7

Integration

7.1 Two examples

Up to now we have been concerned with extracting information about how a function changes from the function itself. Given knowledge about an object's position, for example, we want to know the object's speed. Given information about the height of a curve we want to know its slope. We now consider problems that are, whether obviously or not, the reverse of such problems.

EXAMPLE 7.1.1 An object moves in a straight line so that its speed at time t is given by $v(t) = 3t$ in, say, cm/sec. If the object is at position 10 on the straight line when $t = 0$, where is the object at any time t?

There are two reasonable ways to approach this problem. If $s(t)$ is the position of the object at time t, we know that $s'(t) = v(t)$. Because of our knowledge of derivatives, we know therefore that $s(t) = 3t^2/2 + k$, and because $s(0) = 10$ we easily discover that $k = 10$, so $s(t) = 3t^2/2 + 10$. For example, at $t = 1$ the object is at position $3/2 + 10 = 11.5$. This is certainly the easiest way to deal with this problem. Not all similar problems are so easy, as we will see; the second approach to the problem is more difficult but also more general.

We start by considering how we might approximate a solution. We know that at $t = 0$ the object is at position 10. How might we approximate its position at, say, $t = 1$? We know that the speed of the object at time $t = 0$ is 0; if its speed were constant then in the first second the object would not move and its position would still be 10 when $t = 1$. In fact, the object will not be too far from 10 at $t = 1$, but certainly we can do better. Let's look at the times 0.1, 0.2, 0.3, …, 1.0, and try approximating the location of the object at each, by supposing that during each tenth of a second the object is going at a constant speed. Since the object initially has speed 0, we again suppose it maintains this speed, but only for a tenth of second; during that time the object would not move. During the tenth of a second from $t = 0.1$ to $t = 0.2$, we suppose that the object is traveling at 0.3 cm/sec, namely, its actual speed at $t = 0.1$. In this case the object would travel $(0.3)(0.1) = 0.03$ centimeters: 0.3 cm/sec times 0.1 seconds. Similarly, between $t = 0.2$ and $t = 0.3$ the object would travel $(0.6)(0.1) = 0.06$ centimeters. Continuing, we get as an approximation that the object travels

$$(0.0)(0.1) + (0.3)(0.1) + (0.6)(0.1) + \cdots + (2.7)(0.1) = 1.35$$

centimeters, ending up at position 11.35. This is a better approximation than 10, certainly, but is still just an approximation. (We know in fact that the object ends up at position 11.5, because we've already done the problem using the first approach.) Presumably, we will get a better approximation if we divide the time into one hundred intervals of a hundredth of a second each, and repeat the process:

$$(0.0)(0.01) + (0.03)(0.01) + (0.06)(0.01) + \cdots + (2.97)(0.01) = 1.485.$$

We thus approximate the position as 11.485. Since we know the exact answer, we can see that this is much closer, but if we did not already know the answer, we wouldn't really know how close.

We can keep this up, but we'll never really know the exact answer if we simply compute more and more examples. Let's instead look at a "typical" approximation. Suppose we divide the time into n equal intervals, and imagine that on each of these the object travels at a constant speed. Over the first time interval we approximate the distance traveled as $(0.0)(1/n) = 0$, as before. During the second time interval, from $t = 1/n$ to $t = 2/n$, the object travels approximately $3(1/n)(1/n) = 3/n^2$ centimeters. During time interval number i, the object travels approximately $(3(i-1)/n)(1/n) = 3(i-1)/n^2$ centimeters, that is, its speed at time $(i-1)/n$, $3(i-1)/n$, times the length of time interval number i, $1/n$. Adding these up as before, we approximate the distance traveled as

$$(0)\frac{1}{n} + 3\frac{1}{n^2} + 3(2)\frac{1}{n^2} + 3(3)\frac{1}{n^2} + \cdots + 3(n-1)\frac{1}{n^2}$$

centimeters. What can we say about this? At first it looks rather less useful than the concrete calculations we've already done. But in fact a bit of algebra reveals it to be much more useful. We can factor out a 3 and $1/n^2$ to get

$$\frac{3}{n^2}(0+1+2+3+\cdots+(n-1)),$$

that is, $3/n^2$ times the sum of the first $n-1$ positive integers. Now we make use of a fact you may have run across before:

$$1+2+3+\cdots+k=\frac{k(k+1)}{2}.$$

In our case we're interested in $k=n-1$, so

$$1+2+3+\cdots+(n-1)=\frac{(n-1)(n)}{2}=\frac{n^2-n}{2}.$$

This simplifies the approximate distance traveled to

$$\frac{3}{n^2}\frac{n^2-n}{2}=\frac{3}{2}\frac{n^2-n}{n^2}=\frac{3}{2}\left(\frac{n^2}{n^2}-\frac{n}{n^2}\right)=\frac{3}{2}\left(1-\frac{1}{n}\right).$$

Now this is quite easy to understand: as n gets larger and larger this approximation gets closer and closer to $(3/2)(1-0)=3/2$, so that $3/2$ is the exact distance traveled during one second, and the final position is 11.5.

So for $t=1$, at least, this rather cumbersome approach gives the same answer as the first approach. But really there's nothing special about $t=1$; let's just call it t instead. In this case the approximate distance traveled during time interval number i is $3(i-1)(t/n)(t/n)=3(i-1)t^2/n^2$, that is, speed $3(i-1)(t/n)$ times time t/n, and the total distance traveled is approximately

$$(0)\frac{t}{n}+3(1)\frac{t^2}{n^2}+3(2)\frac{t^2}{n^2}+3(3)\frac{t^2}{n^2}+\cdots+3(n-1)\frac{t^2}{n^2}.$$

As before we can simplify this to

$$\frac{3t^2}{n^2}(0+1+2+\cdots+(n-1))=\frac{3t^2}{n^2}\frac{n^2-n}{2}=\frac{3}{2}t^2\left(1-\frac{1}{n}\right).$$

In the limit, as n gets larger, this gets closer and closer to $(3/2)t^2$ and the approximated position of the object gets closer and closer to $(3/2)t^2+10$, so the actual position is $(3/2)t^2+10$, exactly the answer given by the first approach to the problem. □

EXAMPLE 7.1.2 Find the area under the curve $y=3x$ between $x=0$ and any positive value x. There is here no obvious analogue to the first approach in the previous example, but the second approach works fine. (Because the function $y = 3x$ is so simple, there is another approach that works here, but it is even more limited in potential application than is approach number one.) How might we approximate the desired area? We know how to compute areas of rectangles, so we approximate the area by rectangles. Jumping straight to the general case, suppose we divide the interval between 0 and x into n equal subintervals, and use a rectangle above each subinterval to approximate the area under the curve. There are many ways we might do this, but let's use the height of the curve at the left endpoint of the subinterval as the height of the rectangle, as in figure 7.1.1. The height of rectangle number i is then $3(i-1)(x/n)$, the width is x/n, and the area is $3(i-1)(x^2/n^2)$. The total area of the rectangles is

$$(0)\frac{x}{n} + 3(1)\frac{x^2}{n^2} + 3(2)\frac{x^2}{n^2} + 3(3)\frac{x^2}{n^2} + \cdots + 3(n-1)\frac{x^2}{n^2}.$$

By factoring out $3x^2/n^2$ this simplifies to

$$\frac{3x^2}{n^2}(0+1+2+\cdots+(n-1)) = \frac{3x^2}{n^2}\frac{n^2-n}{2} = \frac{3}{2}x^2\left(1-\frac{1}{n}\right).$$

As n gets larger this gets closer and closer to $3x^2/2$, which must therefore be the true area under the curve. □

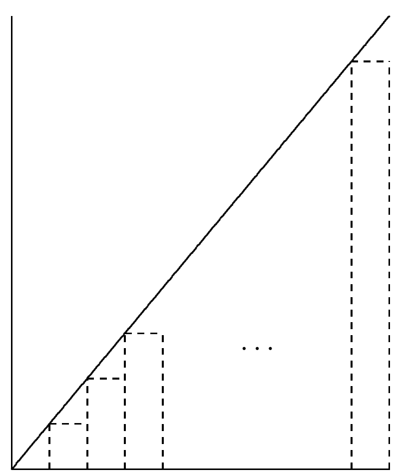

Figure 7.1.1 Approximating the area under $y = 3x$ with rectangles. Drag the slider to change the number of rectangles.

What you will have noticed, of course, is that while the problem in the second example appears to be much different than the problem in the first example, and while the easy approach to problem one does not appear to apply to problem two, the "approximation" approach works in both, and moreover the *calculations are identical.* As we will see, there are many, many problems that appear much different on the surface but that turn out to be the same as these problems, in the sense that when we try to approximate solutions we end up with mathematics that looks like the two examples, though of course the function involved will not always be so simple.

Even better, we now see that while the second problem did not appear to be amenable to approach one, it can in fact be solved in the same way. The reasoning is this: we know that problem one can be solved easily by finding a function whose derivative is $3t$. We also know that mathematically the two problems are the same, because both can be solved by taking a limit of a sum, and the sums are identical. Therefore, we don't really need to compute the limit of either sum because we know that we will get the same answer by computing a function with the derivative $3t$ or, which is the same thing, $3x$.

It's true that the first problem had the added complication of the "10", and we certainly need to be able to deal with such minor variations, but that turns out to be quite simple. The lesson then is this: whenever we can solve a problem by taking the limit of a sum of a certain form, we can instead of computing the (often nasty) limit find a new function with a certain derivative.

Exercises 7.1.

1. Suppose an object moves in a straight line so that its speed at time t is given by $v(t) = 2t+2$, and that at $t = 1$ the object is at position 5. Find the position of the object at $t = 2$. ⇒

2. Suppose an object moves in a straight line so that its speed at time t is given by $v(t) = t^2+2$, and that at $t = 0$ the object is at position 5. Find the position of the object at $t = 2$. ⇒

3. By a method similar to that in example 7.1.2, find the area under $y = 2x$ between $x = 0$ and any positive value for x. ⇒

4. By a method similar to that in example 7.1.2, find the area under $y = 4x$ between $x = 0$ and any positive value for x. ⇒

5. By a method similar to that in example 7.1.2, find the area under $y = 4x$ between $x = 2$ and any positive value for x bigger than 2. ⇒

6. By a method similar to that in example 7.1.2, find the area under $y = 4x$ between any two positive values for x, say $a < b$. ⇒

7. Let $f(x) = x^2 + 3x + 2$. Approximate the area under the curve between $x = 0$ and $x = 2$ using 4 rectangles and also using 8 rectangles. ⇒

8. Let $f(x) = x^2 - 2x + 3$. Approximate the area under the curve between $x = 1$ and $x = 3$ using 4 rectangles. ⇒

7.2 The Fundamental Theorem of Calculus

Let's recast the first example from the previous section. Suppose that the speed of the object is $3t$ at time t. How far does the object travel between time $t = a$ and time $t = b$? We are no longer assuming that we know where the object is at time $t = 0$ or at any other time. It is certainly true that it is *somewhere,* so let's suppose that at $t = 0$ the position is k. Then just as in the example, we know that the position of the object at any time is $3t^2/2+k$. This means that at time $t = a$ the position is $3a^2/2 + k$ and at time $t = b$ the position is $3b^2/2 + k$. Therefore the change in position is $3b^2/2 + k - (3a^2/2 + k) = 3b^2/2 - 3a^2/2$. Notice that the k drops out; this means that it doesn't matter that we don't know k, it doesn't even matter if we use the wrong k, we get the correct answer. In other words, to find the change in position between time a and time b we can use *any* antiderivative of the speed function $3t$—it need not be the one antiderivative that actually gives the location of the object.

What about the second approach to this problem, in the new form? We now want to approximate the change in position between time a and time b. We take the interval of time between a and b, divide it into n subintervals, and approximate the distance traveled during each. The starting time of subinterval number i is now $a + (i - 1)(b - a)/n$, which we abbreviate as t_{i-1}, so that $t_0 = a$, $t_1 = a + (b - a)/n$, and so on. The speed of the object is $f(t) = 3t$, and each subinterval is $(b - a)/n = \Delta t$ seconds long. The distance traveled during subinterval number i is approximately $f(t_{i-1})\Delta t$, and the total change in distance is approximately

$$f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-1})\Delta t.$$

The exact change in position is the limit of this sum as n goes to infinity. We abbreviate this sum using **sigma notation**:

$$\sum_{i=0}^{n-1} f(t_i)\Delta t = f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-1})\Delta t.$$

The notation on the left side of the equal sign uses a large capital sigma, a Greek letter, and the left side is an abbreviation for the right side. The answer we seek is

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\Delta t.$$

Since this must be the same as the answer we have already obtained, we know that

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\Delta t = \frac{3b^2}{2} - \frac{3a^2}{2}.$$

The significance of $3t^2/2$, into which we substitute $t = b$ and $t = a$, is of course that it is a function whose derivative is $f(t)$. As we have discussed, by the time we know that we want to compute

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\Delta t,$$

it no longer matters what $f(t)$ stands for—it could be a speed, or the height of a curve, or something else entirely. We know that the limit can be computed by finding any function with derivative $f(t)$, substituting a and b, and subtracting. We summarize this in a theorem. First, we introduce some new notation and terms.

We write

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\Delta t$$

if the limit exists. That is, the left hand side means, or is an abbreviation for, the right hand side. The symbol $\int$ is called an **integral sign**, and the whole expression is read as "the integral of $f(t)$ from a to b." What we have learned is that this integral can be computed by finding a function, say $F(t)$, with the property that $F'(t) = f(t)$, and then computing $F(b) - F(a)$. The function $F(t)$ is called an **antiderivative** of $f(t)$. Now the theorem:

THEOREM 7.2.1 Fundamental Theorem of Calculus Suppose that $f(x)$ is continuous on the interval $[a, b]$. If $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

□

Let's rewrite this slightly:

$$\int_a^x f(t)\,dt = F(x) - F(a).$$

We've replaced the variable x by t and b by x. These are just different names for quantities, so the substitution doesn't change the meaning. It does make it easier to think of the two sides of the equation as functions. The expression

$$\int_a^x f(t)\,dt$$

is a function: plug in a value for x, get out some other value. The expression $F(x) - F(a)$ is of course also a function, and it has a nice property:

$$\frac{d}{dx}(F(x) - F(a)) = F'(x) = f(x),$$

since $F(a)$ is a constant and has derivative zero. In other words, by shifting our point of view slightly, we see that the odd looking function

$$G(x) = \int_a^x f(t)\,dt$$

has a derivative, and that in fact $G'(x) = f(x)$. This is really just a restatement of the Fundamental Theorem of Calculus, and indeed is often called the Fundamental Theorem of Calculus. To avoid confusion, some people call the two versions of the theorem "The Fundamental Theorem of Calculus, part I" and "The Fundamental Theorem of Calculus, part II", although unfortunately there is no universal agreement as to which is part I and which part II. Since it really is the same theorem, differently stated, some people simply call them both "The Fundamental Theorem of Calculus."

THEOREM 7.2.2 Fundamental Theorem of Calculus Suppose that $f(x)$ is continuous on the interval $[a, b]$ and let

$$G(x) = \int_a^x f(t)\,dt.$$

Then $G'(x) = f(x)$. □

We have not really proved the Fundamental Theorem. In a nutshell, we gave the following argument to justify it: Suppose we want to know the value of

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\Delta t.$$

We can interpret the right hand side as the distance traveled by an object whose speed is given by $f(t)$. We know another way to compute the answer to such a problem: find the position of the object by finding an antiderivative of $f(t)$, then substitute $t = a$ and $t = b$ and subtract to find the distance traveled. This must be the answer to the original problem as well, even if $f(t)$ does not represent a speed.

What's wrong with this? In some sense, nothing. As a practical matter it is a very convincing argument, because our understanding of the relationship between speed and distance seems to be quite solid. From the point of view of mathematics, however, it is unsatisfactory to justify a purely mathematical relationship by appealing to our understanding of the physical universe, which could, however unlikely it is in this case, be wrong.

A complete proof is a bit too involved to include here, but we will indicate how it goes. First, if we can prove the second version of the Fundamental Theorem, theorem 7.2.2, then we can prove the first version from that:

Proof of Theorem 7.2.1. We know from theorem 7.2.2 that

$$G(x) = \int_a^x f(t)\,dt$$

is an antiderivative of $f(x)$, and therefore any antiderivative $F(x)$ of $f(x)$ is of the form $F(x) = G(x) + k$. Then

$$\begin{aligned} F(b) - F(a) = G(b) + k - (G(a) + k) &= G(b) - G(a) \\ &= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt. \end{aligned}$$

It is not hard to see that $\int_a^a f(t)\,dt = 0$, so this means that

$$F(b) - F(a) = \int_a^b f(t)\,dt,$$

which is exactly what theorem 7.2.1 says. ■

So the real job is to prove theorem 7.2.2. We will sketch the proof, using some facts that we do not prove. First, the following identity is true of integrals:

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt.$$

This can be proved directly from the definition of the integral, that is, using the limits of sums. It is quite easy to see that it must be true by thinking of either of the two applications of integrals that we have seen. It turns out that the identity is true no matter what c is, but it is easiest to think about the meaning when $a \le c \le b$.

First, if $f(t)$ represents a speed, then we know that the three integrals represent the distance traveled between time a and time b; the distance traveled between time a and time c; and the distance traveled between time c and time b. Clearly the sum of the latter two is equal to the first of these.

Second, if $f(t)$ represents the height of a curve, the three integrals represent the area under the curve between a and b; the area under the curve between a and c; and the area under the curve between c and b. Again it is clear from the geometry that the first is equal to the sum of the second and third.

Proof sketch for Theorem 7.2.2. We want to compute $G'(x)$, so we start with the definition of the derivative in terms of a limit:

$$\begin{aligned} G'(x) &= \lim_{\Delta x \to 0} \frac{G(x+\Delta x) - G(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(\int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt \right) \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(\int_a^x f(t)\,dt + \int_x^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt \right) \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} f(t)\,dt. \end{aligned}$$

Now we need to know something about

$$\int_x^{x+\Delta x} f(t)\,dt$$

when Δx is small; in fact, it is very close to $\Delta x f(x)$, but we will not prove this. Once again, it is easy to believe this is true by thinking of our two applications: The integral

$$\int_x^{x+\Delta x} f(t)\,dt$$

can be interpreted as the distance traveled by an object over a very short interval of time. Over a sufficiently short period of time, the speed of the object will not change very much, so the distance traveled will be approximately the length of time multiplied by the speed at the beginning of the interval, namely, $\Delta x f(x)$. Alternately, the integral may be interpreted as the area under the curve between x and $x + \Delta x$. When Δx is very small, this will be very close to the area of the rectangle with base Δx and height $f(x)$; again this is $\Delta x f(x)$. If we accept this, we may proceed:

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} f(t)\,dt = \lim_{\Delta x \to 0} \frac{\Delta x f(x)}{\Delta x} = f(x),$$

which is what we wanted to show. ■

It is still true that we are depending on an interpretation of the integral to justify the argument, but we have isolated this part of the argument into two facts that are not too hard to prove. Once the last reference to interpretation has been removed from the proofs of these facts, we will have a real proof of the Fundamental Theorem.

Now we know that to solve certain kinds of problems, those that lead to a sum of a certain form, we "merely" find an antiderivative and substitute two values and subtract. Unfortunately, finding antiderivatives can be quite difficult. While there are a small number of rules that allow us to compute the derivative of any common function, there are no such rules for antiderivatives. There are some techniques that frequently prove useful, but we will never be able to reduce the problem to a completely mechanical process.

Because of the close relationship between an integral and an antiderivative, the integral sign is also used to mean "antiderivative". You can tell which is intended by whether the limits of integration are included:

$$\int_1^2 x^2\,dx$$

is an ordinary integral, also called a **definite integral**, because it has a definite value, namely

$$\int_1^2 x^2\,dx = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}.$$

We use

$$\int x^2\,dx$$

to denote the antiderivative of x^2, also called an **indefinite integral**. So this is evaluated as

$$\int x^2\,dx = \frac{x^3}{3} + C.$$

It is customary to include the constant C to indicate that there are really an infinite number of antiderivatives. We do not need this C to compute definite integrals, but in other circumstances we will need to remember that the C is there, so it is best to get into the habit of writing the C. When we compute a definite integral, we first find an antiderivative and then substitute. It is convenient to first display the antiderivative and then do the substitution; we need a notation indicating that the substitution is yet to be done. A typical solution would look like this:

$$\int_1^2 x^2\,dx = \left.\frac{x^3}{3}\right|_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}.$$

The vertical line with subscript and superscript is used to indicate the operation "substitute and subtract" that is needed to finish the evaluation.

Exercises 7.2.

Find the antiderivatives of the functions:

1. $8\sqrt{x} \Rightarrow$

2. $3t^2 + 1 \Rightarrow$

3. $4/\sqrt{x} \Rightarrow$

4. $2/z^2 \Rightarrow$

5. $(5x+1)^2 \Rightarrow$

6. $(x-6)^2 \Rightarrow$

7. $x^{3/2} \Rightarrow$

8. $\dfrac{2}{x\sqrt{x}} \Rightarrow$

9. $|2t-4| \Rightarrow$

Compute the values of the integrals:

10. $\displaystyle\int_1^4 t^2 + 3t\,dt \Rightarrow$

11. $\displaystyle\int_0^\pi \sin t\,dt \Rightarrow$

12. $\displaystyle\int_0^3 x^3\,dx \Rightarrow$

13. $\displaystyle\int_1^2 x^5\,dx \Rightarrow$

14. Find the derivative of $G(x) = \displaystyle\int_1^x t^2 - 3t\,dt \Rightarrow$

15. Find the derivative of $G(x) = \displaystyle\int_1^{x^2} t^2 - 3t\,dt \Rightarrow$

16. Find the derivative of $G(x) = \displaystyle\int_1^x \tan(t^2)\,dt \Rightarrow$

17. Find the derivative of $G(x) = \displaystyle\int_1^{x^2} \tan(t^2)\,dt \Rightarrow$

7.3 Some Properties of Integrals

Suppose an object moves so that its speed, or more properly velocity, is given by $v(t) = -t^2 + 5t$, as shown in figure 7.3.1. Let's examine the motion of this object carefully. We know that the velocity is the derivative of position, so position is given by $s(t) = -t^3/3 + 5t^2/2 + C$. Let's suppose that at time $t = 0$ the object is at position 0, so $s(t) = -t^3/3 + 5t^2/2$; this function is also pictured in figure 7.3.1.

Between $t = 0$ and $t = 5$ the velocity is positive, so the object moves away from the starting point, until it is a bit past position 20. Then the velocity becomes negative and the object moves back toward its starting point. The position of the object at $t = 5$ is exactly $s(5) = 125/6$, and at $t = 6$ it is $s(6) = 18$. The total distance traveled by the object is therefore $125/6 + (125/6 - 18) = 71/3 \approx 23.7$.

As we have seen, we can also compute distance traveled with an integral; let's try it.

$$\int_0^6 v(t)\,dt = \int_0^6 -t^2 + 5t\,dt = \left.\frac{-t^3}{3} + \frac{5}{2}t^2\right|_0^6 = 18.$$

What went wrong? Well, nothing really, except that it's not really true after all that "we can also compute distance traveled with an integral". Instead, as you might guess from this

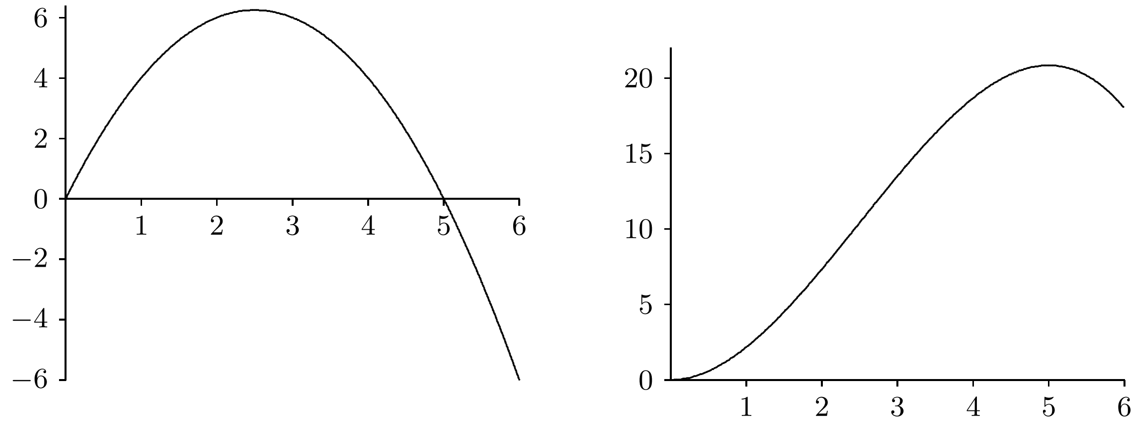

Figure 7.3.1 The velocity of an object and its position.

example, the integral actually computes the *net* distance traveled, that is, the difference between the starting and ending point.

As we have already seen,

$$\int_0^6 v(t)\,dt = \int_0^5 v(t)\,dt + \int_5^6 v(t)\,dt.$$

Computing the two integrals on the right (do it!) gives $125/6$ and $-17/6$, and the sum of these is indeed 18. But what does that negative sign mean? It means precisely what you might think: it means that the object moves backwards. To get the total distance traveled we can add $125/6 + 17/6 = 71/3$, the same answer we got before.

Remember that we can also interpret an integral as measuring an area, but now we see that this too is a little more complicated than we have suspected. The area under the curve $v(t)$ from 0 to 5 is given by

$$\int_0^5 v(t)\,dt = \frac{125}{6},$$

and the "area" from 5 to 6 is

$$\int_5^6 v(t)\,dt = -\frac{17}{6}.$$

In other words, the area between the x-axis and the curve, but under the x-axis, "counts as negative area". So the integral

$$\int_0^6 v(t)\,dt = 18$$

measures "net area", the area above the axis minus the (positive) area below the axis.

If we recall that the integral is the limit of a certain kind of sum, this behavior is not surprising. Recall the sort of sum involved:

$$\sum_{i=0}^{n-1} v(t_i)\Delta t.$$

In each term $v(t)\Delta t$ the Δt is positive, but if $v(t_i)$ is negative then the term is negative. If over an entire interval, like 5 to 6, the function is always negative, then the entire sum is negative. In terms of area, $v(t)\Delta t$ is then a negative height times a positive width, giving a negative rectangle "area".

So now we see that when evaluating

$$\int_5^6 v(t)\,dt = -\frac{17}{6}$$

by finding an antiderivative, substituting, and subtracting, we get a surprising answer, but one that turns out to make sense.

Let's now try something a bit different:

$$\int_6^5 v(t)\,dt = \left.\frac{-t^3}{3} + \frac{5}{2}t^2\right|_6^5 = \frac{-5^3}{3} + \frac{5}{2}5^2 - \frac{-6^3}{3} - \frac{5}{2}6^2 = \frac{17}{6}.$$

Here we simply interchanged the limits 5 and 6, so of course when we substitute and subtract we're subtracting in the opposite order and we end up multiplying the answer by -1. This too makes sense in terms of the underlying sum, though it takes a bit more thought. Recall that in the sum

$$\sum_{i=0}^{n-1} v(t_i)\Delta t,$$

the Δt is the "length" of each little subinterval, but more precisely we could say that $\Delta t = t_{i+1} - t_i$, the difference between two endpoints of a subinterval. We have until now assumed that we were working left to right, but could as well number the subintervals from right to left, so that $t_0 = b$ and $t_n = a$. Then $\Delta t = t_{i+1} - t_i$ is negative and in

$$\int_6^5 v(t)\,dt = \sum_{i=0}^{n-1} v(t_i)\Delta t,$$

the values $v(t_i)$ are negative but also Δt is negative, so all terms are positive again. On the other hand, in

$$\int_5^0 v(t)\,dt = \sum_{i=0}^{n-1} v(t_i)\Delta t,$$

the values $v(t_i)$ are positive but Δt is negative,and we get a negative result:

$$\int_5^0 v(t)\,dt = \left.\frac{-t^3}{3} + \frac{5}{2}t^2\right|_5^0 = 0 - \frac{-5^3}{3} - \frac{5}{2}5^2 = -\frac{125}{6}.$$

Finally we note one simple property of integrals:

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

This is easy to understand once you recall that $(F(x)+G(x))' = F'(x)+G'(x)$. Hence, if $F'(x) = f(x)$ and $G'(x) = g(x)$, then

$$\begin{aligned}\int_a^b f(x) + g(x)\,dx &= \left.(F(x)+G(x))\right|_a^b \\ &= F(b) + G(b) - F(a) - G(a) \\ &= F(b) - F(a) + G(b) - G(a) \\ &= \left.F(x)\right|_a^b + \left.G(x)\right|_a^b \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.\end{aligned}$$

In summary, we will frequently use these properties of integrals:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

and if $a < b$ and $f(x) \le 0$ on $[a,b]$ then

$$\int_a^b f(x)\,dx \le 0$$

and in fact

$$\int_a^b f(x)\,dx = -\int_a^b |f(x)|\,dx.$$

Exercises 7.3.

1. An object moves so that its velocity at time t is $v(t) = -9.8t + 20$ m/s. Describe the motion of the object between $t = 0$ and $t = 5$, find the total distance traveled by the object during that time, and find the net distance traveled. ⇒
2. An object moves so that its velocity at time t is $v(t) = \sin t$. Set up and evaluate a single definite integral to compute the net distance traveled between $t = 0$ and $t = 2\pi$. ⇒
3. An object moves so that its velocity at time t is $v(t) = 1 + 2\sin t$ m/s. Find the net distance traveled by the object between $t = 0$ and $t = 2\pi$, and find the total distance traveled during the same period. ⇒
4. Consider the function $f(x) = (x+2)(x+1)(x-1)(x-2)$ on $[-2, 2]$. Find the total area between the curve and the x-axis (measuring all area as positive). ⇒
5. Consider the function $f(x) = x^2 - 3x + 2$ on $[0, 4]$. Find the total area between the curve and the x-axis (measuring all area as positive). ⇒
6. Evaluate the three integrals:
$$A = \int_0^3 (-x^2 + 9)\,dx \qquad B = \int_0^4 (-x^2 + 9)\,dx \qquad C = \int_4^3 (-x^2 + 9)\,dx,$$
and verify that $A = B + C$. ⇒

7.4 Substitution

We have converted the problem of integration into the problem of finding antiderivatives. Sometimes this is a simple problem, since it will be apparent that the function you wish to integrate is a derivative in some straightforward way. For example, faced with

$$\int x^{10}\,dx$$

you realize immediately that the derivative of x^{11} will supply an x^{10}: $(x^{11})' = 11x^{10}$. We don't want the "11", but constants are easy to alter, because differentiation "ignores" them in certain circumstances, so

$$\frac{d}{dx}\frac{1}{11}x^{11} = \frac{1}{11}11x^{10} = x^{10}.$$

From our knowledge of derivatives, we can immediately write down a number of antiderivatives. Here is a list of those most often used:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad \text{if } n \neq -1$$

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \sin x\, dx = -\cos x + C$$
$$\int \cos x\, dx = \sin x + C$$
$$\int \sec^2 x\, dx = \tan x + C$$
$$\int \sec x \tan x\, dx = \sec x + C$$
$$\int \frac{1}{1+x^2}\, dx = \arctan x + C$$
$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x + C$$

Needless to say, most problems we encounter will not be so simple. Here's a slightly more complicated example: find

$$\int 2x \cos(x^2)\, dx.$$

This is not a "simple" derivative, but a little thought reveals that it must have come from an application of the chain rule. Multiplied on the "outside" is $2x$, which is the derivative of the "inside" function x^2. Checking:

$$\frac{d}{dx} \sin(x^2) = \cos(x^2) \frac{d}{dx} x^2 = 2x \cos(x^2),$$

so

$$\int 2x \cos(x^2)\, dx = \sin(x^2) + C.$$

Even when the chain rule has "produced" a certain derivative, it is not always easy to see. Consider this problem:

$$\int x^3 \sqrt{1-x^2}\, dx.$$

There are two factors in this expression, x^3 and $\sqrt{1-x^2}$, but it is not apparent that the chain rule is involved. Some clever rearrangement reveals that it is:

$$\int x^3 \sqrt{1-x^2}\, dx = \int (-2x) \left(-\frac{1}{2}\right) (1-(1-x^2)) \sqrt{1-x^2}\, dx.$$

This looks messy, but we do now have something that looks like the result of the chain rule: the function $1-x^2$ has been substituted into $-(1/2)(1-x)\sqrt{x}$, and the derivative of $1-x^2$, $-2x$, multiplied on the outside. If we can find a function $F(x)$ whose derivative is $-(1/2)(1-x)\sqrt{x}$ we'll be done, since then

$$\begin{aligned}\frac{d}{dx}F(1-x^2) = -2xF'(1-x^2) &= (-2x)\left(-\frac{1}{2}\right)(1-(1-x^2))\sqrt{1-x^2} \\ &= x^3\sqrt{1-x^2}\end{aligned}$$

But this isn't hard:

$$\begin{aligned}\int -\frac{1}{2}(1-x)\sqrt{x}\,dx &= \int -\frac{1}{2}(x^{1/2}-x^{3/2})\,dx \qquad (7.4.1)\\ &= -\frac{1}{2}\left(\frac{2}{3}x^{3/2}-\frac{2}{5}x^{5/2}\right)+C \\ &= \left(\frac{1}{5}x-\frac{1}{3}\right)x^{3/2}+C.\end{aligned}$$

So finally we have

$$\int x^3\sqrt{1-x^2}\,dx = \left(\frac{1}{5}(1-x^2)-\frac{1}{3}\right)(1-x^2)^{3/2}+C.$$

So we succeeded, but it required a clever first step, rewriting the original function so that it looked like the result of using the chain rule. Fortunately, there is a technique that makes such problems simpler, without requiring cleverness to rewrite a function in just the right way. It sometimes does not work, or may require more than one attempt, but the idea is simple: guess at the most likely candidate for the "inside function", then do some algebra to see what this requires the rest of the function to look like.

One frequently good guess is any complicated expression inside a square root, so we start by trying $u = 1-x^2$, using a new variable, u, for convenience in the manipulations that follow. Now we know that the chain rule will multiply by the derivative of this inner function:

$$\frac{du}{dx} = -2x,$$

so we need to rewrite the original function to include this:

$$\int x^3\sqrt{1-x^2} = \int x^3\sqrt{u}\frac{-2x}{-2x}\,dx = \int \frac{x^2}{-2}\sqrt{u}\,\frac{du}{dx}\,dx.$$

Recall that one benefit of the Leibniz notation is that it often turns out that what looks like ordinary arithmetic gives the correct answer, even if something more complicated is going on. For example, in Leibniz notation the chain rule is

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}.$$

The same is true of our current expression:

$$\int \frac{x^2}{-2}\sqrt{u}\,\frac{du}{dx}\,dx = \int \frac{x^2}{-2}\sqrt{u}\,du.$$

Now we're almost there: since $u = 1 - x^2$, $x^2 = 1 - u$ and the integral is

$$\int -\frac{1}{2}(1-u)\sqrt{u}\,du.$$

It's no coincidence that this is exactly the integral we computed in (7.4.1), we have simply renamed the variable u to make the calculations less confusing. Just as before:

$$\int -\frac{1}{2}(1-u)\sqrt{u}\,du = \left(\frac{1}{5}u - \frac{1}{3}\right)u^{3/2} + C.$$

Then since $u = 1 - x^2$:

$$\int x^3\sqrt{1-x^2}\,dx = \left(\frac{1}{5}(1-x^2) - \frac{1}{3}\right)(1-x^2)^{3/2} + C.$$

To summarize: if we suspect that a given function is the derivative of another via the chain rule, we let u denote a likely candidate for the inner function, then translate the given function so that it is written entirely in terms of u, with no x remaining in the expression. If we can integrate this new function of u, then the antiderivative of the original function is obtained by replacing u by the equivalent expression in x.

Even in simple cases you may prefer to use this mechanical procedure, since it often helps to avoid silly mistakes. For example, consider again this simple problem:

$$\int 2x\cos(x^2)\,dx.$$

Let $u = x^2$, then $du/dx = 2x$ or $du = 2x\,dx$. Since we have exactly $2x\,dx$ in the original integral, we can replace it by du:

$$\int 2x\cos(x^2)\,dx = \int \cos u\,du = \sin u + C = \sin(x^2) + C.$$

This is not the only way to do the algebra, and typically there are many paths to the correct answer. Another possibility, for example, is: Since $du/dx = 2x$, $dx = du/2x$, and then the integral becomes

$$\int 2x\cos(x^2)\,dx = \int 2x\cos u\,\frac{du}{2x} = \int \cos u\,du.$$

The important thing to remember is that you must eliminate all instances of the original variable x.

EXAMPLE 7.4.1 Evaluate $\int (ax+b)^n\,dx$, assuming that a and b are constants, $a \neq 0$, and n is a positive integer. We let $u = ax + b$ so $du = a\,dx$ or $dx = du/a$. Then

$$\int (ax+b)^n\,dx = \int \frac{1}{a}u^n\,du = \frac{1}{a(n+1)}u^{n+1} + C = \frac{1}{a(n+1)}(ax+b)^{n+1} + C.$$

□

EXAMPLE 7.4.2 Evaluate $\int \sin(ax+b)\,dx$, assuming that a and b are constants and $a \neq 0$. Again we let $u = ax + b$ so $du = a\,dx$ or $dx = du/a$. Then

$$\int \sin(ax+b)\,dx = \int \frac{1}{a}\sin u\,du = \frac{1}{a}(-\cos u) + C = -\frac{1}{a}\cos(ax+b) + C.$$

□

EXAMPLE 7.4.3 Evaluate $\int_2^4 x\sin(x^2)\,dx$. First we compute the antiderivative, then evaluate the definite integral. Let $u = x^2$ so $du = 2x\,dx$ or $x\,dx = du/2$. Then

$$\int x\sin(x^2)\,dx = \int \frac{1}{2}\sin u\,du = \frac{1}{2}(-\cos u) + C = -\frac{1}{2}\cos(x^2) + C.$$

Now

$$\int_2^4 x\sin(x^2)\,dx = -\frac{1}{2}\cos(x^2)\Big|_2^4 = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

A somewhat neater alternative to this method is to change the original limits to match the variable u. Since $u = x^2$, when $x = 2$, $u = 4$, and when $x = 4$, $u = 16$. So we can do this:

$$\int_2^4 x\sin(x^2)\,dx = \int_4^{16} \frac{1}{2}\sin u\,du = -\frac{1}{2}(\cos u)\Big|_4^{16} = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

An incorrect, and dangerous, alternative is something like this:

$$\int_2^4 x\sin(x^2)\,dx = \int_2^4 \frac{1}{2}\sin u\,du = -\frac{1}{2}\cos(u)\Big|_2^4 = -\frac{1}{2}\cos(x^2)\Big|_2^4 = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

This is incorrect because $\int_2^4 \frac{1}{2}\sin u\,du$ means that u takes on values between 2 and 4, which is wrong. It is dangerous, because it is very easy to get to the point $-\frac{1}{2}\cos(u)\Big|_2^4$ and forget to substitute x^2 back in for u, thus getting the incorrect answer $-\frac{1}{2}\cos(4)+\frac{1}{2}\cos(2)$. A somewhat clumsy, but acceptable, alternative is something like this:

$$\int_2^4 x\sin(x^2)\,dx = \int_{x=2}^{x=4} \frac{1}{2}\sin u\,du = -\frac{1}{2}\cos(u)\bigg|_{x=2}^{x=4} = -\frac{1}{2}\cos(x^2)\bigg|_2^4 = -\frac{\cos(16)}{2}+\frac{\cos(4)}{2}.$$

□

EXAMPLE 7.4.4 Evaluate $\displaystyle\int_{1/4}^{1/2}\frac{\cos(\pi t)}{\sin^2(\pi t)}\,dt$. Let $u=\sin(\pi t)$ so $du=\pi\cos(\pi t)\,dt$ or $du/\pi=\cos(\pi t)\,dt$. We change the limits to $\sin(\pi/4)=\sqrt{2}/2$ and $\sin(\pi/2)=1$. Then

$$\int_{1/4}^{1/2}\frac{\cos(\pi t)}{\sin^2(\pi t)}\,dt = \int_{\sqrt{2}/2}^{1}\frac{1}{\pi}\frac{1}{u^2}\,du = \int_{\sqrt{2}/2}^{1}\frac{1}{\pi}u^{-2}\,du = \frac{1}{\pi}\frac{u^{-1}}{-1}\bigg|_{\sqrt{2}/2}^{1} = -\frac{1}{\pi}+\frac{\sqrt{2}}{\pi}.$$

□

Exercises 7.4.

Find the antiderivatives or evaluate the definite integral in each problem.

1. $\displaystyle\int (1-t)^9\,dt \Rightarrow$

2. $\displaystyle\int (x^2+1)^2\,dx \Rightarrow$

3. $\displaystyle\int x(x^2+1)^{100}\,dx \Rightarrow$

4. $\displaystyle\int \frac{1}{\sqrt[3]{1-5t}}\,dt \Rightarrow$

5. $\displaystyle\int \sin^3 x\cos x\,dx \Rightarrow$

6. $\displaystyle\int x\sqrt{100-x^2}\,dx \Rightarrow$

7. $\displaystyle\int \frac{x^2}{\sqrt{1-x^3}}\,dx \Rightarrow$

8. $\displaystyle\int \cos(\pi t)\cos\Big(\sin(\pi t)\Big)\,dt \Rightarrow$

9. $\displaystyle\int \frac{\sin x}{\cos^3 x}\,dx \Rightarrow$

10. $\displaystyle\int \tan x\,dx \Rightarrow$

11. $\displaystyle\int_0^{\pi} \sin^5(3x)\cos(3x)\,dx \Rightarrow$

12. $\displaystyle\int \sec^2 x\tan x\,dx \Rightarrow$

13. $\displaystyle\int_0^{\sqrt{\pi}/2} x\sec^2(x^2)\tan(x^2)\,dx \Rightarrow$

14. $\displaystyle\int \frac{\sin(\tan x)}{\cos^2 x}\,dx \Rightarrow$

15. $\displaystyle\int_3^4 \frac{1}{(3x-7)^2}\,dx \Rightarrow$

16. $\displaystyle\int_0^{\pi/6} (\cos^2 x-\sin^2 x)\,dx \Rightarrow$

17. $\displaystyle\int \frac{6x}{(x^2-7)^{1/9}}\,dx \Rightarrow$

18. $\displaystyle\int_{-1}^{1} (2x^3-1)(x^4-2x)^6\,dx \Rightarrow$

19. $\displaystyle\int_{-1}^{1} \sin^7 x\,dx \Rightarrow$

20. $\displaystyle\int f(x)f'(x)\,dx \Rightarrow$