( g \) is a probability density function. It turns out to be very useful, and is called the standard normal probability density function or more informally the bell curve.

### Graph of the standard normal distribution

The function \( F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t) \, dt \) is called the cumulative distribution function or simply (probability) distribution.

EXAMPLE 11.3.5 The exponential distribution has probability density function

\[
f(x) = \begin{cases} \frac{e^{-x}}{\alpha} & x < 0 \\ 0 & x \geq 0 \end{cases}
\]

where \( \alpha \) is a positive constant.

The mean or expected value of a random variable is quite useful, as hinted at in our discussion of dice. Recall that the mean for a discrete random variable is \( E(X) = \sum x \cdot P(x) \). In the more general context we use an integral in place of the sum.

DEFINITION 11.3.6 The mean of a random variable \( X \) with probability density function \( f \) is \( \mu = E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \, dx \) provided the integral converges.

When the mean exists it is unique, since it is the result of an explicit calculation. The mean does not always exist.

The mean might look familiar; it is essentially identical to the center of mass of a one-dimensional beam, as discussed in section 11.1. The probability density function \( f \) plays the role of the physical density function, but now the “beam” has infinite length. If we consider only a finite portion of the beam, say between \( a \) and \( b \), then the center of mass is

\[
\bar{x} = \frac{\int_{a}^{b} x \cdot f(x) \, dx}{\int_{a}^{b} f(x) \, dx}
\]
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is \((X - \mu)^2\), we square the difference so as to measure all distances as positive. To get the typical such squared distance, we compute the mean. For two dice, for example, we get
\[
(2 - 7)^2 \frac{1}{11} + (3 - 7)^2 \frac{1}{11} + \cdots + (7 - 7)^2 \frac{1}{11} + \cdots (11 - 7)^2 \frac{1}{11} = \frac{35}{11}.
\]

Because we squared the differences this does not directly measure the typical distance we seek; if we take the square root of this we do get such a measure, \(\sqrt{35/11} \approx 2.42\). Doing the computation for the strange 11-sided die we get
\[
(2 - 7)^2 \frac{1}{11} + (3 - 7)^2 \frac{1}{11} + \cdots + (7 - 7)^2 \frac{1}{11} + \cdots (11 - 7)^2 \frac{1}{11} + (12 - 7)^2 \frac{1}{11} = 10,
\]

with square root approximately 3.16. Comparing 2.42 to 3.16 tells us that the two-dice rolls clump somewhat more closely near 7 than the rolls of the weird die, which of course with square root approximately 3.16.

To perform the same computation for a probability density function the sum is replaced by an integral, just as in the computation of the mean. The expected value of the squared distances is
\[
V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx,
\]

called the variance. The square root of the variance is the standard deviation, denoted \(\sigma\).

**Example 11.3.8** We compute the standard deviation of the standard normal distribution. The variance is
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} \, dx.
\]

To compute the antiderivative, use integration by parts, with \(u = x\) and \(dv = e^{-x^2/2} \, dx\). This gives
\[
\int x e^{-x^2/2} \, dx = -xe^{-x^2/2} + \int e^{-x^2/2} \, dx.
\]

We cannot do the new integral, but we know its value when the limits are \(-\infty\) to \(\infty\), from our discussion of the standard normal distribution. Thus
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = 0 + \frac{1}{\sqrt{2\pi}} \sqrt{2\pi} = 1.
\]

The standard deviation is then \(\sqrt{1} = 1\).

### Exercises 11.3.

1. Verify that \(\int_{-\infty}^{\infty} e^{-x^2/2} \, dx = 2\sqrt{\pi}\).
2. Show that the function in example 11.3.5 is a probability density function. Compute the mean and standard deviation. \(\Rightarrow\)
3. Compute the mean and standard deviation of the uniform distribution on \([a, b]\). (See example 11.3.5) \(\Rightarrow\)
4. What is the expected value of one roll of a fair six-sided die? \(\Rightarrow\)
5. What is the expected sum of one roll of three fair six-sided dice? \(\Rightarrow\)
6. Let \(\mu = 0\) and \(\sigma = 2\) be real numbers with \(\sigma > 0\). Show that
\[
N(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]
is a probability density function. You will not be able to compute this integral directly; use a substitution to convert the integral into the one from example 11.3.4. The function \(N\)

### 11.4 Arc Length

Here is another geometric application of the integral: find the length of a portion of a curve. As usual, we need to think about how we might approximate the length, and turn the approximation into an integral.

We already know how to compute one simple arc length, that of a line segment. If the endpoints are \(P_0(x_0, y_0)\) and \(P_1(x_1, y_1)\) then the length of the segment is the distance between the points, \(\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}\), from the Pythagorean theorem, as illustrated in figure 11.4.1:

\[
\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}
\]

Figure 11.4.1 The length of a line segment.
Now if the graph of $f$ is “nice” (say, differentiable) it appears that we can approximate the length of a portion of the curve with line segments, and that as the number of segments increases, and their lengths decrease, the sum of the lengths of the line segments will approach the true arc length; see figure 11.4.2.

Now we need to write a formula for the sum of the lengths of the line segments, in a form that we know becomes an integral in the limit. So we suppose we have divided the interval $[a, b]$ into $n$ subintervals as usual, each with length $\Delta x = (b - a)/n$, and endpoints $a = x_0, x_1, x_2, \ldots, x_n = b$. The length of a typical line segment, joining $(x_i, f(x_i))$ to $(x_{i+1}, f(x_{i+1}))$, is $\sqrt{(\Delta x)^2 + (f(x_{i+1}) - f(x_i))^2}$. By the Mean Value Theorem (6.5.2), there is a number $t_i$ in $(x_i, x_{i+1})$ such that $f'(t_i)\Delta x = f(x_{i+1}) - f(x_i)$, so the length of the line segment can be written as

$$\sqrt{(\Delta x)^2 + (f'(t_i))\Delta x^2}.\,$$

The arc length is then

$$\lim_{n \to \infty} \sum_{i=0}^{n-1} \sqrt{1 + (f'(t_i))^2} \Delta x = \int_a^b \sqrt{1 + (f'(x))^2} \, dx.$$

Note that the sum looks a bit different than others we have encountered, because the approximation contains a $t_i$ instead of an $x_i$. In the past we have always used left endpoints (namely, $x_i$) to get a representative value of $f$ on $[x_i, x_{i+1}]$; now we are using a different point, but the principle is the same.

To summarize, to compute the length of a curve on the interval $[a, b]$, we compute the integral

$$\int_a^b \sqrt{1 + (f'(x))^2} \, dx.$$

Unfortunately, integrals of this form are typically difficult or impossible to compute exactly, because usually none of our methods for finding antiderivatives will work. In practice this means that the integral will usually have to be approximated.

As usual, the question is: how might we approximate the surface area? For a surface obtained by rotating a curve around an axis, we can take a polygonal approximation to the curve, as in the last section, and rotate it around the same axis. This gives a surface composed of many “truncated cones,” a truncated cone is called a frustum of a cone. Figure 11.5.1 illustrates this approximation.

![Figure 11.5.1 Approximating a surface (left) by portions of cones (right).](image)

So we need to be able to compute the area of a frustum of a cone. Since the frustum can be formed by removing a small cone from the top of a larger one, we can compute the desired area if we know the surface area of a cone. Suppose a right circular cone has base radius $r$ and slant height $h$. If we cut the cone from the vertex to the base circle and flatten it out, we obtain a sector of a circle with radius $h$ and arc length $2\pi r$, as in figure 11.5.2. The angle at the center, in radians, is $2\pi r/h$, and the area of the frustum of a right circular cone is $\pi r_1 h_1 + A$, and the area of the small cone is $\pi r_0 h_0$, thus, the area of the frustum is $\pi (r_1 h_1 - r_0 h_0) = \pi ((r_1 - r_0)h_0 + r_1 h_0)$. By similar triangles, $h_0 = \frac{h}{r_0}$, $h_1 = \frac{h}{r_1}$.

With a bit of algebra this becomes $r_1 - r_0 = \frac{r_1 h_0}{r_0}$. Substitution into the area gives

$$\pi ((r_1 - r_0)h_0 + r_1 h_0) = \pi (r_0 h_0 + r_1 h) = \pi (h_0 + r_1) = 2\pi h_0 = 2\pi \frac{h_0 + h_1}{2} = 2\pi h.$$

The final form is particularly easy to remember, with $r$ equal to the average of $r_0$ and $r_1$, as it is also the formula for the area of a cylinder. (Think of a cylinder of radius $r$ and height $h$ as the frustum of a cone of infinite height.)

![Figure 11.5.2 The area of a cone.](image)

![Figure 11.5.3 The area of a frustum.](image)
The surface area is given by

\[ A = 2\pi \int_a^b x \sqrt{1 + (f'(x))^2} \, dx \]

EXAMPLE 11.5.2 Compute the area of the surface formed when \( f(x) = x^2 \) between 0 and 2 is rotated around the \( y \)-axis.

We compute \( f'(x) = 2x \), and then

\[ 2\pi \int_0^2 x \sqrt{1 + 4x^2} \, dx = \frac{\pi}{6} (17^{3/2} - 1), \]

by a simple substitution.

Exercises 11.5.

1. Compute the area of the surface formed when \( f(x) = 2 \sqrt{x} \) between \(-1\) and 0 is rotated around the \( x \)-axis. \( \Rightarrow \)

2. Compute the surface area of example 11.5.2 by rotating \( f(x) = \sqrt{7} \) around the \( x \)-axis. \( \Rightarrow \)

3. Compute the area of the surface formed when \( f(x) = x^3 \) between 1 and 3 is rotated around the \( x \)-axis. \( \Rightarrow \)

4. Compute the area of the surface formed when \( f(x) = 2 + \cosh(x) \) between 0 and 1 is rotated around the \( x \)-axis. \( \Rightarrow \)

5. Consider the surface obtained by rotating the graph of \( f(x) = 1/x \), \( x \geq 1 \), around the \( x \)-axis. This surface is called Gabriel’s horn or Torricelli’s trumpet. In exercise 13 in section 11.2 we saw that Gabriel’s horn has finite volume. Show that Gabriel’s horn has infinite surface area.

6. Consider the circle \((x - 2)^2 + y^2 = 1\). Sketch the surface obtained by rotating this circle about the \( y \)-axis. (The surface is called a torus.) What is the surface area? \( \Rightarrow \)

7. Consider the ellipse with equation \( x^2/4 + y^2 = 1 \). If the ellipse is rotated around the \( x \)-axis it forms an ellipsoid. Compute the surface area. \( \Rightarrow \)

8. Generalize the preceding result: rotate the ellipse given by \( x^2/a^2 + y^2/b^2 = 1 \) about the \( x \)-axis and find the surface area of the resulting ellipsoid. You should consider two cases, when \( a > b \) and when \( a < b \). Compare to the area of a sphere. \( \Rightarrow \)

\[ 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} \, dx \]