1

Analytic Geometry

Much of the mathematics in this chapter will be review for you. However, the examples will be oriented toward applications and so will take some thought.

In the (x, y) coordinate system we normally write the x-axis horizontally, with positive numbers to the right of the origin, and the y-axis vertically, with positive numbers above the origin. That is, unless stated otherwise, we take "rightward" to be the positive x-direction and "upward" to be the positive y-direction. In a purely mathematical situation, we normally choose the same scale for the x- and y-axes. For example, the line joining the origin to the point (a, a) makes an angle of 45° with the x-axis (and also with the y-axis).

In applications, often letters other than x and y are used, and often different scales are chosen in the horizontal and vertical directions. For example, suppose you drop something from a window, and you want to study how its height above the ground changes from second to second. It is natural to let the letter t denote the time (the number of seconds since the object was released) and to let the letter h denote the height. For each t (say, at one-second intervals) you have a corresponding height h. This information can be tabulated, and then plotted on the (t, h) coordinate plane, as shown in figure 1.0.1.

We use the word "quadrant" for each of the four regions into which the plane is divided by the axes: the first quadrant is where points have both coordinates positive, or the "northeast" portion of the plot, and the second, third, and fourth quadrants are counted off counterclockwise, so the second quadrant is the northwest, the third is the southwest, and the fourth is the southeast.

Suppose we have two points A and B in the (x, y)-plane. We often want to know the change in x-coordinate (also called the "horizontal distance") in going from A to B. This

seconds	0	1	2	3	4
meters	80	75.1	60.4	35.9	1.6

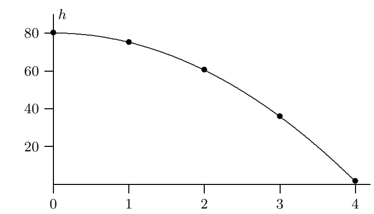


Figure 1.0.1 A data plot, height versus time.

is often written Δx, where the meaning of Δ (a capital delta in the Greek alphabet) is "change in". (Thus, Δx can be read as "change in x" although it usually is read as "delta x". The point is that Δx denotes a single number, and should not be interpreted as "delta times x".) For example, if $A = (2, 1)$ and $B = (3, 3)$, $\Delta x = 3 - 2 = 1$. Similarly, the "change in y" is written Δy. In our example, $\Delta y = 3 - 1 = 2$, the difference between the y-coordinates of the two points. It is the vertical distance you have to move in going from A to B. The general formulas for the change in x and the change in y between a point (x_1, y_1) and a point (x_2, y_2) are:

$$\Delta x = x_2 - x_1, \qquad \Delta y = y_2 - y_1.$$

Note that either or both of these might be negative.

1.1 Lines

If we have two points $A(x_1, y_1)$ and $B(x_2, y_2)$, then we can draw one and only one line through both points. By the *slope* of this line we mean the ratio of Δy to Δx. The slope is often denoted m: $m = \Delta y/\Delta x = (y_2 - y_1)/(x_2 - x_1)$. For example, the line joining the points $(1, -2)$ and $(3, 5)$ has slope $(5 + 2)/(3 - 1) = 7/2$.

EXAMPLE 1.1.1 According to the 1990 U.S. federal income tax schedules, a head of household paid 15% on taxable income up to \$26050. If taxable income was between \$26050 and \$134930, then, in addition, 28% was to be paid on the amount between \$26050 and \$67200, and 33% paid on the amount over \$67200 (if any). Interpret the tax bracket

information (15%, 28%, or 33%) using mathematical terminology, and graph the tax on the y-axis against the taxable income on the x-axis.

The percentages, when converted to decimal values 0.15, 0.28, and 0.33, are the *slopes* of the straight lines which form the graph of the tax for the corresponding tax brackets. The tax graph is what's called a *polygonal line*, i.e., it's made up of several straight line segments of different slopes. The first line starts at the point (0,0) and heads upward with slope 0.15 (i.e., it goes upward 15 for every increase of 100 in the x-direction), until it reaches the point above $x = 26050$. Then the graph "bends upward," i.e., the slope changes to 0.28. As the horizontal coordinate goes from $x = 26050$ to $x = 67200$, the line goes upward 28 for each 100 in the x-direction. At $x = 67200$ the line turns upward again and continues with slope 0.33. See figure 1.1.1. □

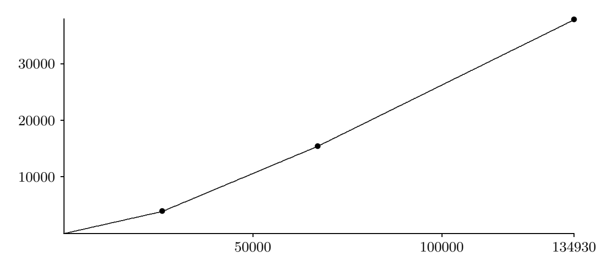


Figure 1.1.1 Tax vs. income.

The most familiar form of the equation of a straight line is: $y = mx + b$. Here m is the slope of the line: if you increase x by 1, the equation tells you that you have to increase y by m. If you increase x by Δx, then y increases by $\Delta y = m\Delta x$. The number b is called the **y-intercept**, because it is where the line crosses the y-axis. If you know two points on a line, the formula $m = (y_2 - y_1)/(x_2 - x_1)$ gives you the slope. Once you know a point and the slope, then the y-intercept can be found by substituting the coordinates of either point in the equation: $y_1 = mx_1 + b$, i.e., $b = y_1 - mx_1$. Alternatively, one can use the "point-slope" form of the equation of a straight line: start with $(y - y_1)/(x - x_1) = m$ and then multiply to get $(y - y_1) = m(x - x_1)$, the point-slope form. Of course, this may be further manipulated to get $y = mx - mx_1 + y_1$, which is essentially the "$mx + b$" form.

It is possible to find the equation of a line between two points directly from the relation $(y - y_1)/(x - x_1) = (y_2 - y_1)/(x_2 - x_1)$, which says "the slope measured between the point (x_1, y_1) and the point (x_2, y_2) is the same as the slope measured between the point (x_1, y_1)

and any other point (x, y) on the line." For example, if we want to find the equation of the line joining our earlier points $A(2, 1)$ and $B(3, 3)$, we can use this formula:

$$\frac{y-1}{x-2} = \frac{3-1}{3-2} = 2, \qquad \text{so that} \qquad y - 1 = 2(x - 2), \qquad \text{i.e.,} \qquad y = 2x - 3.$$

Of course, this is really just the point-slope formula, except that we are not computing m in a separate step.

The slope m of a line in the form $y = mx + b$ tells us the direction in which the line is pointing. If m is positive, the line goes into the 1st quadrant as you go from left to right. If m is large and positive, it has a steep incline, while if m is small and positive, then the line has a small angle of inclination. If m is negative, the line goes into the 4th quadrant as you go from left to right. If m is a large negative number (large in absolute value), then the line points steeply downward; while if m is negative but near zero, then it points only a little downward. These four possibilities are illustrated in figure 1.1.2.

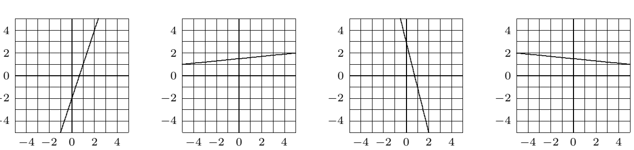


Figure 1.1.2 Lines with slopes 3, 0.1, −4, and −0.1.

If $m = 0$, then the line is horizontal: its equation is simply $y = b$.

There is one type of line that cannot be written in the form $y = mx + b$, namely, vertical lines. A vertical line has an equation of the form $x = a$. Sometimes one says that a vertical line has an "infinite" slope.

Sometimes it is useful to find the x-intercept of a line $y = mx + b$. This is the x-value when $y = 0$. Setting $mx + b$ equal to 0 and solving for x gives: $x = -b/m$. For example, the line $y = 2x - 3$ through the points $A(2, 1)$ and $B(3, 3)$ has x-intercept $3/2$.

EXAMPLE 1.1.2 Suppose that you are driving to Seattle at constant speed, and notice that after you have been traveling for 1 hour (i.e., $t = 1$), you pass a sign saying it is 110 miles to Seattle, and after driving another half-hour you pass a sign saying it is 85 miles to Seattle. Using the horizontal axis for the time t and the vertical axis for the distance y from Seattle, graph and find the equation $y = mt + b$ for your distance from Seattle. Find the slope, y-intercept, and t-intercept, and describe the practical meaning of each.

The graph of y versus t is a straight line because you are traveling at constant speed. The line passes through the two points $(1, 110)$ and $(1.5, 85)$, so its slope is $m = (85 -$

$110)/(1.5-1) = -50$. The meaning of the slope is that you are traveling at 50 mph; m is negative because you are traveling *toward* Seattle, i.e., your distance y is *decreasing*. The word "velocity" is often used for $m = -50$, when we want to indicate direction, while the word "speed" refers to the magnitude (absolute value) of velocity, which is 50 mph. To find the equation of the line, we use the point-slope formula:

$$\frac{y-110}{t-1} = -50, \qquad \text{so that} \qquad y = -50(t-1) + 110 = -50t + 160.$$

The meaning of the y-intercept 160 is that when $t = 0$ (when you started the trip) you were 160 miles from Seattle. To find the t-intercept, set $0 = -50t+160$, so that $t = 160/50 = 3.2$. The meaning of the t-intercept is the duration of your trip, from the start until you arrive in Seattle. After traveling 3 hours and 12 minutes, your distance y from Seattle will be 0. □

Exercises 1.1.

1. Find the equation of the line through $(1,1)$ and $(-5,-3)$ in the form $y = mx + b$. ⇒

2. Find the equation of the line through $(-1,2)$ with slope -2 in the form $y = mx + b$. ⇒

3. Find the equation of the line through $(-1,1)$ and $(5,-3)$ in the form $y = mx + b$. ⇒

4. Change the equation $y - 2x = 2$ to the form $y = mx + b$, graph the line, and find the y-intercept and x-intercept. ⇒

5. Change the equation $x+y = 6$ to the form $y = mx+b$, graph the line, and find the y-intercept and x-intercept. ⇒

6. Change the equation $x = 2y - 1$ to the form $y = mx + b$, graph the line, and find the y-intercept and x-intercept. ⇒

7. Change the equation $3 = 2y$ to the form $y = mx+b$, graph the line, and find the y-intercept and x-intercept. ⇒

8. Change the equation $2x + 3y + 6 = 0$ to the form $y = mx + b$, graph the line, and find the y-intercept and x-intercept. ⇒

9. Determine whether the lines $3x + 6y = 7$ and $2x + 4y = 5$ are parallel. ⇒

10. Suppose a triangle in the x,y–plane has vertices $(-1,0)$, $(1,0)$ and $(0,2)$. Find the equations of the three lines that lie along the sides of the triangle in $y = mx + b$ form. ⇒

11. Suppose that you are driving to Seattle at constant speed. After you have been traveling for an hour you pass a sign saying it is 130 miles to Seattle, and after driving another 20 minutes you pass a sign saying it is 105 miles to Seattle. Using the horizontal axis for the time t and the vertical axis for the distance y from your starting point, graph and find the equation $y = mt + b$ for your distance from your starting point. How long does the trip to Seattle take? ⇒

12. Let x stand for temperature in degrees Celsius (centigrade), and let y stand for temperature in degrees Fahrenheit. A temperature of 0°C corresponds to 32°F, and a temperature of 100°C corresponds to 212°F. Find the equation of the line that relates temperature Fahrenheit y to temperature Celsius x in the form $y = mx + b$. Graph the line, and find the point at which this line intersects $y = x$. What is the practical meaning of this point? ⇒

13. A car rental firm has the following charges for a certain type of car: \$25 per day with 100 free miles included, \$0.15 per mile for more than 100 miles. Suppose you want to rent a car for one day, and you know you'll use it for more than 100 miles. What is the equation relating the cost y to the number of miles x that you drive the car? ⇒

14. A photocopy store advertises the following prices: 5¢ per copy for the first 20 copies, 4¢ per copy for the 21st through 100th copy, and 3¢ per copy after the 100th copy. Let x be the number of copies, and let y be the total cost of photocopying. (a) Graph the cost as x goes from 0 to 200 copies. (b) Find the equation in the form $y = mx + b$ that tells you the cost of making x copies when x is more than 100. ⇒

15. In the Kingdom of Xyg the tax system works as follows. Someone who earns less than 100 gold coins per month pays no tax. Someone who earns between 100 and 1000 gold coins pays tax equal to 10% of the amount over 100 gold coins that he or she earns. Someone who earns over 1000 gold coins must hand over to the King all of the money earned over 1000 in addition to the tax on the first 1000. (a) Draw a graph of the tax paid y versus the money earned x, and give formulas for y in terms of x in each of the regions $0 \le x \le 100$, $100 \le x \le 1000$, and $x \ge 1000$. (b) Suppose that the King of Xyg decides to use the second of these line segments (for $100 \le x \le 1000$) for $x \le 100$ as well. Explain in practical terms what the King is doing, and what the meaning is of the y-intercept. ⇒

16. The tax for a single taxpayer is described in the figure 1.1.3. Use this information to graph tax versus taxable income (i.e., x is the amount on Form 1040, line 37, and y is the amount on Form 1040, line 38). Find the slope and y-intercept of each line that makes up the polygonal graph, up to $x = 97620$. ⇒

1990 Tax Rate Schedules

Schedule X—Use if your filing status is **Single**

If the amount on Form 1040 line 37 is over:	But not over:	Enter on Form 1040 line 38	of the amount over:
\$0	\$19,450	**15%**	**\$0**
19,450	47,050	**\$2,917.50+28%**	**19,450**
47,050	97,620	**\$10,645.50+33%**	**47,050**
97,620	....	Use **Worksheet** below to figure your tax	

Schedule Z—Use if your filing status is **Head of household**

If the amount on Form 1040 line 37 is over:	But not over:	Enter on Form 1040 line 38	of the amount over:
\$0	\$26,050	**15%**	**\$0**
26,050	67,200	**\$3,907.50+28%**	**26,050**
67,200	134,930	**\$15,429.50+33%**	**67,200**
134,930	....	Use **Worksheet** below to figure your tax	

Figure 1.1.3 Tax Schedule.

17. Market research tells you that if you set the price of an item at \$1.50, you will be able to sell 5000 items; and for every 10 cents you lower the price below \$1.50 you will be able to sell another 1000 items. Let x be the number of items you can sell, and let P be the price of an item. (a) Express P linearly in terms of x, in other words, express P in the form $P = mx+b$. (b) Express x linearly in terms of P. ⇒

18. An instructor gives a 100-point final exam, and decides that a score 90 or above will be a grade of 4.0, a score of 40 or below will be a grade of 0.0, and between 40 and 90 the grading

will be linear. Let x be the exam score, and let y be the corresponding grade. Find a formula of the form $y = mx + b$ which applies to scores x between 40 and 90. ⇒

1.2 Distance Between Two Points; Circles

Given two points (x_1, y_1) and (x_2, y_2), recall that their horizontal distance from one another is $\Delta x = x_2 - x_1$ and their vertical distance from one another is $\Delta y = y_2 - y_1$. (Actually, the word "distance" normally denotes "positive distance". Δx and Δy are *signed* distances, but this is clear from context.) The actual (positive) distance from one point to the other is the length of the hypotenuse of a right triangle with legs $|\Delta x|$ and $|\Delta y|$, as shown in figure 1.2.1. The Pythagorean theorem then says that the distance between the two points is the square root of the sum of the squares of the horizontal and vertical sides:

$$\text{distance} = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

For example, the distance between points $A(2,1)$ and $B(3,3)$ is $\sqrt{(3-2)^2 + (3-1)^2} = \sqrt{5}$.

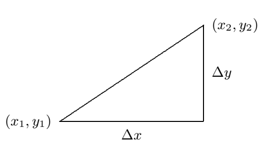


Figure 1.2.1 Distance between two points, Δx and Δy positive.

As a special case of the distance formula, suppose we want to know the distance of a point (x,y) to the origin. According to the distance formula, this is $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$.

A point (x,y) is at a distance r from the origin if and only if $\sqrt{x^2 + y^2} = r$, or, if we square both sides: $x^2 + y^2 = r^2$. This is the equation of the circle of radius r centered at the origin. The special case $r = 1$ is called the unit circle; its equation is $x^2 + y^2 = 1$.

Similarly, if $C(h,k)$ is any fixed point, then a point (x,y) is at a distance r from the point C if and only if $\sqrt{(x-h)^2 + (y-k)^2} = r$, i.e., if and only if

$$(x-h)^2 + (y-k)^2 = r^2.$$

This is the equation of the circle of radius r centered at the point (h,k). For example, the circle of radius 5 centered at the point $(0,-6)$ has equation $(x-0)^2 + (y - -6)^2 = 25$, or $x^2 + (y+6)^2 = 25$. If we expand this we get $x^2+y^2+12y+36 = 25$ or $x^2+y^2+12y+11 = 0$, but the original form is usually more useful.

EXAMPLE 1.2.1 Graph the circle $x^2 - 2x + y^2 + 4y - 11 = 0$. With a little thought we convert this to $(x-1)^2 + (y+2)^2 - 16 = 0$ or $(x-1)^2 + (y+2)^2 = 16$. Now we see that this is the circle with radius 4 and center $(1,-2)$, which is easy to graph. □

Exercises 1.2.

1. Find the equation of the circle of radius 3 centered at:

a) $(0,0)$ **d)** $(0,3)$
b) $(5,6)$ **e)** $(0,-3)$
c) $(-5,-6)$ **f)** $(3,0)$

⇒

2. For each pair of points $A(x_1, y_1)$ and $B(x_2, y_2)$ find (i) Δx and Δy in going from A to B, (ii) the slope of the line joining A and B, (iii) the equation of the line joining A and B in the form $y = mx + b$, (iv) the distance from A to B, and (v) an equation of the circle with center at A that goes through B.

a) $A(2,0)$, $B(4,3)$ **d)** $A(-2,3)$, $B(4,3)$
b) $A(1,-1)$, $B(0,2)$ **e)** $A(-3,-2)$, $B(0,0)$
c) $A(0,0)$, $B(-2,-2)$ **f)** $A(0.01,-0.01)$, $B(-0.01,0.05)$

⇒

3. Graph the circle $x^2 + y^2 + 10y = 0$.

4. Graph the circle $x^2 - 10x + y^2 = 24$.

5. Graph the circle $x^2 - 6x + y^2 - 8y = 0$.

6. Find the standard equation of the circle passing through $(-2,1)$ and tangent to the line $3x - 2y = 6$ at the point $(4,3)$. Sketch. (Hint: The line through the center of the circle and the point of tangency is perpendicular to the tangent line.) ⇒

1.3 Functions

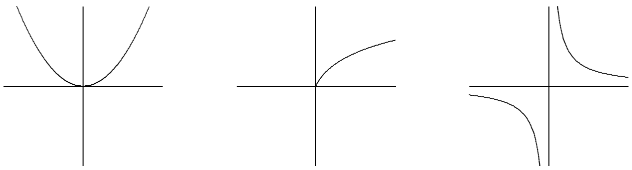
A **function** $y = f(x)$ is a rule for determining y when we're given a value of x. For example, the rule $y = f(x) = 2x + 1$ is a function. Any line $y = mx + b$ is called a **linear** function. The graph of a function looks like a curve above (or below) the x-axis, where for any value of x the rule $y = f(x)$ tells us how far to go above (or below) the x-axis to reach the curve.

Functions can be defined in various ways: by an algebraic formula or several algebraic formulas, by a graph, or by an experimentally determined table of values. (In the latter case, the table gives a bunch of points in the plane, which we might then interpolate with a smooth curve, if that makes sense.)

Given a value of x, a function must give at most one value of y. Thus, vertical lines are not functions. For example, the line $x = 1$ has infinitely many values of y if $x = 1$. It is also true that if x is any number not 1 there is no y which corresponds to x, but that is not a problem—only multiple y values is a problem.

In addition to lines, another familiar example of a function is the parabola $y = f(x) = x^2$. We can draw the graph of this function by taking various values of x (say, at regular intervals) and plotting the points $(x, f(x)) = (x, x^2)$. Then connect the points with a smooth curve. (See figure 1.3.1.)

The two examples $y = f(x) = 2x + 1$ and $y = f(x) = x^2$ are both functions which can be evaluated at *any* value of x from negative infinity to positive infinity. For many functions, however, it only makes sense to take x in some interval or outside of some "forbidden" region. The set of x-values at which we're allowed to evaluate the function is called the **domain** of the function.



$y = f(x) = x^2$ $y = f(x) = \sqrt{x}$ $y = f(x) = 1/x$

Figure 1.3.1 Some graphs.

For example, the square-root function $y = f(x) = \sqrt{x}$ is the rule which says, given an x-value, take the nonnegative number whose square is x. This rule only makes sense if x is positive or zero. We say that the domain of this function is $x \ge 0$, or more formally $\{x \in \mathbb{R} \mid x \ge 0\}$. Alternately, we can use interval notation, and write that the domain is $[0, \infty)$. (In interval notation, square brackets mean that the endpoint is included, and a parenthesis means that the endpoint is not included.) The fact that the domain of $y = \sqrt{x}$ is $[0, \infty)$ means that in the graph of this function (see figure 1.3.1) we have points (x, y) only above x-values on the right side of the x-axis.

Another example of a function whose domain is not the entire x-axis is: $y = f(x) = 1/x$, the reciprocal function. We cannot substitute $x = 0$ in this formula. The function makes sense, however, for any nonzero x, so we take the domain to be: $\{x \in \mathbb{R} \mid x \ne 0\}$. The graph of this function does not have any point (x, y) with $x = 0$. As x gets close to 0 from either side, the graph goes off toward infinity. We call the vertical line $x = 0$ an **asymptote**.

To summarize, two reasons why certain x-values are excluded from the domain of a function are that (i) we cannot divide by zero, and (ii) we cannot take the square root

of a negative number. We will encounter some other ways in which functions might be undefined later.

Another reason why the domain of a function might be restricted is that in a given situation the x-values outside of some range might have no practical meaning. For example, if y is the area of a square of side x, then we can write $y = f(x) = x^2$. In a purely mathematical context the domain of the function $y = x^2$ is all of $\mathbb{R}$. But in the story-problem context of finding areas of squares, we restrict the domain to positive values of x, because a square with negative or zero side makes no sense.

In a problem in pure mathematics, we usually take the domain to be all values of x at which the formulas can be evaluated. But in a story problem there might be further restrictions on the domain because only certain values of x are of interest or make practical sense.

In a story problem, often letters different from x and y are used. For example, the volume V of a sphere is a function of the radius r, given by the formula $V = f(r) = {}^4/_3\pi r^3$. Also, letters different from f may be used. For example, if y is the velocity of something at time t, we may write $y = v(t)$ with the letter v (instead of f) standing for the velocity function (and t playing the role of x).

The letter playing the role of x is called the **independent variable**, and the letter playing the role of y is called the **dependent variable** (because its value "depends on" the value of the independent variable). In story problems, when one has to translate from English into mathematics, a crucial step is to determine what letters stand for variables. If only words and no letters are given, then we have to decide which letters to use. Some letters are traditional. For example, almost always, t stands for time.

EXAMPLE 1.3.1 An open-top box is made from an $a \times b$ rectangular piece of cardboard by cutting out a square of side x from each of the four corners, and then folding the sides up and sealing them with duct tape. Find a formula for the volume V of the box as a function of x, and find the domain of this function.

The box we get will have height x and rectangular base of dimensions $a - 2x$ by $b - 2x$. Thus,

$$V = f(x) = x(a - 2x)(b - 2x).$$

Here a and b are constants, and V is the variable that depends on x, i.e., V is playing the role of y.

This formula makes mathematical sense for any x, but in the story problem the domain is much less. In the first place, x must be positive. In the second place, it must be less than half the length of either of the sides of the cardboard. Thus, the domain is

$$\{x \in \mathbb{R} \mid 0 < x < \frac{1}{2}(\text{minimum of } a \text{ and } b)\}.$$

In interval notation we write: the domain is the interval $(0, \min(a, b)/2)$. (You might think about whether we could allow 0 or $\min(a, b)/2$ to be in the domain. They make a certain physical sense, though we normally would not call the result a box. If we were to allow these values, what would the corresponding volumes be? Does that make sense?) □

EXAMPLE 1.3.2 Circle of radius r centered at the origin The equation for this circle is usually given in the form $x^2 + y^2 = r^2$. To write the equation in the form $y = f(x)$ we solve for y, obtaining $y = \pm\sqrt{r^2 - x^2}$. But *this is not a function*, because when we substitute a value in $(-r, r)$ for x there are two corresponding values of y. To get a function, we must choose one of the two signs in front of the square root. If we choose the positive sign, for example, we get the upper semicircle $y = f(x) = \sqrt{r^2 - x^2}$ (see figure 1.3.2). The domain of this function is the interval $[-r, r]$, i.e., x must be between $-r$ and r (including the endpoints). If x is outside of that interval, then $r^2 - x^2$ is negative, and we cannot take the square root. In terms of the graph, this just means that there are no points on the curve whose x-coordinate is greater than r or less than $-r$. □

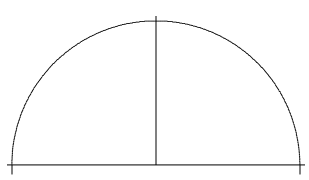


Figure 1.3.2 Upper semicircle $y = \sqrt{r^2 - x^2}$

EXAMPLE 1.3.3 Find the domain of

$$y = f(x) = \frac{1}{\sqrt{4x - x^2}}.$$

To answer this question, we must rule out the x-values that make $4x - x^2$ negative (because we cannot take the square root of a negative number) and also the x-values that make $4x - x^2$ zero (because if $4x - x^2 = 0$, then when we take the square root we get 0, and we cannot divide by 0). In other words, the domain consists of all x for which $4x - x^2$ is strictly positive. We give two different methods to find out when $4x - x^2 > 0$.

First method. Factor $4x - x^2$ as $x(4 - x)$. The product of two numbers is positive when either both are positive or both are negative, i.e., if either $x > 0$ and $4 - x > 0$,

or else $x < 0$ and $4 - x < 0$. The latter alternative is impossible, since if x is negative, then $4 - x$ is greater than 4, and so cannot be negative. As for the first alternative, the condition $4 - x > 0$ can be rewritten (adding x to both sides) as $4 > x$, so we need: $x > 0$ and $4 > x$ (this is sometimes combined in the form $4 > x > 0$, or, equivalently, $0 < x < 4$). In interval notation, this says that the domain is the interval $(0, 4)$.

Second method. Write $4x - x^2$ as $-(x^2 - 4x)$, and then complete the square, obtaining $-\Big((x - 2)^2 - 4\Big) = 4 - (x - 2)^2$. For this to be positive we need $(x - 2)^2 < 4$, which means that $x - 2$ must be less than 2 and greater than -2: $-2 < x - 2 < 2$. Adding 2 to everything gives $0 < x < 4$. Both of these methods are equally correct; you may use either in a problem of this type. □

A function does not always have to be given by a single formula, as we have already seen (in the income tax problem, for example). Suppose that $y = v(t)$ is the velocity function for a car which starts out from rest (zero velocity) at time $t = 0$; then increases its speed steadily to 20 m/sec, taking 10 seconds to do this; then travels at constant speed 20 m/sec for 15 seconds; and finally applies the brakes to decrease speed steadily to 0, taking 5 seconds to do this. The formula for $y = v(t)$ is different in each of the three time intervals: first $y = 2x$, then $y = 20$, then $y = -4x + 120$. The graph of this function is shown in figure 1.3.3.

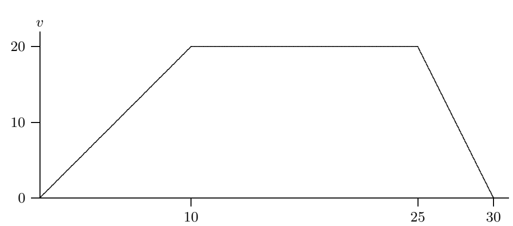


Figure 1.3.3 A velocity function.

Not all functions are given by formulas at all. A function can be given by an experimentally determined table of values, or by a description other than a formula. For example, the population y of the U.S. is a function of the time t: we can write $y = f(t)$. This is a perfectly good function—we could graph it (up to the present) if we had data for various t—but we can't find an algebraic formula for it.

Exercises 1.3.

Find the domain of each of the following functions:

1. $y = f(x) = \sqrt{2x-3} \Rightarrow$

2. $y = f(x) = 1/(x+1) \Rightarrow$

3. $y = f(x) = 1/(x^2-1) \Rightarrow$

4. $y = f(x) = \sqrt{-1/x} \Rightarrow$

5. $y = f(x) = \sqrt[3]{x} \Rightarrow$

6. $y = f(x) = \sqrt[4]{x} \Rightarrow$

7. $y = f(x) = \sqrt{r^2-(x-h)^2}$, where r is a positive constant. $\Rightarrow$

8. $y = f(x) = \sqrt{1-(1/x)} \Rightarrow$

9. $y = f(x) = 1/\sqrt{1-(3x)^2} \Rightarrow$

10. $y = f(x) = \sqrt{x} + 1/(x-1) \Rightarrow$

11. $y = f(x) = 1/(\sqrt{x}-1) \Rightarrow$

12. Find the domain of $h(x) = \begin{cases} (x^2-9)/(x-3) & x \neq 3 \\ 6 & \text{if } x = 3. \end{cases} \Rightarrow$

13. Suppose $f(x) = 3x-9$ and $g(x) = \sqrt{x}$. What is the domain of the composition $(g \circ f)(x)$? (Recall that **composition** is defined as $(g \circ f)(x) = g(f(x))$.) What is the domain of $(f \circ g)(x)$? $\Rightarrow$

14. A farmer wants to build a fence along a river. He has 500 feet of fencing and wants to enclose a rectangular pen on three sides (with the river providing the fourth side). If x is the length of the side perpendicular to the river, determine the area of the pen as a function of x. What is the domain of this function? $\Rightarrow$

15. A can in the shape of a cylinder is to be made with a total of 100 square centimeters of material in the side, top, and bottom; the manufacturer wants the can to hold the maximum possible volume. Write the volume as a function of the radius r of the can; find the domain of the function. $\Rightarrow$

16. A can in the shape of a cylinder is to be made to hold a volume of one liter (1000 cubic centimeters). The manufacturer wants to use the least possible material for the can. Write the surface area of the can (total of the top, bottom, and side) as a function of the radius r of the can; find the domain of the function. $\Rightarrow$

1.4 Shifts and Dilations

Many functions in applications are built up from simple functions by inserting constants in various places. It is important to understand the effect such constants have on the appearance of the graph.

Horizontal shifts. *If we replace x by $x-C$ everywhere it occurs in the formula for $f(x)$, then the graph shifts over C to the right.* (If C is negative, then this means that the graph shifts over $|C|$ to the left.) For example, the graph of $y = (x-2)^2$ is the x^2-parabola shifted over to have its vertex at the point 2 on the x-axis. The graph of $y = (x+1)^2$ is the same

parabola shifted over to the left so as to have its vertex at -1 on the x-axis. Note well: when replacing x by $x-C$ we must pay attention to meaning, not merely appearance. Starting with $y = x^2$ and literally replacing x by $x-2$ gives $y = x-2^2$. This is $y = x-4$, a line with slope 1, not a shifted parabola.

Vertical shifts. *If we replace y by $y-D$, then the graph moves up D units.* (If D is negative, then this means that the graph moves down $|D|$ units.) If the formula is written in the form $y = f(x)$ and if y is replaced by $y-D$ to get $y-D = f(x)$, we can equivalently move D to the other side of the equation and write $y = f(x)+D$. Thus, this principle can be stated: *to get the graph of $y = f(x)+D$, take the graph of $y = f(x)$ and move it D units up.* For example, the function $y = x^2-4x = (x-2)^2-4$ can be obtained from $y = (x-2)^2$ (see the last paragraph) by moving the graph 4 units down. The result is the x^2-parabola shifted 2 units to the right and 4 units down so as to have its vertex at the point $(2,-4)$.

Warning. Do not confuse $f(x)+D$ and $f(x+D)$. For example, if $f(x)$ is the function x^2, then $f(x)+2$ is the function x^2+2, while $f(x+2)$ is the function $(x+2)^2 = x^2+4x+4$.

EXAMPLE 1.4.1 Circles An important example of the above two principles starts with the circle $x^2+y^2 = r^2$. This is the circle of radius r centered at the origin. (As we saw, this is not a single function $y = f(x)$, but rather two functions $y = \pm\sqrt{r^2-x^2}$ put together; in any case, the two shifting principles apply to equations like this one that are not in the form $y = f(x)$.) If we replace x by $x-C$ and replace y by $y-D$—getting the equation $(x-C)^2+(y-D)^2 = r^2$—the effect on the circle is to move it C to the right and D up, thereby obtaining the circle of radius r centered at the point (C,D). This tells us how to write the equation of any circle, not necessarily centered at the origin. □

We will later want to use two more principles concerning the effects of constants on the appearance of the graph of a function.

Horizontal dilation. *If x is replaced by x/A in a formula and $A > 1$, then the effect on the graph is to expand it by a factor of A in the x-direction (away from the y-axis).* If A is between 0 and 1 then the effect on the graph is to contract by a factor of $1/A$ (towards the y-axis). We use the word "dilate" to mean expand or contract.

For example, replacing x by $x/0.5 = x/(1/2) = 2x$ has the effect of contracting toward the y-axis by a factor of 2. If A is negative, we dilate by a factor of $|A|$ and then flip about the y-axis. Thus, replacing x by $-x$ has the effect of taking the mirror image of the graph with respect to the y-axis. For example, the function $y = \sqrt{-x}$, which has domain $\{x \in \mathbb{R} \mid x \leq 0\}$, is obtained by taking the graph of $\sqrt{x}$ and flipping it around the y-axis into the second quadrant.

Vertical dilation. *If y is replaced by y/B in a formula and $B > 0$, then the effect on the graph is to dilate it by a factor of B in the vertical direction.* As before, this is an expansion or contraction depending on whether B is larger or smaller than one. Note that if we have a function $y = f(x)$, replacing y by y/B is equivalent to multiplying the function on the right by B: $y = Bf(x)$. The effect on the graph is to expand the picture away from the x-axis by a factor of B if $B > 1$, to contract it toward the x-axis by a factor of $1/B$ if $0 < B < 1$, and to dilate by $|B|$ and then flip about the x-axis if B is negative.

EXAMPLE 1.4.2 Ellipses A basic example of the two expansion principles is given by an **ellipse of semimajor axis** a **and semiminor axis** b. We get such an ellipse by starting with the unit circle—the circle of radius 1 centered at the origin, the equation of which is $x^2+y^2 = 1$—and dilating by a factor of a horizontally and by a factor of b vertically. To get the equation of the resulting ellipse, which crosses the x-axis at $\pm a$ and crosses the y-axis at $\pm b$, we replace x by x/a and y by y/b in the equation for the unit circle. This gives

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad \text{or} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

□

Finally, if we want to analyze a function that involves both shifts and dilations, it is usually simplest to work with the dilations first, and then the shifts. For instance, if we want to dilate a function by a factor of A in the x-direction and then shift C to the right, we do this by replacing x first by x/A and then by $(x-C)$ in the formula. As an example, suppose that, after dilating our unit circle by a in the x-direction and by b in the y-direction to get the ellipse in the last paragraph, we then wanted to shift it a distance h to the right and a distance k upward, so as to be centered at the point (h,k). The new ellipse would have equation

$$\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1.$$

Note well that this is different than first doing shifts by h and k and then dilations by a and b:

$$\left(\frac{x}{a}-h\right)^2 + \left(\frac{y}{b}-k\right)^2 = 1.$$

See figure 1.4.1.

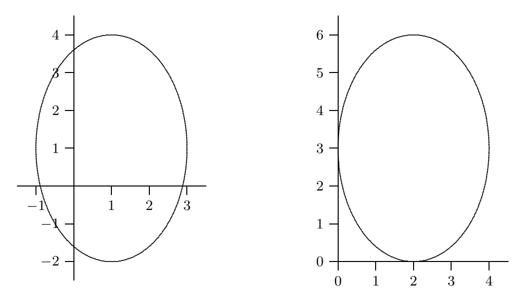


Figure 1.4.1 Ellipses: $\left(\frac{x-1}{2}\right)^2 + \left(\frac{y-1}{3}\right)^2 = 1$ on the left, $\left(\frac{x}{2}-1\right)^2 + \left(\frac{y}{3}-1\right)^2 = 1$ on the right.

Exercises 1.4.

Starting with the graph of $y = \sqrt{x}$, the graph of $y = 1/x$, and the graph of $y = \sqrt{1-x^2}$ (the upper unit semicircle), sketch the graph of each of the following functions:

1. $f(x) = \sqrt{x-2}$

2. $f(x) = -1-1/(x+2)$

3. $f(x) = 4+\sqrt{x+2}$

4. $y = f(x) = x/(1-x)$

5. $y = f(x) = -\sqrt{-x}$

6. $f(x) = 2+\sqrt{1-(x-1)^2}$

7. $f(x) = -4+\sqrt{-(x-2)}$

8. $f(x) = 2\sqrt{1-(x/3)^2}$

9. $f(x) = 1/(x+1)$

10. $f(x) = 4+2\sqrt{1-(x-5)^2/9}$

11. $f(x) = 1+1/(x-1)$

12. $f(x) = \sqrt{100-25(x-1)^2}+2$

The graph of $f(x)$ is shown below. Sketch the graphs of the following functions.

13. $y = f(x-1)$

14. $y = 1+f(x+2)$

15. $y = 1+2f(x)$

16. $y = 2f(3x)$

17. $y = 2f(3(x-2))+1$

18. $y = (1/2)f(3x-3)$

19. $y = f(1+x/3)+2$