Applications of the Derivative

6.1 Optimization

Many important applied problems involve finding the best way to accomplish some task. Often this involves finding the maximum or minimum value of some function: the minimum time to make a certain journey, the minimum cost for doing a task, the maximum power that can be generated by a device, and so on. Many of these problems can be solved by finding the appropriate function and then using techniques of calculus to find the maximum or the minimum value required.

Generally such a problem will have the following mathematical form. Find the largest (or smallest) value of \( f(x) \) when \( a \leq x \leq b \). Sometimes \( a \) or \( b \) are infinite, but frequently the real world imposes some constraint on the values that \( x \) may have.

Such a problem differs in two ways from the local maximum and minimum problems we encountered when graphing functions: We are interested only in the function between \( a \) and \( b \), and we want to know the largest or smallest value that \( f(x) \) takes on, not merely values that are the largest or smallest in a small interval. That is, we seek not a local maximum or minimum but a global maximum or minimum, sometimes also called an absolute maximum or minimum.

Any global maximum or minimum must of course be a local maximum or minimum. If we find all possible local extrema, then the global maximum, if it exists, must be the largest of the local maxima and the global minimum, if it exists, must be the smallest of the local minima. We already know where local extrema can occur: only at those points at which \( f'(x) \) is zero or undefined. Actually, there are two additional points at which a maximum or minimum can occur if the endpoints \( a \) and \( b \) are not infinite, namely, at \( a \) and \( b \). We have not previously considered such points because we have not been interested in limiting a function to a small interval. An example should make this clear.

**EXAMPLE 6.1.1** Find the maximum and minimum values of \( f(x) = x^2 \) on the interval \([-2, 1]\], shown in figure 6.1.1. We compute \( f'(x) = 2x \), which is zero at \( x = 0 \) and is always defined.

Since \( f'(1) = 2 \) we would not normally flag \( x = 1 \) as a point of interest, but it is clear from the graph that when \( f(x) \) is restricted to \([-2, 1]\) there is a local maximum at \( x = 1 \). Likewise we would not normally pay attention to \( x = -2 \), but since we have truncated \( f \) at \(-2 \) we have introduced a new local maximum there as well. In a technical sense nothing new is going on here: When we truncate \( f \) we actually create a new function, let's call it \( g \), that is defined only on the interval \([-2, 1]\). If we try to compute the derivative of this new function we actually find that it does not have a derivative at \(-2 \) or \( 1 \). Why? Because to compute the derivative at \( 1 \) we must compute the limit

\[
\lim_{\Delta x \to 0} \frac{g(1 + \Delta x) - g(1)}{\Delta x}
\]

This limit does not exist because when \( \Delta x > 0 \), \( g(1 + \Delta x) \) is not defined. It is simpler, however, simply to remember that we must always check the endpoints.

So the function \( g \), that is, \( f \) restricted to \([-2, 1]\), has one critical value and two finite endpoints, any of which might be the global maximum or minimum. We could first determine which of these are local maximum or minimum points (or neither); then the largest local maximum must be the global maximum and the smallest local minimum must be the global minimum. It is usually easier, however, to compute the value of \( f \) at every point at which the global maximum or minimum might occur; the largest of these is the global maximum, the smallest is the global minimum.

So we compute \( f(-2) = 4 \), \( f(0) = 0 \), \( f(1) = 1 \). The global maximum is 4 at \( x = -2 \) and the global minimum is 0 at \( x = 0 \).

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It is possible that there is no global maximum or minimum. It is difficult, and not particularly useful, to express a complete procedure for determining whether this is the case. Generally, the best approach is to gain enough understanding of the shape of the graph to decide. Fortunately, only a rough idea of the shape is usually needed.

There are some particularly nice cases that are easy. A continuous function on a closed interval \([a, b]\) always has both a global maximum and a global minimum, so examining the critical values and the endpoints is enough.

**THEOREM 6.1.2 Extreme value theorem** If \( f \) is continuous on a closed interval \([a, b]\), then it has both a minimum and a maximum value. That is, there are real numbers \( c \) and \( d \) in \([a, b]\) so that for every \( x \) in \([a, b]\), \( f(x) \leq f(c) \) and \( f(x) \geq f(d) \).

Another easy case: If a function is continuous and has a single critical value, then if there is a local maximum at the critical value it is a global maximum, and if it is a local minimum it is a global minimum. There may also be a global minimum in the first case, or a global maximum in the second case, but that will generally require more effort to determine.

**EXAMPLE 6.1.2** Let \( f(x) = -x^2 + 4x - 3 \). Find the maximum value of \( f(x) \) on the interval \([0, 4]\). First note that \( f'(x) = -2x + 4 = 0 \) when \( x = 2 \), and \( f''(2) = -4 < 0 \). Next observe that \( f''(x) \) is defined for all \( x \), so there are no other critical values. Finally, \( f(0) = 3 \) and \( f(4) = 7 \). The largest value of \( f(x) \) on the interval \([0, 4]\) is \( f(4) = 7 \).

**EXAMPLE 6.1.3** Let \( f(x) = -x^3 + 4x - 3 \). Find the maximum value of \( f(x) \) on the interval \([-1, 1]\).

First note that \( f'(x) = -2x + 4 = 0 \) when \( x = 2 \). But \( x = 2 \) is not in the interval, so we don't use it. Thus the only two points to be checked are the endpoints: \( f(-1) = -8 \) and \( f(1) = 0 \). So the largest value of \( f(x) \) on \([-1, 1]\) is \( f(1) = 0 \).

**EXAMPLE 6.1.4** Find the maximum and minimum values of the function \( f(x) = 7 + |x| - 2| | \) for \( x \) between 1 and 4 inclusive. The derivative \( f'(x) \) is never zero, but \( f'(x) \) is undefined at \( x = 2 \), so we compute \( f(2) = 5 \). Checking the end points we get \( f(1) = 6 \) and \( f(4) = 9 \). The smallest of these numbers is \( f(2) = 7 \), which is, therefore, the minimum value of \( f(x) \) on the interval \( 1 \leq x \leq 4 \), and the maximum is \( f(4) = 9 \).

**EXAMPLE 6.1.5** Find all local maxima and minima for \( f(x) = x^3 - x \), and determine whether there is a global maximum or minimum on the open interval \((-2, 2)\). In example 5.1.2 we found a local maximum at \((-\sqrt{3}/3, 2\sqrt{3}/9)\) and a local minimum at \((\sqrt{3}/3, -2\sqrt{3}/9)\). Since the endpoints are not in the interval \((-2, 2)\) they cannot be considered. Is the local maximum a global maximum? Here we must look more closely at the graph. We know that on the closed interval \([-\sqrt{3}/3, \sqrt{3}/3]\) there is a global maximum at \( x = -\sqrt{3}/3 \) and a global minimum at \( x = \sqrt{3}/3 \). So the question becomes: what happens between \(-2 \) and \(-\sqrt{3}/3 \), and between \(\sqrt{3}/3 \) and 2? Since there is a local minimum at \( x = \sqrt{3}/3 \), the graph must continue up to the right, since there are no more critical values. This means no value of \( f \) will be less than \(-2\sqrt{3}/9 \) or greater than \(2\sqrt{3}/9 \). How can we tell? Since the function increases to the right of \(\sqrt{3}/3 \), we need to know what the function does “close to” 2. Here the easiest test is to pick a number and do a computation to get some idea of what’s going on. Since \( f(1.9) = 4.959 > 2\sqrt{3}/9 \), there is no global maximum at \(-\sqrt{3}/3 \), and hence no global maximum at all. (How can we tell that \(4.959 > 2\sqrt{3}/9 \)?) We can use a calculator to approximate the right hand side; if it is not even close to 4.959 we can take this as decisive. Since \(2\sqrt{3}/9 \approx 0.3849\), there’s really no question. Funny things can happen in the rounding done by computers and calculators, however, so we might be a little more careful, especially if the values come out quite close. In this case we can convert the relation \(4.959 > 2\sqrt{3}/9 \) into \((2/2)4.959 > \sqrt{3}\) and ask whether this is true. Since the left side is clearly larger than \(4 \) which is clearly larger than \(\sqrt{3} \), this settles the question.

A similar analysis shows that there is also no global minimum. The graph of \( f(x) \) on \((-2, 2) \) is shown in figure 6.1.2.

**EXAMPLE 6.1.6** Of all rectangles of area 100, which has the smallest perimeter?

First we must translate this into a purely mathematical problem in which we want to find the minimum value of a function. If \( x \) denotes one of the sides of the rectangle, then the adjacent side must be \(100/x \) in order that the area be 100. So the function we want

\[
f(x) = 2(x + 100/x)
\]
to minimize is $$f(x) = 2x + 2\frac{100}{x}$$ since the perimeter is twice the length plus twice the width of the rectangle. Not all values of $$x$$ make sense in this problem: lengths of sides of rectangles must be positive, so $$x > 0$$. If $$x > 0$$ then is $$100/x$$, so we need no second condition on $$x$$.

We next find $$f'(x)$$ and set it equal to zero: $$0 = f'(x) = 2 - 200/x^2$$. Solving $$f'(x) = 0$$ for $$x$$ gives us $$x = \pm 10$$. We are interested only in $$x > 0$$, so only the value $$x = 10$$ is of interest. Since $$f''(x)$$ is defined everywhere on the interval $$(0,\infty)$$, there are no more critical values, and there are no endpoints. Is there a local maximum, minimum, or neither at $$x = 10$$? The second derivative is $$f''(x) = 400/x^3$$, and $$f''(10) > 0$$, so there is a local minimum. Since there is only one critical value, this is also the global minimum, so the rectangle with smallest perimeter is the $$10 \times 10$$ square.

**Example 6.1.8** You want to sell a certain number $$n$$ of items in order to maximize your profit. Market research tells you that if you set the price at $$51.50$$, you will be able to sell 5000 items, and for every 10 cents you lower the price below 51.50 you will be able to sell another 1000 items. Suppose that your fixed costs ("start-up costs") total $28000, and the per item cost of production ("marginal cost") is $80. Find the price to set per item and the number of items sold in order to maximize profit, and also determine the maximum profit you can get.

The first step is to convert the problem into a function maximization problem. Since we want to maximize profit by setting the price per item, we should look for a function $$P(x)$$ representing the profit when the price per item is $$x$$. Profit is revenue minus costs, and revenue is number of items sold times the price per item, so we get $$P = nx - 20000 - 0.5nx$$. The number of items sold is itself a function of $$x$$; $$n = 5000 + (1000(1.5 - x))/0.10$$, because $$(1.5 - x)/0.10$$ is the number of 10 cents of 10 cents that the price is below $1.50. Now we substitute for $$n$$ in the profit function: $$P(x) = (5000 + 1000(1.5 - x)/0.10)x - 20000 - 0.5(5000 + 1000(1.5 - x)/0.10) = - 10000x^2 + 25000x - 12000$$ We want to know the maximum value of this function when $$x$$ is between 0 and 1.5. The derivative is $$P'(x) = - 20000x + 25000$$, which is zero when $$x = 1.25$$. Since $$P''(x) = - 20000 < 0$$, there must be a local maximum at $$x = 1.25$$, and since this is the only critical value it must be a global maximum as well. (Alternatively, we could compute $$P(0) = - 12000$$, $$P(1.25) = 3625$$, and $$P(1.5) = 3000$$ and note that $$P(1.25)$$ is the maximum of these.) Thus the maximum profit is $3625, attained when we set the price at $1.25 and sell 7500 items.

**Figure 6.1.4** We have situated the picture in a convenient way relative to the axis $$a$$ and $$y$$ axes, namely, the center of the sphere at the origin and the vertex of the cone at the far left on the $$x$$-axis.

Notice that the function we want to maximize, $$\pi x^2 h/3$$, depends on two variables. This is frequently the case, but often the two variables are related in some way so that "really" there is only one variable. So our next step is to find the relationship and use it to solve for one of the variables in terms of the other, so as to have a function of only one variable to maximize. In this problem, the condition is apparent in the figure: the upper corner of the triangle, whose coordinates are $$(h - R, r)$$, must be on the circle of radius $$R$$. That is, $$(h - R)^2 + r^2 = R^2$$. We can solve for $$h$$ in terms of $$r$$ or for $$r$$ in terms of $$h$$. Either involves taking a square root, but we notice that the volume function contains $$r^2$$, not $$r$$ by itself, so it is easiest to solve for $$r^2$$ directly: $$r^2 = R^2 - (h - R)^2$$. Then we substitute the result into $$\pi x^2 h/3$$:

$$V(h) = \pi (R^2 - (h - R)^2)h/3 = \pi R^3 - \frac{2}{3} \pi R^2$$

We want to maximize $$V(h)$$ when $$h$$ is between 0 and 2$$R$$. Now we solve $$0 = f'(h) = -\pi h^2 + (4/3)\pi Rh$$; getting $$h = 0$$ or $$h = 4R/3$$. We compute $$V(0) = V(2R) = 0$$ and $$V(4R/3) = (32/81)\pi R^3$$. The maximum is the latter; since the volume of the sphere is $$(4/3)\pi R^3$$, the fraction of the sphere occupied by the cone is

$$\frac{(32/81)\pi R^3}{(4/3)\pi R^3} = \frac{8}{27} \approx 30\%$$

**Figure 6.1.3** Rectangle in a parabola.

**Example 6.1.9** Find the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by "cone" we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.) Let $$R$$ be the radius of the sphere, and let $$r$$ and $$h$$ be the base radius and height of the cone inside the sphere. What we want to maximize is the volume of the cone: $$\pi r^2 h/3$$. Here $$R$$ is a fixed value, but $$r$$ and $$h$$ can vary. Namely, we could choose $$r$$ to be as large as possible—equal to $$R$$—by taking the height equal to $$R$$; or we could make the cone’s height $$h$$ larger at the expense of making $$r$$ a little less than $$R$$. See the cross-section depicted in

**Figure 6.1.5** Minimizing travel time.

**Example 6.1.10** If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by “cone” we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.) Let $$R$$ be the radius of the sphere, and let $$r$$ and $$h$$ be the base radius and height of the cone inside the sphere. What we want to maximize is the volume of the cone: $$\pi r^2 h/3$$. Here $$R$$ is a fixed value, but $$r$$ and $$h$$ can vary. Namely, we could choose $$r$$ to be as large as possible—equal to $$R$$—by taking the height equal to $$R$$; or we could make the cone’s height $$h$$ larger at the expense of making $$r$$ a little less than $$R$$. See the cross-section depicted in

**Figure 6.1.5** Minimizing travel time.
EXAMPLE 6.1.12 Suppose you want to reach a point A that is located across the sand from a nearby road (see figure 6.1.5). Suppose that the road is straight, and b is the distance from A to the closest point C on the road. Let \( v \) be your speed on the road, and let \( w \), which is less than \( v \), be your speed on the sand. Right now you are at the point \( B \), which is a distance \( a \) from \( C \). At what point \( B \) should you turn off the road and head across the sand in order to minimize your travel time to \( A \)?

Let \( x \) be the distance short of \( C \) where you turn off, i.e., the distance from \( B \) to \( C \). We want to minimize the total travel time. Recall that when traveling at constant velocity, time is distance divided by velocity.

You travel the distance \( \sqrt{a^2 + x^2} \) at speed \( v \), and then the distance \( \sqrt{x^2 + b^2} \) at speed \( w \). The total time for the trip is

\[
T(x) = \frac{\sqrt{a^2 + x^2}}{v} + \frac{\sqrt{x^2 + b^2}}{w}
\]

We want to find the minimum value of \( f \) when \( x \) is between 0 and \( a \). As usual we set

\[
f'(x) = \frac{-x}{v\sqrt{a^2 + x^2}} + \frac{-x}{w\sqrt{x^2 + b^2}}
\]

The critical values occur where the derivative is zero, so for values of \( x \) where

\[
f'(x) = 0 \quad \Rightarrow \quad \frac{-x}{v\sqrt{a^2 + x^2}} = \frac{-x}{w\sqrt{x^2 + b^2}}
\]

Solving for \( x \) gives

\[
x = \frac{v^2}{\sqrt{a^2 + v^2}} \quad \text{and} \quad x = \frac{w^2}{\sqrt{w^2 + b^2}}
\]

Notice that \( a \) does not appear in the last expression, but \( a \) is not irrelevant, since we are interested only in critical values that are in \([0, a] \) and \( w\sqrt{\sqrt{a^2 + v^2} - \sqrt{x^2 + b^2}} \) is either in this interval or not. If it is, we can use the second derivative to test it:

\[
f''(x) = \frac{2v^2}{(a^2 + v^2)^\frac{3}{2}} - \frac{2w^2}{(w^2 + b^2)^\frac{3}{2}}<0
\]

Since this is always positive there is a local minimum at the critical point, and so it is a global minimum as well.

If the critical value is not in \([0, a] \), it is larger than \( a \). In this case the minimum must occur at one of the endpoints. We compute

\[
f(a) = \frac{v}{\sqrt{a^2 + v^2}} \quad \text{and} \quad f(0) = \frac{w}{\sqrt{b^2 + w^2}}
\]

8. You have 1 foot of fence to make a rectangular play area alongside the wall of your house.

9. Marketing tells you that if you set the price of an item at $10 then you will be unable to sell it, but that you can sell 500 items for each dollar below $10 that you set the price. Suppose your fixed costs total $3000, and your marginal cost is $2 per item. What is the most profit you can make?"
30. If you fit the cone with the largest possible surface area (lateral area plus area of base) into a sphere, what percent of the volume of the sphere is occupied by the cone?

31. Two electrical charges, one a positive charge A of magnitude $a$ and the other a negative charge B of magnitude $b$, are located a distance $c$ apart. A positively charged particle $P$ is situated on the line between A and B. Find where $P$ should be put so that the pull away from A towards B is minimal. Here assume that the force from each charge is proportional to the strength of the source and inversely proportional to the square of the distance from the source.

32. Find the fraction of the area of a triangle that is occupied by the largest rectangle that can be drawn in the triangle (with one of its sides along a side of the triangle). Show that this fraction does not depend on the dimensions of the given triangle.

33. How are your answers to Problem 6 affected if the cost per item for the volume discounts) by 1 cent for each 25 items produced?

34. Two electrical charges, one a positive charge $A$ of magnitude $a$ and the other a negative charge $B$ of magnitude $b$, are located a distance $c$ apart. A positively charged particle $P$ is situated on the line between A and B. Find where $P$ should be put so that the pull away from A towards B is minimal. Here assume that the force from each charge is proportional to the strength of the source and inversely proportional to the square of the distance from the source.

We are interested in the time at which $y$-fast is its radius increasing when the radius is 4 cm?

Thus, $\dot{r} = 400$ mph.
EXAMPLE 6.2.6  A road running north to south crosses a road going east to west at the point \( P \). Car \( A \) is driving north along the first road, and car \( B \) is driving east along the second road. At a particular time car \( A \) is 10 kilometers to the north of \( P \) and traveling at 80 km/hr, while car \( B \) is 15 kilometers to the east of \( P \) and traveling at 108 km/hr. How fast is the distance between the two cars changing?

7. Sand is poured onto a surface at 15 cm\(^3\)/sec, forming a conical pile whose base diameter is always equal to its altitude. How fast is the altitude of the pile increasing when the pile is 3 cm high? 

8. A boat is pulled in to a dock by a rope with one end attached to the front of the boat and the other end passing through a ring attached to the dock at a point 5 ft higher than the front of the boat. The rope is being pulled through the ring at a rate of 0.6 ft/sec. How fast is the boat approaching the dock when 13 ft of rope are out? 

9. A balloon is at a height of 50 meters, and is rising at the constant rate of 5 m/sec. A bicyclist passes beneath it, traveling in a straight line at the constant speed of 10 m/sec. How fast is the distance between the bicyclist and the balloon increasing 2 seconds later? 

10. A pyramid-shaped vat has square cross-section and stands on its tip. The dimensions at the horizon, as depicted in figure 6.2.5. How fast is the shadow of a 25 meter wall lengthening at the time of interest? 

Notice how this problem differs from example 6.2.2. In both cases we started with the Pythagorean Theorem and took derivatives on both sides. However, in example 6.2.2 one of the sides was a constant (the altitude of the plane), and so the derivative of the square of that side of the triangle was simply zero. In this example, on the other hand, all three sides of the right triangle are variables, even though we are interested in a specific value of each side of the triangle (namely, when the sides have lengths 10 and 15). Make sure that you understand at the start of the problem what are the variables and what are the constants.

Exercises 6.2.

1. A cylindrical tank standing upright (with one circular base on the ground) has radius 20 cm. How fast does the water level in the tank drop when the water is being drained at 3 cm\(^3\)/sec? 

2. A cylindrical tank standing upright (with one circular base on the ground) has radius 1 meter. How fast does the water level in the tank drop when the water is being drained at 3 liters per second? 

3. A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.6 m/sec. How fast is the top sliding down the wall when the foot of the ladder is 5 m from the wall? 

4. A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The top of the ladder is being pulled up the wall at 0.1 meters per second. How fast is the foot of the ladder approaching the wall when the foot of the ladder is 5 m from the wall? 

5. A rotating beacon is located 2 miles out in the water. Let \( P \) be the point on the shore that is closest to the beacon. As the beacon rotates at 10 rev/min, the beam of light sweeps down the shore once each time it revolves. Assume that the shore is straight. How fast is the point \( P \) moving? 

6. A baseball diamond is a square 90 ft on a side. A player runs from first base to second base at 15 ft/sec. At what rate is the player’s distance from third base decreasing when she is half way from first to second base? 

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Let \( a(t) \) be the distance of car \( A \) north of \( P \) at time \( t \), and let \( b(t) \) be the distance of car \( B \) east of \( P \) at time \( t \), and let \( c(t) \) be the distance from car \( A \) to car \( B \) at time \( t \). By the Pythagorean Theorem, \( c(t)^2 = a(t)^2 + b(t)^2 \). Taking derivatives we get \( 2c(t)c'(t) = 2a(t)a'(t) + 2b(t)b'(t) \), so 

\[
\frac{dc}{dt} = \frac{a(t)a'(t) + b(t)b'(t)}{c(t)}.
\]

Substituting known values we get: 

\[
\frac{dc}{dt} = \frac{2a(t)b(t)}{c(t)}.
\]

at the time of interest. 

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14. A man 1.8 meters tall walks at the rate of 1 meter per second toward a streetlight that is 4 meters above the ground. At what rate is the tip of his shadow moving? At what rate is her shadow lengthening? 

15. A police helicopter is flying at 150 mph at a constant altitude of 0.5 mile above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 1 mile from the helicopter, and that this distance is decreasing at 180 mph. Find the speed of the car. 

16. A police helicopter is flying at 200 kilometers per hour at a constant altitude of 1 km above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 2 kilometers from the helicopter, and that this distance is decreasing at 250 kph. Find the speed of the car. 

14. A light shines from the top of a pole 20 m high. An object is dropped from the same height from a point 10 m away, so that its height at time \( t \) is \( h(t) = 20 - 9.8t^2/2 \). How fast is the object’s shadow moving on the ground one second later?

9. Do example 6.2.6 assuming that the angle between the two roads is 120° instead of 90° (that is, the “north–south” road actually goes in a somewhat northwest–southeast direction from \( P \)). Recall the law of cosines: \( c^2 = a^2 + b^2 - 2ab \cos \theta \). 

10. Do example 6.2.6 assuming that car \( A \) is 300 meters north of \( P \), car \( B \) is 400 meters east of \( P \), both cars are going at constant speed toward \( P \), and the two cars will collide in 10 seconds. 

11. Do example 6.2.6 assuming that 8 seconds ago car \( A \) started from rest at \( P \) and has been picking up speed at the steady rate of 5 m/sec\(^2 \), and 6 seconds after car \( A \) started car \( B \) passed \( P \) moving east at constant speed 60 m/sec. 

12. Referring again to example 6.2.6, suppose that instead of car \( B \) an airplane is flying at speed 200 km/hr to the east of \( P \) at an altitude of 2 km, as depicted in figure 6.2.8. How fast is the distance between car and airplane changing? 

13. Referring again to example 6.2.6, suppose that instead of car \( B \) an airplane is flying at speed 200 km/hr to the east of \( P \) at an altitude of 2 km, that it is gaining altitude at 10 km/hr. How fast is the distance between car and airplane changing? 

15. A police helicopter is flying at 150 mph at a constant altitude of 0.5 mile above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 1 mile from the helicopter, and that this distance is decreasing at 180 mph. Find the speed of the car.

16. A police helicopter is flying at 200 kilometers per hour at a constant altitude of 1 km above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 2 kilometers from the helicopter, and that this distance is decreasing at 250 kph. Find the speed of the car.
6.3 Newton's Method

Suppose you have a function \( f(x) \), and you want to find as accurately as possible where it crosses the \( x \)-axis; in other words, you want to solve \( f(x) = 0 \). Suppose you know of no way to find an exact solution by algebraic procedure, but you are able to use an approximation, provided it can be made quite close to the true value. Newton's method is a way to find a solution to the equation to as many decimal places as you want. It is what is called an "iterative procedure," meaning that it can be repeated again and again to get an answer of greater and greater accuracy. Iterative procedures like Newton's method are well suited to programming for a computer. Newton's method uses the fact that the tangent line to a curve is a good approximation to the curve near the point of tangency.

**EXAMPLE 6.3.1** Approximate \( \sqrt{3} \). Since \( \sqrt{3} \) is a simple equation to \( x^2 - 3 = 0 \), we use \( f(x) = x^2 - 3 \). We start by guessing something reasonably close to the true value; this is usually easy to do; let's use \( \sqrt{3} \) to be our first approximation to the curve, as shown in figure 6.3.1. Since \( f'(x) = 2x \), the slope of this tangent line is 4 and its equation is \( y = 4x - 7 \). The tangent line is quite close to \( f(x) \), so it crosses the \( x \)-axis near the point where \( f(x) \) crosses, that is, near \( \sqrt{3} \). It is easy to find where the tangent line crosses the \( x \)-axis: solve \( 0 = 4x - 7 \) to get \( x = \frac{7}{4} = 1.75 \). This is certainly a better approximation than 2, but let us say not close enough. We can improve it by doing the same thing again: find the tangent line at \( x = 1.75 \), find where this new tangent line crosses the \( x \)-axis, and use that value as a better approximation. We can continue indefinitely, though it gets a bit tedious. Let's see if we can shortcut the process. Suppose the host approximation to the intercept we have so far is \( x_0 \). To find a better approximation we will always do the same thing: find the slope of the tangent line at \( x_0 \), find the equation of the tangent line, find the intercept. The slope is \( 2x_0 \). The tangent line is \( y = (2x_0)(x-x_0) + (x_0^2 - 3) \), using the point-slope formula for a line. Finally, the intercept is found by solving \( 0 = (2x_0)(x-x_0) + (x_0^2 - 3) \). With a little algebra this turns into \( x = (x_0^2 + 3)/(2x_0) \), this is the next approximation, which we naturally call \( x_{i+1} \). Instead of doing the whole tangent line computation every time we can simply use this formula to get as many approximations as we want. Starting with \( x_0 = 2 \), we get \( x_1 = (x_0^2 + 3)/(2x_0) = (2^2 + 3)/4 = 7/4 \) (the same approximation we got above, of course), \( x_2 = (x_1^2 + 3)/(2x_1) = (49/16 + 3)/7/2 = 97/56 \approx 1.73214 \). We might guess already that 1.73205 is accurate to two decimal places, and in fact it turns out that it is accurate to 5 places.

Let's think about this process in more general terms. We want to approximate a solution to \( f(x) = 0 \). We start with a rough guess, which we call \( x_0 \). We use the tangent line to \( f(x) \) to get a new approximation that we hope will be closer to the true value. What is the equation of the tangent line when \( x = x_0 \) ? The slope is \( f'(x_0) \) and the line goes through \((x_0, f(x_0))\), so the equation of the line is \( y = f(x_0)(x-x_0) + f(x_0) \).

We note that we want to know where \( 2x = \tan x \) or \( f(x) = \tan x - 2x = 0 \). We compute \( f'(x) = \sec^2 x - 2 \) and set up the formula:

\[
x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}
\]

From the graph in figure 6.3.2 we guess \( x_0 = 1 \) as a starting point, then using the formula we compute \( x_1 = 1.310476030 \), \( x_2 = 1.233290696 \), \( x_3 = 1.176565900 \), \( x_4 = 1.167528093 \), \( x_5 = 1.165561636 \). So we guess that the first three places are correct, but that is not the same as saying 1.165 is correct to three decimal places; 1.165 might be correct, rounded approximation. How can we tell? We can substitute 1.165 and 1.1655 and 1.166 into \( tan x - 2x \); this gives −0.002495852, −0.002712471, 0.013986242. Since the first two are negative and the third is positive, \( tan x - 2x \) crosses the \( x \)-axis between 1.165 and 1.166, so the correct value to three places is 1.166.

**Exercises 6.3.**

1. Approximate the fifth root of 7, using \( x_0 = 1.5 \) as a first guess. Use Newton's method to find \( x_1 \) as your approximation. ⇒
2. Use Newton's Method to approximate the cube root of 10 to 4 decimal places. ⇒
3. The function \( f(x) = x^5 - 3x^3 - 2x + 6 \) has a root between 3 and 4, because \( f(3) = -3 \) and \( f(4) = 10 \). Approximate the root to two decimal places. ⇒
4. A rectangular piece of cardboard of dimensions 8 \times 12 is used to make an open-top box by cutting out a small square of side \( x \) from each corner and bending up the sides. (See exercise 20 in 6.1.) If \( x = 2 \), then the volume of the box is \( 2 \cdot 4 \cdot 13 = 104 \). Use Newton's method to find a value of \( x \) for which the box has volume 100, accurate to 3 significant figures. ⇒

RAW TEXT END
6.4 Linear Approximations

Newton’s method is one example of the usefulness of the tangent line as an approximation to a curve. Here we explore another such application.

Recall that the tangent line to \( f(x) \) at a point \( x = a \) is given by \( L(x) = f(a)(x-a) + f(a) \). The tangent line in this context is also called the linear approximation to \( f \) at \( a \).

If \( f \) is differentiable at \( a \), then under a microscope \( f \) will look very much like a straight line. Figure 6.4.1 shows a tangent line to \( y = x^2 \) at three different magnifications.

If we want to approximate \((b)\), because computing it exactly is difficult, we can approximate the value using a linear approximation, provided that we can compute the tangent line at some close to \( b \).

\[
\text{Figure 6.4.1 The linear approximation to } y = x^2.
\]

**Example 6.4.1** Let \( f(x) = \sqrt{x} \). Then \( f(x) = \frac{1}{2}x^{-1/2} \). The linear approximation to \( f \) at \( x = 5 \) is \( L(x) = 1/2\sqrt{5} + \frac{x-5}{2\sqrt{5}} \). As an immediate application we can approximate square roots of numbers near 9 by hand.

To estimate \( \sqrt{10} \), we substitute 6 into the linear approximation instead of into \( f(x) \), so \( \sqrt{10} \approx 3.17 \). This rounds to 3.17 while the square root of 10 is actually 3.16 to two decimal places, so this estimate is only accurate to one decimal place. This is not too surprising, as 10 is really not very close to 9 on the other hand, for many calculations, 3.2 would be accurate enough.

With modern calculators and computing software it may not appear necessary to use linear approximations. But in fact they are quite useful. In cases requiring an explicit numerical approximation, they allow us to get a quick rough estimate which can be used as a “reality check” on a more complex calculation. In some complex calculations involving

**Exercises 6.4**

1. Let \( f(x) = x^3 \). If \( a = 1 \) and \( dx = \Delta x = 1/2 \), what are \( \Delta y \) and \( dy \)?

2. Let \( f(x) = \sqrt{x} \). If \( a = 1 \) and \( dx = \Delta x = 1/10 \), what are \( \Delta y \) and \( dy \)?

3. Let \( f(x) = \sin(2x) \). If \( a = \pi \) and \( dx = \Delta x = \pi/100 \), what are \( \Delta y \) and \( dy \)?

4. Differentials to estimate the amount of paint needed to apply a coat of paint 0.02 cm thick to a sphere with diameter 40 meters. (Recall that the volume of a sphere of radius \( r \) is \( V = (4/3)\pi r^3 \). Notice that you are given that \( dr = 0.0002 \).)

5. Show in detail that the linear approximation of \( \sin x \) at \( x = 0 \) is \( L(x) = x \) and the linear approximation of \( \cos x \) at \( x = 0 \) is \( L(x) = 1 \).

6.5 The Mean Value Theorem

Here are two interesting questions involving derivatives:

1. Suppose two different functions have the same derivative; what can you say about the relationship between the two functions?

2. Suppose you drive a car from toll booth on a toll road to another toll booth at an average speed of 70 miles per hour. What can be concluded about your actual speed during the trip? In particular, did you exceed the 65 mile per hour speed limit?

While those sound very different, it turns out that the two problems are very closely related. We know that “speed” is really the derivative by a different name; let’s start by translating the second question into something that may be easier to visualize. Suppose that \( f(t) \) gives the position of your car on the toll road at time \( t \). Your change in position between one toll booth and the next is given by \( f(t_2) - f(t_1) \), assuming that at time \( t_1 \) you were at the first booth and at time \( t_1 \) you arrived at the second booth.

Your average speed for the trip is \( \frac{f(t_2) - f(t_1)}{t_2 - t_1} \). If we think about the graph of \( f(t) \), the average speed is the slope of the line that connects the points \((t_1, f(t_1))\) and \((t_2, f(t_2))\). Your speed at any particular time \( t \) between \( t_1 \) and \( t_2 \) is \( f'(t) \), the slope of the curve. Now question (2) becomes a question about slope. In particular, if the slope between endpoints is \( 70 \), what can be said of the slopes at points between the endpoints?

As a general rule, when faced with a new problem it is often a good idea to examine one or more simplified versions of the problem, in the hope that this will lead to an understanding of the original problem. In this case, the problem in its “slopes” form is somewhat easier to simplify than the original, but equivalent, problem.

Here is a special instance of the problem. Suppose that \( f(t_0) = f(t_1) \). Then the two endpoints have the same height and the slope of the line connecting the endpoints is zero. What can we say about the slope between the endpoints? It shouldn’t take much experimentation before you are convinced of the truth of this statement: Somewhere

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functions, the linear approximation makes an otherwise intractable calculation possible, without serious loss of accuracy.

**Example 6.4.2** Consider the trigonometric function \( \sin x \). Its linear approximation at \( x = 0 \) is simply \( L(x) = x \). When \( x \) is small this is quite a good approximation and is used frequently by engineers and scientists to simplify some calculations.

**Definition 6.4.3** Let \( y = f(x) \) be a differentiable function. We define a new independent variable \( dx \), and a new dependent variable \( dy = f'(x)dx \). Notice that \( dy \) is a function both of \( x \) (since \( f'(x) \) is a function of \( x \)) and of \( dx \). We say that \( dx \) and \( dy \) are differentials.

Let \( \Delta x = x - a \) and \( \Delta y = f(x) - f(a) \). If \( x \) is near \( a \) then \( \Delta x \) is small. If we set \( dx = \Delta x \) then

\[
dy = f'(a)dx \approx \frac{\Delta y}{\Delta x} \Delta x = \Delta y.
\]

Thus, \( dy \) can be used to approximate \( \Delta y \), the actual change in the function \( f \) between \( a \) and \( x \). This is exactly the approximation given by the tangent line:

\[
dy = f'(a)(x-a) \approx f(a) - f(a) = L(x) - f(a).
\]

While \( L(x) \) approximates \( f(x) \), \( dy \) approximates how \( f(x) \) has changed from \( f(a) \). Figure 6.4.2 illustrates the relationships.

\[
\text{Figure 6.4.2 Differentials.}
\]
So the height of \( g(x) \) is the same at both endpoints. This means, by Rolle’s Theorem, that at some \( c \), \( g'(c) = 0 \). But we know that \( g'(c) = f'(c) - m \), so

\[
0 = f'(c) - m = f'(c) - f(b) - f(a)
\]

which turns into

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

exactly what we want.

Returning to the original formulation of question (2), we see that if \( f(t) \) gives the position of your car at time \( t \), then the Mean Value Theorem says that at some time \( c \), \( f'(c) = 0 \), that is, at some time you must have been traveling at exactly your average speed for the trip, and that indeed you exceeded the speed limit.

Now let’s return to question (1). Suppose, for example, that two functions are known to have derivative equal to 5 everywhere, \( f'(x) = g'(x) = 5 \). It is easy to find such functions:

\[ 5x, 5x + 47, 5x - 132, \text{etc.} \]

Are there other, more complicated, examples? No—the only functions that work are the “obvious” ones, namely, 5x plus some constant. How can we see that this is true?

Although “5” is a very simple derivative, let’s look at an even simpler one. Suppose that \( f'(x) = g'(x) = 0 \). Again we can find examples: \( f(x) = 0, f(x) = 47, f(x) = -511 \) all have \( f'(x) = 0 \). Are there non-constant functions \( f \) with derivative 0? No, and here’s why: Suppose that \( f(x) \) is not a constant function. This means that there are two points on the function with different heights, say \( f(a) \neq f(b) \). The Mean Value Theorem tells us that at some point \( c \), \( f'(c) = \frac{(f(b) - f(a))/(b-a)} \)

So any non-constant function does not have a derivative that is zero everywhere; this is the same as saying that the only functions with zero derivative are the constant functions.

Let’s go back to the slightly less easy example: suppose that \( f'(x) = g'(x) = 5 \). Then

\[
(f(x) - x')' = f'(x) - g'(x) = 5 - 5 = 0
\]

So what we discovered in the previous paragraph, we know that \( f(x) - g(x) = k \), for some constant \( k \). So any two functions with derivative 5 must differ by a constant; since 5x is known to work, the only other examples must look like 5x + k.

Now we can extend this to more complicated functions, without any extra work. Suppose that \( f'(x) = g'(x) \). Then as before \( (f(x) - g(x))' = f'(x) - g'(x) = 0 \), so \( f(x) - g(x) = k \). Again this means that if we find just a single function \( g(x) \) with a certain derivative, then every other function with the same derivative must be of the form \( g(x) + k \).

**EXAMPLE 6.5.3** Describe all functions that have derivative \( 5x - 3 \). It’s easy to find one: \( g(x) = (5/2)x^2 - 3x \) has \( g'(x) = 5x - 3 \). The only other functions with the same derivative are therefore of the form \( f(x) = (5/2)x^2 - 3x + k \).

**EXAMPLE 6.5.5** Describe all functions with derivative \( \sin(x) \). One such function is \( -\cos x \); so all such functions have the form \( -\cos x + k \).

**Exercises 6.5.**

1. Let \( f(x) = x^2 \). Find a value \( c \in (-1,2) \) so that \( f'(c) \) equals the slope between the endpoints of \( f(x) \) on \([-1,2]\).

2. Verify that \( f(x) = x^2 + 3x + 2 \) satisfies the hypotheses of the Mean Value Theorem on the interval \([-1,2]\), and then find all of the values, \( c \), that satisfy the conclusion of the theorem.

3. Verify that \( f(x) = 3x^2 + 7 \) satisfies the hypotheses of the Mean Value Theorem on the interval \([-2,0]\), and then find all of the values, \( c \), that satisfy the conclusion of the theorem.

4. Let \( f(x) = \tan x \). Show that \( f(\pi/2) = 0 \) but there is no number \( c \in (\pi/2,\pi) \) such that \( f'(c) = 0 \). Why does this not contradict Rolle’s theorem?

5. Let \( f(x) = (x - 3)^2 \). Show that there is no value \( x \in (1,4) \) such that \( f'(c) = (f(4) - f(1))/(4-1) \). Why is this not a contradiction of the Mean Value Theorem?

6. Describe all functions with derivative \( x^2 + 4x - 5 \).

7. Describe all functions with derivative \( 3x^2 + 2x \).

8. Show that the equation \( x^2 + 2x = 1 \) does not have more than two distinct real roots.

9. Let \( f \) be differentiable on \( R \). Suppose that \( f'(c) \neq 0 \) for every \( x \). Prove that \( f \) has at most one real root.

10. Prove that for any real \( x \) and \( y \), \( \cos x - \cos y \leq |x - y| \). State and prove an analogous result involving \( \sin \).

11. Show that \( \sqrt[n]{x} \leq 1 + (n/2) \) if \(-1 < x \leq 1 \).