1

Analytic Geometry

Much of the mathematics in this chapter will be review for you. However, the examples will be oriented toward applications and so will take some thought.

In the \((x, y)\) coordinate system we normally write the \(x\)-axis horizontally, with positive numbers to the right of the origin, and the \(y\)-axis vertically, with positive numbers above the origin. That is, unless stated otherwise, we take “rightward” to be the positive \(x\)-direction and “upward” to be the positive \(y\)-direction. In a purely mathematical situation, we normally choose the same scale for the \(x\)- and \(y\)-axes. For example, the line joining the origin to the point \((a, a)\) makes an angle of 45° with the \(x\)-axis (and also with the \(y\)-axis).

In applications, often letters other than \(x\) and \(y\) are used, and often different scales are chosen in the horizontal and vertical directions. For example, suppose you drop something from a window, and you want to study how its height above the ground changes from second to second. It is natural to let the letter \(t\) denote the time (the number of seconds since the object was released) and to let the letter \(h\) denote the height. For each \(t\) (say, at one-second intervals) you have a corresponding height \(h\). This information can be tabulated, and then plotted on the \((t, h)\) coordinate plane, as shown in figure 1.0.1.

We use the word “quadrant” for each of the four regions into which the plane is divided by the axes: the first quadrant is where points have both coordinates positive, or the “northeast” portion of the plot, and the second, third, and fourth quadrants are counted counter-clockwise, so the second quadrant is the northeast, the third is the southwest, and the fourth is the southeast.

Suppose we have two points \(A\) and \(B\) in the \((x, y)\)-plane. We often want to know the change in \(y\)-coordinate (also called the “horizontal distance”) in going from \(A\) to \(B\). This is often written \(\Delta y\), where the meaning of \(\Delta\) (a capital delta in the Greek alphabet) is “change in.” (Thus, \(\Delta\) can be read as “change in” although it usually is read as “delta \(x\).” The point is that \(\Delta y\) denotes a single number, and should not be interpreted as “delta times \(x\)”.) For example, if \(A = (2, 1)\) and \(B = (3, 3)\), \(\Delta x = 3 - 2 = 1\). Similarly, the “change in \(y\)” is written \(\Delta y\). In our example, \(\Delta y = 3 - 1 = 2\), the difference between the \(y\)-coordinates of the two points. It is the vertical distance you have to move in going from \(A\) to \(B\).

The general formulas for the change in \(x\) and the change in \(y\) between a point \((x_1, y_1)\) and a point \((x_2, y_2)\) are:

\[
\Delta x = x_2 - x_1, \quad \Delta y = y_2 - y_1.
\]

Note that either or both of these might be negative.

1.1 Lines

If we have two points \(A(x_1, y_1)\) and \(B(x_2, y_2)\), then we can draw one and only one line through both points. By the slope of this line we mean the ratio of \(\Delta y \) to \(\Delta x\). The slope is often denoted \(m\): \(m = \Delta y / \Delta x = (y_2 - y_1) / (x_2 - x_1)\). For example, the line joining the points \((1, -2)\) and \((3, 5)\) has slope \((5 + 2)/(3 - 1) = 7/2\).

**EXAMPLE 1.1.1**
According to the 1990 U.S. federal income tax schedules, a head of household paid 15% on taxable income up to $26050. If taxable income was between $26050 and $313930, then, in addition, 28% was to be paid on the amount between $26050 and $67280, and 33% paid on the amount over $67280 (if any). Interpret the tax bracket.

![Figure 1.0.1](image)

**Figure 1.0.1** A data plot, height versus time.

The line passes through the two points \((1, 0)\) and \((5, b)\), with slope 0.28. As the horizontal coordinate goes from \(x = 26050\) to \(x = 67200\), the line turns upward again. Note that either or both of these might be negative.

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![Figure 1.1.1](image)

**Figure 1.1.1** Tax vs. income.

The most familiar form of the equation of a straight line is: \(y = mx + b\). Here \(m\) is the slope of the line: if you increase \(x\) by 1, the equation tells you that you have to increase \(y\) by \(m\). If you increase \(x\) by \(\Delta x\), then \(y\) increases by \(\Delta y = m\Delta x\). The number \(b\) is called the \(y\)-intercept, because it is where the line crosses the \(y\)-axis. If you know two points on a line, a formula \(m = (y_2 - y_1)/(x_2 - x_1)\) gives you the slope. Once you know a point and the slope, then the \(y\)-intercept can be found by substituting the coordinates of either point in the equation: \(y_1 = mx_1 + b\), i.e., \(b = y_1 - mx_1\). Alternatively, one can use the “point-slope” form of the equation of a straight line: start with \((y - y_1)/(x - x_1) = m\), and then multiply to get \((y - y_1) = m(x - x_1)\), the point-slope form. Of course, this may be further manipulated to get \(y = mx - mx_1 + y_1\); which is essentially the ‘\(mx + b\)’ form.

It is possible to find the equation of a line between two points directly from the relation \((y - y_2)/(x - x_1) = (y_2 - y_1)/(x_2 - x_1)\), which says “the slope measured between the point \((x_1, y_1)\) and the point \((x_2, y_2)\) is the same as the slope measured between the point \((x_1, y_1)\) and...”
110/(1.5 - 1) = −50. The meaning of the slope is that you are traveling at 50 mph, m is negative because you are traveling toward Seattle, i.e., your distance y is decreasing. The word “velocity” is often used for m = −50, when we want to indicate direction, while the word “speed” refers to the magnitude (absolute value) of velocity, which is 50 mph. To find the equation of the line, we use the point-slope formula:

\[ y - y_1 = m(x - x_1) \]

\[ y - (-10) = -50(x - 0) \]

\[ y + 10 = -50x \]

The meaning of the y-intercept is that when \( t = 0 \) (when you started the trip) you were 110 miles from Seattle. To find the x-intercept, set \( y = 0 \), so that \( -50t + 110 = 0 \) or \( t = 2.2 \).

The meaning of the y-intercept is the duration of your trip, from the start until you arrive in Seattle. After traveling 3 hours and 12 minutes, your distance y from Seattle will be 0.

**Exercises 1.1.**

1. Find the equation of the line through (1, 1) and (−5, −3) in the form \( y = mx + b \).

2. Find the equation of the line through (−1, 2) with slope −2 in the form \( y = mx + b \).

3. Find the equation of the line through (−3, 1) and (5, −3) in the form \( y = mx + b \).

4. Given two points \((x_1, y_1)\) and \((x_2, y_2)\), recall that their horizontal distance from one another is \(2|x_2 - x_1|\) and their vertical distance from one another is \(|y_2 - y_1|\).

5. Use the point-slope formula:

\[ y = mx + b \]

\[ \frac{y-y_1}{x-x_1} = m \]

**1.2 Distance Between Two Points: Circles**

Two given points \((x_1, y_1)\) and \((x_2, y_2)\), recall that their horizontal distance from one another is \(\Delta x = x_2 - x_1\) and their vertical distance from one another is \(\Delta y = y_2 - y_1\). (Actually, the word “distance” normally denotes “positive distance”, and \(\Delta x\) and \(\Delta y\) are signed distances, but this is clear from context.) The actual (positive) distance from one point to the other is the length of the hypotenuse of a right triangle with legs \(\Delta x\) and \(\Delta y\), as shown in figure 1.2.1. The Pythagorean theorem says that the distance between the two points is the square root of the sum of the squares of the horizontal and vertical sides:

\[ \text{distance} = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

For example, the distance between points \(A(2, 1)\) and \(B(3, 3)\) is \(\sqrt{(3 - 2)^2 + (3 - 1)^2} = \sqrt{2} \).

A point \((x, y)\) is at a distance \(r\) from the origin if and only if \(\sqrt{x^2 + y^2} = r\) or, if we square both sides: \(x^2 + y^2 = r^2\). This is the equation of the circle of radius \(r\) centered at the origin. The special case \(r = 1\) is called the unit circle; its equation is \(x^2 + y^2 = 1\).

Similarly, if \(C(h, k)\) is any fixed point, then a point \((x, y)\) is a distance \(r\) from the point \(C\) if and only if \(\sqrt{(x - h)^2 + (y - k)^2} = r\), i.e., if and only if

\[ (x - h)^2 + (y - k)^2 = r^2 \]

This is the equation of the circle of radius \(r\) centered at the point \((h, k)\). For example, the circle of radius 5 centered at the point \((0, 0)\) has equation \((x - 0)^2 + (y - 0)^2 = 25\), or \(x^2 + y^2 = 25\). If we expand this we get \(x^2 + y^2 + 12y + 36 = 25\) or \(x^2 + y^2 + 12y + 11 = 0\), but the original form is usually more useful.

**1.3 Functions**

A function \(y = f(x)\) is a rule for determining \(y\) when we’re given a value of \(x\). For example, the rule \(y = f(x) = 2x + 1\) is a function. Any line \(y = mx + b\) is called a linear function. The graph of a function looks like a curve above (or below) the \(x\)-axis, where for any value of \(x\) the rule \(y = f(x)\) tells us how far to go above (or below) the \(x\)-axis to reach the curve.

Functions can be defined in various ways: by an algebraic formula or several algebraic formulas, by a graph, or by an experimentally determined table of values. (In the latter case, the table gives a bunch of points in the plane, which we might then interpolate with a smooth curve, if that makes sense.)

Given a value of \(x\), a function must give at most one value of \(y\). Thus, vertical lines are not functions. For example, the line \(x = 1\) has infinitely many values of \(y\) if \(y = 1\).
is also true that if \( x \) is any number not 1 there is no \( y \) which corresponds to \( x \), but that is not a problem—only multiple \( y \) values is a problem.

In addition to lines, another familiar example of a function is the parabola \( y = f(x) = x^2 \). We can draw the graph of this function by taking various values of \( x \) (say, at regular intervals) and plotting the points \((x, f(x)) = (x, x^2)\). Then connect the points with a smooth curve. (See figure 1.3.1.)

The two examples \( y = f(x) = 2x + 1 \) and \( y = f(x) = x^2 \) are both functions which can be evaluated at any value of \( x \) from negative infinity to positive infinity. For many functions, however, it only makes sense to take the interval or outside of some “forbidden” region. The set of \( x \)-values at which we’re allowed to evaluate the function is called the domain of the function.

To answer this question, we must rule out the \( x \)-values that make \( 4 - x^2 \leq 0 \), or more formally

\[
\{ x \in \mathbb{R} \mid x \leq -2 \} \cup \{ x \in \mathbb{R} \mid x \geq 2 \}
\]

This formula makes mathematical sense for any \( x \), but in the story problem the domain is much less. In the first place, \( x \) must be positive. In the second place, it must be less than half the length of either of the sides of the cardboard. Thus, the domain is

\[
\{ x \in \mathbb{R} \mid 0 < x < \frac{1}{2} \min(\text{of } \text{a} \text{ and } \text{b}) \}
\]

or else \( x < 0 \) and \( 4 - x < 0 \). The latter alternative is impossible, since if \( x \) is negative, then \( 4 - x \) is greater than \( 4 \), and \( 4 - x < 0 \) cannot be negative. As for the first alternative, the condition \( 4 - x > 0 \) can be rewritten (adding \( x \) to both sides) as \( x < 4 \), so we need: \( x > 0 \) and \( x < 4 \) (this is sometimes combined in the form \( 0 < x < 4 \), or, equivalently, \( 0 < x < 4 \)).

In interval notation, this says that the domain is the interval \((0, 4)\).

Second method. Write \( 4x - x^2 = -(x^2 - 4x) \), and then complete the square, obtaining

\[
-(x^2 - 2x^2 - 4) = -(x - 2)^2 - 4
\]

For this to be positive we need \((x - 2)^2 < 4\), which means that \( x - 2 \) must be less than 2 and greater than \(-2 \). Thus \( 2 < x < 4 \). Both of these methods are equally correct; you may use either in a problem of this type.

A function does not always have to be given by a single formula, as we have already seen (in the income tax problem, for example). Suppose that \( y = f(t) \) is the velocity function for a car which starts out from rest (zero velocity) at time \( t = 0 \); then increases its speed steadily to \( 20 \) m/sec, taking \( 10 \) seconds to do this; then travels at constant speed \( 20 \) m/sec for \( 15 \) seconds; and finally applies the brakes to decrease speed steadily to 0, taking \( 5 \) seconds to do this. The function \( y = f(t) \) is different in each of the three time intervals: first \( y = 2t \), then \( y = 20 \), then \( y = -4t + 120 \). The graph of this function is shown in figure 1.3.3.
Exercises 1.3.

Find the domain of each of the following functions:
1. \( y = f(x) = \sqrt{x - 2} \)  
2. \( y = f(x) = 1/(x + 1) \)  
3. \( y = f(x) = 1/(x^2 - 1) \)  
4. \( y = f(x) = \sqrt{x + 1} \)  
5. \( y = f(x) = \sqrt{7} \)  
6. \( y = f(x) = \sqrt{7}/x \)  
7. \( y = f(x) = \sqrt{1 - (x - 2)^2} \), where \( r \) is a positive constant.
8. \( y = f(x) = \sqrt{1 - (x - 2)^2} \)  
9. \( y = f(x) = 1/(\sqrt{7} - x) \)  
10. \( y = f(x) = 1/(\sqrt{7} - 1) \)  
11. \( y = f(x) = 1/(\sqrt{7} - 1) \)  
12. Find the domain of \( f(x) = (x^2 - 9)/(x - 3) \)  
13. Suppose \( f(x) = 3x - 9 \) and \( g(x) = \sqrt{x} \). What is the domain of the composition \( g(f(x)) \)? (Recall that composition is defined as \( (g \circ f)(x) ) \). What is the domain of \( (f \circ g)(x) \)?
14. A farmer wants to build a fence along a river. He has 500 feet of fencing and wants to enclose a rectangular pen on three sides (with the river providing the fourth side). If \( x \) is the length of the side perpendicular to the river, determine the area of the pen as a function of \( x \). What is the domain of this function?  
15. A can in the shape of a cylinder is to be made with a total of 100 square centimeters of material in the sides, top, and bottom, the manufacturer wants the can to hold the maximum possible volume. Write the volume as a function of the radius \( r \) of the can; find the domain of the function.
16. A can in the shape of a cylinder is to be made to hold a volume of one liter (1000 cubic centimeters). The manufacturer wants to use the least possible material for the can. Write the surface area of the can (total of the top, bottom, and side) as a function of the radius \( r \) of the can, find the domain of the function.

1.4 Shifts and Dilations

Many functions in applications are built up from simple functions by inserting constants in various places. It is important to understand the effect such constants have on the appearance of the graph.

Horizontal shifts. If we replace \( x \) by \( x - C \) everywhere it occurs in the formula for \( f(x) \), then the effect on the graph is to dilate it by a factor of \( B \) in the vertical direction. As before, this is an expansion or contraction depending on whether \( B \) is larger or smaller than one. Note that if we have a function \( y = f(x) \), replacing \( y \) by \( y/B \) is equivalent to multiplying the function on the right by \( B \). \( y = B f(x) \). The effect on the graph is to expand the picture away from the \( x \)-axis by a factor of \( B \) if \( B > 1 \), to contract it toward the \( x \)-axis by a factor of \( 1/B \) if \( 0 < B < 1 \), and to dilate by \( B \) and then flip about the \( y \)-axis if \( B \) is negative.

EXAMPLE 1.4.2 Ellipses

A basic example of the two expansion principles is given by an ellipse of semimajor axis \( a \) and semiminor axis \( b \). We get such an ellipse by starting with the unit circle—the circle of radius 1 centered at the origin, the equation of which is \( x^2 + y^2 = 1 \)—and dilating by a factor of \( a \) horizontally and by a factor of \( b \) vertically. To get the equation of the resulting ellipse, which crosses the \( x \)-axis at \( a \) and crosses the \( y \)-axis at \( b \), we replace \( x/\sqrt{a} \) by \( x \) and \( y/\sqrt{b} \) by \( y \) in the equation for the unit circle. This gives

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

Finally, if we want to analyze a function that involves both shifts and dilations, it is usually simplest to work with the dilations first, and then the shifts. For instance, if we want to dilate a function by a factor of \( A \) in the \( x \)-direction and then shift \( C \) to the right, we do this by replacing \( x \) first by \( x/A \) and then by \( (x - C) \) in the formula. As an example, suppose that, after diluting our unit circle by \( a \) in the \( x \)-direction and by \( b \) in the \( y \)-direction to get the ellipse in the last paragraph, we then wanted to shift it a distance \( h \) to the right and a distance \( k \) upward, so as to be centered at the point \((h, k)\). The new ellipse would have equation

\[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1
\]

Note well that this is different than first doing shifts by \( h \) and \( k \) and then dilations by \( a \) and \( b \).

See figure 1.4.1.

Exercises 1.4.

Starting with the graph of \( y = \sqrt{x} \), graph the function of \( y = 1/x \), and the graph of \( y = \sqrt{1-x} \) (the upper unit semicircle), sketch the graph of each of the following functions:
1. \( f(x) = \sqrt{x - 2} \)
2. \( f(x) = -1/(x + 2) \)
3. \( f(x) = 4 + \sqrt{x - 2} \)
4. \( y = f(x) = x^2 - 1 \)
5. \( y = f(x) = \sqrt{x} \)
6. \( f(x) = 2 + \sqrt{1-x} \)
7. \( f(x) = -4 + \sqrt{x} \)
8. \( f(x) = 2\sqrt{x} \)
9. \( f(x) = 1/(x + 1) \)
10. \( f(x) = 4 + 2\sqrt{1 - (x - 1)^2} \)
11. \( f(x) = 1 + 1/(x - 1) \)
12. \( f(x) = \sqrt{100 - 25(x - 1)^2} + 2 \)

The graph of \( f(x) \) is shown below. Sketch the graphs of the following functions.
13. \( y = f(x - 1) \)
14. \( y = 1/f(x + 2) \)
15. \( y = 1 + 2f(x) \)
16. \( y = 2f(3x) \)
17. \( y = 2f(3(x - 2)) + 1 \)
18. \( y = (1/2)f(3x - 3) \)
19. \( y = f(1 + x/3) + 2 \)