10

Techniques of Integration

10.1 Powers of Sine and Cosine

Functions consisting of products of the sine and cosine can be integrated by using substitution and trigonometric identities. These can sometimes be tedious, but the technique is straightforward. Some examples will suffice to explain the approach.

EXAMPLE 10.1.1 Evaluate \( \int \sin^5 x \, dx \). Rewrite the function:
\[
\int \sin^5 x \, dx = \int \sin x \sin^4 x \, dx = \int \sin x (\sin^2 x)^2 \, dx.
\]
Now use \( u = \cos x, \, du = -\sin x \, dx \):
\[
\int \sin x (\cos^2 x)^2 \, dx = \int -(1 - u^2)^2 \, du
= \int -(1 - 2u^2 + u^4) \, du
= -u + \frac{2}{3} u^3 - \frac{1}{5} u^5 + C
= -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + C.
\]

EXAMPLE 10.1.2 Evaluate \( \int \sin^6 x \, dx \). Use \( \sin^2 x = (1 - \cos(2x))/2 \) to rewrite the function:
\[
\int \sin^6 x \, dx = \int (\sin^2 x)^3 \, dx = \int \left( \frac{1 - \cos(2x)}{2} \right)^3 \, dx
= \frac{1}{8} \int 1 - 3 \cos 2x + 3 \cos^2 2x - \cos^3 2x \, dx.
\]
Now we have four integrals to evaluate:
\[
\int 1 \, dx = x
\]
and
\[
\int -3 \cos 2x \, dx = -\frac{3}{2} \sin 2x
\]
are easy. The \( \cos^3 2x \) integral is like the previous example:
\[
\int -\cos^3 2x \, dx = \int -\cos 2x \cos^2 2x \, dx
= \int -\cos 2x \left( \frac{1 + \cos(4x)}{2} \right) \, dx
= -\frac{1}{2} \left( \frac{1 + \cos(4x)}{2} \right)
= -\frac{1}{2} \left( \sin 2x - \sin 3x \right).
\]
And finally we use another trigonometric identity, \( \cos^2 x = (1 + \cos(2x))/2 \):
\[
\int 3 \cos^2 2x \, dx = 3 \int \frac{1 + \cos 4x}{2} \, dx = \frac{3}{2} \left( x + \sin 4x \right).
\]
So at long last we get
\[
\int \sin^6 x \, dx = \frac{x}{8} - \frac{3}{16} \sin 2x - \frac{1}{16} \left( \sin 2x - \frac{3}{4} \sin 3x \right) + \frac{3}{16} \left( x + \frac{\sin 4x}{4} \right) + C.
\]

EXAMPLE 10.1.3 Evaluate \( \int \sin^2 x \cos^2 x \, dx \). Use the formulas \( \sin^2 x = (1 - \cos(2x))/2 \)
and \( \cos^2 x = (1 + \cos(2x))/2 \) to get:
\[
\int \sin^2 x \cos^2 x \, dx = \int \frac{1 - \cos(2x)}{2} \cdot \frac{1 + \cos(2x)}{2} \, dx.
\]
The remainder is left as an exercise.


10.2 Trigonometric Substitutions

Exercises 10.1.

Find the antiderivatives.

1. $\int \sin^2 x \, dx$ ⇒
2. $\int \sin^3 x \, dx$ ⇒
3. $\int \sin^4 x \, dx$ ⇒
4. $\int \cos^2 x \sin^3 x \, dx$ ⇒
5. $\int \cos^3 x \, dx$ ⇒
6. $\int \sin^2 x \cos^2 x \, dx$ ⇒
7. $\int \cos^3 x \sin^2 x \, dx$ ⇒
8. $\int \sin x (\cos x)^{1/2} \, dx$ ⇒
9. $\int \sec^3 x \tan x \, dx$ ⇒
10. $\int \tan^3 x \sec x \, dx$ ⇒

10.2 Trigonometric Substitutions

So far we have seen that it sometimes helps to replace a subexpression of a function by a single variable. Occasionally it can help to replace the original variable by something more complicated. This seems like a “reverse” substitution, but it is really no different in principle than ordinary substitution.

EXAMPLE 10.2.1 Evaluate $\int \sqrt{1 - x^2} \, dx$. Let $x = \sin u$ so $dx = \cos u \, du$. Then

$$\int \sqrt{1 - x^2} \, dx = \int \sqrt{1 - \sin^2 u} \cos u \, du = \int \sqrt{\cos^2 u} \cos u \, du.$$ 

We would like to replace $\sqrt{\cos^2 u}$ by $\cos u$, but this is valid only if $\cos u$ is positive, since $\sqrt{\cos^2 u}$ is positive. Consider again the substitution $x = \sin u$. We could just as well think of this as $u = \arcsin x$. If we do, then by the definition of the arcsine, $-\pi/2 \leq u \leq \pi/2$, so $\cos u \geq 0$. Then we continue:

$$\int \sqrt{\cos^2 u} \cos u \, du = \int \cos^2 u \, du = \int \left(\frac{1 + \cos 2u}{2}\right) \, du = \frac{u}{2} + \frac{\sin 2u}{4} + C$$

$$= \frac{\arcsin x}{2} + \frac{\sin(2\arcsin x)}{4} + C.$$ 

This is a perfectly good answer, though the term $\sin(2\arcsin x)$ is a bit unpleasant. It is possible to simplify this. Using the identity $\sin 2x = 2\sin x \cos x$, we can write $\sin 2u = 2\sin u \cos u = 2\sin(2\arcsin x)\sqrt{1 - \sin^2 u} = 2x\sqrt{1 - x^2}$. Then the full antiderivative is

$$\int \sqrt{\cos^2 u} \cos u \, du = \frac{\arcsin x}{2} + \frac{2x\sqrt{1 - x^2}}{4} + C.$$ 

EXAMPLE 10.2.2 Evaluate $\int \sqrt{4 - 9x^2} \, dx$. We start by rewriting this so that it looks more like the previous example:

$$\int \sqrt{4 - 9x^2} \, dx = \int \sqrt{4(1 - (3x/2)^2)} \, dx = \int 2\sqrt{1 - (3x/2)^2} \, dx.$$ 

Now let $3x/2 = \sin u$ so $(3/2) \, dx = \cos u \, du$ or $dx = (2/3) \cos u \, du$. Then

$$\int 2\sqrt{1 - (3x/2)^2} \, dx = \int 2\sqrt{1 - \sin^2 u} \, (2/3) \cos u \, du = \frac{4u}{3} + \frac{\sin 2u}{12} + C$$

$$= \frac{2\arcsin(3x/2)}{3} + \frac{2\sin u \cos u}{3} + C$$

$$= \frac{2\arcsin(3x/2)}{3} + \frac{2\sin(\arcsin(3x/2)) \cos(\arcsin(3x/2))}{3} + C$$

$$= \frac{2\arcsin(3x/2)}{3} + \frac{2(3x/2)^2}{3} \arcsin(3x/2) + 4 = \frac{2\arcsin(3x/2)}{3} + \frac{2(3x/2)^2}{3} \arcsin(3x/2) + 2 + \frac{\sqrt{1 - 9x^2}}{2} + C,$$ 

using some of the work from example 10.2.1.

EXAMPLE 10.2.3 Evaluate $\int \sqrt{1 + x^2} \, dx$. Let $x = \tan u$, $dx = \sec^2 u \, du$, so

$$\int \sqrt{1 + x^2} \, dx = \int \sqrt{1 + \tan^2 u} \, \sec^2 u \, du = \int \sec^3 u \, du.$$ 

Since $u = \arctan(x)$, $-\pi/2 \leq u \leq \pi/2$ and $\sec u \geq 0$, so $\sqrt{\sec^2 u} = \sec u$. Then

$$\int \sec^3 u \, du = \int \sec^3 u \, du.$$ 

In problems of this type, two integrals come up frequently: $\int \sec^3 u \, du$ and $\int \sec u \, du$. Both have relatively nice expressions but they are a bit tricky to discover.
First we do \( \int \sec u \, du \), which we will need to compute \( \int \sec^3 u \, du \):

\[
\int \sec u \, du = \int \sec u \frac{\sec u + \tan u}{\sec u + \tan u} \, du
= \int \sec^2 u + \sec u \tan u \, du.
\]

Now let \( w = \sec u + \tan u \), \( dw = \sec u \tan u \, du \), exactly the numerator of the function we are integrating. Thus

\[
\int \sec^3 u \, du = \int \sec u \tan u \, du = \int 1 \, dw = \ln |w| + C
= \ln |\sec u + \tan u| + C.
\]

So putting these together we get

\[
\int \sec^3 u \, du = \sec u \tan u + \frac{\ln |\sec u + \tan u|}{2} + C,
\]

and reverting to the original variable \( x \):

\[
\int \sqrt{1 + x^2} \, dx = \frac{\sec u \tan u}{2} + \frac{\ln |\sec u + \tan u|}{2} + C
= \frac{\sec(\arctan x) \tan(\arctan x)}{2} + \frac{\ln |\sec(\arctan x) + \tan(\arctan x)|}{2} + C
= \frac{x \sqrt{1 + x^2}}{2} + \frac{\ln |\sqrt{1 + x^2} + x|}{2} + C,
\]

using \( \tan(\arctan x) = x \) and \( \sec(\arctan x) = \sqrt{1 + \tan^2(\arctan x)} = \sqrt{1 + x^2} \). 

\[\square\]
10.3 Integration by Parts 211
du = f’(x) dx and dv = g’(x) dx and
\[ \int u \, dv = uv - \int v \, du. \]

To use this technique we need to identify likely candidates for \( u = f(x) \) and \( dv = g’(x) \) dx.

**EXAMPLE 10.3.1** Evaluate \( \int x \ln x \, dx \). Let \( u = \ln x \) so \( du = 1/x \) dx and \( dv = x \) dx so \( v = x^2/2 \) and
\[ \int x \ln x \, dx = \frac{x^2 \ln x}{2} - \frac{1}{2} x^2 \ln x - \frac{1}{2} x^2 + C. \]

**EXAMPLE 10.3.2** Evaluate \( \int x \sin x \, dx \). Let \( u = x \) so \( du = dx \) and \( dv = \sin x \) dx so \( v = -\cos x \) and
\[ \int x \sin x \, dx = -x \cos x - \int -\cos x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C. \]

**EXAMPLE 10.3.3** Evaluate \( \int \sec^3 x \, dx \). Of course we already know the answer to this, but we needed to be clever to discover it. Here we’ll use the new technique to discover the antiderivative. Let \( u = \sec x \) and \( dv = \sec^2 x \) dx. Then \( du = \sec x \tan x \) dx and \( v = \tan x \) and
\[ \int \sec^3 x \, dx = \sec x \tan x - \int \tan^2 x \sec x \, dx 
= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx 
= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx. \]

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At first this looks useless—we’re right back to \( \int \sec^3 x \, dx \). But looking more closely:
\[ \int \sec^3 x \, dx = \sec x \tan x - \int \sec x \, dx + \int \sec x \, dx \]
\[ = 2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx \]
\[ = \frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec x \, dx \]
\[ = \frac{\sec x \tan x}{2} + \ln |\sec x + \tan x| + C. \]

**EXAMPLE 10.3.4** Evaluate \( \int x^2 \sin x \, dx \). Let \( u = x^2 \), \( dv = \sin x \) dx; then \( du = 2x \) dx and \( v = -\cos x \). Now \( \int x^2 \sin x \, dx = -x^2 \cos x + \int 2x \cos x \, dx \). This is better than the original integral, but we need to do integration by parts again. Let \( u = 2x \), \( dv = \cos x \) dx; then \( du = 2 \) dx and \( v = \sin x \), and
\[ \int x^2 \sin x \, dx = -x^2 \cos x + \int 2x \cos x \, dx \]
\[ = -x^2 \cos x + 2x \sin x - \int 2 \sin x \, dx \]
\[ = -x^2 \cos x + 2x \sin x + 2 \cos x + C. \]

Such repeated use of integration by parts is fairly common, but it can be a bit tedious to accomplish, and it is easy to make errors, especially sign errors involving the subtraction in the formula. There is a nice tabular method to accomplish the calculation that minimizes the chance for error and speeds up the whole process. We illustrate with the previous example. Here is the table:

<table>
<thead>
<tr>
<th>( \text{sign} )</th>
<th>( u )</th>
<th>( dv )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( + )</td>
<td>( x^2 )</td>
<td>( \sin x )</td>
</tr>
<tr>
<td>( - )</td>
<td>( 2x )</td>
<td>( -\cos x )</td>
</tr>
<tr>
<td>( + )</td>
<td>( 2 )</td>
<td>( -\sin x )</td>
</tr>
<tr>
<td>( - )</td>
<td>( 0 )</td>
<td>( \cos x )</td>
</tr>
</tbody>
</table>

\( \text{or} \)

<table>
<thead>
<tr>
<th>( u )</th>
<th>( dv )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td>( \sin x )</td>
</tr>
<tr>
<td>( -2x )</td>
<td>( -\cos x )</td>
</tr>
<tr>
<td>( 2 )</td>
<td>( -\sin x )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( \cos x )</td>
</tr>
</tbody>
</table>
To form the first table, we start with \( u \) at the top of the second column and repeatedly compute the derivative; starting with \( dv \) at the top of the third column, we repeatedly compute the antiderivative. In the first column, we place a “−” in every second row. To form the second table we combine the first and second columns by ignoring the boundary; if you do this by hand, you may simply start with two columns and add a “−” to every second row.

To compute with this second table we begin at the top. Multiply the first entry in column \( u \) by the second entry in column \( dv \) to get \( -x^2 \cos x \), and add this to the integral of the product of the second entry in column \( u \) and second entry in column \( dv \). This gives:

\[-x^2 \cos x + \int 2x \cos x \, dx,
\]
or exactly the result of the first application of integration by parts. Since this integral is not yet easy, we return to the table. Now we multiply twice on the diagonal, \((x^2)(-\cos x)\) and \((-2x)(-\sin x)\) and then once straight across, \((2)(-\sin x)\), and combine these as

\[-x^2 \cos x + 2x \sin x - \int 2 \sin x \, dx,
\]
giving the same result as the second application of integration by parts. While this integral is easy, we may return yet once more to the table. Now multiply three times on the diagonal to get \((x^2)(-\cos x), (-2x)(-\sin x), (2)(\cos x)\), and once straight across, \((0)(\cos x)\). We combine these as before to get

\[-x^2 \cos x + 2x \sin x + 2 \cos x + \int 0 \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.
\]

Typically we would fill in the table one line at a time, until the “straight across” multiplication gives an easy integral. If we can see that the \( u \) column will eventually become zero, we can instead fill in the whole table; computing the products as indicated will then give the entire integral, including the “+C”, as above.

**Exercises 10.3.**

Find the antiderivatives.

1. \( \int x \cos x \, dx \Rightarrow \)
2. \( \int x^2 \cos x \, dx \Rightarrow \)
3. \( \int xe^x \, dx \Rightarrow \)
4. \( \int xe^{x^2} \, dx \Rightarrow \)
5. \( \int \sin^2 x \, dx \Rightarrow \)
6. \( \int \ln x \, dx \Rightarrow \)

**10.4 Rational Functions**

A rational function is a fraction with polynomials in the numerator and denominator. For example,

\[
\frac{x^3}{x^2 + x - 6}, \quad \frac{1}{(x - 3)^2}, \quad \frac{x^2 + 1}{x^2 - 1},
\]

are all rational functions of \( x \). There is a general technique called “partial fractions” that, in principle, allows us to integrate any rational function. The algebraic steps in the technique are rather cumbersome if the polynomial in the denominator has degree more than 2, and the technique requires that we factor the denominator, something that is not always possible. However, in practice one does not often run across rational functions with high degree polynomials in the denominator for which one has to find the antiderivative. So we shall explain how to find the antiderivative of a rational function only when the denominator is a quadratic polynomial \( ax^2 + bx + c \).

We should mention a special type of rational function that we already know how to integrate: If the denominator has the form \((ax + b)^n\), the substitution \( u = ax + b \) will always work. The denominator becomes \( u^n \), and each \( x \) in the numerator is replaced by \((u - b)/a\), and \( dx = du/a \). While it may be tedious to complete the integration if the numerator has high degree, it is merely a matter of algebra.

6. \( \int \frac{x^3}{x^2 + x - 6} \, dx \Rightarrow \)
7. \( \int x \arctan x \, dx \Rightarrow \)
8. \( \int x^3 \sin x \, dx \Rightarrow \)
9. \( \int x^3 \cos x \, dx \Rightarrow \)
10. \( \int x \sin^2 x \, dx \Rightarrow \)
11. \( \int x \sin x \cos x \, dx \Rightarrow \)
12. \( \int \arctan(\sqrt{x}) \, dx \Rightarrow \)
13. \( \int \sin(\sqrt{x}) \, dx \Rightarrow \)
14. \( \int \sec^2 x \cos^2 x \, dx \Rightarrow \)
EXAMPLE 10.4.1 Find \( \int \frac{x^3}{(3-2x)^2} \, dx \). Using the substitution \( u = 3 - 2x \) we get

\[
\int \frac{x^3}{(3-2x)^2} \, dx = \frac{1}{-2} \int \frac{(u-3)^3}{u^5} \, du = \frac{1}{16} \int \frac{9u^2 + 27u - 27}{u^5} \, du
\]

Then we have the special case we have already seen, that can

\[
= \frac{1}{16} \int u^{-2} - 9u^{-3} + 27u^{-4} - 27u^{-5} \, du
\]

\[
= \frac{1}{16} \left( \frac{u^{-1}}{-1} - 9u^{-2} + 27u^{-3} - 27u^{-4} \right) + C
\]

\[
= \frac{1}{16} \left( (3-2x)^{-1} - \frac{9(3-2x)^{-2}}{2} + \frac{27(3-2x)^{-3}}{-3} - \frac{27(3-2x)^{-4}}{-4} \right) + C
\]

\[
= -\frac{1}{16(3-2x)} + \frac{9}{32(3-2x)^2} - \frac{9}{16(3-2x)^3} + \frac{27}{64(3-2x)^4} + C.
\]

We now proceed to the case in which the denominator is a quadratic polynomial. We can always factor out the coefficient of \( x^2 \) and put it outside the integral, so we can assume that the denominator has the form \( x^2 + bx + c \). There are three possible cases, depending on how the quadratic factors: either \( x^2 + bx + c = (x-\alpha)(x-\beta) \), or \( x^2 + bx + c = (x-\beta)^2 \), or it doesn’t factor. We can use the quadratic formula to decide which of these we have, and to factor the quadratic if it is possible.

EXAMPLE 10.4.2 Determine whether \( x^2 + x + 1 \) factors, and factor it if possible. The quadratic formula tells us that \( x^2 + x + 1 = 0 \) when

\[
x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1}}{2}.
\]

Since there is no square root of \(-3\), this quadratic does not factor.

EXAMPLE 10.4.3 Determine whether \( x^2 - x - 1 \) factors, and factor it if possible. The quadratic formula tells us that \( x^2 - x - 1 = 0 \) when

\[
x = \frac{1 \pm \sqrt{1^2 - 4 \cdot (-1)}}{2} = 1 \pm \sqrt{2}.
\]

Therefore

\[
x^2 - x - 1 = \left( x - \frac{1 + \sqrt{2}}{2} \right) \left( x - \frac{1 - \sqrt{2}}{2} \right).
\]

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If \( x^2 + bx + c = (x-\alpha)^2 \) then we have the special case we have already seen, that can be handled with a substitution. The other two cases require different approaches.

If \( x^2 + bx + c = (x-\alpha)(x-\beta) \), we have an integral of the form

\[
\int \frac{p(x)}{(x-\alpha)(x-\beta)} \, dx
\]

where \( p(x) \) is a polynomial. The first step is to make sure that \( p(x) \) has degree less than 2.

EXAMPLE 10.4.4 Rewrite \( \int \frac{x^3}{(x-2)(x+3)} \, dx \) in terms of an integral with a numerator that has degree less than 2. To do this we use long division of polynomials to discover that

\[
\frac{x^3}{(x-2)(x+3)} = \frac{x^3}{x^2 + x - 6} = x - 1 + \frac{7x - 6}{x^2 + x - 6} = x - 1 + \frac{7x - 6}{(x-2)(x+3)}
\]

so

\[
\int \frac{x^3}{(x-2)(x+3)} \, dx = \int x - 1 \, dx + \int \frac{7x - 6}{(x-2)(x+3)} \, dx.
\]

The first integral is easy, so only the second requires some work.

Now consider the following simple algebra of fractions:

\[
\frac{A}{x-\alpha} + \frac{B}{x-\beta} = \frac{A(x-\beta) + B(x-\alpha)}{(x-\alpha)(x-\beta)} = \frac{(A+B)x - \alpha B - \beta A}{(x-\alpha)(x-\beta)}.
\]

That is, adding two fractions with constant numerator and denominators \((x-\alpha)\) and \((x-\beta)\) produces a fraction with denominator \((x-\alpha)(x-\beta)\) and a polynomial of degree less than 2 for the numerator. We want to reverse this process: starting with a single fraction, we want to write it as a sum of two simpler fractions. An example should make it clear how to proceed.

EXAMPLE 10.4.5 Evaluate \( \int \frac{x^3}{(x-2)(x+3)} \, dx \). We start by writing \( \frac{7x - 6}{(x-2)(x+3)} \) as the sum of two fractions. We want to end up with

\[
\frac{7x - 6}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}
\]

If we go ahead and add the fractions on the right hand side we get

\[
\frac{7x - 6}{(x-2)(x+3)} = \frac{(A+B)x + 3A - 2B}{(x-2)(x+3)}
\]

So all we need to do is find \( A \) and \( B \) so that \( 7x - 6 = (A+B)x + 3A - 2B \), which is to say, we need \( 7 = A + B \) and \(-6 = 3A - 2B \). This is a problem you’ve seen before: solve a
system of two equations in two unknowns. There are many ways to proceed; here’s one: If

\[ 7 = A + B \text{ then } B = 7 - A \text{ and so } -6 = 3A - 2B = 3A - 2(7 - A) = 3A - 14 + 2A = 5A - 14. \]

This is easy to solve for \( A \): \( A = 8/5 \), and then \( B = 7 - A = 7 - 8/5 = 27/5. \) Thus

\[
\int \frac{7x - 6}{(x - 2)(x + 3)} \, dx = \int \frac{8}{5} \frac{1}{x - 2} + \frac{27}{5} \frac{1}{x + 3} \, dx = \frac{8}{5} \ln |x - 2| + \frac{27}{5} \ln |x + 3| + C.
\]

The answer to the original problem is now

\[
\int \frac{x^3}{(x - 2)(x + 3)} \, dx = \int x - 1 \, dx + \int \frac{7x - 6}{(x - 2)(x + 3)} \, dx = \frac{x^2}{2} - x + \frac{8}{5} \ln |x - 2| + \frac{27}{5} \ln |x + 3| + C.
\]

Now suppose that \( x^2 + bx + c \) doesn’t factor. Again we can use long division to ensure that the numerator has degree less than 2, then we complete the square.

**EXAMPLE 10.4.6** Evaluate

\[
\int \frac{x + 1}{x^2 + 4x + 8} \, dx.
\]

The quadratic denominator does not factor. We could complete the square and use a trigonometric substitution, but it is simpler to rearrange the integrand:

\[
\int \frac{x + 1}{x^2 + 4x + 8} \, dx = \int \frac{x + 2}{x^2 + 4x + 8} \, dx - \int \frac{1}{x^2 + 4x + 8} \, dx.
\]

The first integral is an easy substitution problem, using \( u = x^2 + 4x + 8 \):

\[
\int \frac{x + 2}{x^2 + 4x + 8} \, dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |x^2 + 4x + 8|.
\]

For the second integral we complete the square:

\[
x^2 + 4x + 8 = (x + 2)^2 + 4 = \left( \frac{x + 2}{2} \right)^2 + 4,
\]

making the integral

\[
\frac{1}{4} \int \frac{1}{\left( \frac{x + 2}{2} \right)^2 + 1} \, dx.
\]

Using \( u = \frac{x + 2}{2} \) we get

\[
\frac{1}{4} \int \frac{1}{u^2 + 1} \, du = \frac{1}{4} \int \frac{2}{u^2 + 1} \, du = \frac{1}{2} \arctan \left( \frac{x + 2}{2} \right).
\]

The final answer is now

\[
\int \frac{x + 1}{x^2 + 4x + 8} \, dx = \frac{1}{2} \ln |x^2 + 4x + 8| - \frac{1}{2} \arctan \left( \frac{x + 2}{2} \right) + C.
\]
the areas of all trapezoids we get

\[
\frac{f(x_0) + f(x_1)}{2} \Delta x + \frac{f(x_1) + f(x_2)}{2} \Delta x + \cdots + \frac{f(x_{n-1}) + f(x_n)}{2} \Delta x = \\
\left( \frac{f(x_0)}{2} + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2} \right) \Delta x.
\]

This is usually known as the Trapezoid Rule. For a modest number of subintervals this is not too difficult to do with a calculator; a computer can easily do many subintervals.

**EXAMPLE 10.5.2** Approximate \( \int_0^1 e^{-x^2} \, dx \) to two decimal places. The second derivative of \( f = e^{-x^2} \) is \((4x^2 - 2)e^{-x^2}\), and it is not hard to see that on \([0, 1] \), \(|(4x^2 - 2)e^{-x^2}| \leq 2\).

We begin by estimating the number of subintervals we are likely to need. To get two decimal places of accuracy, we will certainly need \( E(\Delta x) < 0.005 \) or

\[
\frac{1}{12} \left( \frac{1}{n^2} \right) < 0.005
\]

\[
\frac{1}{6} \left( \frac{1}{100} \right) < n^2
\]

\[
5.77 \approx \sqrt{\frac{100}{3}} < n
\]

With \( n = 6 \), the error estimate is thus \( 1/6^2 < 0.0047 \). We compute the trapezoid approximation for six intervals:

\[
\left( \frac{f(0)}{2} + f(1/6) + f(2/6) + \cdots + f(5/6) + \frac{f(1)}{2} \right) \frac{1}{6} \approx 0.74512.
\]

So the true value of the integral is between 0.74512 – 0.0047 = 0.74042 and 0.74512 + 0.0047 = 0.74982. Unfortunately, the first rounds to 0.74 and the second rounds to 0.75, so we can’t be sure of the correct value in the second decimal place; we need to pick a larger \( n \). As it turns out, we need to go to \( n = 12 \), which means that both round to the same value, which turns out to be 0.75. For comparison, using 12 rectangles to approximate the area gives 0.7727, which is considerably less accurate than the approximation using six trapezoids.

In practice it generally pays to start by requiring better than the maximum possible error; for example, we might have initially required \( E(\Delta x) < 0.001 \), or

\[
\frac{1}{12} \left( \frac{1}{n^2} \right) < 0.001
\]

\[
\frac{1}{6} \left( \frac{1}{1000} \right) < n^2
\]

\[
12.91 \approx \sqrt{\frac{1000}{3}} < n
\]

Had we immediately tried \( n = 13 \) this would have given us the desired answer.

**THEOREM 10.5.1** Suppose \( f \) has a second derivative \( f'' \) everywhere on the interval \([a, b]\), and \(|f''(x)| \leq M\) for all \( x \) in the interval. With \( \Delta x = (b - a)/n \), an error estimate for the trapezoid approximation is

\[
E(\Delta x) = \frac{b - a}{12} M (\Delta x)^2 = \frac{(b - a)^3}{12n^2} M.
\]

Let’s see how we can use this.
by using something other than a straight line? The obvious candidate is a parabola: if we can approximate a short piece of the curve with a parabola with equation \( y = ax^2 + bx + c \), we can easily compute the area under the parabola.

There are an infinite number of parabolas through any two given points, but only one through three given points. If we find a parabola through three consecutive points \((x_i, f(x_i)), (x_{i+1}, f(x_{i+1})), (x_{i+2}, f(x_{i+2}))\) on the curve, it should be quite close to the curve over the whole interval \([x_i, x_{i+2}]\), as in Figure 10.5.3. If we divide the interval \([a, b]\) into an even number of subintervals, we can then approximate the curve by a sequence of parabolas, each covering two of the subintervals. For this to be practical, we would like a fourth derivative of \(f\) everywhere on the interval \([a, b]\). Since \(|f(x)| \leq M\) for all \(x\) in the interval, with \(\Delta x = (b - a)/n\), an error estimate for Simpson’s approximation is

\[
E(\Delta x) = \frac{b - a}{180} M (\Delta x)^4 = \frac{(b - a)^5}{180n^4} M.
\]

**EXAMPLE 10.5.4** Let us again approximate \(\int_0^1 e^{-x^2} \, dx\) to two decimal places. The fourth derivative of \(f = e^{-x^2}\) is \((16x^2 - 48x^2 + 12)e^{-x^2}\); on \([0, 1]\) this is at most 12 in absolute value. We begin by estimating the number of subintervals we are likely to need.

To get two decimal places of accuracy, we will certainly need \(E(\Delta x) < 0.005\), but taking a cue from our earlier example, let’s require \(E(\Delta x) < 0.001\):

\[
\frac{1}{180} \left( \frac{1}{n^4} \right) < 0.001
\]

\[
\frac{200}{3} < n^4
\]

\[
2.86 \approx \sqrt[4]{200} < n
\]

So we try \(n = 4\), since we need an even number of subintervals. Then the error estimate is \(12/180/4^4 < 0.0003\) and the approximation is

\[
(f(0) + 4 f(1/4) + 2 f(1/2) + 4 f(3/4) + f(1)) \cdot \frac{1}{3} \approx 0.746855.
\]

So the true value of the integral is between 0.746855 − 0.0003 = 0.746555 and 0.746855 + 0.0003 = 0.7471555, both of which round to 0.75.
10.6 Additional exercises

**Exercises 10.5.**

In the following problems, compute the trapezoid and Simpson approximations using 4 subintervals, and compute the error estimate for each. (Finding the maximum values of the second and fourth derivatives can be challenging for some of these; you may use a graphing calculator or computer software to estimate the maximum values.) If you have access to Sage or similar software, approximate each integral to two decimal places. You can use this Sage worksheet to get started.

1. \[\int_1^3 x \, dx \Rightarrow\]
2. \[\int_0^\infty x^2 \, dx \Rightarrow\]
3. \[\int_2^4 x^3 \, dx \Rightarrow\]
4. \[\int_1^3 \frac{1}{x} \, dx \Rightarrow\]
5. \[\int_1^2 \frac{1}{x + 2} \, dx \Rightarrow\]
6. \[\int_0^\infty \frac{1}{x} \, dx \Rightarrow\]
7. \[\int_0^\infty \frac{x}{x + 4} \, dx \Rightarrow\]
8. \[\int_0^\infty \frac{1}{x^2 + 1} \, dx \Rightarrow\]
9. \[\int_0^\infty \sqrt{x^2 + 1} \, dx \Rightarrow\]
10. \[\int_0^\infty \sqrt{x^2 + 1} \, dx \Rightarrow\]
11. Using Simpson’s rule on a parabola \(f(x)\), even with just two subintervals, gives the exact value of the integral, because the parabolas used to approximate \(f\) will be \(f\) itself. Remarkably, Simpson’s rule also computes the integral of a cubic function \(f(x) = ax^3 + bx^2 + cx + d\) exactly. Show this is true by showing that

\[
\int_{x_0}^{x_1} f(x) \, dx = \frac{x_0^2 - x_0^2}{2} (f(x_0) + 4f((x_0 + x_1)/2) + f(x_1)).
\]

This does require a bit of messy algebra, so you may prefer to use Sage.

**10.6 Additional exercises**

These problems require the techniques of this chapter, and are in no particular order. Some problems may be done in more than one way.

1. \[\int (t + 4)^3 \, dt \Rightarrow\]
2. \[\int (t^2 - 9)^{1/2} \, dt \Rightarrow\]
3. \[\int (e^{t^2} + 16)e^t \, dt \Rightarrow\]
4. \[\int \sin t \cos 2t \, dt \Rightarrow\]
5. \[\int \tan t \sec^2 t \, dt \Rightarrow\]
6. \[\int \frac{2t + 1}{t^2 + t + 3} \, dt \Rightarrow\]
7. \[\int \frac{1}{t^2 - 4} \, dt \Rightarrow\]
8. \[\int \frac{1}{(25 - t^2)^{1/2}} \, dt \Rightarrow\]
9. \[\int \frac{\cos 3t}{\sqrt{\sin 3t}} \, dt \Rightarrow\]
10. \[\int \sec^4 t \, dt \Rightarrow\]
11. \[\int \frac{e^t}{\sqrt{e^{2t} + 1}} \, dt \Rightarrow\]
12. \[\int \cos^4 t \, dt \Rightarrow\]