9

Transcendental Functions

9.1 Inverse functions

Informally, two functions f and g are **inverses** if each reverses, or undoes, the other. More precisely:

DEFINITION 9.1.1 Two functions f and g are inverses if for all x in the domain of g, $f(g(x)) = x$, and for all x in the domain of f, $g(f(x)) = x$. □

EXAMPLE 9.1.2 $f = x^3$ and $g = x^{1/3}$ are inverses, since $(x^3)^{1/3} = x$ and $(x^{1/3})^3 = x$. □

EXAMPLE 9.1.3 $f = x^2$ and $g = x^{1/2}$ are not inverses. While $(x^{1/2})^2 = x$, it is not true that $(x^2)^{1/2} = x$. For example, with $x = -2$, $((-2)^2)^{1/2} = 4^{1/2} = 2$. □

The problem in the previous example can be traced to the fact that there are two different numbers with square equal to 4. This turns out to be precisely descriptive of functions without inverses.

DEFINITION 9.1.4 Let A and B be sets and let $f : A \to B$ be a function. We say that f is **injective** or **one-to-one** if $f(x) = f(y)$ implies that $x = y$.

We say that f is **surjective** or **onto** if for every $b \in B$ there is an $a \in A$ such that $f(a) = b$.

If f is both injective and surjective, then f is **bijective** or **one-to-one and onto**. □

We are interested only in the case that A and B are sets of real numbers, and in this case there is a nice geometric interpretation of injectivity. It is often easy to use this interpretation to decide whether a function is or is not 1-1.

THEOREM 9.1.5 Horizontal line test If f is a function defined on some subset of the real numbers, then f is injective if and only if every horizontal line intersects the graph of f at most once. ■

EXAMPLE 9.1.6 The function $f = x^2$ fails this test: horizontal lines $y = k$ for $k > 0$ intersect the graph of f twice. (The horizontal line $y = 0$ does intersect it only once, and lines $y = k$, $k < 0$, do not intersect the graph at all.) □

EXAMPLE 9.1.7 In each of these cases, we assume that $f: A \to \mathbb{R}$, where A is the set of all real numbers for which f makes sense.

The function $f(x) = x$ is bijective.

The function $f(x) = x^2$ is neither injective nor surjective. If we think of f as a function from $\mathbb{R}$ to the non-negative real numbers, then f is surjective; in other words, if a function is not surjective this is not a major stumbling block.

The function $f(x) = 1/x$ is injective but not surjective since there is no value of x such that $f(x) = 0$.

The function $f(x) = x(x-1)(x+1)$ is surjective but not injective; $f(x) = 0$ for three different values of x. On the other hand $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. Since f is continuous on $\mathbb{R}$, the intermediate value theorem (2.5.6) guarantees that f takes all values between $-\infty$ and ∞. □

The derivative furnishes us with a convenient criterion for injectivity without explicitly looking for points where injectivity may fail.

THEOREM 9.1.8 If f is differentiable and $f'(x) > 0$ then f is injective.

Proof. Suppose that $f(a) = f(b)$ for some $a < b$. By Rolle's theorem (6.5.1) there exists $c \in (a, b)$ such that $f'(c) = (f(b) - f(a))/(b - a) = 0$, which contradicts the hypothesis that $f'(x) > 0$. Hence, if $f(a) = f(b)$ then $a = b$. ■

In the same way, we can see that if $f'(x) < 0$ then f is injective.

EXAMPLE 9.1.9 Let $f(x) = x^5 + x$. Since $f'(x) = 5x^4 + 1 > 0$, f is injective. □

EXAMPLE 9.1.10 Let $f(x) = 2x + \sin x$. Then $f'(x) = 2 + \cos x \geq 1$ for every x. Hence, f is injective. □

EXAMPLE 9.1.11 Let $f(x) = x^3$. This f is injective although the above theorem does not apply, since $f'(0) = 0$. Therefore, the conditions in the theorem are sufficient but not necessary. □

Our knowledge of derivatives can also lead us to conclude that a function is not injective.

THEOREM 9.1.12 If f is continuous and has a local maximum or minimum then f is not injective.

Proof. Suppose that f has a local maximum at $x = c$. Then in some interval $(c-h, c+h)$, $f(x) \leq f(c)$. Let $a \in (c - h, c)$. If $f(a) = f(c)$ then f is not injective; otherwise, $f(a) < f(c)$.

Let $b \in (c, c + h)$. If $f(b) = f(c)$ or $f(b) = f(a)$ then f is not injective. Otherwise, either $f(b) < f(a) < f(c)$ or $f(a) < f(b) < f(c)$. If $f(b) < f(a)$ then by the intermediate value theorem, there is a number $d \in (c, b)$ such that $f(d) = f(a)$ and so f is not injective. Likewise, if $f(a) < f(b)$ then there is a number d in (a, c) such that $f(d) = f(b)$ and so f is not injective.

In every case, we see that f is not injective. ■

To return to our principal interest, inverse functions, we now connect bijections and inverses.

THEOREM 9.1.13 Suppose $f: A \to B$ is a bijection. Then f has an inverse function $g: B \to A$.

Proof. Suppose $b \in B$. Since f is onto, there is an $a \in A$ such that $f(a) = b$. Since f is 1–1, a is the only element of A with this property. We let $g(b) = a$. Now it is easy to see that for all $a \in A$, $g(f(a)) = a$ and for all $b \in B$, $f(g(b)) = b$. ■

We really don't have any choice about how to define g in this proof; if f is a bijection, its inverse is completely determined. Thus, instead of using a new symbol g, we normally refer to the inverse of f as f^{-1}.

Unfortunately, it is often difficult to find an explicit formula for the inverse of a given function, f, even if it is known that f is bijective. Generally, we attempt to find an inverse in this way:

1. Write $y = f(x)$.
2. Interchange x and y.
3. Solve for y.
4. Replace y with $f^{-1}(x)$.

Step 3 is the hard part; indeed it is sometimes impossible to perform using algebraic operations.

EXAMPLE 9.1.14 Find the inverse of $f(x) = (2x - 6)/(3x + 7)$. First we write $x = (2y - 6)/(3y + 7)$. Now we solve for y:

$$\begin{aligned} x &= \frac{2y-6}{3y+7} \\ x(3y+7) &= 2y-6 \\ 3xy + 7x &= 2y - 6 \\ 7x + 6 &= 2y - 3xy \\ 7x + 6 &= y(2-3x) \\ \frac{7x+6}{2-3x} &= y \end{aligned}$$

Finally, we say $f^{-1}(x) = (7x + 6)/(2 - 3x)$ □

EXAMPLE 9.1.15 Find the inverse function of $f(x) = x^2 - 4x + 8$ where $x \geq 2$. What are the domain and range of the inverse function?

First, $y = x^2 - 4x + 8$ becomes $x = y^2 - 4y + 8$. Now we complete the square: $x = (y-2)^2+4$ and rearrange to get $x-4 = (y-2)^2$. Since in the original function $x-2 \geq 0$, and we have switched x and y, we know that $y - 2 \geq 0$. Thus taking the square root, we know $y - 2 = \sqrt{x-4}$, **not** $y - 2 = -\sqrt{x-4}$. Finally we write $y = f^{-1}(x) = 2 + \sqrt{x-4}$. The domain of f is $x \geq 4$ and the range is $y \geq 2$.

□

While simple in principle, this method is sometimes difficult or impossible to apply. For example, consider $f(x) = x^3 + x$. Since $f'(x) = 3x^2 + 1 > 0$ for every x, f is injective. (In fact it is bijective.) To find the inverse as above, we would need to solve $x = y^3 + y$ for y; while possible, this is considerably more difficult than solving the quadratic of the previous example. Some simple looking equations are impossible to solve using algebraic manipulation.

For example, consider $f(x) = x^5+x^3+x+1$ a "quintic" polynomial (i.e., a fifth degree polynomial). Since $f'(x) = 5x^4 + 3x^2 + 1 > 0$, f is injective (and indeed f is bijective). If there were a quintic formula, analogous to the quadratic formula, we could use that to compute f^{-1}. Unfortunately, no such formula exists—fifth degree equations cannot in general be solved. (There are exceptions; $x^5 = 1$ can be solved, for example.)

Fortunately, it is often more important to know that a function has an inverse then to be able to come up with an explicit formula. Once an inverse is known to exist, numerical

techniques can often be employed to obtain approximations of the inverse function. Thus, theorem 9.1.8 and proposition 9.1.12 provide useful criteria for deciding whether a function is invertible.

We now turn to the calculus of inverse functions.

THEOREM 9.1.16 Let A be an open interval and let $f : A \to \mathbb{R}$ be injective and continuous. Then f^{-1} is continuous on $f(A)$.

Proof. Since A is an open interval and f is injective and continuous it follows by proposition 9.1.12 that f has no local maxima or minima. Hence, f is either strictly increasing or strictly decreasing. Without loss of generality, f is strictly increasing.

Fix $b \in f(A)$. Then there exists a unique $a \in A$ such that $f(a) = b$. Let $\epsilon > 0$ and we may assume that $(a-\epsilon, a+\epsilon) \subseteq A$. Let $\delta = \min\{b - f(a-\epsilon), f(a+\epsilon) - b\}$ and note that $\delta > 0$ since f is increasing. Then the interval $(b-\delta, b+\delta)$ is mapped by f^{-1} into $(a-\epsilon, a+\epsilon)$. Since ϵ was arbitrary, it follows that f^{-1} is continuous at b. ■

Our principal interest in inverses is the simple relationship between the derivative of a function and its inverse.

THEOREM 9.1.17 Inverse function theorem Let A be an open interval and let $f : A \to \mathbb{R}$ be injective and differentiable. If $f'(x) \neq 0$ for every $x \in A$ then f^{-1} is differentiable on $f(A)$ and $(f^{-1})'(x) = 1/f'(f^{-1}(x))$.

Proof. Fix $b \in f(A)$. Then there exists a unique $a \in A$ such that $f(a) = b$. For $y \neq b$, let $x = f^{-1}(y)$. Since f is differentiable, it follows that f and hence f^{-1} are continuous.

Then

$$\lim_{y\to b} \frac{f^{-1}(b) - f^{-1}(y)}{b-y} = \lim_{x\to a} \frac{a-x}{f(a)-f(x)} = \frac{1}{f'(a)}.$$

■

In Leibniz notation, this can be written as $\dfrac{dx}{dy} = \dfrac{1}{dy/dx}$, which is easy to remember since it looks like ordinary fractional algebra.

EXAMPLE 9.1.18 Let $f(x) = 3x^3 + 5x - 7$. Since $f(0) = -7$, $f^{-1}(-7) = 0$. Since $f'(x) = 9x^2 + 5$, $f'(0) = 5$ and so $(f^{-1})'(-7) = 1/f'(0) = 1/5$. □

Exercises 9.1.

1. Which of the following functions are injective? Which are surjective? Which are bijective? Sketch the graph of each function to illustrate your answers.
   a. $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

   b. $f\colon [0,\infty) \to \mathbb{R}$, $f(x) = x^2$.
   c. $f\colon \mathbb{R} \to [0,\infty)$, $f(x) = x^2$.
   d. $f\colon (-\infty, 0] \to [0,\infty)$, $f(x) = x^2$.
2. Which of the following functions are injective? Which are surjective? Which are bijective? Sketch the graph of each function to illustrate your answers.
   a. $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$.
   b. $f\colon [0,\infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$.
   c. $f\colon \mathbb{R} \to [-1,1]$, $f(x) = \sin x$.
   d. $f\colon [0,\pi] \to [-1,1]$, $f(x) = \cos x$.
3. Define
$$f(x) = \begin{cases} -\dfrac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$
   Show that f is not injective on $\mathbb{R}$. Show that $f'(x) > 0$ for $x \neq 0$. Why does this not contradict theorem 9.1.8?
4. Define
$$f(x) = \begin{cases} 1-x & -2 \le x < 0 \\ 0 & x = 0 \\ 10+x & 0 < x \le 2 \end{cases}$$
   Show that f is injective and has a local minimum. Why does this not contradict theorem 9.1.12?
5. If $A = \mathbb{R}$ sketch the graph of the identity function on A.
6. Find the inverse function of $f(x) = (4x-6)/(7x+5)$. What are the domain and range of f^{-1}?
7. Find the inverse function of $f(x) = 11x/(13x-6)$. What are the domain and range of f^{-1}?
8. Find the inverse function of $f(x) = ax + b$ when $a \neq 0$. What are the domain and range of f^{-1}?
9. Find the inverse function of $f(x) = 1/(cx+d)$ when $c \neq 0$. What are the domain and range of f^{-1}?
10. Suppose that $ad - bc \neq 0$. Find the inverse function of $f(x) = (ax+b)/(cx+d)$. What are the domain and range of f^{-1}? (The domain and range will depend on which if any of a, b, c, and d are zero.)
    Note: The condition $ad - bc \neq 0$ is a technical condition which ensures that both the domain of f will be all real numbers with perhaps one exception and that that the range of f will be all real numbers with perhaps one exception.
11. Find the inverse function of $f(x) = |x-4|$ for $x \le 4$. What are the domain and range of f^{-1}?
12. Find the inverse function of $f(x) = \sqrt{x-5}$. What are the domain and range of f^{-1}?
13. Find the inverse function of $f(x) = x^3 - 5$. What are the domain and range of f^{-1}?
14. Find the inverse function of $f(x) = x^7 - 2$. What are the domain and range of f^{-1}?
15. Find the inverse function of $f(x) = 2x^2 + 8x - 4$ for $x \ge -2$. What are the domain and range of f^{-1}?

16. Find the inverse function of $f(x) = x^2 - 9x + 10$ for $x \le 3$. What are the domain and range of f^{-1}?
17. Find the inverse function of $f(x) = x^2 + bx + c$ for $x \ge -b/2$. What are the domain and range of f^{-1}?
18. Find the inverse function of $f(x) = x^2 + bx + c$ for $x \le -b/2$. What are the domain and range of f^{-1}?
19. Find the inverse function of $f(x) = (1+\sqrt{x})/(1-\sqrt{x})$. What are the domain and range of f^{-1}?
20. Show that $f(x) = x^7 + 3x$ has an inverse function on $\mathbb{R}$.
21. Show that $f(x) = x^{19/9} + x^5$ has an inverse function on $\mathbb{R}$.
22. Let $A \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$ be injective. Then f^{-1} exists on $f(A)$.
    Note that the point $P(a, f(a))$ is on the graph of f and that $Q(f(a), a)$ is the corresponding point on the graph of f^{-1}.
    a. Show that if $a \neq f(a)$ then the slope of the line segment PQ is -1.
    b. Conclude that if $a \neq f(a)$ the line segment PQ is perpendicular to the graph, L, of the identity function on $\mathbb{R}$.
    c. Show that the midpoint of PQ is on L.
    d. Conclude that the graph of f^{-1} is the graph of f reflected through L.
23. Let $f(x) = x^3 + x$. Sketch the graph of f and f^{-1} on the same diagram.
24. Let $f(x) = x^5 + x^3 + 1$. Sketch the graph of f and f^{-1} on the same diagram.
25.
    a. Suppose that f is an increasing function on $\mathbb{R}$. What can you say about f^{-1}?
    b. Suppose that f is a concave up function on $\mathbb{R}$. What can you say about f^{-1}?
    In both parts, use exercise 22 to illustrate your claim.
26. Let $f(x) = 3x^3 + 9x + 4$. Compute $(f^{-1})'(4)$.
27. Let $f(x) = 2x^2 + 11$. Show that f is increasing at $x = 0$. Thus, there is an interval I containing 0 such that f is injective on I. Compute $(f^{-1})'(11)$.
28. Let $f(x) = ax + b$ with $a \neq 0$. Compute $(f^{-1})'(b)$. Why do we need the condition $a \neq 0$?
29. Suppose that f is injective on some interval containing 3. If $f(3) = 4$ and $f'(3) = 6$ what is $(f^{-1})'(4)$?

9.2 The natural logarithm

The function $f(t) = 1/t$ is continuous on $(0,\infty)$. By the fundamental theorem of calculus, f has an antiderivative on on the interval with end points x and 1 whenever $x > 0$. This observation allows us to make the following definition.

DEFINITION 9.2.1 The **natural logarithm** $\ln(x)$ is an antiderivative of $1/x$, given by

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

□

Figure 9.2.1 gives a geometric interpretation of ln. Note that when $x < 1$, $\ln x$ is negative.

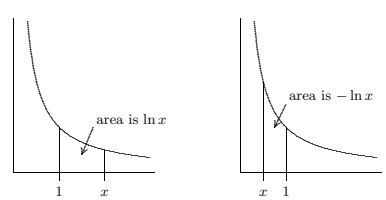

Figure 9.2.1 $\ln(x)$ is an area.

Some properties of this function $\ln x$ are now easy to see.

THEOREM 9.2.2 Suppose that $x, y > 0$ and $q \in \mathbb{Q}$.

a. $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$.
b. $\ln(1) = 0$.
c. $\ln(xy) = \ln x + \ln y$
d. $\ln(x/y) = \ln x - \ln y$
e. $\ln x^q = q \ln x$.

Proof. Part (a) is simply the Fundamental Theorem of Calculus (7.2.2). Part (b) follows directly from the definition, since

$$\ln(1) = \int_1^1 \frac{1}{t}\,dt.$$

Part (c) is a bit more involved; start with:

$$\ln(xy) = \int_1^{xy} \frac{1}{t}\,dt = \int_1^x \frac{1}{t}\,dt + \int_x^{xy} \frac{1}{t}\,dt = \ln(x) + \int_x^{xy} \frac{1}{t}\,dt.$$

In the remaining integral, use the substitution $u = t/x$ to get

$$\int_x^{xy} \frac{1}{t}\,dt = \int_1^y \frac{1}{xu} x\,du = \int_1^y \frac{1}{u}\,du = \ln(y).$$

Parts (d) and (e) are left as exercises. ■

Part (e) is in fact true for any real number q (not just rationals) but one of the points of our approach here is to give a rigorous definition of real powers which so far we have not done.

We now turn to the task of sketching the graph of $\ln x$.

THEOREM 9.2.3 $\ln x$ is increasing and its graph is concave down everywhere.

Proof. Since $\frac{d}{dx}\ln x = 1/x$ is positive for $x > 0$, the Mean Value Theorem (6.5.2) implies that $\ln x$ is increasing. The second derivative of $\ln x$ is then $-1/x^2$ which is negative, so the graph is concave down. ■

Notice that this theorem implies that $\ln x$ is injective.

THEOREM 9.2.4 $\lim_{x\to\infty} \ln x = \infty$

Proof. Note that $\ln 2 > 0$ and for $n \in \mathbb{N}$, $\ln 2^n = n \ln 2$. Since $\ln x$ is increasing, when $x > 2^n$, $\ln(x) > n\ln 2$. Since $\lim_{n\to\infty} n\ln 2 = \infty$, also $\lim_{x\to\infty} \ln x = \infty$. ■

COROLLARY 9.2.5 $\lim_{x\to 0^+} \ln x = -\infty$

Proof. If $0 < x < 1$, then $(1/x) > 1$ and $\lim_{x\to 0^+}(1/x) = \infty$. Let $y = 1/x$; then $\lim_{x\to 0^+} \ln x = \lim_{y\to\infty} \ln(1/y) = \lim_{y\to\infty} \ln(1) - \ln(y) = \lim_{y\to\infty} -\ln(y) = -\infty$. ■

Thus, the domain of ln is $(0, \infty)$ and the range is $\mathbb{R}$; $\ln(x)$ is shown in figure 9.2.2.

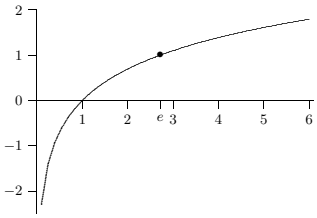

Figure 9.2.2 The graph of $\ln(x)$.

By the intermediate value theorem (2.5.6) there is a number e such that $\ln e = 1$. The number e is also known as **Napier's constant**.

It turns out that e is not rational. In fact, e is not the root of a polynomial with rational coefficients which means that e is a **transcendental number**. We will not prove these assertions here. The value of e is approximately 2.718.

EXAMPLE 9.2.6 Let $f(x) = \ln(x^5 + 7x + 12)$. Compute $f'(x)$.

Using the chain rule: $f'(x) = \dfrac{1}{x^5 + 7x + 12}(5x + 7)$. □

EXAMPLE 9.2.7 Let $f(x) = \ln(-x)$ for $x < 0$. Compute $f'(x)$.

$$f'(x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

So the derivatives of $\ln(x)$ and $\ln(-x)$ are the same. Thus, you will often see $\int \frac{1}{x}\,dx = \ln|x| + C$ as the general antiderivative of $1/x$. □

EXAMPLE 9.2.8 Compute $\int \tan x\,dx$.

Use $u = \cos x$:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int -\frac{1}{u}\,du = -\ln|u| + C = -\ln|\cos x| + C.$$

Using one of the properties of the logarithm, we could go further:

$$-\ln|\cos x| + C = \ln|(\cos x)^{-1}| + C = \ln|\sec x| + C.$$ □

EXAMPLE 9.2.9 Let $f(x) = \dfrac{x^{18}(x+2)^{6/7}}{(x^{10} + 4x^2 + 1)^6}$. Compute $f'(x)$

Computing the derivative directly is straightforward but irritating. We therefore take an indirect approach. Note that $f(x) > 0$ for every x. Let $g(x) = \ln f(x)$. Then $g'(x) = f'(x)/f(x)$ and so $f'(x) = f(x)g'(x)$. Now

$$\begin{aligned} g(x) &= \ln\left(\frac{x^{18}(x+2)^{6/7}}{(x^{10} + 4x^2 + 1)^6}\right) \\ &= 18\ln x + \frac{6}{7}\ln(x+2) - 6\ln(x^{10} + 4x^2 + 1) \end{aligned}$$

Hence,

$$g'(x) = \frac{18}{x} + \frac{6}{7(x+2)} - \frac{6(10x^9 + 8x)}{x^{10} + 4x^2 + 1}.$$

Therefore,

$$f'(x) = \frac{x^{18}(x+2)^{6/7}}{(x^{10} + 4x^2 + 1)^6}\left(\frac{18}{x} + \frac{6}{7(x+2)} - \frac{6(10x^9 + 8x)}{x^{10} + 4x^2 + 1}\right).$$ □

Exercises 9.2.

1. Prove parts (d) and (e) of theorem 9.2.2.

In subsequent exercises, it is understood that the arguments in any logarithms are positive unless otherwise stated.

2. Expand $\ln((x+45)^7(x-2))$.
3. Expand $\ln \dfrac{x^3}{3x - 5 + (7/x)}$.
4. Sketch the graph of $y = \ln(x-7)^3 + 14$.
5. Sketch the graph of $y = \ln|x|$ for $x \neq 0$.
6. Write $\ln 3x + 17\ln(x-2) - 2\ln(x^2 + 4x + 1)$ as a single logarithm.
7. Differentiate $f(x) = x\ln x$.
8. Differentiate $f(x) = \ln(\ln(3x))$.
9. Sketch the graph of $\ln(x^2 - 2x)$.
10. Differentiate $f(x) = \dfrac{1 + \ln(3x^2)}{1 + \ln(4x)}$.
11. Differentiate $f(x) = \ln|\sec x + \tan x|$.
12. Find the second derivative of $f(x) = \sqrt{\ln(x^4 - 2)}$.
13. Find the equation of the tangent line to $f(x) = \ln x$ at $x = a$.
14. Differentiate $f(x) = \dfrac{x^8(x-23)^{1/2}}{27x^6(4x-6)^8}$.
15. If $f(x) = \ln(x^3 + 2)$ compute $f'(e^{1/3})$.
16. Compute $\int_1^e \frac{1}{x}\,dx$.
17. Compute the derivative with respect to x of $\int_1^{\ln x} \ln t\,dt$. (Assume that $x > 1$.)
18. Compute $\int_0^{\pi/6} \tan(2x)\,dx$.
19. Compute $\int \frac{\ln x}{x}\,dx$.
20. Compute $\int \frac{\sin(2x)}{1 + \cos^2 x}\,dx$.
21. Find the volume of the solid obtained by rotating the region under $y = 1/\sqrt{x}$ from 1 to e about the x-axis.

9.3 The exponential function

In this section, we define what is arguably the single most important function in all of mathematics. We have already noted that the function $\ln x$ is injective, and therefore it has an inverse.

DEFINITION 9.3.1 The inverse function of $\ln(x)$ is $y = \exp(x)$, called the **natural exponential function**. □

The domain of $\exp(x)$ is all real numbers and the range is $(0, \infty)$. Note that because $\exp(x)$ is the inverse of $\ln(x)$, $\exp(\ln x) = x$ for $x > 0$, and $\ln(\exp x) = x$ for all x. Also, our knowledge of $\ln(x)$ tells us immediately that $\exp(1) = e$, $\exp(0) = 1$, $\lim_{x\to\infty} \exp x = \infty$, and $\lim_{x\to-\infty} \exp x = 0$.

THEOREM 9.3.2 $\dfrac{d}{dx}\exp(x) = \exp(x)$.

Proof. By the Inverse Function Theorem (9.1.17), $\exp(x)$ has a derivative everywhere. The theorem also tells us what the derivative is. Alternately, we may compute the derivative using implicit differentiation: Let $y = \exp x$, so $\ln y = x$. Differentiating with respect to x we get $\dfrac{1}{y}\dfrac{dy}{dx} = 1$. Hence, $\frac{dy}{dx} = y = \exp x$. ■

COROLLARY 9.3.3 Since $\exp x > 0$, $\exp x$ is an increasing function whose graph is concave up. ■

The graph of the natural exponential function is indicated in figure 9.3.1. Compare this to the graph of $\ln x$, figure 9.2.2.

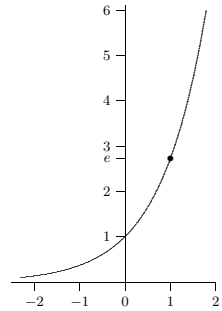

Figure 9.3.1 The graph of $\exp(x)$.

COROLLARY 9.3.4 The general antiderivative of $\exp x$ is $\exp x + C$. ■

Of course, the word "exponential" already has a mathematical meaning, and this meaning extends in a natural way to the exponential function $\exp(x)$.

LEMMA 9.3.5 For any rational number q, $\exp(q) = e^q$.

Proof. Let $y = e^q$. Then $\ln y = \ln(e^q) = q \ln e = q$, and so $y = \exp(q)$. ■

In view of this lemma, we usually write $\exp(x)$ as e^x for any real number x. Conveniently, it turns out that the usual laws of exponents apply to e^x.

THEOREM 9.3.6 For every $x, y \in \mathbb{R}$ and $q \in \mathbb{Q}$:

(a) $e^{x+y} = e^x e^y$

(b) $e^{x-y} = e^x / e^y$

(c) $(e^x)^q = e^{xq}$

Proof. Parts (b) and (c) are left as exercises. For part (a), $\ln(e^x e^y) = \ln e^x + \ln e^y = x + y$, so $e^x e^y = e^{x+y}$. ■

EXAMPLE 9.3.7 Solve $e^{4x+5} - 3 = 0$ for x.

If $e^{4x+5} - 3 = 0$ then $4x + 5 = \ln 3$ and so $x = \dfrac{\ln 3 - 5}{4}$. □

EXAMPLE 9.3.8 Find the derivative of $f(x) = e^{x^3} \sin(4x)$.

By the product and chain rules, $f'(x) = 3x^2 e^{x^3} \sin(4x) + 4e^{x^3} \cos(4x)$. □

EXAMPLE 9.3.9 Evaluate $\int x e^{x^2} dx$.

Let $u = x^2$, so $du = 2x\, dx$. Then

$$\int x e^{x^2}\, dx = \frac{1}{2} \int e^u\, du = \frac{1}{2} e^u = \frac{1}{2} e^{x^2} + C.$$

□

Exercises 9.3.

1. Prove parts (b) and (c) of theorem 9.3.6.
2. Solve $\ln(1 + \sqrt{x}) = 6$ for x.
3. Solve $e^{x^2} = 8$ for x.
4. Solve $\ln(\ln(x)) = 1$ for x.
5. Sketch the graph of $f(x) = e^{4x-5} + 6$.
6. Sketch the graph of $f(x) = 3e^{x+6} - 4$.
7. Find the equation of the tangent line to $f(x) = e^x$ at $x = a$.
8. Compute the derivative of $f(x) = 3x^2 e^{5x-6}$.

9. Compute the derivative of $f(x) = e^x - \left(1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!}\right)$.
10. Prove that $e^x > 1$ for $x \geq 0$. Then prove that $e^x > 1 + x$ for $x \geq 0$.
11. Using the previous two exercises, prove (using mathematical induction) that $e^x > 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!} = \displaystyle\sum_{k=0}^{n} \frac{x^k}{k!}$ for $x \geq 0$.
12. Use the preceding exercise to show that $e > 2.7$.
13. Differentiate $\dfrac{e^{kx} + e^{-kx}}{2}$ with respect to x.
14. Compute $\displaystyle\lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}}$.
15. Integrate $5x^4 e^{x^5}$ with respect to x.
16. Compute $\displaystyle\int_0^{\pi/3} \cos(2x) e^{\sin 2x}\, dx$.
17. Compute $\displaystyle\int \frac{e^{1/x^2}}{x^3}\, dx$.
18. Let $F(x) = \displaystyle\int_0^{e^x} e^{t^4}\, dt$. Compute $F'(0)$.
19. If $f(x) = e^{kx}$ what is $f^{(940)}(x)$?

9.4 Other bases

Notice that if $q \in \mathbb{Q}$ and $a > 0$ then $a^q = e^{\ln(a^q)} = e^{q \ln a}$. This equation motivates the following definition.

DEFINITION 9.4.1 For $a > 0$ and $x \in \mathbb{R}$, we define $a^x = e^{x \ln a}$. The function $f(x) = a^x$ is the exponential function with base a. Separately, we define $0^x = 0$ for $x > 0$. □

Notice also that for $x \in \mathbb{R}$ and $a > 0$, $\ln a^x = \ln(e^{x \ln a}) = x \ln a$. Hence, the power rule for the natural logarithm works even when the power is irrational.

We now show that the familiar rules for exponents are valid.

THEOREM 9.4.2 For $x, y \in \mathbb{R}$ and $a, b > 0$:

a $a^{x+y} = a^x a^y$

b $a^{x-y} = a^x / a^y$

c $(a^x)^y = a^{xy}$

d $(ab)^x = a^x b^x$

Proof.

(a) We compute: $a^{x+y} = e^{(x+y)\ln a} = e^{x \ln a + y \ln a} = e^{x \ln a} e^{y \ln a} = a^x a^y$.

The proof of (b) is similar and left as an exercise.

(c) We compute: $(a^x)^y = e^{y \ln(a^x)} = e^{yx \ln a} = a^{xy}$.

(d) We compute: $(ab)^x = e^{x \ln(ab)} = e^{x \ln a + x \ln b} = e^{x \ln a} e^{x \ln b} = a^x b^x$. ■

THEOREM 9.4.3 If $f(x) = a^x$ (with $a > 0$) then $f'(x) = a^x \ln a$.

Proof.

$$f'(x) = \frac{d}{dx}(e^{x \ln a}) = e^{x \ln a} \ln a = a^x \ln a.$$

■

COROLLARY 9.4.4 For $a > 0$ and $a \neq 1$, $\displaystyle\int a^x\, dx = \frac{a^x}{\ln a} + C$. ■

We are now in a position to prove the general power rule.

THEOREM 9.4.5 Power Rule If $f(x) = x^n$, $x > 0$, and n is any real number, then $f'(x) = nx^{n-1}$.

Proof. $f'(x) = \dfrac{d}{dx} x^n = \dfrac{d}{dx} e^{n \ln x} = e^{n \ln x} \dfrac{n}{x} = \dfrac{nx^n}{x} = nx^{n-1}$. ■

The restriction that $x > 0$ is necessary since we have not defined exponential expressions with negative bases and arbitrary real powers.

We now turn to logarithms base a. Note that if $a > 0$ and $a \neq 1$ then $a^x \ln a \neq 0$ for every x. Hence, the function $f(x) = a^x$ is injective.

DEFINITION 9.4.6 If $a > 0$ and $a \neq 1$, the inverse of a^x is called the **logarithmic function base** a. In symbols, we write this function as $\log_a x$. □

We exclude $a = 1$ because $1^x = 1$ is not injective on any domain containing more than one point.

Remark. If $a = 10$ we usually write log instead of $\log_{10}$, and of course $\log_e = \ln$. In more advanced texts, "log" refers to the natural logarithm.

THEOREM 9.4.7 The following hold for $a, x, y > 0$, $a \neq 1$, and $q \in \mathbb{R}$:

a. $\log_a(xy) = \log_a x + \log_a y$

b. $\log_a \dfrac{x}{y} = \log_a x - \log_a y$

c. $\log_a x^q = q \log_a x$.

Proof. (a) Let $u = \log_a x$ and $v = \log_a y$. Then $a^u = x$ and $a^v = y$, and $xy = a^u a^v = a^{u+v}$, so $\log_a(xy) = u + v = \log_a x + \log_a y$.

The other parts are left as exercises. ■

When computing decimal approximations to logs of arbitrary bases with a calculator or a computer algebra system the following result comes in handy.

LEMMA 9.4.8 If $a, b > 0$, $a, b \neq 1$, and $x > 0$, ¡$\log_a x = \dfrac{\log_b x}{\log_b a}$.

Proof. Let $y = \log_a x$, so $a^y = x$. Then $\log_b x = \log_b(a^y) = y \log_b a = \log_a x \log_b a$. ■

Typically this is useful when $b = e$ and $b = 10$, since calculators can typically compute logarithms to those bases.

THEOREM 9.4.9 $\dfrac{d}{dx} \log_a x = \dfrac{1}{x \ln a}$.

Proof. By the preceding lemma, $f(x) = \ln x / \ln a$, and the derivative is then easy. ■

Finally, we express e^x as a limit. When $x = 1$ we get a limit expression for e which is sometimes taken as the definition of e.

THEOREM 9.4.10 If $x \geq 0$, $e^x = \displaystyle\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$.

Proof. If $x = 0$ both expressions are 1.

If $x > 0$ we begin by rewriting the right side as we have before:

$$\left(1 + \frac{x}{n}\right)^n = \left(e^{\ln(1+x/n)}\right)^n = e^{n \ln(1+x/n)}.$$

Now because e^x is continuous,

$$\lim_{n\to\infty} e^{n \ln(1+x/n)} = e^{\lim_{n\to\infty} n \ln(1+x/n)}.$$

So really we need to compute $\displaystyle\lim_{n\to\infty} n \ln(1 + x/n)$, for which we use L'Hôpitals rule:

$$\lim_{n\to\infty} n \ln(1 + x/n) = \lim_{n\to\infty} \frac{\ln(1 + x/n)}{1/n} = \lim_{n\to\infty} \frac{\frac{1}{1+x/n} \frac{-x}{n^2}}{-1/n^2} = \lim_{n\to\infty} \frac{1}{1 + x/n} x = x.$$

■

This same simple fact, $a = e^{\ln a}$, is useful in many similar situations.

EXAMPLE 9.4.11 Let $f(x) = x^x$, $x > 0$. Compute $f'(x)$ and $\displaystyle\lim_{x\to 0^+} f(x)$.

Start with $f(x) = x^x = e^{x \ln x}$. Then

$$f'(x) = e^{x \ln x}\left(x\frac{1}{x} + \ln x\right) = x^x(1 + \ln x).$$

For the limit, we again notice that

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^{\lim_{x\to 0^+} x\ln x}.$$

Then we compute the limit by L'Hôpital's rule again:

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0.$$

Thus $\lim_{x\to 0^+} x^x = e^0 = 1$. □

EXAMPLE 9.4.12 Compute $\int_{\pi/6}^{\pi/3} 2^{\cos x}\sin x\, dx$.

Let $u = \cos x$, so $du = -\sin x\, dx$. Changing the limits, when $x = \pi/6$, $u = \sqrt{3}/2$, and when $x = \pi/3$, $u = 1/2$. Then

$$\int_{\pi/6}^{\pi/3} 2^{\cos x}\sin x\, dx = -\int_{\sqrt{3}/2}^{1/2} 2^u du = -\frac{2^u}{\ln 2}\bigg|_{\sqrt{3}/2}^{1/2} = \frac{-2^{1/2} + 2^{\sqrt{3}/2}}{\ln 2}.$$

□

Exercises 9.4.

1. Prove part (b) of theorem 9.4.2.
2. Sketch the graph of $y = a^x$ in the three cases $a > 1$, $a = 1$, and $0 < a < 1$. What happens to the graph as $a \to 0^+$? What happens to the graph as $a \to \infty$?
3. Sketch the graph of $y = \log_a x$ in the two cases $a > 1$ and $0 < a < 1$. What happens to the graph as $a \to 0^+$? What happens to the graph as $a \to \infty$? (Use the previous exercise together with exercise 22 in section 9.1.)
4. Prove parts (b) and (c) of theorem 9.4.7.
5. Sketch the graph of $y = 3^{6x-1} + 5$.
6. Sketch the graph of $y = -(1/2)^{-3x}$.
7. Sketch the graph of $y = 4\log_2(12x + 6) - 2$.
8. Compute the second derivative of $f(x) = x^x$.
9. Compute $f'(\pi/4)$ when $f(x) = 5^{\sin 3x} + \log_7 x$.
10. Compute the derivative of $f(x) = 3^x - 4x^2 + \sin(3x) - \pi^e$.

11. Compute $\int_1^2 3^x - x^3\, dx$.
12. Compute $\int \sin(2^x)2^x dx$.
13. Find the area of the region given by $\{(x, y) \mid 1 \le x \le 2, 2^x \le y \le 3^x\}$.
14. Find the average of the function $f(x) = 2x \cdot 5^{x^2}$ on the interval $[4, 9]$.
15. Find the volume of the solid obtained by rotating the region $\{(x, y) \mid 2 \le x \le 4, (\log_2 x)/x \le y \le 2^x\}$ about the line $y = -1$.
16. Show that $\log_a x = -\log_{1/a} x$ for any $a > 0, a \ne 1$. Interpret this result geometrically; that is, sketch the graph of $y = \log_a x$ and $y = \log_{1/a} x$ on the same diagram and point out how the graphs are related to each other.

9.5 Inverse Trigonometric Functions

The trigonometric functions frequently arise in problems, and often it is necessary to invert the functions, for example, to find an angle with a specified sine. Of course, there are many angles with the same sine, so the sine function doesn't actually have an inverse that reliably "undoes" the sine function. If you know that $\sin x = 0.5$, you can't reverse this to discover x, that is, you can't solve for x, as there are infinitely many angles with sine 0.5. Nevertheless, it is useful to have something like an inverse to the sine, however imperfect. The usual approach is to pick out some collection of angles that produce all possible values of the sine exactly once. If we "discard" all other angles, the resulting function does have a proper inverse.

The sine takes on all values between -1 and 1 exactly once on the interval $[-\pi/2, \pi/2]$. If we truncate the sine, keeping only the interval $[-\pi/2, \pi/2]$, as shown in figure 9.5.1, then this truncated sine has an inverse function. We call this the inverse sine or the arcsine, and write $y = \arcsin(x)$.

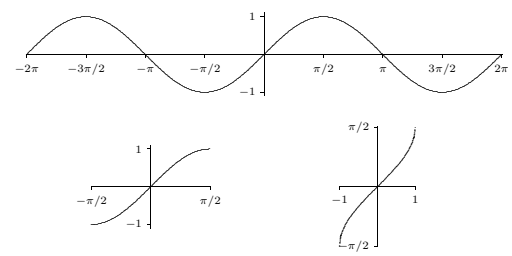

Figure 9.5.1 The sine, the truncated sine, the inverse sine.

Recall that a function and its inverse undo each other in either order, for example, $(\sqrt[3]{x})^3 = x$ and $\sqrt[3]{x^3} = x$. This does not work with the sine and the "inverse sine" because the inverse sine is the inverse of the truncated sine function, not the real sine function. It is true that $\sin(\arcsin(x)) = x$, that is, the sine undoes the arcsine. It is not true that the arcsine undoes the sine, for example, $\sin(5\pi/6) = 1/2$ and $\arcsin(1/2) = \pi/6$, so doing first the sine then the arcsine does not get us back where we started. This is because $5\pi/6$ is not in the domain of the truncated sine. If we start with an angle between $-\pi/2$ and $\pi/2$ then the arcsine does reverse the sine: $\sin(\pi/6) = 1/2$ and $\arcsin(1/2) = \pi/6$.

What is the derivative of the arcsine? Since this is an inverse function, we can discover the derivative by using implicit differentiation. Suppose $y = \arcsin(x)$. Then

$$\sin(y) = \sin(\arcsin(x)) = x.$$

Now taking the derivative of both sides, we get

$$y'\cos y = 1$$
$$y' = \frac{1}{\cos y}$$

As we expect when using implicit differentiation, y appears on the right hand side here. We would certainly prefer to have y' written in terms of x, and as in the case of $\ln x$ we can actually do that here. Since $\sin^2 y + \cos^2 y = 1$, $\cos^2 y = 1 - \sin^2 y = 1 - x^2$. So $\cos y = \pm\sqrt{1 - x^2}$, but which is it—plus or minus? It could in general be either, but this isn't "in general": since $y = \arcsin(x)$ we know that $-\pi/2 \le y \le \pi/2$, and the cosine of an angle in this interval is always positive. Thus $\cos y = \sqrt{1 - x^2}$ and

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1 - x^2}}.$$

Note that this agrees with figure 9.5.1: the graph of the arcsine has positive slope everywhere.

We can do something similar for the cosine. As with the sine, we must first truncate the cosine so that it can be inverted, as shown in figure 9.5.2. Then we use implicit differentiation to find that

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1 - x^2}}.$$

Note that the truncated cosine uses a different interval than the truncated sine, so that if $y = \arccos(x)$ we know that $0 \le y \le \pi$. The computation of the derivative of the arccosine is left as an exercise.

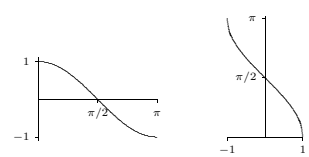

Figure 9.5.2 The truncated cosine, the inverse cosine.

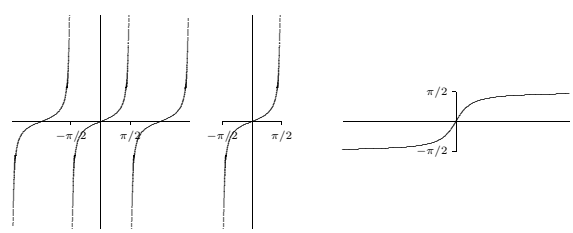

Figure 9.5.3 The tangent, the truncated tangent, the inverse tangent.

Finally we look at the tangent; the other trigonometric functions also have "partial inverses" but the sine, cosine and tangent are enough for most purposes. The tangent, truncated tangent and inverse tangent are shown in figure 9.5.3; the derivative of the arctangent is left as an exercise.

Exercises 9.5.

1. Show that the derivative of $\arccos x$ is $-\frac{1}{\sqrt{1 - x^2}}$.
2. Show that the derivative of $\arctan x$ is $\frac{1}{1 + x^2}$.
3. The inverse of cot is usually defined so that the range of arccot is $(0, \pi)$. Sketch the graph of $y = \operatorname{arccot} x$. In the process you will make it clear what the domain of arccot is. Find the derivative of the arccotangent. ⇒
4. Show that $\operatorname{arccot} x + \arctan x = \pi/2$.
5. Find the derivative of $\arcsin(x^2)$. ⇒
6. Find the derivative of $\arctan(e^x)$. ⇒
7. Find the derivative of $\arccos(\sin x^3)$ ⇒
8. Find the derivative of $\ln((\arcsin x)^2)$ ⇒
9. Find the derivative of $\arccos e^x$ ⇒

10. Find the derivative of $\arcsin x + \arccos x \Rightarrow$

11. Find the derivative of $\log_5(\arctan(x^x)) \Rightarrow$

12. Compute $\displaystyle\int \frac{\operatorname{arcsec} x}{x\sqrt{x^2-1}}\,dx$

13. Compute $\displaystyle\int \frac{\ln(\arcsin x)}{\arcsin x\sqrt{1-x^2}}\,dx$

14. Compute $\displaystyle\int_0^{\sqrt{2}} \left(1 + x^3 + xe^{x^2} - \frac{1}{1+x^2}\right) dx$

15. Compute $\displaystyle\int \frac{dx}{1+9x^2}$

16. Find the equation of the tangent line to $f(x) = \operatorname{arccsc} x$ at $x = \pi/6$.

17. Let

$$A = \left\{(x,y) \mid \frac{1}{2} \le x \le \frac{\sqrt{3}}{2}, 0 \le y \le \frac{1}{(1-x^2)^{1/4}}\right\}.$$

Sketch the region A. Let S be the solid obtained from rotating A about the x-axis. Compute the volume of S.

9.6 Hyperbolic Functions

The hyperbolic functions appear with some frequency in applications, and are quite similar in many respects to the trigonometric functions. This is a bit surprising given our initial definitions.

DEFINITION 9.6.1 The **hyperbolic cosine** is the function

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

and the **hyperbolic sine** is the function

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

□

Notice that cosh is even (that is, $\cosh(-x) = \cosh(x)$) while sinh is odd ($\sinh(-x) = -\sinh(x)$), and $\cosh x + \sinh x = e^x$. Also, for all x, $\cosh x > 0$, while $\sinh x = 0$ if and only if $e^x - e^{-x} = 0$, which is true precisely when $x = 0$.

LEMMA 9.6.2 The range of $\cosh x$ is $[1, \infty)$.

Proof. Let $y = \cosh x$. We solve for x:

$$\begin{aligned} y &= \frac{e^x + e^{-x}}{2} \\ 2y &= e^x + e^{-x} \\ 2ye^x &= e^{2x} + 1 \\ 0 &= e^{2x} - 2ye^x + 1 \\ e^x &= \frac{2y \pm \sqrt{4y^2-4}}{2} \\ e^x &= y \pm \sqrt{y^2-1} \end{aligned}$$

From the last equation, we see $y^2 \ge 1$, and since $y \ge 0$, it follows that $y \ge 1$.

Now suppose $y \ge 1$, so $y \pm \sqrt{y^2-1} > 0$. Then $x = \ln(y \pm \sqrt{y^2-1})$ is a real number, and $y = \cosh x$, so y is in the range of $\cosh(x)$. ■

DEFINITION 9.6.3 The other hyperbolic functions are

$$\begin{aligned} \tanh x &= \frac{\sinh x}{\cosh x} \\ \coth x &= \frac{\cosh x}{\sinh x} \\ \operatorname{sech} x &= \frac{1}{\cosh x} \\ \operatorname{csch} x &= \frac{1}{\sinh x} \end{aligned}$$

The domain of coth and csch is $x \ne 0$ while the domain of the other hyperbolic functions is all real numbers. Graphs are shown in figure 9.6.1 □

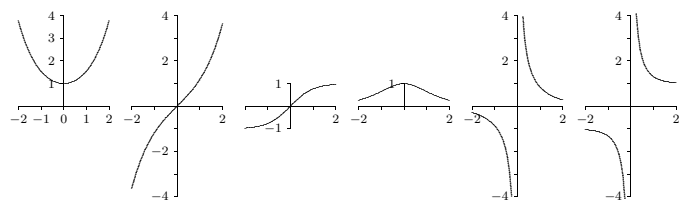

Figure 9.6.1 The hyperbolic functions: cosh, sinh, tanh, sech, csch, coth.

Certainly the hyperbolic functions do not closely resemble the trigonometric functions graphically. But they do have analogous properties, beginning with the following identity.

THEOREM 9.6.4 For all x in $\mathbb{R}$, $\cosh^2 x - \sinh^2 x = 1$.

Proof. The proof is a straightforward computation:

$$\cosh^2 x - \sinh^2 x = \frac{(e^x+e^{-x})^2}{4} - \frac{(e^x-e^{-x})^2}{4} = \frac{e^{2x}+2+e^{-2x}-e^{2x}+2-e^{-2x}}{4} = \frac{4}{4} = 1.$$

■

This immediately gives two additional identities:

$$1 - \tanh^2 x = \operatorname{sech}^2 x \qquad \text{and} \qquad \coth^2 x - 1 = \operatorname{csch}^2 x.$$

The identity of the theorem also helps to provide a geometric motivation. Recall that the graph of $x^2 - y^2 = 1$ is a hyperbola with asymptotes $x = \pm y$ whose x-intercepts are ± 1. If (x, y) is a point on the right half of the hyperbola, and if we let $x = \cosh t$, then $y = \pm\sqrt{x^2-1} = \pm\sqrt{\cosh^2 t - 1} = \pm \sinh t$. So for some suitable t, $\cosh t$ and $\sinh t$ are the coordinates of a typical point on the hyperbola. In fact, it turns out that t is twice the area shown in the first graph of figure 9.6.2. Even this is analogous to trigonometry; $\cos t$ and $\sin t$ are the coordinates of a typical point on the unit circle, and t is twice the area shown in the second graph of figure 9.6.2.

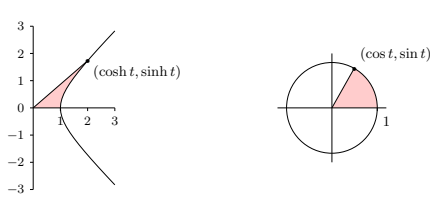

Figure 9.6.2 Geometric definitions of sin, cos, sinh, cosh: t is twice the shaded area in each figure.

Given the definitions of the hyperbolic functions, finding their derivatives is straightforward. Here again we see similarities to the trigonometric functions.

THEOREM 9.6.5 $\displaystyle\frac{d}{dx}\cosh x = \sinh x$ and $\displaystyle\frac{d}{dx}\sinh x = \cosh x$.

Proof. $\displaystyle\frac{d}{dx}\cosh x = \frac{d}{dx}\frac{e^x+e^{-x}}{2} = \frac{e^x-e^{-x}}{2} = \sinh x$, and $\displaystyle\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x-e^{-x}}{2} = \frac{e^x+e^{-x}}{2} = \cosh x$. ■

Of course, this immediately gives us two anti-derivatives as well.

Since $\cosh x > 0$, $\sinh x$ is increasing and hence injective, so $\sinh x$ has an inverse, $\operatorname{arcsinh} x$. Also, $\sinh x > 0$ when $x > 0$, so $\cosh x$ is injective on $[0, \infty)$ and has a (partial) inverse, $\operatorname{arccosh} x$. The other hyperbolic functions have inverses as well, though $\operatorname{arcsech} x$ is only a partial inverse. We may compute the derivatives of these functions as we have other inverse functions.

THEOREM 9.6.6 $\displaystyle\frac{d}{dx}\operatorname{arcsinh} x = \frac{1}{\sqrt{1+x^2}}$.

Proof. Let $y = \operatorname{arcsinh} x$, so $\sinh y = x$. Then $\displaystyle\frac{d}{dx}\sinh y = \cosh(y)\cdot y' = 1$, and so $\displaystyle y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$. ■

The other derivatives are left to the exercises.

Exercises 9.6.

1. Show that the range of $\sinh x$ is all real numbers. (Hint: show that if $y = \sinh x$ then $x = \ln(y + \sqrt{y^2+1})$.)

2. Compute the following limits:

a. $\displaystyle\lim_{x\to\infty} \cosh x$

b. $\displaystyle\lim_{x\to\infty} \sinh x$

c. $\displaystyle\lim_{x\to\infty} \tanh x$

d. $\displaystyle\lim_{x\to\infty} (\cosh x - \sinh x)$

$\Rightarrow$

3. Show that the range of $\tanh x$ is $(-1, 1)$. What are the ranges of coth, sech, and csch? (Use the fact that they are reciprocal functions.)

4. Prove that for every $x, y \in \mathbb{R}$, $\sinh(x+y) = \sinh x\cosh y + \cosh x \sinh y$. Obtain a similar identity for $\sinh(x-y)$.

5. Prove that for every $x, y \in \mathbb{R}$, $\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$. Obtain a similar identity for $\cosh(x-y)$.

6. Use exercises 4 and 5 to show that $\sinh(2x) = 2\sinh x\cosh x$ and $\cosh(2x) = \cosh^2 x + \sinh^2 x$ for every x. Conclude also that $(\cosh(2x)-1)/2 = \sinh^2 x$ and $(\cosh(2x)+1)/2 = \cosh^2 x$.

7. Show that $\displaystyle\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$. Compute the derivatives of the remaining hyperbolic functions as well.

8. What are the domains of the six inverse hyperbolic functions?

9. Sketch the graphs of all six inverse hyperbolic functions.

10. Compute $\int \sqrt{x^2+1}\,dx$. (Hint: make the substitution $x = \sinh(u)$ and then use exercise 6.)

11. Fix $t > 0$. The shaded region in the left-hand graph in figure 9.6.2 is bounded by $y = x \tanh t$, $y = 0$, and $x^2 - y^2 = 1$. Prove that twice the area of this region is t.