2

Instantaneous Rate of Change: The Derivative

2.1 The slope of a function

Suppose that y is a function of x, say $y = f(x)$. It is often necessary to know how sensitive the value of y is to small changes in x.

EXAMPLE 2.1.1 Take, for example, $y = f(x) = \sqrt{625 - x^2}$ (the upper semicircle of radius 25 centered at the origin). When $x = 7$, we find that $y = \sqrt{625 - 49} = 24$. Suppose we want to know how much y changes when x increases a little, say to 7.1 or 7.01.

In the case of a straight line $y = mx+b$, the slope $m = \Delta y/\Delta x$ measures the change in y per unit change in x. This can be interpreted as a measure of "sensitivity"; for example, if $y = 100x + 5$, a small change in x corresponds to a change one hundred times as large in y, so y is quite sensitive to changes in x.

Let us look at the same ratio $\Delta y/\Delta x$ for our function $y = f(x) = \sqrt{625 - x^2}$ when x changes from 7 to 7.1. Here $\Delta x = 7.1 - 7 = 0.1$ is the change in x, and

$$\begin{aligned}\Delta y = f(x + \Delta x) - f(x) &= f(7.1) - f(7)\\ &= \sqrt{625 - 7.1^2} - \sqrt{625 - 7^2} \approx 23.9706 - 24 = -0.0294.\end{aligned}$$

Thus, $\Delta y/\Delta x \approx -0.0294/0.1 = -0.294$. This means that y changes by less than one third the change in x, so apparently y is not very sensitive to changes in x at $x = 7$. We say "apparently" here because we don't really know what happens between 7 and 7.1. Perhaps y changes dramatically as x runs through the values from 7 to 7.1, but at 7.1 y just happens to be close to its value at 7. This is not in fact the case for this particular function, but we don't yet know why. □

One way to interpret the above calculation is by reference to a line. We have computed the slope of the line through $(7, 24)$ and $(7.1, 23.9706)$, called a **chord** of the circle. In general, if we draw the chord from the point $(7, 24)$ to a nearby point on the semicircle $(7 + \Delta x,\ f(7 + \Delta x))$, the slope of this chord is the so-called **difference quotient**

$$\text{slope of chord} = \frac{f(7 + \Delta x) - f(7)}{\Delta x} = \frac{\sqrt{625 - (7 + \Delta x)^2} - 24}{\Delta x}.$$

For example, if x changes only from 7 to 7.01, then the difference quotient (slope of the chord) is approximately equal to $(23.997081 - 24)/0.01 = -0.2919$. This is slightly less steep than the chord from $(7, 24)$ to $(7.1, 23.9706)$.

As the second value $7 + \Delta x$ moves in towards 7, the chord joining $(7, f(7))$ to $(7 + \Delta x, f(7+\Delta x))$ shifts slightly. As indicated in figure 2.1.1, as Δx gets smaller and smaller, the chord joining $(7, 24)$ to $(7 + \Delta x, f(7 + \Delta x))$ gets closer and closer to the **tangent line** to the circle at the point $(7, 24)$. (Recall that the tangent line is the line that just grazes the circle at that point, i.e., it doesn't meet the circle at any second point.) Thus, as Δx gets smaller and smaller, the slope $\Delta y/\Delta x$ of the chord gets closer and closer to the slope of the tangent line. This is actually quite difficult to see when Δx is small, because of the scale of the graph. The values of Δx used for the figure are 1, 5, 10 and 15, not really very small values. The tangent line is the one that is uppermost at the right hand endpoint.

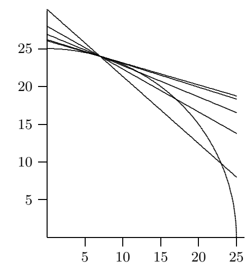


Figure 2.1.1 Chords approximating the tangent line. (AP)

So far we have found the slopes of two chords that should be close to the slope of the tangent line, but what is the slope of the tangent line exactly? Since the tangent line touches the circle at just one point, we will never be able to calculate its slope directly, using two "known" points on the line. What we need is a way to capture what happens to the slopes of the chords as they get "closer and closer" to the tangent line.

Instead of looking at more particular values of Δx, let's see what happens if we do some algebra with the difference quotient using just Δx. The slope of a chord from $(7, 24)$ to a nearby point is given by

$$\begin{aligned}\frac{\sqrt{625 - (7 + \Delta x)^2} - 24}{\Delta x} &= \frac{\sqrt{625 - (7 + \Delta x)^2} - 24}{\Delta x}\,\frac{\sqrt{625 - (7 + \Delta x)^2} + 24}{\sqrt{625 - (7 + \Delta x)^2} + 24}\\ &= \frac{625 - (7 + \Delta x)^2 - 24^2}{\Delta x(\sqrt{625 - (7 + \Delta x)^2} + 24)}\\ &= \frac{49 - 49 - 14\Delta x - \Delta x^2}{\Delta x(\sqrt{625 - (7 + \Delta x)^2} + 24)}\\ &= \frac{\Delta x(-14 - \Delta x)}{\Delta x(\sqrt{625 - (7 + \Delta x)^2} + 24)}\\ &= \frac{-14 - \Delta x}{\sqrt{625 - (7 + \Delta x)^2} + 24}\end{aligned}$$

Now, can we tell by looking at this last formula what happens when Δx gets very close to zero? The numerator clearly gets very close to -14 while the denominator gets very close to $\sqrt{625 - 7^2} + 24 = 48$. Is the fraction therefore very close to $-14/48 = -7/24 \cong -0.29167$? It certainly seems reasonable, and in fact it is true: as Δx gets closer and closer to zero, the difference quotient does in fact get closer and closer to $-7/24$, and so the slope of the tangent line is exactly $-7/24$.

What about the slope of the tangent line at $x = 12$? Well, 12 can't be all that different from 7; we just have to redo the calculation with 12 instead of 7. This won't be hard, but it will be a bit tedious. What if we try to do all the algebra without using a specific value for x? Let's copy from above, replacing 7 by x. We'll have to do a bit more than that—for

example, the "24" in the calculation came from $\sqrt{625 - 7^2}$, so we'll need to fix that too.

$$\begin{aligned}&\frac{\sqrt{625 - (x + \Delta x)^2} - \sqrt{625 - x^2}}{\Delta x} =\\ &= \frac{\sqrt{625 - (x + \Delta x)^2} - \sqrt{625 - x^2}}{\Delta x}\,\frac{\sqrt{625 - (x + \Delta x)^2} + \sqrt{625 - x^2}}{\sqrt{625 - (x + \Delta x)^2} + \sqrt{625 - x^2}}\\ &= \frac{625 - (x + \Delta x)^2 - 625 + x^2}{\Delta x(\sqrt{625 - (x + \Delta x)^2} + \sqrt{625 - x^2})}\\ &= \frac{625 - x^2 - 2x\Delta x - \Delta x^2 - 625 + x^2}{\Delta x(\sqrt{625 - (x + \Delta x)^2} + \sqrt{625 - x^2})}\\ &= \frac{\Delta x(-2x - \Delta x)}{\Delta x(\sqrt{625 - (x + \Delta x)^2} + \sqrt{625 - x^2})}\\ &= \frac{-2x - \Delta x}{\sqrt{625 - (x + \Delta x)^2} + \sqrt{625 - x^2}}\end{aligned}$$

Now what happens when Δx is very close to zero? Again it seems apparent that the quotient will be very close to

$$\frac{-2x}{\sqrt{625 - x^2} + \sqrt{625 - x^2}} = \frac{-2x}{2\sqrt{625 - x^2}} = \frac{-x}{\sqrt{625 - x^2}}.$$

Replacing x by 7 gives $-7/24$, as before, and now we can easily do the computation for 12 or any other value of x between -25 and 25.

So now we have a single, simple formula, $-x/\sqrt{625 - x^2}$, that tells us the slope of the tangent line for any value of x. This slope, in turn, tells us how sensitive the value of y is to changes in the value of x.

What do we call such a formula? That is, a formula with one variable, so that substituting an "input" value for the variable produces a new "output" value? This is a function. Starting with one function, $\sqrt{625 - x^2}$, we have derived, by means of some slightly nasty algebra, a new function, $-x/\sqrt{625 - x^2}$, that gives us important information about the original function. This new function in fact is called the **derivative** of the original function. If the original is referred to as f or y then the derivative is often written f' or y' and pronounced "f prime" or "y prime", so in this case we might write $f'(x) = -x/\sqrt{625 - x^2}$. At a particular point, say $x = 7$, we say that $f'(7) = -7/24$ or "f prime of 7 is $-7/24$" or "the derivative of f at 7 is $-7/24$."

To summarize, we compute the derivative of $f(x)$ by forming the difference quotient

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}, \tag{2.1.1}$$

which is the slope of a line, then we figure out what happens when Δx gets very close to 0.

We should note that in the particular case of a circle, there's a simple way to find the derivative. Since the tangent to a circle at a point is perpendicular to the radius drawn to the point of contact, its slope is the negative reciprocal of the slope of the radius. The radius joining $(0,0)$ to $(7,24)$ has slope $24/7$. Hence, the tangent line has slope $-7/24$. In general, a radius to the point $(x, \sqrt{625-x^2})$ has slope $\sqrt{625-x^2}/x$, so the slope of the tangent line is $-x/\sqrt{625-x^2}$, as before. It is **NOT** always true that a tangent line is perpendicular to a line from the origin—don't use this shortcut in any other circumstance.

As above, and as you might expect, for different values of x we generally get different values of the derivative $f'(x)$. Could it be that the derivative always has the same value? This would mean that the slope of f, or the slope of its tangent line, is the same everywhere. One curve that always has the same slope is a line; it seems odd to talk about the tangent line to a line, but if it makes sense at all the tangent line must be the line itself. It is not hard to see that the derivative of $f(x) = mx + b$ is $f'(x) = m$; see exercise 6.

Exercises 2.1.

1. Draw the graph of the function $y = f(x) = \sqrt{169 - x^2}$ between $x = 0$ and $x = 13$. Find the slope $\Delta y/\Delta x$ of the chord between the points of the circle lying over (a) $x = 12$ and $x = 13$, (b) $x = 12$ and $x = 12.1$, (c) $x = 12$ and $x = 12.01$, (d) $x = 12$ and $x = 12.001$. Now use the geometry of tangent lines on a circle to find (e) the exact value of the derivative $f'(12)$. Your answers to (a)–(d) should be getting closer and closer to your answer to (e). ⇒
2. Use geometry to find the derivative $f'(x)$ of the function $f(x) = \sqrt{625 - x^2}$ in the text for each of the following x: (a) 20, (b) 24, (c) -7, (d) -15. Draw a graph of the upper semicircle, and draw the tangent line at each of these four points. ⇒
3. Draw the graph of the function $y = f(x) = 1/x$ between $x = 1/2$ and $x = 4$. Find the slope of the chord between (a) $x = 3$ and $x = 3.1$, (b) $x = 3$ and $x = 3.01$, (c) $x = 3$ and $x = 3.001$. Now use algebra to find a simple formula for the slope of the chord between $(3, f(3))$ and $(3 + \Delta x, f(3 + \Delta x))$. Determine what happens when Δx approaches 0. In your graph of $y = 1/x$, draw the straight line through the point $(3, 1/3)$ whose slope is this limiting value of the difference quotient as Δx approaches 0. ⇒
4. Find an algebraic expression for the difference quotient $\big(f(1+\Delta x) - f(1)\big)/\Delta x$ when $f(x) = x^2 - (1/x)$. Simplify the expression as much as possible. Then determine what happens as Δx approaches 0. That value is $f'(1)$. ⇒
5. Draw the graph of $y = f(x) = x^3$ between $x = 0$ and $x = 1.5$. Find the slope of the chord between (a) $x = 1$ and $x = 1.1$, (b) $x = 1$ and $x = 1.001$, (c) $x = 1$ and $x = 1.00001$. Then use algebra to find a simple formula for the slope of the chord between 1 and $1 + \Delta x$. (Use the expansion $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.) Determine what happens as Δx approaches 0, and in your graph of $y = x^3$ draw the straight line through the point $(1, 1)$ whose slope is equal to the value you just found. ⇒
6. Find an algebraic expression for the difference quotient $(f(x+\Delta x) - f(x))/\Delta x$ when $f(x) = mx + b$. Simplify the expression as much as possible. Then determine what happens as Δx approaches 0. That value is $f'(x)$. ⇒

7. Sketch the unit circle. Discuss the behavior of the slope of the tangent line at various angles around the circle. Which trigonometric function gives the slope of the tangent line at an angle θ? Why? Hint: think in terms of ratios of sides of triangles.
8. Sketch the parabola $y = x^2$. For what values of x on the parabola is the slope of the tangent line positive? Negative? What do you notice about the graph at the point(s) where the sign of the slope changes from positive to negative and vice versa?

2.2 An example

We started the last section by saying, "It is often necessary to know how sensitive the value of y is to small changes in x." We have seen one purely mathematical example of this: finding the "steepness" of a curve at a point is precisely this problem. Here is a more applied example.

With careful measurement it might be possible to discover that a dropped ball has height $h(t) = h_0 - kt^2$, t seconds after it is released. (Here h_0 is the initial height of the ball, when $t = 0$, and k is some number determined by the experiment.) A natural question is then, "How fast is the ball going at time t?" We can certainly get a pretty good idea with a little simple arithmetic. To make the calculation more concrete, let's say $h_0 = 100$ meters and $k = 4.9$ and suppose we're interested in the speed at $t = 2$. We know that when $t = 2$ the height is $100 - 4 \cdot 4.9 = 80.4$. A second later, at $t = 3$, the height is $100 - 9 \cdot 4.9 = 55.9$, so in that second the ball has traveled $80.4 - 55.9 = 24.5$ meters. This means that the *average* speed during that time was 24.5 meters per second. So we might guess that 24.5 meters per second is not a terrible estimate of the speed at $t = 2$. But certainly we can do better. At $t = 2.5$ the height is $100 - 4.9(2.5)^2 = 69.375$. During the half second from $t = 2$ to $t = 2.5$ the ball dropped $80.4 - 69.375 = 11.025$ meters, at an average speed of $11.025/(1/2) = 22.05$ meters per second; this should be a better estimate of the speed at $t = 2$. So it's clear now how to get better and better approximations: compute average speeds over shorter and shorter time intervals. Between $t = 2$ and $t = 2.01$, for example, the ball drops 0.19649 meters in one hundredth of a second, at an average speed of 19.649 meters per second.

We can't do this forever, and we still might reasonably ask what the actual speed precisely at $t = 2$ is. If Δt is some tiny amount of time, what we want to know is what happens to the average speed $(h(2) - h(2+\Delta t))/\Delta t$ as Δt gets smaller and smaller. Doing

a bit of algebra:

$$\begin{aligned}
\frac{h(2) - h(2+\Delta t)}{\Delta t} &= \frac{80.4 - (100 - 4.9(2+\Delta t)^2)}{\Delta t} \\
&= \frac{80.4 - 100 + 19.6 + 19.6\Delta t + 4.9\Delta t^2}{\Delta t} \\
&= \frac{19.6\Delta t + 4.9\Delta t^2}{\Delta t} \\
&= 19.6 + 4.9\Delta t
\end{aligned}$$

When Δt is very small, this is very close to 19.6, and indeed it seems clear that as Δt goes to zero, the average speed goes to 19.6, so the exact speed at $t = 2$ is 19.6 meters per second. This calculation should look very familiar. In the language of the previous section, we might have started with $f(x) = 100 - 4.9x^2$ and asked for the slope of the tangent line at $x = 2$. We would have answered that question by computing

$$\frac{f(2+\Delta x) - f(2)}{\Delta x} = \frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x} = -19.6 - 4.9\Delta x$$

The algebra is the same, except that following the pattern of the previous section the subtraction would be reversed, and we would say that the slope of the tangent line is -19.6. Indeed, in hindsight, perhaps we should have subtracted the other way even for the dropping ball. At $t = 2$ the height is 80.4; one second later the height is 55.9. The usual way to compute a "distance traveled" is to subtract the earlier position from the later one, or $55.9 - 80.4 = -24.5$. This tells us that the distance traveled is 24.5 meters, and the negative sign tells us that the height went down during the second. If we continue the original calculation we then get -19.6 meters per second as the exact speed at $t = 2$. If we interpret the negative sign as meaning that the motion is downward, which seems reasonable, then in fact this is the same answer as before, but with even more information, since the numerical answer contains the direction of motion as well as the speed. Thus, the speed of the ball is the value of the derivative of a certain function, namely, of the function that gives the position of the ball. (More properly, this is the *velocity* of the ball; velocity is signed speed, that is, speed with a direction indicated by the sign.)

The upshot is that this problem, finding the speed of the ball, is *exactly* the same problem mathematically as finding the slope of a curve. This may already be enough evidence to convince you that whenever some quantity is changing (the height of a curve or the height of a ball or the size of the economy or the distance of a space probe from earth or the population of the world) the rate at which the quantity is changing can, in principle, be computed in exactly the same way, by finding a derivative.

Exercises 2.2.

1. An object is traveling in a straight line so that its position (that is, distance from some fixed point) is given by this table:

time (seconds)	0	1	2	3
distance (meters)	0	10	25	60

   Find the average speed of the object during the following time intervals: $[0, 1]$, $[0, 2]$, $[0, 3]$, $[1, 2]$, $[1, 3]$, $[2, 3]$. If you had to guess the speed at $t = 2$ just on the basis of these, what would you guess? ⇒
2. Let $y = f(t) = t^2$, where t is the time in seconds and y is the distance in meters that an object falls on a certain airless planet. Draw a graph of this function between $t = 0$ and $t = 3$. Make a table of the average speed of the falling object between (a) 2 seconds and 3 seconds, (b) 2 seconds and 2.1 seconds, (c) 2 seconds and 2.01 seconds, (d) 2 seconds and 2.001 seconds. Then use algebra to find a simple formula for the average speed between time 2 and time $2+\Delta t$. (If you substitute $\Delta t = 1, 0.1, 0.01, 0.001$ in this formula you should again get the answers to parts (a)–(d).) Next, in your formula for average speed (which should be in simplified form) determine what happens as Δt approaches zero. This is the instantaneous speed. Finally, in your graph of $y = t^2$ draw the straight line through the point $(2, 4)$ whose slope is the instantaneous velocity you just computed; it should of course be the tangent line. ⇒
3. If an object is dropped from an 80-meter high window, its height y above the ground at time t seconds is given by the formula $y = f(t) = 80 - 4.9t^2$. (Here we are neglecting air resistance; the graph of this function was shown in figure 1.0.1.) Find the average velocity of the falling object between (a) 1 seconds and 1.1 seconds, (b) 1 seconds and 1.01 seconds, (c) 1 seconds and 1.001 seconds. Now use algebra to find a simple formula for the average velocity of the falling object between 1 seconds and $1+\Delta t$ seconds. Determine what happens to this average velocity as Δt approaches 0. That is the instantaneous velocity at time $t = 1$ second (it will be negative, because the object is falling). ⇒

2.3 Limits

In the previous two sections we computed some quantities of interest (slope, velocity) by seeing that some expression "goes to" or "approaches" or "gets really close to" a particular value. In the examples we saw, this idea may have been clear enough, but it is too fuzzy to rely on in more difficult circumstances. In this section we will see how to make the idea more precise.

There is an important feature of the examples we have seen. Consider again the formula

$$\frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x}.$$

We wanted to know what happens to this fraction as "Δx goes to zero." Because we were able to simplify the fraction, it was easy to see the answer, but it was not quite as simple as "substituting zero for Δx," as that would give

$$\frac{-19.6\cdot 0 - 4.9\cdot 0}{0},$$

which is meaningless. The quantity we are really interested in does not make sense "at zero," and this is why the answer to the original problem (finding a velocity or a slope) was not immediately obvious. In other words, we are generally going to want to figure out what a quantity "approaches" in situations where we can't merely plug in a value. If you would like to think about a hard example (which we will analyze later) consider what happens to $(\sin x)/x$ as x approaches zero. Try evaluating this expression on a calculator for smaller and smaller values of x; do they seem to approach some fixed value?

EXAMPLE 2.3.1 Does $\sqrt{x}$ approach 1.41 as x approaches 2? In this case it is possible to compute the actual value $\sqrt{2}$ to a high precision to answer the question. But since in general we won't be able to do that, let's not. We might start by computing $\sqrt{x}$ for values of x close to 2, as we did in the previous sections. Here are some values: $\sqrt{2.05} = 1.431782106$, $\sqrt{2.04} = 1.428285686$, $\sqrt{2.03} = 1.424780685$, $\sqrt{2.02} = 1.421267040$, $\sqrt{2.01} = 1.417744688$, $\sqrt{2.005} = 1.415980226$, $\sqrt{2.004} = 1.415627070$, $\sqrt{2.003} = 1.415273825$, $\sqrt{2.002} = 1.414920492$, $\sqrt{2.001} = 1.414567072$. So it looks at least possible that indeed these values "approach" 1.41—already $\sqrt{2.001}$ is quite close. If we continue this process, however, at some point we will appear to "stall." In fact, $\sqrt{2} = 1.414213562\ldots$, so we will never even get as far as 1.4142, no matter how long we continue the sequence. $\square$

So in a fuzzy, everyday sort of sense, it is true that $\sqrt{x}$ "gets close to" 1.41, but it does not "approach" 1.41 in the sense we want. To compute an exact slope or an exact velocity, what we want to know is that a given quantity becomes "arbitrarily close" to a fixed value, meaning that the first quantity can be made "as close as we like" to the fixed value. Consider again the quantities

$$\frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x} = -19.6 - 4.9\Delta x.$$

These two quantities are equal as long as Δx is not zero; if Δx is zero, the left hand quantity is meaningless, while the right hand one is -19.6. Can we say more than we did before about why the right hand side "approaches" -19.6, in the desired sense? Can we really make it "as close as we want" to -19.6? Let's try a test case. Can we make $-19.6 - 4.9\Delta x$ within one millionth (0.000001) of -19.6? The values within a millionth of -19.6 are those in the interval $(-19.600001, -19.599999)$. As Δx approaches zero, does $-19.6 - 4.9\Delta x$ eventually reside inside this interval? If Δx is positive, this would require

that $-19.6 - 4.9\Delta x > -19.600001$. This is something we can manipulate with a little algebra:

$$\begin{aligned}
-19.6 - 4.9\Delta x &> -19.600001\\
-4.9\Delta x &> -0.000001\\
\Delta x &< -0.000001/-4.9\\
\Delta x &< 0.0000002040816327\ldots
\end{aligned}$$

Thus, we can say with certainty that if Δx is positive and less than 0.0000002, then $\Delta x < 0.0000002040816327\ldots$ and so $-19.6 - 4.9\Delta x > -19.600001$. We could do a similar calculation if Δx is negative.

So now we know that we can make $-19.6 - 4.9\Delta x$ within one millionth of -19.6. But can we make it "as close as we want"? In this case, it is quite simple to see that the answer is yes, by modifying the calculation we've just done. It may be helpful to think of this as a game. I claim that I can make $-19.6 - 4.9\Delta x$ as close as you desire to -19.6 by making Δx "close enough" to zero. So the game is: you give me a number, like 10^{-6}, and I have to come up with a number representing how close Δx must be to zero to guarantee that $-19.6 - 4.9\Delta x$ is at least as close to -19.6 as you have requested.

Now if we actually play this game, I could redo the calculation above for each new number you provide. What I'd like to do is somehow see that I will always succeed, and even more, I'd like to have a simple strategy so that I don't have to do all that algebra every time. A strategy in this case would be a formula that gives me a correct answer no matter what you specify. So suppose the number you give me is ϵ. How close does Δx have to be to zero to guarantee that $-19.6 - 4.9\Delta x$ is in $(-19.6 - \epsilon, -19.6 + \epsilon)$? If Δx is positive, we need:

$$\begin{aligned}
-19.6 - 4.9\Delta x &> -19.6 - \epsilon\\
-4.9\Delta x &> -\epsilon\\
\Delta x &< -\epsilon/-4.9\\
\Delta x &< \epsilon/4.9
\end{aligned}$$

So if I pick any number δ that is less than $\epsilon/4.9$, the algebra tells me that whenever $\Delta x < \delta$ then $\Delta x < \epsilon/4.9$ and so $-19.6 - 4.9\Delta x$ is within ϵ of -19.6. (This is exactly what I did in the example: I picked $\delta = 0.0000002 < 0.0000002040816327\ldots$.) A similar calculation again works for negative Δx. The important fact is that this is now a completely general result—it shows that I can always win, no matter what "move" you make.

Now we can codify this by giving a precise definition to replace the fuzzy, "gets closer and closer" language we have used so far. Henceforward, we will say something like "the limit of $(-19.6\Delta x - 4.9\Delta x^2)/\Delta x$ as Δx goes to zero is -19.6," and abbreviate this mouthful

as

$$\lim_{\Delta x\to 0} \frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x} = -19.6.$$

Here is the actual, official definition of "limit".

DEFINITION 2.3.2 Limit Suppose f is a function. We say that $\lim_{x\to a} f(x) = L$ if for every $\epsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x - a| < \delta$, $|f(x) - L| < \epsilon$. $\square$

The ϵ and δ here play exactly the role they did in the preceding discussion. The definition says, in a very precise way, that $f(x)$ can be made as close as desired to L (that's the $|f(x) - L| < \epsilon$ part) by making x close enough to a (the $0 < |x - a| < \delta$ part). Note that we specifically make no mention of what must happen if $x = a$, that is, if $|x - a| = 0$. This is because in the cases we are most interested in, substituting a for x doesn't even make sense.

Make sure you are not confused by the names of important quantities. The generic definition talks about $f(x)$, but the function and the variable might have other names. In the discussion above, the function we analyzed was

$$\frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x}.$$

and the variable of the limit was not x but Δx. The x was the variable of the original function; when we were trying to compute a slope or a velocity, x was essentially a fixed quantity, telling us at what point we wanted the slope. (In the velocity problem, it was literally a fixed quantity, as we focused on the time 2.) The quantity a of the definition in all the examples was zero: we were always interested in what happened as Δx became very close to zero.

Armed with a precise definition, we can now prove that certain quantities behave in a particular way. The bad news is that even proofs for simple quantities can be quite tedious and complicated; the good news is that we rarely need to do such proofs, because most expressions act the way you would expect, and this can be proved once and for all.

EXAMPLE 2.3.3 Let's show carefully that $\lim_{x\to 2} x + 4 = 6$. This is not something we "need" to prove, since it is "obviously" true. But if we couldn't prove it using our official definition there would be something very wrong with the definition.

As is often the case in mathematical proofs, it helps to work backwards. We want to end up showing that under certain circumstances $x + 4$ is close to 6; precisely, we want to show that $|x + 4 - 6| < \epsilon$, or $|x - 2| < \epsilon$. Under what circumstances? We want this to be true whenever $0 < |x - 2| < \delta$. So the question becomes: can we choose a value for δ that

guarantees that $0 < |x - 2| < \delta$ implies $|x - 2| < \epsilon$? Of course: no matter what ϵ is, $\delta = \epsilon$ works. $\square$

So it turns out to be very easy to prove something "obvious," which is nice. It doesn't take long before things get trickier, however.

EXAMPLE 2.3.4 It seems clear that $\lim_{x\to 2} x^2 = 4$. Let's try to prove it. We will want to be able to show that $|x^2 - 4| < \epsilon$ whenever $0 < |x - 2| < \delta$, by choosing δ carefully. Is there any connection between $|x - 2|$ and $|x^2 - 4|$? Yes, and it's not hard to spot, but it is not so simple as the previous example. We can write $|x^2 - 4| = |(x+2)(x-2)|$. Now when $|x-2|$ is small, part of $|(x+2)(x-2)|$ is small, namely $(x-2)$. What about $(x+2)$? If x is close to 2, $(x+2)$ certainly can't be too big, but we need to somehow be precise about it. Let's recall the "game" version of what is going on here. You get to pick an ϵ and I have to pick a δ that makes things work out. Presumably it is the really tiny values of ϵ I need to worry about, but I have to be prepared for anything, even an apparently "bad" move like $\epsilon = 1000$. I expect that ϵ is going to be small, and that the corresponding δ will be small, certainly less than 1. If $\delta \le 1$ then $|x + 2| < 5$ when $|x - 2| < \delta$ (because if x is within 1 of 2, then x is between 1 and 3 and $x + 2$ is between 3 and 5). So then I'd be trying to show that $|(x + 2)(x - 2)| < 5|x - 2| < \epsilon$. So now how can I pick δ so that $|x - 2| < \delta$ implies $5|x - 2| < \epsilon$? This is easy: use $\delta = \epsilon/5$, so $5|x - 2| < 5(\epsilon/5) = \epsilon$. But what if the ϵ you choose is not small? If you choose $\epsilon = 1000$, should I pick $\delta = 200$? No, to keep things "sane" I will never pick a δ bigger than 1. Here's the final "game strategy:" When you pick a value for ϵ I will pick $\delta = \epsilon/5$ or $\delta = 1$, whichever is smaller. Now when $|x - 2| < \delta$, I know both that $|x + 2| < 5$ and that $|x - 2| < \epsilon/5$. Thus $|(x + 2)(x - 2)| < 5(\epsilon/5) = \epsilon$.

This has been a long discussion, but most of it was explanation and scratch work. If this were written down as a proof, it would be quite short, like this:

Proof that $\lim_{x\to 2} x^2 = 4$. Given any ϵ, pick $\delta = \epsilon/5$ or $\delta = 1$, whichever is smaller. Then when $|x-2| < \delta$, $|x+2| < 5$ and $|x-2| < \epsilon/5$. Hence $|x^2-4| = |(x+2)(x-2)| < 5(\epsilon/5) = \epsilon$. $\square$

It probably seems obvious that $\lim_{x\to 2} x^2 = 4$, and it is worth examining more closely why it seems obvious. If we write $x^2 = x \cdot x$, and ask what happens when x approaches 2, we might say something like, "Well, the first x approaches 2, and the second x approaches 2, so the product must approach $2 \cdot 2$." In fact this is pretty much right on the money, except for that word "must." Is it really true that if x approaches a and y approaches b then xy approaches ab? It is, but it is not really obvious, since x and y might be quite complicated. The good news is that we can see that this is true once and for all, and then we don't have to worry about it ever again. When we say that x might be "complicated" we really mean that in practice it might be a function. Here is then what we want to know:

THEOREM 2.3.5 Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then $\lim_{x\to a} f(x)g(x) = LM$.

Proof. We have to use the official definition of limit to make sense of this. So given any ϵ we need to find a δ so that $0 < |x-a| < \delta$ implies $|f(x)g(x) - LM| < \epsilon$. What do we have to work with? We know that we can make $f(x)$ close to L and $g(x)$ close to M, and we have to somehow connect these facts to make $f(x)g(x)$ close to LM.

We use, as is so often the case, a little algebraic trick:

$$\begin{aligned}|f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM|\\ &= |f(x)(g(x)-M) + (f(x)-L)M|\\ &\le |f(x)(g(x)-M)| + |(f(x)-L)M|\\ &= |f(x)||g(x)-M| + |f(x)-L||M|.\end{aligned}$$

This is all straightforward except perhaps for the "$\le$". That is an example of the **triangle inequality**, which says that if a and b are any real numbers then $|a+b| \le |a| + |b|$. If you look at a few examples, using positive and negative numbers in various combinations for a and b, you should quickly understand why this is true; we will not prove it formally.

Since $\lim_{x\to a} f(x) = L$, there is a value δ_1 so that $0 < |x-a| < \delta_1$ implies $|f(x) - L| < |\epsilon/(2M)|$, This means that $0 < |x-a| < \delta_1$ implies $|f(x)-L||M| < \epsilon/2$. You can see where this is going: if we can make $|f(x)||g(x)-M| < \epsilon/2$ also, then we'll be done.

We can make $|g(x) - M|$ smaller than any fixed number by making x close enough to a; unfortunately, $\epsilon/(2f(x))$ is not a fixed number, since x is a variable. Here we need another little trick, just like the one we used in analyzing x^2. We can find a δ_2 so that $|x-a| < \delta_2$ implies that $|f(x)-L| < 1$, meaning that $L-1 < f(x) < L+1$. This means that $|f(x)| < N$, where N is either $|L-1|$ or $|L+1|$, depending on whether L is negative or positive. The important point is that N doesn't depend on x. Finally, we know that there is a δ_3 so that $0 < |x-a| < \delta_3$ implies $|g(x)-M| < \epsilon/(2N)$. Now we're ready to put everything together. Let δ be the smallest of δ_1, δ_2, and δ_3. Then $|x-a| < \delta$ implies that $|f(x)-L| < |\epsilon/(2M)|$, $|f(x)| < N$, and $|g(x)-M| < \epsilon/(2N)$. Then

$$\begin{aligned}|f(x)g(x) - LM| &\le |f(x)||g(x)-M| + |f(x)-L||M|\\ &< N\frac{\epsilon}{2N} + \left|\frac{\epsilon}{2M}\right||M|\\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

This is just what we needed, so by the official definition, $\lim_{x\to a} f(x)g(x) = LM$. ∎

A handful of such theorems give us the tools to compute many limits without explicitly working with the definition of limit.

THEOREM 2.3.6 Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ and k is some constant. Then

$$\begin{aligned}&\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x) = kL\\ &\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L+M\\ &\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L-M\\ &\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x) = LM\\ &\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M},\ \text{if } M \text{ is not } 0\end{aligned}$$

∎

Roughly speaking, these rules say that to compute the limit of an algebraic expression, it is enough to compute the limits of the "innermost bits" and then combine these limits. This often means that it is possible to simply plug in a value for the variable, since $\lim_{x\to a} x = a$.

EXAMPLE 2.3.7 Compute $\lim_{x\to 1}\frac{x^2-3x+5}{x-2}$. If we apply the theorem in all its gory detail, we get

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-3x+5}{x-2} &= \frac{\lim_{x\to 1}(x^2-3x+5)}{\lim_{x\to 1}(x-2)}\\ &= \frac{(\lim_{x\to 1} x^2) - (\lim_{x\to 1} 3x) + (\lim_{x\to 1} 5)}{(\lim_{x\to 1} x) - (\lim_{x\to 1} 2)}\\ &= \frac{(\lim_{x\to 1} x)^2 - 3(\lim_{x\to 1} x) + 5}{(\lim_{x\to 1} x) - 2}\\ &= \frac{1^2 - 3\cdot 1 + 5}{1-2}\\ &= \frac{1-3+5}{-1} = -3\end{aligned}$$

□

It is worth commenting on the trivial limit $\lim_{x\to 1} 5$. From one point of view this might seem meaningless, as the number 5 can't "approach" any value, since it is simply a fixed

number. But 5 can, and should, be interpreted here as the function that has value 5 everywhere, $f(x) = 5$, with graph a horizontal line. From this point of view it makes sense to ask what happens to the height of the function as x approaches 1.

Of course, as we've already seen, we're primarily interested in limits that aren't so easy, namely, limits in which a denominator approaches zero. There are a handful of algebraic tricks that work on many of these limits.

EXAMPLE 2.3.8 Compute $\lim_{x\to 1}\frac{x^2+2x-3}{x-1}$. We can't simply plug in $x=1$ because that makes the denominator zero. However:

$$\begin{aligned}\lim_{x\to 1}\frac{x^2+2x-3}{x-1} &= \lim_{x\to 1}\frac{(x-1)(x+3)}{x-1}\\ &= \lim_{x\to 1}(x+3) = 4\end{aligned}$$

□

While theorem 2.3.6 is very helpful, we need a bit more to work easily with limits. Since the theorem applies when some limits are already known, we need to know the behavior of some functions that cannot themselves be constructed from the simple arithmetic operations of the theorem, such as $\sqrt{x}$. Also, there is one other extraordinarily useful way to put functions together: composition. If $f(x)$ and $g(x)$ are functions, we can form two functions by composition: $f(g(x))$ and $g(f(x))$. For example, if $f(x) = \sqrt{x}$ and $g(x) = x^2+5$, then $f(g(x)) = \sqrt{x^2+5}$ and $g(f(x)) = (\sqrt{x})^2 + 5 = x+5$. Here is a companion to theorem 2.3.6 for composition:

THEOREM 2.3.9 Suppose that $\lim_{x\to a} g(x) = L$ and $\lim_{x\to L} f(x) = f(L)$. Then

$$\lim_{x\to a} f(g(x)) = f(L).$$

∎

Note the special form of the condition on f: it is not enough to know that $\lim_{x\to L} f(x) = M$, though it is a bit tricky to see why. Many of the most familiar functions do have this property, and this theorem can therefore be applied. For example:

THEOREM 2.3.10 Suppose that n is a positive integer. Then

$$\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{a},$$

provided that a is positive if n is even. ∎

This theorem is not too difficult to prove from the definition of limit.

Another of the most common algebraic tricks was used in section 2.1. Here's another example:

EXAMPLE 2.3.11 Compute $\lim_{x\to -1}\frac{\sqrt{x+5}-2}{x+1}$.

$$\begin{aligned}\lim_{x\to -1}\frac{\sqrt{x+5}-2}{x+1} &= \lim_{x\to -1}\frac{\sqrt{x+5}-2}{x+1}\,\frac{\sqrt{x+5}+2}{\sqrt{x+5}+2}\\ &= \lim_{x\to -1}\frac{x+5-4}{(x+1)(\sqrt{x+5}+2)}\\ &= \lim_{x\to -1}\frac{x+1}{(x+1)(\sqrt{x+5}+2)}\\ &= \lim_{x\to -1}\frac{1}{\sqrt{x+5}+2} = \frac{1}{4}\end{aligned}$$

At the very last step we have used theorems 2.3.9 and 2.3.10. □

Occasionally we will need a slightly modified version of the limit definition. Consider the function $f(x) = \sqrt{1-x^2}$, the upper half of the unit circle. What can we say about $\lim_{x\to 1} f(x)$? It is apparent from the graph of this familiar function that as x gets close to 1 from the left, the value of $f(x)$ gets close to zero. It does not even make sense to ask what happens as x approaches 1 from the right, since $f(x)$ is not defined there. The definition of the limit, however, demands that $f(1+\Delta x)$ be close to $f(1)$ whether Δx is positive or negative. Sometimes the limit of a function exists from one side or the other (or both) even though the limit does not exist. Since it is useful to be able to talk about this situation, we introduce the concept of **one sided limit**:

DEFINITION 2.3.12 One-sided limit Suppose that $f(x)$ is a function. We say that $\lim_{x\to a^-} f(x) = L$ if for every $\epsilon > 0$ there is a $\delta > 0$ so that whenever $0 < a - x < \delta$, $|f(x) - L| < \epsilon$. We say that $\lim_{x\to a^+} f(x) = L$ if for every $\epsilon > 0$ there is a $\delta > 0$ so that whenever $0 < x - a < \delta$, $|f(x) - L| < \epsilon$. □

Usually $\lim_{x\to a^-} f(x)$ is read "the limit of $f(x)$ from the left" and $\lim_{x\to a^+} f(x)$ is read "the limit of $f(x)$ from the right".

EXAMPLE 2.3.13 Discuss $\lim_{x\to 0}\frac{x}{|x|}$, $\lim_{x\to 0^-}\frac{x}{|x|}$, and $\lim_{x\to 0^+}\frac{x}{|x|}$.

The function $f(x) = x/|x|$ is undefined at 0; when $x > 0$, $|x| = x$ and so $f(x) = 1$; when $x < 0$, $|x| = -x$ and $f(x) = -1$. Thus $\lim_{x\to 0^-}\frac{x}{|x|} = \lim_{x\to 0^-} -1 = -1$ while $\lim_{x\to 0^+}\frac{x}{|x|} =$

$\lim_{x\to 0^+} 1 = 1$. The limit of $f(x)$ must be equal to both the left and right limits; since they are different, the limit $\lim_{x\to 0} \frac{x}{|x|}$ does not exist. □

Exercises 2.3.

Compute the limits. If a limit does not exist, explain why.

1. $\lim_{x\to 3} \frac{x^2+x-12}{x-3} \Rightarrow$

2. $\lim_{x\to 1} \frac{x^2+x-12}{x-3} \Rightarrow$

3. $\lim_{x\to -4} \frac{x^2+x-12}{x-3} \Rightarrow$

4. $\lim_{x\to 2} \frac{x^2+x-12}{x-2} \Rightarrow$

5. $\lim_{x\to 1} \frac{\sqrt{x+8}-3}{x-1} \Rightarrow$

6. $\lim_{x\to 0^+} \sqrt{\frac{1}{x}+2} - \sqrt{\frac{1}{x}} \Rightarrow$

7. $\lim_{x\to 2} 3 \Rightarrow$

8. $\lim_{x\to 4} 3x^3 - 5x \Rightarrow$

9. $\lim_{x\to 0} \frac{4x-5x^2}{x-1} \Rightarrow$

10. $\lim_{x\to 1} \frac{x^2-1}{x-1} \Rightarrow$

11. $\lim_{x\to 0^+} \frac{\sqrt{2-x^2}}{x} \Rightarrow$

12. $\lim_{x\to 0^+} \frac{\sqrt{2-x^2}}{x+1} \Rightarrow$

13. $\lim_{x\to a} \frac{x^3-a^3}{x-a} \Rightarrow$

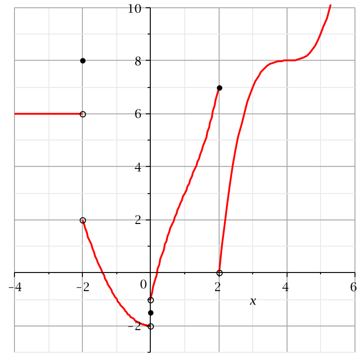
14. $\lim_{x\to 2} (x^2+4)^3 \Rightarrow$

15. $\lim_{x\to 1} \begin{cases} x-5 & x \neq 1, \\ 7 & x = 1. \end{cases} \Rightarrow$

16. $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$ (Hint: Use the fact that $|\sin a| \le 1$ for any real number a. You should probably use the definition of a limit here.) $\Rightarrow$

17. Give an ϵ–δ proof, similar to example 2.3.3, of the fact that $\lim_{x\to 4} (2x-5) = 3$.

18. Evaluate the expressions by reference to this graph:



(a) $\lim_{x\to 4} f(x)$ (b) $\lim_{x\to -3} f(x)$ (c) $\lim_{x\to 0} f(x)$
(d) $\lim_{x\to 0^-} f(x)$ (e) $\lim_{x\to 0^+} f(x)$ (f) $f(-2)$
(g) $\lim_{x\to 2^-} f(x)$ (h) $\lim_{x\to -2^-} f(x)$ (i) $\lim_{x\to 0} f(x+1)$
(j) $f(0)$ (k) $\lim_{x\to 1^-} f(x-4)$ (l) $\lim_{x\to 0^+} f(x-2)$
$\Rightarrow$

19. Use a calculator to estimate $\lim_{x\to 0} \frac{\sin x}{x}$.

20. Use a calculator to estimate $\lim_{x\to 0} \frac{\tan(3x)}{\tan(5x)}$.

2.4 The Derivative Function

We have seen how to create, or derive, a new function $f'(x)$ from a function $f(x)$, summarized in the paragraph containing equation 2.1.1. Now that we have the concept of limits, we can make this more precise.

DEFINITION 2.4.1 The derivative of a function f, denoted f', is

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

□

We know that f' carries important information about the original function f. In one example we saw that $f'(x)$ tells us how steep the graph of $f(x)$ is; in another we saw that $f'(x)$ tells us the velocity of an object if $f(x)$ tells us the position of the object at time x. As we said earlier, this same mathematical idea is useful whenever $f(x)$ represents some changing quantity and we want to know something about how it changes, or roughly, the "rate" at which it changes. Most functions encountered in practice are built up from a small collection of "primitive" functions in a few simple ways, for example, by adding or multiplying functions together to get new, more complicated functions. To make good use of the information provided by $f'(x)$ we need to be able to compute it for a variety of such functions.

We will begin to use different notations for the derivative of a function. While initially confusing, each is often useful so it is worth maintaining multiple versions of the same thing.

Consider again the function $f(x) = \sqrt{625-x^2}$. We have computed the derivative $f'(x) = -x/\sqrt{625-x^2}$, and have already noted that if we use the alternate notation $y = \sqrt{625-x^2}$ then we might write $y' = -x/\sqrt{625-x^2}$. Another notation is quite different, and in time it will become clear why it is often a useful one. Recall that to compute the the derivative of f we computed

$$\lim_{\Delta x\to 0} \frac{\sqrt{625-(7+\Delta x)^2} - 24}{\Delta x}.$$

The denominator here measures a distance in the x direction, sometimes called the "run", and the numerator measures a distance in the y direction, sometimes called the "rise," and "rise over run" is the slope of a line. Recall that sometimes such a numerator is abbreviated Δy, exchanging brevity for a more detailed expression. So in general, a derivative is given by

$$y' = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}.$$

To recall the form of the limit, we sometimes say instead that

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}.$$

In other words, dy/dx is another notation for the derivative, and it reminds us that it is related to an actual slope between two points. This notation is called **Leibniz notation**, after Gottfried Leibniz, who developed the fundamentals of calculus independently, at about the same time that Isaac Newton did. Again, since we often use f and $f(x)$ to mean the original function, we sometimes use df/dx and $df(x)/dx$ to refer to the derivative. If

the function $f(x)$ is written out in full we often write the last of these something like this

$$f'(x) = \frac{d}{dx}\sqrt{625-x^2}$$

with the function written to the side, instead of trying to fit it into the numerator.

EXAMPLE 2.4.2 Find the derivative of $y = f(t) = t^2$.

We compute

$$\begin{aligned} y' = \lim_{\Delta t\to 0} \frac{\Delta y}{\Delta t} &= \lim_{\Delta t\to 0} \frac{(t+\Delta t)^2 - t^2}{\Delta t} \\ &= \lim_{\Delta t\to 0} \frac{t^2 + 2t\Delta t + \Delta t^2 - t^2}{\Delta t} \\ &= \lim_{\Delta t\to 0} \frac{2t\Delta t + \Delta t^2}{\Delta t} \\ &= \lim_{\Delta t\to 0} 2t + \Delta t = 2t. \end{aligned}$$

Remember that Δt is a single quantity, not a "Δ" times a "t", and so Δt^2 is $(\Delta t)^2$ not $\Delta(t^2)$. □

EXAMPLE 2.4.3 Find the derivative of $y = f(x) = 1/x$.

The computation:

$$\begin{aligned} y' = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x\to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\frac{x}{x(x+\Delta x)} - \frac{x+\Delta x}{x(x+\Delta x)}}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\frac{x-(x+\Delta x)}{x(x+\Delta x)}}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{x - x - \Delta x}{x(x+\Delta x)\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{-\Delta x}{x(x+\Delta x)\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{-1}{x(x+\Delta x)} = \frac{-1}{x^2} \end{aligned}$$

□

Note. If you happen to know some "derivative formulas" from an earlier course, for the time being you should pretend that you do not know them. In examples like the ones above and the exercises below, you are required to know how to find the derivative formula starting from basic principles. We will later develop some formulas so that we do not always need to do such computations, but we will continue to need to know how to do the more involved computations.

Sometimes one encounters a point in the domain of a function $y = f(x)$ where there is *no derivative*, because there is no tangent line. In order for the notion of the tangent line at a point to make sense, the curve must be "smooth" at that point. This means that if you imagine a particle traveling at some steady speed along the curve, then the particle does not experience an abrupt change of direction. There are two types of situations you should be aware of—corners and cusps—where there's a sudden change of direction and hence no derivative.

EXAMPLE 2.4.4 Discuss the derivative of the absolute value function $y = f(x) = |x|$.

If x is positive, then this is the function $y = x$, whose derivative is the constant 1. (Recall that when $y = f(x) = mx + b$, the derivative is the slope m.) If x is negative, then we're dealing with the function $y = -x$, whose derivative is the constant -1. If $x = 0$, then the function has a corner, i.e., there is no tangent line. A tangent line would have to point in the direction of the curve—but there are *two* directions of the curve that come together at the origin. We can summarize this as

$$y' = \begin{cases} 1 & \text{if } x > 0; \\ -1 & \text{if } x < 0; \\ \text{undefined} & \text{if } x = 0. \end{cases}$$

□

EXAMPLE 2.4.5 Discuss the derivative of the function $y = x^{2/3}$, shown in figure 2.4.1. We will later see how to compute this derivative; for now we use the fact that $y' = (2/3)x^{-1/3}$. Visually this looks much like the absolute value function, but it technically has a cusp, not a corner. The absolute value function has no tangent line at 0 because there are (at least) two obvious contenders—the tangent line of the left side of the curve and the tangent line of the right side. The function $y = x^{2/3}$ does not have a tangent line at 0, but unlike the absolute value function it can be said to have a single direction: as we approach 0 from either side the tangent line becomes closer and closer to a vertical line; the curve is vertical at 0. But as before, if you imagine traveling along the curve, an abrupt change in direction is required at 0: a full 180 degree turn. □

In practice we won't worry much about the distinction between these examples; in both cases the function has a "sharp point" where there is no tangent line and no derivative.

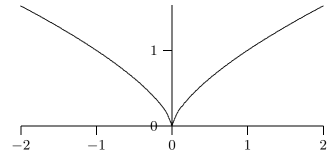


Figure 2.4.1 A cusp on $x^{2/3}$.

Exercises 2.4.

1. Find the derivative of $y = f(x) = \sqrt{169 - x^2}$. ⇒

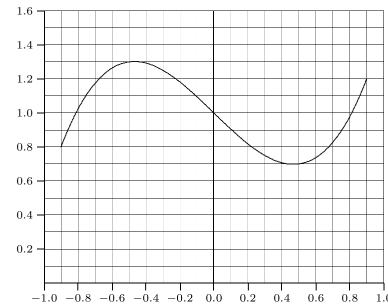
2. Find the derivative of $y = f(t) = 80 - 4.9t^2$. ⇒

3. Find the derivative of $y = f(x) = x^2 - (1/x)$. ⇒

4. Find the derivative of $y = f(x) = ax^2 + bx + c$ (where a, b, and c are constants). ⇒

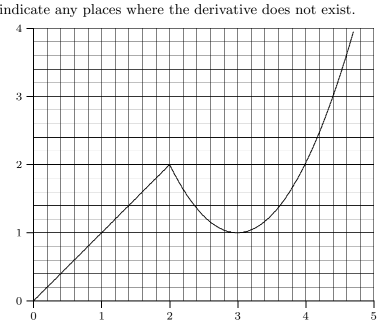
5. Find the derivative of $y = f(x) = x^3$. ⇒

6. Shown is the graph of a function $f(x)$. Sketch the graph of $f'(x)$ by estimating the derivative at a number of points in the interval: estimate the derivative at regular intervals from one end of the interval to the other, and also at "special" points, as when the derivative is zero. Make sure you indicate any places where the derivative does not exist.



7. Shown is the graph of a function $f(x)$. Sketch the graph of $f'(x)$ by estimating the derivative at a number of points in the interval: estimate the derivative at regular intervals from one end of the interval to the other, and also at "special" points, as when the derivative is zero.

Make sure you indicate any places where the derivative does not exist.



8. Find the derivative of $y = f(x) = 2/\sqrt{2x+1}$ ⇒

9. Find the derivative of $y = g(t) = (2t-1)/(t+2)$ ⇒

10. Find an equation for the tangent line to the graph of $f(x) = 5 - x - 3x^2$ at the point $x = 2$ ⇒

11. Find a value for a so that the graph of $f(x) = x^2 + ax - 3$ has a horizontal tangent line at $x = 4$. ⇒

2.5 Properties of Functions

As we have defined it in Section 1.3, a function is a very general object. At this point, it is useful to introduce a collection of properties of functions; knowing that a function has one of these properties can be very useful. Consider the graphs of the functions in Figure 2.5.1. It would clearly be useful to have words to help us describe the distinct features of each of them. We will point out and define a few properties (there are many more) of the functions pictured here. For the sake of the discussion, we will assume that the graphs do not exhibit any unusual behavior off-stage (i.e., outside the view of the graphs).

Functions. Each graph in Figure 2.5.1 certainly represents a function—since each passes the *vertical line test*. In other words, as you sweep a vertical line across the graph of each function, the line never intersects the graph more than once. If it did, then the graph would not represent a function.

Bounded. The graph in (c) appears to approach zero as x goes to both positive and negative infinity. It also never exceeds the value 1 or drops below the value 0. Because the graph never increases or decreases without bound, we say that the function represented by the graph in (c) is a **bounded** function.

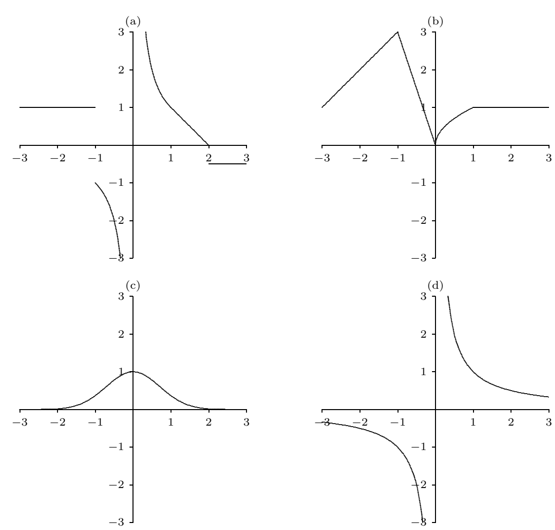


Figure 2.5.1 Function Types: (a) a discontinuous function, (b) a continuous function, (c) a bounded, differentiable function, (d) an unbounded, differentiable function

DEFINITION 2.5.1 Bounded A function f is bounded if there is a number M such that $|f(x)| < M$ for every x in the domain of f. □

For the function in (c), one such choice for M would be 10. However, the smallest (optimal) choice would be $M = 1$. In either case, simply finding an M is enough to establish boundedness. No such M exists for the hyperbola in (d) and hence we say that it is **unbounded**.

Continuity. The graphs shown in (b) and (c) both represent **continuous** functions. Geometrically, this is because there are no jumps in the graphs. That is, if you pick a point on the graph and approach it from the left and right, the values of the function approach the value of the function at that point. For example, we can see that this is not true for function values near $x = -1$ on the graph in (a) which is not continuous at that location.

DEFINITION 2.5.2 Continuous at a Point A function f is continuous at a point a if $\lim_{x\to a} f(x) = f(a)$. □

DEFINITION 2.5.3 Continuous A function f is continuous if it is continuous at every point in its domain. □

Strangely, we also say that (d) is continuous even though there is a vertical asymptote. A careful reading of the definition of continuous reveals the phrase "*at every point in its domain*." Because the location of the asymptote, $x = 0$, is not in the domain of the function, and because the rest of the function is continuous, we say that (d) is continuous.

Differentiability. If a function has a derivative at every point, we say the function is **differentiable**. We can see that the tangent line is well-defined at every point on the graph in (c). Therefore, we say that (c) is a differentiable function.

DEFINITION 2.5.4 Differentiable at a Point A function f is differentiable at point a if $f'(a)$ exists. □

DEFINITION 2.5.5 Differentiable A function f is differentiable if is differentiable at every point (excluding endpoints and isolated points in the domain of f) in the domain of f. □

Take note that, for technical reasons not discussed here, both of these definitions exclude endpoints and isolated points in the domain from consideration.

We close with a useful theorem about continuous functions:

THEOREM 2.5.6 Intermediate Value Theorem If f is continuous on the interval $[a, b]$ and d is between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ such that $f(c) = d$. ■

This is most frequently used when $d = 0$.

EXAMPLE 2.5.7 Explain why the function $f = x^3 + 3x^2 + x - 2$ has a root between 0 and 1.

By theorem 2.3.6, f is continuous. Since $f(0) = -2$ and $f(1) = 3$, and 0 is between -2 and 3, there is a $c \in [0, 1]$ such that $f(c) = 0$. □

This example also points the way to a simple method for approximating roots.

EXAMPLE 2.5.8 Approximate the root of the previous example to one decimal place.

If we compute $f(0.1)$, $f(0.2)$, and so on, we find that $f(0.6) < 0$ and $f(0.7) > 0$, so by the Intermediate Value Theorem, f has a root between 0.6 and 0.7. Repeating the process

with $f(0.61)$, $f(0.62)$, and so on, we find that $f(0.61) < 0$ and $f(0.62) > 0$, so f has a root between 0.61 and 0.62, and the root is 0.6 rounded to one decimal place. □

Exercises 2.5.

1. Along the lines of Figure 2.5.1, for each part below sketch the graph of a function that is:

- **a.** bounded, but not continuous.
- **b.** differentiable and unbounded.
- **c.** continuous at $x = 0$, not continuous at $x = 1$, and bounded.
- **d.** differentiable everywhere except at $x = -1$, continuous, and unbounded.

2. Is $f(x) = \sin(x)$ a bounded function? If so, find the smallest M. ⇒

3. Is $s(t) = 1/(1 + t^2)$ a bounded function? If so, find the smallest M. ⇒

4. Is $v(u) = 2 \ln |u|$ a bounded function? If so, find the smallest M. ⇒

5. Consider the function

$$h(x) = \begin{cases} 2x - 3, & \text{if } x < 1 \\ 0, & \text{if } x \geq 1. \end{cases}$$

Show that it is continuous at the point $x = 0$. Is h a continuous function? ⇒

6. Approximate a root of $f = x^3 - 4x^2 + 2x + 2$ to one decimal place. ⇒

7. Approximate a root of $f = x^4 + x^3 - 5x + 1$ to one decimal place. ⇒