6 Applications of the Derivative

6.1 Optimization

Many important applied problems involve finding the best way to accomplish some task. Often this involves finding the maximum or minimum value of some function: the minimum time to make a certain journey, the minimum cost for doing a task, the maximum power that can be generated by a device, and so on. Many of these problems can be solved by finding the appropriate function and then using techniques of calculus to find the maximum or the minimum value required.

Generally such a problem will have the following mathematical form: Find the largest (or smallest) value of \( f(x) \) when \( a \leq x \leq b \). Sometimes \( a \) or \( b \) are infinite, but frequently the real world imposes some constraint on the values that \( x \) may have.

Such a problem differs in two ways from the local maximum and minimum problems we encountered when graphing functions: We are interested only in the function between \( a \) and \( b \), and we want to know the largest or smallest value that \( f(x) \) takes on, not merely values that are the largest or smallest in a small interval. That is, we seek not a local maximum or minimum but a global maximum or minimum, sometimes also called an absolute maximum or minimum.

Any global maximum or minimum must of course be a local maximum or minimum. If we find all possible local extrema, then the global maximum, if it exists, must be the largest of the local maxima and the global minimum, if it exists, must be the smallest of the local minima. We already know where local extrema can occur: only at those points at which \( f'(x) \) is zero or undefined. Actually, there are two additional points at which a maximum or minimum can occur if the endpoints \( a \) and \( b \) are not infinite, namely, at \( a \) and \( b \). We have not previously considered such points because we have not been interested in limiting a function to a small interval. An example should make this clear.

EXAMPLE 6.1.1 Find the maximum and minimum values of \( f(x) = x^2 \) on the interval \([-1, 1]\). (See figure 6.1.2.)

Since \( f'(x) = 2x \) is never zero, but \( f'(x) \) is undefined at \( x = 0 \), we compute \( f(0) = 0 \). So the largest value of \( f(x) \) on \([-1, 1]\) is \( f(1) = 1 \).

EXAMPLE 6.1.2 Extreme value theorem If \( f \) is continuous on a closed interval \([a, b]\), then it has both a minimum and a maximum point. That is, there are real numbers \( c \) and \( d \) in \([a, b]\) so that for every \( x \) in \([a, b]\), \( f(x) \leq f(c) \) and \( f(x) \geq f(d) \).

Another easy case: if \( f \) is continuous and has a single critical value, then if there is a local maximum at the critical value it is a global maximum, and if it is a local minimum it is a global minimum. There may also be a critical value in the first case, or a global maximum in the second case, but that will generally require more effort to determine.

EXAMPLE 6.1.3 Let \( f(x) = -x^2 + 4x - 3 \). Find the maximum value of \( f(x) \) on the interval \([0, 4]\). First note that \( f'(x) = -2x + 4 = 0 \) when \( x = 2 \) and \( f(2) = 1 \). Next observe that \( f'(x) \) is defined for all \( x \), so there are no other critical values. Finally, \( f(0) = -3 \) and \( f(4) = -3 \). The largest value of \( f(x) \) on the interval \([0, 4]\) is \( f(2) = 1 \).

EXAMPLE 6.1.4 Let \( f(x) = -x^2 + 4x - 3 \). Find the maximum value of \( f(x) \) on the interval \([-1, 1]\).

First note that \( f'(x) = -2x + 4 = 0 \) when \( x = 2 \). But \( x = 2 \) is not in the interval, so we don’t use it. Thus the only two points to be checked are the endpoints: \( f(-1) = -8 \) and \( f(1) = 0 \). So the largest value of \( f(x) \) on \([-1, 1]\) is \( f(1) = 0 \).

EXAMPLE 6.1.5 Find the maximum and minimum values of the function \( f(x) = 7 + |x - 2| \) for \( x \) between 1 and 4 inclusive. The derivative \( f'(x) \) is never zero, but \( f'(x) \) is undefined at \( x = 2 \). So we compute \( f(2) = 7 \). Checking the end points we get \( f(1) = 8 \) and \( f(4) = 9 \). The smallest of these numbers is \( f(2) = 7 \), which is, therefore, the minimum value of \( f(x) \) on the interval \([1, 4]\). The maximum is \( f(4) = 9 \).

EXAMPLE 6.1.6 Find all local maxima and minima for \( f(x) = x^3 - x \), and determine whether there is a global maximum or minimum on the open interval \((-2, 2)\). In example 5.1.2 we found a local maximum at \((-\sqrt[3]{3}, 2\sqrt[3]{3})\) and a local minimum at \((\sqrt[3]{3}, -2\sqrt[3]{3})\). Since the endpoints are not in the interval \((-2, 2)\), they cannot be considered.

Is the lone local maximum a global maximum? Here we must look more closely at the graph. We know that on the closed interval \([-\sqrt[3]{3}, \sqrt[3]{3}]\), there is a global maximum at \( x = -\sqrt[3]{3} \) and a global minimum at \( x = \sqrt[3]{3} \). So the question becomes: what happens between \(-2 \approx -1.262 \) and \( -\sqrt[3]{3} \), and between \( \sqrt[3]{3} \) and \( 2 \)? Since there is a local minimum at \( x = \sqrt[3]{3} \), the graph must continue up to the right, since there are no more critical values. This means no value of \( f \) will be less than \(-2\sqrt[3]{3} \approx -1.59 \). And \( 2 \approx 1.262 \) is not the smallest possible value of \( x \). As a result, \( f \) must increase to the right of \( \sqrt[3]{3} \), we need to know that the function does “close to” \( 2 \). Here the easiest test is to pick a number and do a computation to get some idea of what’s going on. Since \( f(1.9) = 4.95 \approx 4.59 \), \( 2 \approx 2.3 \), there is a global maximum at \( x = -\sqrt[3]{3} \) and no global maximum at \( -\sqrt[3]{3} \). And hence no global maximum at all. (How can we tell that \( 4.95 > 2\sqrt[3]{3} \)?

We can use a calculator to approximate the right hand side; if it is not even close to \( 4.95 \), then we can take this as decisive. Since \( 2\sqrt[3]{3} \approx 2.08 \), it’s really no question. Finding things can happen in the rounding done by computers and calculators, however, so we might be a little more careful, especially if the values come out quite close. In this case we can convert the relation \( 4.95 > 2\sqrt[3]{3} \) into \( 0.95 > 2/3 \) and ask whether this is true. Since the left side is clearly larger than \( 0.4 \), which is clearly larger than \( 1/3 \), this settles the question.

A similar analysis shows that there is also no global minimum. The graph of \( f(x) \) on \((-2, 2)\) is shown in figure 6.1.2.

EXAMPLE 6.1.7 Of all rectangles of area 100, which has the smallest perimeter?

First we must translate this into a purely mathematical problem in which we want to find the minimum value of a function. If \( a \) denotes one of the sides of the rectangle, then the adjacent side must be \( 100/a \) in order that the area be 100. So the function we want

\[ \text{minimize } P(a) = 2a + 2(100/a) \]
to minimize is
\[ f(x) = 2x + \frac{100}{x} \]
since the perimeter is twice the length plus twice the width of the rectangle. Not all values of \( x \) make sense in this problem: lengths of sides of rectangles must be positive, so \( x > 0 \).
If \( x > 0 \) then so is \( 100/x \), so we need no second condition on \( x \).
We next find \( f'(x) \) and set it equal to zero: 0 = \( f'(x) = 2 - 100/x^2 \). Solving \( f'(x) = 0 \) for \( x \) gives us \( x = \pm 10 \). We are interested only in \( x > 0 \), so only the value \( x = 10 \) is of interest. Since \( f''(x) \) is defined everywhere on the interval \((0, \infty)\), there are no more critical values, and there are no endpoints. Is there a local maximum, minimum, or neither at \( x = 10^+ \)? The second derivative is \( f''(x) = 200/x^3 \), and \( f''(10) > 0 \), so there is a local minimum. Since there is only one critical value, this is also the global minimum, so the rectangle with smallest perimeter is the \( 10 \times 10 \) square.

**EXAMPLE 6.1.8** You want to sell a certain number \( n \) of items in order to maximize your profit. Market research tells you that if you set the price at $1.50, you will be able to sell 5000 items, and for every 10 cents you lower the price below $1.50, you will be able to sell another 1000 items. Suppose that your fixed costs ("start-up costs") total $2800, and the per unit cost of production ("marginal cost") is $0.50. Find the price to set per item and the number of items sold in order to maximize profit, and also determine the maximum profit you can get.

The first step is to convert the problem into a function maximization problem. Since we want to maximize profit by setting the price per item, we should look for a function \( P(x) \) representing the profit when the price per item is \( x \). Profit is revenue minus cost, and revenue is number of items sold times the price per item, so we get \( P = nx - 2000 - 0.5nx \). The number of items sold is itself a function of \( x \), \( n = 5000 + 1000(1.5 - x)/0.10 \), because \((1.5 - x)/0.10 \) is the number of 10 cents of the price below $1.50 that is the price below $1.50. Now we substitute for \( n \) in the profit function:
\[
P(x) = (5000 + 1000(1.5 - x)/0.10)x - 2000 - 0.5(5000 + 1000(1.5 - x)/0.10)
\]
\[
= -10000x + 250000 + 12000.
\]
We want to know the maximum value of this function when \( x \) is between 0 and 1.5. The derivative is \( P'(x) = -20000 + 250000 \), which is zero when \( x = 1.25 \). Since \( P''(x) = 20000 > 0 \), there must be a local maximum at \( x = 1.25 \). and hence a global maximum as well. (Alternatively, we could compute \( P(0) = -12000, P(1.25) = 3625 \), and \( P(1.5) = 3000 \) and note that \( P(1.25) \) is the maximum of these.) Thus the maximum profit is $3625, attained when we set the price at $1.25 and sell 7500 items.

**EXAMPLE 6.1.9** Find the largest rectangle (that is, the rectangle with largest area) that fits inside the graph of the parabola \( y = x^2 \) below the line \( y = a \) (a is an unspecified constant value), with the top side of the rectangle on the horizontal line \( y = a \); see figure 6.1.3.

We want to find the maximum value of some function \( A(x) \) representing area. Perhaps the hardest part of this problem is deciding what \( x \) should represent. The lower right corner of the rectangle is at \( (x, x^2) \), and once this is chosen the rectangle is completely determined. So we can let the \( x \) in \( A(x) \) be the \( x \) of the parabola \( f(x) = x^2 \). Then the area is \( A(x) = (2x)(a - x^2) - 2x^2 + 2x \). We want the maximum value of \( A(x) \) when \( x \) is in \([0, \sqrt{a}]\). (You might object to allowing \( x \) to be more than \( \sqrt{a} \), since then the "rectangle" has either no width or no height, so is not "really" a rectangle. But the problem is somewhat easier if we simply allow such rectangles, which have zero area.)

Setting 0 = \( A'(x) = -4x + 2a \) we get \( x = \sqrt{a/3} \). Testing this and the two endpoints, we have \( A(0) = A(\sqrt{a/3}) = 0 \) and \( A(\sqrt{a/3}) = (4/9)a^{3/2} \). The maximum area thus occurs when the rectangle has dimensions \( \sqrt{a/3} \times (2/3)a \).

**EXAMPLE 6.1.10** If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by "cone" we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.) Let \( R \) be the radius of the sphere, and let \( r \) and \( h \) be the base radius and height of the cone inside the sphere. What we want to maximize is the volume of the cone: \( \pi r^2 h/3 \). Here \( R \) is a fixed value, but \( r \) and \( h \) can vary. Namely, we could choose \( r \) to be as large as possible—equal to \( R \) by taking the height equal to \( R \). or we could make the cone's height \( h \) larger at the expense of making \( r \) a little less than \( R \). See the cross-section depicted in figure 6.1.4. We have situated the picture in a convenient way relative to the \( x \) and \( y \) axes, namely, with the center of the sphere at the origin and the vertex of the cone at the far left on the \( x \)-axis.

Notice that if \( r = h = R \), then \( \pi r^2 h/3 \) is the maximum volume. So our next step is to find the relationship and use it to solve for one of the variables in terms of the other, so as to have a function of only one variable to maximize. In this problem, the condition is apparent in the figure: the upper corner of the triangle, whose coordinates are \((h - R, r)\), must be on the circle of radius \( R \). That is,
\[
(h - R)^2 + r^2 = R^2.
\]
We can solve for \( h \) in terms of \( r \) or for \( r \) in terms of \( h \). Either involves taking a square root, but we notice that the volume function contains \( r^2 \), not \( h^2 \), so it is easiest to solve for \( r^2 \) directly:
\[
r^2 = R^2 - (h - R)^2 = R^2 - h^2 + 2Rr.
\]
Then we substitute the result into \( \pi r^2 h/3 \):
\[
V(h) = \pi r^2 h/3 = \pi \left( R^2 - h^2 + 2Rr \right) h/3
\]
\[
= \frac{\pi}{3} \left( R^3 - h^3 + 2R \sqrt{R^2 - h^2} \right).
\]
We want to maximize \( V(h) \) when \( h \) is between 0 and 2\( R \). Now we solve 0 = \( f'(h) = -3h^2 + (4/3)\pi R \), getting \( h = 0 \) or \( h = 4R/3 \). We compute \( V(0) = V(2R) = 0 \) and \( V(4R/3) = (32/81)\pi R^2 \). The maximum is the latter; since the volume of the sphere is \((4/3)\pi R^3 \), the fraction of the sphere occupied by the cone is
\[
\frac{(32/81)\pi R^2}{(4/3)\pi R^3} = \frac{8}{27} \approx 0.300.
\]
EXAMPLE 6.1.12 Suppose you want to reach a point A that is located across the sand from a nearby road (see figure 6.1.5). Suppose that the road is straight, and h is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from C. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A?

Let x be the distance short of C where you turn off, i.e., the distance from B to C. We want to minimize the total travel time. Recall that when traveling at constant velocity, time is elapsed directly by distance.

You travel the distance $\sqrt{x^2 + a^2}$ at speed v, and then the distance $\sqrt{x^2 + b^2}$ at speed w. Since $\sqrt{x^2 + a^2} - \sqrt{x^2 + b^2} = x$, and by the Pythagorean theorem, $\sqrt{x^2 + a^2} + \sqrt{x^2 + b^2} = C$, the total time for the trip is

$$f(x) = \frac{x}{v} + \frac{\sqrt{x^2 + b^2}}{w}.$$  

We want to find the minimum value of $f$ when $x$ is between 0 and a. As usual we set $f'(x) = 0$ and solve for $x$:

$$0 = f'(x) = \frac{1}{v} - \frac{x}{v\sqrt{x^2 + b^2}}$$  

$$w\sqrt{x^2 + b^2} - xv = 0$$  

Since it is always positive there is a local minimum at the critical point, and so it is a global minimum as well.

If the critical value is not in $[0, a]$, then it is larger than $a$. In this case the minimum must occur at one of the endpoints. We can compute:

$$f(0) = \frac{b}{w}$$  

$$f(a) = \frac{\sqrt{a^2 + b^2}}{w}$$

Exercise 6.1.12

1. Let $f(x) = 1 + 4x - x^2$ for $x \leq 3$ and $f(x) = (x+5)/2$ for $x > 3$. Find the maximum value and minimum values of $f(x)$ for $x$ in $[0, 4]$. Graph $f(x)$ to check your answers.

2. Find the dimensions of the rectangle of largest area having fixed perimeter 100. ⇒

3. Find the dimensions of the rectangle of largest area having fixed perimeter $P$. ⇒

4. A box with square base and no top is to hold a volume 100. Find the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base. ⇒

5. A box with square base and no top is to hold a volume 200. Find the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base. (This ratio will not involve $V$) ⇒

6. You have 100 feet of fence to make a rectangular play area alongside the wall of your house. The wall of the house bounds one side. What is the largest size possible (in square feet) for the play area? ⇒

9. Marketing tells you that if you set the price of an item at $10 then you will be unable to sell it, but that you can sell 500 items for each dollar below $10 that you set the price. Suppose your fixed costs total $3000, and your marginal cost is 82 per item. What is the most profit you can make? ⇒

10. Find the area of the largest rectangle that fits inside a semicircle of radius 10 (one side of the rectangle is along the diameter of the semicircle). ⇒

11. Find the area of the largest rectangle that fits inside a semicircle of radius $r$ (one side of the rectangle is along the diameter of the semicircle). ⇒

12. For a cylinder with surface area 50, including the top and the bottom, find the ratio of height to base radius that maximizes the volume. ⇒

13. For a cylinder with given surface area $S$, including the top and the bottom, find the ratio of height to base radius that maximizes the volume. ⇒

14. You want to make cylindrical containers to hold 1 liter (1000 cubic centimeters) using the least amount of construction material. The side is made from a rectangular piece of material, and this can be done with no material wasted. However, the top and bottom are cut from squares of side $2r$, so that 2($2r)^2 = 8r^2$ of material is needed (rather than $2\pi r^2$, which is the total area of the top and bottom). Find the dimensions of the container using the least amount of material, and also find the ratio of height to radius for this container. ⇒

15. You want to make cylindrical containers of a given volume $V$ using the least amount of construction material. The side is made from a rectangular piece of material, and this can be done with no material wasted. However, the top and bottom are cut from squares of side $2r$, so that 2($2r)^2 = 8r^2$ of material is needed (rather than $2\pi r^2$, which is the total area of the top and bottom). Find the optimal ratio of height to radius. ⇒

16. Given a right circular cone, you put an upside-down cone inside it so that its vertex is at the center of the base of the larger cone and its base is parallel to the base of the larger cone. If you choose the upside-down cone to have the largest possible volume, what fraction of the volume of the larger cone does it occupy? (Let H and R be the height and base radius of the larger cone, and let h and r be the height and base radius of the smaller cone. Hint: Use similar triangles to get an equation relating $H$ and $h$.) ⇒

17. In example 6.1.12, what happens if $w \geq v$ (i.e., your speed on sand is at least your speed on the road)? ⇒

18. A container holding a fixed volume is being made in the shape of a cylinder with a hemispherical top. (The hemispherical top has the same radius as the cylinder.) Find the ratio of height to radius of the cylinder which minimizes the cost of the container if (a) the cost per unit area of the top is twice as great as the cost per unit area of the side, and (b) the container is made with uniform round edges, for a circular cardboard of radius 1 meter. ⇒

19. A piece of cardboard is 1 meter by 1.2 meter. A square is to be cut from each corner and the sides folded up to make an open-top box. What are the dimensions of the box with maximum possible volume? ⇒

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but it is difficult to determine which of these is smaller by direct comparison. If, as is likely in practice, we know the values of $v, w, a,$ and $b,$ then it is easy to determine this. With a little cleverness, however, we can determine the minimum in general. We have seen that $f''(x)$ is always positive, so the derivative $f'(x)$ is always increasing. We know that at $w\sqrt{\pi r^2}$ the derivative is zero, so for values of $x$ less than that critical value, the derivative is negative. This means that $f(0) > f(x)$, so the minimum occurs when $x = a$.

So the upshot is this: If you start farther away from C than $w\sqrt{\pi r^2}$ you always want to cut across the sand when you are at a distance $w\sqrt{\pi r^2}$ from point C. If you start closer than this to C, you should cut directly across the sand.

Summary—Steps to solve an optimization problem.

1. Decide what the variables are and what the constants are, draw a diagram if appropriate, understand clearly what it is that you are to be maximized or minimized.

2. Write a formula for the function for which you wish to find the maximum or minimum.

3. Express that formula in terms of only one variable, that is, in the form $f(x)$.

4. Set $f'(x) = 0$ and solve. Check all critical values and endpoints to determine the extreme values.
30. If you fit the cone with the largest possible surface area (lateral area plus area of base) into a sphere, what percent of the volume of the sphere is occupied by the cone?

31. Two electrical charges, one a positive charge $A$ of magnitude $a$ and the other a negative charge $B$ of magnitude $b$, are located a distance $c$ apart. A positively charged particle $P$ is situated on the line between $A$ and $B$. Find where $P$ should be put so that the pull away from $A$ towards $B$ is minimal. Here assume that the force from each charge is proportional to the square of the distance from the source and inversely proportional to the square of the distance from the source.

32. Find the fraction of the area of a triangle that is occupied by the largest rectangle that can be drawn in the triangle (with one of its sides along a side of the triangle). Show that this fraction does not depend on the dimensions of the given triangle.

33. How are your answers to Problem 32 affected if the cost per item for the $x$ items, instead of being simply $2$, decreases below $2$ in proportion to $x$ (because of economy of scale and volume discounts) by $1$ cent for each $25$ items produced? 

34. You are standing near the side of a large wading pool of uniform depth when you see a child in trouble. You can run at a speed $v$ on land and at a slower speed $v$ in the water. Your perpendicular distance from the side of the pool to a child, the child’s perpendicular distance is $b$, and the distance along the side of the pool between the closest point to you and the closest point to the child is $c$ (see the figure below). Without stopping to do any calculations, you instinctively choose the quickest route (shown in the figure) and save the child. Our purpose is to derive a relation between the angle $\theta$ your path makes with the perpendicular to the side of the pool when you're on land, and the angle $\theta$ your path makes with the perpendicular when you're in the water. To do this, let $x$ be the distance between the closest point to you at the side of the pool and the point where you enter the water. Write the total time you run (on land and in the water) in terms of $x$ (and also the constants $a, b, c, v, v$). Then set the derivative equal to zero. The result, called "Snell's law" or the "law of refraction," also governs the bending of light when it goes into water.

![Figure 6.1.7 Wading pool rescue.](image)

**6.2 Related Rates**

Suppose we have two variables $x$ and $y$ (in most problems the letters will be different, but for now let's use $x$ and $y$) which are both changing with time. A "related rates" problem is a problem in which we know one of the rates of change at a given instant -- say, $\frac{dx}{dt}$, and what is the geometric quantity whose rate of change we're being asked about? Note that $\frac{dy}{dt}$ is, it is moving on a parabolic path. At a particular time, say $t = 5$, the $x$ coordinate is 6 and we measure the speed at which the $x$ coordinate of the object is changing and find that $\frac{dx}{dt} = 3$. At the same time, how fast is the $y$ coordinate changing?

Using the chain rule, $\frac{dy}{dt} = 2\frac{dx}{dt}$. At $t = 5$ we know that $x = 6$ and $\frac{dx}{dt} = 3$, so $\frac{dy}{dt} = 2 \cdot 6 = 12$.

In many cases, particularly interesting ones, $x$ and $y$ will be related in some other way, for example $x = f(y)$, or $F(x, y) = k$, or perhaps $F(x, y) = G(x, y)$, where $F(x, y)$ and $G(x, y)$ are expressions involving both variables. In all cases, you can solve the related rates problem by taking the derivative of both sides, plugging in all the known values (namely, $x$, $y$, and $\frac{dy}{dt}$), and then solving for $\frac{dx}{dt}$.

To summarize, here are the steps in doing a related rates problem:

1. Decide what the two variables are.
2. Find an equation relating them.
3. Take $\frac{d}{dt}$ of both sides.
4. Plug in all known values at the instant in question.
5. Solve for the unknown rate.

**EXAMPLE 6.2.1** Suppose an object is moving along a path described by $y = x^2$, that is, it is moving on a parabolic path. At a particular time, say $t = 5$, the $x$ coordinate is 6 and we measure the speed at which the $x$ coordinate of the object is changing and find that $\frac{dx}{dt} = 5$. At the same time, how fast is the $y$ coordinate changing?

Using the chain rule, $\frac{dy}{dt} = 2\frac{dx}{dt}$. At $t = 5$ we know that $x = 6$ and $\frac{dx}{dt} = 3$, so $\frac{dy}{dt} = 2 \cdot 6 = 12$.

**EXAMPLE 6.2.2** A plane is flying directly away from you at 500 mph at an altitude of 3 miles. How fast is the plane’s distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

To see what’s going on, we first draw a schematic representation of the situation, as in figure 6.2.1:

Because the plane is in level flight directly away from you, at the rate at which $x$ changes is the speed of the plane, $\frac{dx}{dt} = 500$. The distance between you and the plane is $y$; it is $\frac{dy}{dt}$ that we wish to know. By the Pythagorean Theorem we know that $x^2 + y^2 = 10^2$.

**EXAMPLE 6.2.3** You are injecting a spherical balloon at the rate of 7 cm$^3$/sec. How fast is its radius increasing when the radius is 4 cm?

Here the variables are the radius $r$ and the volume $V$. We know $\frac{dV}{dt}$, and we want $\frac{dr}{dt}$. The two variables are related by means of the equation $V = \frac{4}{3}\pi r^3$. Taking the derivative of both sides gives $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. We now substitute the values we know at the instant in question: $7 = 4\pi \cdot 4^2 \frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{7}{16\pi} \text{ cm/sec}$.\n
**EXAMPLE 6.2.4** Water is poured into a conical container at the rate of 10 cm$^3$/sec. The cone points directly down, and it has a height of 30 cm and a base radius of 10 cm; see figure 6.2.2.

How fast is the water level rising when the water is 4 cm deep (at its deepest point)?

The water forms a conical shape within the big cone; its height and base radius and volume are all increasing as water is poured into the container. This means that we actually have three things varying with time: the water level $h$ (the height of the cone of water), the radius $r$ of the circular top surface of water (the base radius of the cone of water), and the volume of water $V$. The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$. We know $\frac{dV}{dt}$, and we want $\frac{dh}{dt}$. At first something seems to be wrong: we have a third variable $r$ whose rate we don’t know.

But the dimensions of the cone of water must have the same proportions as those of the container. That is, because of similar triangles, $r/h = 10/30$ so $r = h/3$. Now we can eliminate $r$ from the problem entirely: $V = \frac{1}{3}\pi (h^2/3)^2 h = \pi h^3/27$. We take the derivative of both sides and plug in $h = 4$ and $\frac{dV}{dt} = 10$, obtaining $10 = (3\pi \cdot 4^2/27)(\frac{dh}{dt})$. Thus $\frac{dh}{dt} = \frac{90}{16\pi} \text{ cm/sec}$.\n
**EXAMPLE 6.2.5** A swing consists of a board at the end of a 10 ft long rope. Think of the board as a point $P$ at the end of the rope, and let $Q$ be the point of attachment at the other end. Suppose that the swing is directly below $Q$ at time $t = 0$, and is being pushed by someone who walks at 6 ft/sec from left to right. Find (a) how fast the swing is rising after 1 sec; (b) the angular speed of the rope in deg/sec after 1 sec.

We start out by asking: What is the geometric quantity whose rate of change we know, and what is the geometric quantity whose rate of change we’re being asked about? Note that the person pushing the swing is moving horizontally at a rate we know. In other words, the horizontal coordinate of $P$ is increasing at 6 ft/sec. In the xy-plane let us make the convenient choice of putting the origin at the location of $P$ at time $t = 0$, i.e., a distance 10 directly below the point of attachment. Then the rate we know is $\frac{dx}{dt}$, and in part (a) the rate we want is $\frac{dy}{dt}$ (the rate at which $P$ is rising). In part (b) the rate we want is $\dot{\theta} = \frac{d\theta}{dt}$, where $\theta$ stands for the angle in radians through which the swing has swung from the vertical. (Actually, since we want our answer in deg/sec, at the end we must convert $\frac{d\theta}{dt}$ from rad/sec by multiplying by $180/\pi$.)

(a) From the diagram we see that we have a right triangle whose legs are $x = 10$ and $y$, and whose hypotenuse is 10. Hence $x^2 + (10 - y)^2 = 100$. Taking the derivative of both sides we obtain: $2x\frac{dx}{dt} + 2(10-y)\frac{dy}{dt} = 0$. Now we look at what we know after 1 second, namely $x = 6$ (because $x$ started at 0 and has been increasing at the rate of 6 ft/sec for 1 sec), $y = 2$ (because we get 10 $- 8$ from the Pythagorean theorem applied to the triangle with hypotenuse 10 and leg 6), and $\dot{\theta} = 6$. Putting these values gives us $2 \cdot 6 - 6 \cdot 2 \frac{dy}{dt} = 0$, from which we can easily solve for $\frac{dy}{dt} = 4.5$ ft/sec.

(b) Here our two variables are $x$ and $\theta$, so we want to use the same right triangle as in part (a), but this time relate $\theta$ to $x$. Since the hypotenuse is constant (equal to 10), the best way to do this is to use the sine: $\sin \theta = x/10$. Taking derivatives we obtain: $\cos \theta \dot{\theta} = \frac{1}{10}$. We have $\cos \theta = \sqrt{1 - (x/10)^2}$, so $\dot{\theta} = \frac{1}{10\sqrt{1 - x^2/100}}$.
EXAMPLE 6.2.6 A road running north to south crosses a road going east to west at the point \( P \). Car A is driving north along the first road, and car B is driving east along the second road. At a particular time car A is 10 kilometers to the north of \( P \) and traveling at 80 km/hr, while car B is 15 kilometers to the east of \( P \) and traveling at 108 km/hr. How fast is the distance between the two cars changing?

![Figure 6.2.4 Cars moving apart.](image)

\[ \frac{dt}{dx} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\frac{108}{80}} = \frac{40}{9}. \]

Figure 6.2.4

7. Sand is poured onto a surface at 15 cm^3/sec, forming a conical pile whose base diameter is always equal to its altitude. How fast is the altitude of the pile increasing when the pile is 3 cm high? \( \Rightarrow \)

8. A boat is pulled into a dock by a rope with one end attached to the front of the boat and the other end passing through a ring attached to the dock at a point 5 ft higher than the front of the boat. The rope is being pulled through the ring at the rate of 0.6 ft/sec. How fast is the boat approaching the dock when 12 ft of rope are out? \( \Rightarrow \)

9. A balloon is at a height of 50 meters, and is rising at the constant rate of 5 m/sec. A bicyclist passes beneath it, traveling in a straight line at the constant speed of 14 km/hr. How fast is the distance between the bicyclist and balloon increasing 2 seconds later? \( \Rightarrow \)

10. A pyramid-shaped vat has square cross-section and stands on its tip. The dimensions at the top are 2 m \times 2 m, and the depth is 5 m. If water is flowing into the vat at 3 m^3/min, how fast is the water level rising when the depth of water at (the deepest point) is 4 m? \( \Rightarrow \)

11. The sun is rising at the rate of 1/4 deg/min, and appears to be climbing into the sky perpendicular to the horizon, as depicted in figure 6.2.5. How fast is the shadow of a 200 meter building shrinking at the moment when the shadow is 500 meters long? \( \Rightarrow \)

12. The sun is setting at the rate of 1/4 deg/min, and appears to be dropping perpendicular to the horizon, as depicted in figure 6.2.5. How fast is the shadow of a 25 meter wall lengthening at the moment when the shadow is 50 meters long? \( \Rightarrow \)

![Figure 6.2.6 Trough.](image)

Figure 6.2.6

13. The trough shown in figure 6.2.6 is constructed by fastening together three slabs of wood of dimensions 10 ft \times 1 ft. and then attaching the construction to a wooden wall at each end. The angle \( \theta \) was originally 30°, but because of poor construction the sides are collapsing. The trough is full of water. At what rate (in ft^3/sec) is the water spilling out over the top of the trough if the sides have each fallen to an angle of 45°, and are collapsing at the rate of 1° per second? \( \Rightarrow \)

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Let \( u(t) \) be the distance of car A north of \( P \) at time \( t \), and \( b(t) \) the distance of car B east of \( P \) at time \( t \), and let \( c(t) \) be the distance from car A to car B at time \( t \). By the Pythagorean Theorem, \( c(t)^2 = u(t)^2 + b(t)^2 \). Taking derivatives we get \( 2c(t) \frac{dc(t)}{dt} = 2u(t) \frac{du(t)}{dt} + 2b(t) \frac{db(t)}{dt} \), so

\[ \frac{dc(t)}{dt} = \frac{u(t) \frac{du(t)}{dt} + b(t) \frac{db(t)}{dt}}{c(t)}. \]

Substituting known values we get:

\[ \frac{dc(t)}{dt} = \frac{10 \times 15 + 10 \times 80}{\sqrt{10^2 + 15^2 + 80^2}} \approx 127\text{km/hr} \] at the time of interest. \( \Rightarrow \)

Notice how this problem differs from example 6.2.2. In both cases we started with the Pythagorean Theorem and took derivatives on both sides. However, in example 6.2.2 one of the sides was a constant (the altitude of the plane), and so the derivative of the square of that side of the triangle was simply zero. In this example, on the other hand, all three sides of the right triangle are variables, even though we are interested in a specific value of each side of the triangle (namely, when the sides have lengths 10 and 15). Make sure that you understand at the start of the problem what are the variables and what are the constants.

Exercises 6.2.

1. A cylindrical tank standing upright (with one circular base on the ground) has radius 20 cm. How fast does the water level in the tank drop when the water is being drained at 25 cm^3/sec? \( \Rightarrow \)

2. A cylindrical tank standing upright (with one circular base on the ground) has radius 1 meter. How fast does the water level in the tank drop when the water is being drained at 3 liters per second? \( \Rightarrow \)

3. A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.6 m/sec. How fast is the top sliding down the wall when the foot of the ladder is 5 m from the wall? \( \Rightarrow \)

4. A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The top of the ladder is being pulled up the wall at 0.1 meters per second. How fast is the foot of the ladder approaching the wall when the foot of the ladder is 5 m from the wall? \( \Rightarrow \)

5. A rotating beacon is located 2 miles out in the water. Let \( A \) be the point on the shore that is closest to the beacon. As the beacon rotates at 10 rev/min, the beam of light covers down the shore once each time it revolves. Assume that the shore is straight. How fast is the distance where the beam hits the shore moving at an instant when the beam is lighting up a point 2 miles along the shore from the point \( A \)? \( \Rightarrow \)

6. A baseball diamond is a square 90 ft on a side. A player runs from first base to second base at 15 ft/sec. At what rate is the player’s distance from third base decreasing when she is half way from first to second base? \( \Rightarrow \)

7. A woman 5 ft tall walks at the rate of 3.5 ft/sec away from a streetlight that is 12 ft above the ground. At what rate is the tip of her shadow moving? At what rate is her shadow lengthening? \( \Rightarrow \)

8. A man 1.8 meters tall walks at the rate of 1 meter per second toward a straightlight that is 4 meters above the ground. At what rate is the tip of his shadow moving? At what rate is his shadow shortening? \( \Rightarrow \)

9. A police helicopter is flying at 150 mph at a constant altitude of 0.5 mile above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 1 mile from the helicopter, and that this distance is decreasing at 100 mph. Find the speed of the car. \( \Rightarrow \)

10. A police helicopter is flying at 200 kilometers per hour at a constant altitude of 1 km above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 2 kilometers from the helicopter, and that this distance is decreasing at 250 kph. Find the speed of the car. \( \Rightarrow \)

11. A police helicopter is flying at 200 kilometers per hour at a constant altitude of 1 km above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 1 kilometer from the helicopter, and that this distance is decreasing at 200 mph. Find the speed of the car. \( \Rightarrow \)

12. A police helicopter is flying at 200 kilometers per hour at a constant altitude of 1 km above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 1 kilometer from the helicopter, and that this distance is decreasing at 200 mph. Find the speed of the car. \( \Rightarrow \)

13. A light shines from the top of a pole 20 m high. An object is dropped from the same height from a point 10 m away, so that its height at time \( t \) seconds is \( h(t) = 20 - 9.8t^2/2 \). How fast is the object’s shadow moving on the ground one second later? \( \Rightarrow \)
6.3 Newton’s Method

Suppose you have a function \( f(x) \), and you want to find as accurately as possible where it crosses the \( x \)-axis; in other words, you want to solve \( f(x) = 0 \). Suppose you know of no way to find an exact solution by any algebraic procedure, but you are able to use an approximation, provided it can be made quite close to the true value. Newton’s method is a way to find a solution to the equation to as many decimal places as you want. It is what is called an “iterative procedure,” meaning that it can be repeated again and again to get an answer of greater and greater accuracy. Iterative procedures like Newton’s method are well suited to programming for a computer. Newton’s method uses the fact that the tangent line to a curve is a good approximation to the curve near the point of tangency.

**Example 6.3.1** Approximate \( \sqrt{3} \). Since \( \sqrt{3} \) is a solution to \( x^2 - 3 = 0 \), we use \( f(x) = x^2 - 3 \). We start by guessing something reasonably close to the true value; this is usually easy to do; let’s use \( x_0 = 2 \). Now use the tangent line to the curve when \( x = 2 \) as an approximation to the curve, as shown in figure 6.3.1. Since \( f'(x) = 2x \), the slope of this tangent line is 4 and its equation is \( y = 4x - 7 \). The tangent line is quite close to \( f(x) \), so it crosses the \( x \)-axis near the point at which \( f(x) \) crosses, that is, near \( \sqrt{3} \). It is easy to find where the tangent line crosses the \( x \)-axis: solve \( 0 = 4x - 7 \) to get \( x = 7/4 = 1.75 \). This is certainly a better approximation than 2, but let us say not close enough. We can improve it by doing the same thing again: find the tangent line at \( x = 1.75 \), find where this new tangent line crosses the \( x \)-axis, and use that value as a better approximation. We can continue indefinitely, though it gets a bit tedious. Let see if we can shortcut the process. Suppose the host approximation to the intercept we have so far is \( x_1 \). To find a better approximation we will always do the same thing: find the slope of the tangent line at \( x_1 \), find the equation of the tangent line, find the \( x \)-intercept. The slope is 2\( x_1 \). The tangent line is \( y = (2x_1)(x - x_1) + (x_1^2 - 3) \), using the point-slope formula for a line. Finally, the intercept is found by solving \( 0 = (2x_1)(x - x_1) + (x_1^2 - 3) \). With a little algebra this turns into \( x - (x_1^2 + 3)/(2x_1) \), this is the next approximation, which we naturally call \( x_{i+1} \). Instead of doing the whole tangent line computation every time we can simply use this formula to get as many approximations as we want. Starting with \( x_1 = 2 \), we get \( x_2 = (x_1^2 + 3)/(2x_1) = (x_1^2 + 3)/4 \) (the same approximation we got above, of course). \( x_3 = (x_2^2 + 3)/(2x_2) = (17/4)/3 = 1.72814 \), \( x_3 \approx 1.72814 \), and so on. This is still a bit tedious by hand, but with a calculator or, even better, a good computer program, it is quite easy to get many, many approximations. We might guess already that \( 1.73205 \) is accurate to two decimal places, and in fact it turns out that it is accurate to 5 places.

Let’s think about this process in more general terms. We want to approximate a solution to \( f(x) = 0 \). We start with a rough guess, which we call \( x_0 \). We use the tangent line to \( f(x) \) to get a new approximation that we hope will be closer to the true value. What is the equation of the tangent line when \( x = x_0 \)? The slope is \( f'(x_0) \) and the line goes through \((x_0, f(x_0))\), so the equation of the line is

\[
y - f(x_0) = f'(x_0)(x - x_0)
\]

**Exercise 6.3.1** You may want to use this Sage worksheet:

1. Approximate the fifth root of 7, using \( x_0 = 1.5 \) as a first guess. Use Newton’s method to find \( x_1 \) as your approximation. \( \Rightarrow \)
2. Use Newton’s Method to approximate the cube root of 10 to 2 decimal places. \( \Rightarrow \)
3. The function \( f(x) = x^3 - 3x - 2 \) has a root between 3 and 4, because \( f(3) = -3 \) and \( f(4) = 10 \) Approximate the root to two decimal places. \( \Rightarrow \)
4. A rectangular piece of cardboard of dimensions 8 \( \times \) 12 is used to make an open-top box by cutting out a small square of side \( x \) from each corner and bending up the sides. (See exercise 20 in 6.1.) If \( x = 2 \), then the volume of the box is 2 \( \times \) 4 \( \times \) 13 \( = \) 104. Use Newton’s method to find a value of \( x \) for which the box has volume 100, accurate to 3 significant figures. \( \Rightarrow \)
Newton’s method is one example of the usefulness of the tangent line as an approximation to a curve. Here we explore another such application.

Recall that the tangent line to \( f(x) \) at a point \( x=a \) is given by \( L(x) = f'(a)(x-a) + f(a) \). The tangent line in this context is also called the linear approximation to \( f \) at \( a \). While \( f \) is differentiable at \( a \) then under a microscope \( f \) will look very much like a straight line. Figure 6.4.1 shows a tangent line to \( y = x^2 \) at three different magnifications.

If we want to approximate \( f(b) \), because computing it exactly is difficult, we can approximate the value using a linear approximation, provided that we can compute the tangent line at some close to \( b \).

Example 6.4.1

Let \( f(x) = \sqrt{x} \). Then \( f'(x) = 1/(2\sqrt{x}) \). The linear approximation to \( f \) at \( x = 5 \) is \( L(x) = 1/(2\sqrt{5})(x-5) + \sqrt{5} = (x-5)/6 + \sqrt{5} \). As an immediate application we can approximate square roots of numbers near 9 by hand.

To estimate \( \sqrt{10} \), we substitute 6 into the linear approximation instead of into \( f(x) \), so \( \sqrt{6} + 1/6 \approx 3 + 1/6 \approx 3.17 \). This rounds to 3.17 while the square root of 10 is actually 3.16 to two decimal places, so this estimate is only accurate to one decimal place. This is not too surprising, as 10 is really not very close to 9; on the other hand, for many actually 3 \( \sqrt{2} \) would be accurate enough.

With modern calculators and computing software it may not appear necessary to use linear approximations. But in fact they are quite useful. In cases requiring an explicit numerical approximation, they allow us to get a quick rough estimate which can be used as a “reality check” on a more complex calculation. In some complex calculations involving

**Exercises 6.4.**

1. Let \( f(x) = x^3 \). If \( a = 1 \) and \( dx = \Delta x = 1/2 \), what are \( \Delta y \) and \( dy \)?

2. Let \( f(x) = \sqrt{x} \). If \( a = 1 \) and \( dx = \Delta x = 1/10 \), what are \( \Delta y \) and \( dy \)?

3. Let \( f(x) = \sin(2x) \). If \( a = \pi/2 \) and \( dx = \Delta x = \pi/100 \), what are \( \Delta y \) and \( dy \)?

4. Use differentials to estimate the amount of paint needed to apply a coat of paint \( 0.02 \) cm thick to a sphere with diameter \( 40 \) meters. (Recall that the volume of a sphere of radius \( r \) is \( V = (4/3)\pi r^3 \). Notice that you are given that \( dr = 0.0002 \) ?

5. Show in detail that the linear approximation of \( \sin(x) \) at \( x = 0 \) is \( L(x) = x \) and the linear approximation of \( \cos(x) \) at \( x = 0 \) is \( L(x) = 1 \).

**6.5 The Mean Value Theorem**

Here are two interesting questions involving derivatives:

1. Suppose two different functions have the same derivative; what can you say about the relationship between the two functions?

2. Suppose you drive a car from toll booth a on a toll road to another toll booth at an average speed of 70 miles per hour. What can be concluded about your actual speed during the trip? In particular, did you exceed the 65 mile per hour speed limit?

While those sound very different, it turns out that the two problems are very closely related. We know that “speed” is really the derivative by a different name: let’s start by translating the second question into something that may be easier to visualize. Suppose that if the function \( f(t) \) gives the position of your car on the toll road at time \( t \). Your change in position between one toll booth and the next is given by \( f(t_1) - f(t_0) \), assuming that at time \( t_0 \) you were at the first booth and at time \( t_1 \) you arrived at the second booth. Your average speed for the trip is \( f(t_1) - f(t_0) / (t_1 - t_0) \). If we think about the graph of \( f(t) \), the average speed is the slope of the line that connects the two points \( (t_0, f(t_0)) \) and \( (t_1, f(t_1)) \). Your speed at any particular time \( t \) between \( t_0 \) and \( t_1 \) is \( f'(t) \), the slope of curve. Now question 2 becomes a question about slope. In particular, if the slope between endpoints is 70, what can be said of the slopes at points between the endpoints?

As a general rule, when faced with a new problem it is often a good idea to examine one or more simplified versions of the problem, in the hope that this will lead to an understanding of the original problem. In this case, the problem in its “slope” form is somewhat easier to simplify than the original, but equivalent, problem.

Here is a special instance of the problem. Suppose that \( f(t_0) = f(t_1) \). Then the two endpoints have the same height and the slope of the line connecting the endpoints is zero. What can we say about the slope between the endpoints? It shouldn’t take much experimentation before you are convinced of the truth of this statement: Somewhere between \( t_0 \) and \( t_1 \) the slope is exactly zero, that is, somewhere between \( t_0 \) and \( t_1 \) the slope is equal to the slope of the line between the endpoints. This suggests that perhaps the same is true even if the endpoints are at different heights, and again a bit of experimentation will probably convince you that this is so. But we can do better than “experimentation” — we can prove that this is so.

We start with the simplified version:

**Theorem 6.5.1 Rolle’s Theorem**

Suppose that \( f(x) \) has a derivative on the interval \( [a, b] \) and is continuous on the interval \( [a, b] \), Then at some value \( c \) in \( (a, b) \).

**Proof.** We know that \( f(x) \) has a maximum and minimum value on \( [a, b] \) (because it is continuous), and we also know that the maximum and minimum must occur at an endpoint, at a point at which the derivative is zero, or at a point where the derivative is undefined. Since the derivative is never undefined, that possibility is removed.

If the maximum or minimum occurs at a point \( c \), other than an endpoint, where \( f'(c) = 0 \), then we have found the point we seek. Otherwise, the maximum and minimum both occur at an endpoint, and since the endpoints have the same height, the maximum and minimum are the same. This means that \( f(x) = f(a) \) at every \( x \in [a, b] \), so the function is a horizontal line, and it has derivative zero everywhere in \( (a, b) \). Then we may choose any \( c \) at all to get \( f'(c) = 0 \).

Perhaps remarkably, this special case is all we need to prove the more general one as well.

**Theorem 6.5.2 Mean Value Theorem**

Suppose that \( f(x) \) has a derivative on the interval \( (a, b) \) and is continuous on the interval \( [a, b] \). Then at some value \( c \) in \( (a, b) \),

**Proof.** Let \( m = (f(b) - f(a)) / (b - a) \) and consider a new function \( g(x) = f(x) - m(x - a) - f(a) \). We know that \( g(x) \) has a derivative everywhere, since \( g'(x) = f'(x) - m \). We can compute \( g(a) = f(a) - m(a - a) - f(a) = 0 \)

and \( \Delta y = \frac{dy}{dx} \Delta x = f'(c) \Delta x = f'(a) \Delta x \approx \frac{f(b) - f(a)}{b - a} \Delta x = \frac{f(b) - f(a)}{b - a} \Delta x = \frac{f(b) - f(a)}{b - a} \Delta x \).

This is not too surprising, as 10 is really not very close to 9; on the other hand, for many actually 3 \( \sqrt{2} \) would be accurate enough.
So the height of \( g(x) \) is the same at both endpoints. This means, by Rolle’s Theorem, that at some \( c \), \( g'(c) = 0 \). But we know that \( g'(c) = f'(c) - m \), so
\[
0 = f'(c) - m = f'(c) - (f(b) - f(a)) / (b - a),
\]
which turns into
\[
f'(c) = (f(b) - f(a)) / (b - a),
\]
which exactly what we want.

Returning to the original formulation of question (2), we see that if \( f(t) \) gives the position of your car at time \( t \), then the Mean Value Theorem says that at some time \( c \), \( f'(c) = 70 \), that is, at some time you must have been traveling at exactly your average speed for the trip, and that indeed you exceeded the speed limit.

Now let’s return to question (1). Suppose, for example, that two functions are known to have derivative equal to 5 everywhere, \( f \) and \( g \), and that indeed you exceeded the speed limit.

**EXAMPLE 6.5.3** Describe all functions with derivative \( 5x - 3 \). It’s easy to find one: \( g(x) = (5/2)x^2 - 3x \) has \( g'(x) = 5x - 3 \). The only other functions with the same derivative are therefore of the form \( f(x) = (5/2)x^2 - 3x + k \).

**EXAMPLE 6.5.4** If \( f(x) = g'(x) \) for every \( x \in (a,b) \), then for some constant \( k \), \( f(x) = g(x) + k \) on the interval \( (a,b) \).

**THEOREM 6.5.4** If \( f'(x) = g'(x) \) for every \( x \in (a,b) \), then for some constant \( k \), \( f(x) = g(x) + k \) on the interval \( (a,b) \).

**Exercises 6.5.**

1. Let \( f(x) = x^2 \). Find a value \( c \in [-1,2] \) so that \( f'(c) \) equals the slope between the endpoints of \( f(x) \) on \([-1,2] \).
2. Verify that \( f(x) = x/(x + 2) \) satisfies the hypotheses of the Mean Value Theorem on the interval \([1,4]\) and then find all of the values, \( c \), that satisfy the conclusion of the theorem.
3. Verify that \( f(x) = 3x/(x + 7) \) satisfies the hypotheses of the Mean Value Theorem on the interval \([-2,6]\) and then find all of the values, \( c \), that satisfy the conclusion of the theorem.
4. Let \( f(x) = \tan x \). Show that \( f'(x) = f(2x) = 0 \) but there is no number \( c \in (\pi,2\pi) \) such that \( f'(c) = 0 \). Why does this not contradict Rolle’s theorem?
5. Let \( f(x) = \cos x + k \). Show that there is no value \( c \in (1,4) \) such that \( f'(c) = f(4) - f(1)/(4 - 1) \). Why is this not a contradiction of the Mean Value Theorem?
6. Describe all functions with derivative \( x^2 + 47x - 5 \).
7. Describe all functions with derivative \( \sin(2x) \).
8. Show that the equation \( \sin x = -x + 1 \) does not have more than two distinct real roots.
9. Let \( f \) be differentiable on \( R \). Suppose that \( f(x) \neq 0 \) for every \( x \). Prove that \( f \) has at most one real root.
10. Prove that for all real \( x \) and \( y \), \( x - \cos x - \cos y \leq |x - y| \). State and prove an analogous result involving sine.
11. Show that \( \sqrt{1 + x^2} \leq 1 + (x/2) \) if \(-1 < x < 1\).