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Analytic Geometry

Much of the mathematics in this chapter will be review for you. However, the examples will be oriented toward applications and so will take some thought.

In the \((x, y)\) coordinate system we normally write the \(x\)-axis horizontally, with positive numbers to the right of the origin, and the \(y\)-axis vertically, with positive numbers above the origin. That is, unless stated otherwise, we take "rightward" to be the positive \(x\)-direction and "upward" to be the positive \(y\)-direction. In a purely mathematical situation, we normally choose the same scale for the \(x\) and \(y\)-axes. For example, the line joining the origin to the point \((a, b)\) makes an angle of \(45^\circ\) with the \(x\)-axis (and also with the \(y\)-axis).

In applications, often letters other than \(x\) and \(y\) are used, and often different scales are chosen in the horizontal and vertical directions. For example, suppose you drop something from a window, and you want to study how its height above the ground changes from second to second. It is natural to let the letter \(t\) denote the time (the number of seconds since the object was released) and to let the letter \(h\) denote the height. For each \(t\) (say, at one-second intervals) you have a corresponding height \(h\). This information can be tabulated, and then plotted on the \((t, h)\) coordinate plane, as shown in figure 1.1.1.

We use the word "quadrant" for each of the four regions into which the plane is divided by the axes: the first quadrant is where both coordinates are positive, or the "northeast" portion of the plot, and the second, third, and fourth quadrants are counted off counterclockwise, so the second quadrant is the northwest, the third is the southeast, and the fourth is the southeast.

Suppose we have two points \(A\) and \(B\) in the \((x, y)\)-plane. We often want to know the change in \(x\)-coordinate (also called the "horizontal distance") in going from \(A\) to \(B\). This is often written \(\Delta x\), where the meaning of \(\Delta\) (a capital delta in the Greek alphabet) is "change in". (Thus, \(\Delta x\) can be read as "change in \(x\)" although it usually is read as "delta \(x\)." The point is that \(\Delta x\) denotes a single number, and should not be interpreted as "delta times \(x\)".) For example, if \(A = (2, 1)\) and \(B = (3, 3)\), \(\Delta x = 3 - 2 = 1\). Similarly, the "change in \(y\)" is written \(\Delta y\). In our example, \(\Delta y = 3 - 1 = 2\), the difference between the \(y\)-coordinates of the two points. It is the vertical distance you have to move in going from \(A\) to \(B\). The general formulas for the change in \(x\) and the change in \(y\) between a point \((x_1, y_1)\) and a point \((x_2, y_2)\) are:

\[
\Delta x = x_2 - x_1, \quad \Delta y = y_2 - y_1.
\]

Note that either or both of these might be negative.

1.1 Lines

If we have two points \(A(x_1, y_1)\) and \(B(x_2, y_2)\), then we can draw one and only one line through both points. By the slope of this line we mean the ratio of \(\Delta y\) to \(\Delta x\). The slope is often denoted \(m = \Delta y/\Delta x = (y_2 - y_1)/(x_2 - x_1)\). For example, the line joining the points \((1, -2)\) and \((3, 5)\) has slope \(5 - (-2)/(3 - 1) = 7/2\).

EXAMPLE 1.1.1 According to the 1990 U.S. federal income tax schedules, a head of household paid 15% on taxable income up to $26050. If taxable income was between $26050 and $114960, then, in addition, 26% was to be paid on the amount between $26050 and $67260, and 33% paid on the amount over $67260 (if any). Interpret the tax bracket information (15%, 28%, or 33%) using mathematical terminology, and graph the tax on the \(y\)-axis against the taxable income on the \(x\)-axis.

The percentages, when converted to decimal values 0.15, 0.28, and 0.33, are the slopes of the straight lines which form the graph of the tax for the corresponding tax brackets. The tax graph is what’s called a polygonal line, i.e., it’s made up of several straight line segments of different slopes. The first line starts at the point \((0, 0)\) and heads upward with slope 0.15 (i.e., it goes upward 15 for every increase of 100 in the \(x\)-direction), until it reaches the point above \(x = 26050\). Then the graph "bends upward," i.e., the slope changes to \(b = 0.33\). As the horizontal coordinate goes from \(x = 26050\) to \(x = 67260\), the line goes upward 28 for each 100 in the \(x\)-direction. At \(x = 67260\) the line turns upward again and continues with slope 0.33. See figure 1.1.1.

\[
\begin{align*}
\text{Figure 1.1.1} & \quad \text{Tax vs. income.} \\
\end{align*}
\]

The most familiar form of the equation of a straight line is:

\[
y = mx + b
\]

where \(m\) is the slope of the line: if you increase \(x\) by 1, the equation tells you that you have to increase \(y\) by \(m\). If you increase \(x\) by \(\Delta x\), then your \(y\) increases by \(\Delta y = m \Delta x\). The number \(b\) is called the \(y\)-\(intercept\), because it is where the line crosses the \(y\)-axis. If you know two points on a line, the formula \(m = (y_2 - y_1)/(x_2 - x_1)\) gives you the slope. Once you know a point and the slope, then the \(y\)-intercept can be found by substituting the coordinates of either point in the equation \(y_1 = mx_1 + b\), i.e., \(b = y_1 - mx_1\). Alternatively, one can use the "point-slope" form of the equation of a straight line: start with \((y - y_1)/(x - x_1) = m\) and then multiply to get \((y - y_1) = m(x - x_1)\), the point-slope form of the equation. Of course, this may be further manipulated to get \(y = mx - mx_1 + y_1\), which is essentially the "\(m\) \(x\) + \(b\)" form.

It is possible to find the equation of a line between two points directly from the relation \((y - y_1)/(x - x_1) = (y_2 - y_1)/(x_2 - x_1)\), which says "the slope measured between the point \((x_1, y_1)\) and the point \((x_2, y_2)\) is the same as the slope measured between the point \((x_1, y_1)\) and \((x_2, y_2)\)."
1.1 Lines

110)/1(5.5 − 1) = −50. The meaning of the slope is that you are traveling at 50 mph; m is negative because you are traveling toward Seattle, i.e., your distance y is decreasing. The word “velocity” is often used for m = −50, when we want to indicate direction, while the word “speed” refers to the magnitude (absolute value) of velocity, which is 50 mph. To find the equation of the line, we use the point-slope formula:

\[ y − y_1 = m(x − x_1) \]

\[ \frac{y − 110}{x} = −50, \quad \text{so that} \quad y = −50(x − 1) + 110 = −50x + 160. \]

The meaning of the \( x \)-intercept is that when \( t = 0 \) (when you started the trip) you were 160 miles from Seattle. To find the \( t \)-intercept, set \( 0 = −50t + 160 \), so that \( t = 160/50 = 3.2 \). The meaning of the \( y \)-intercept is the duration of your trip, from the start until you arrive in Seattle. After traveling 3 hours and 12 minutes, your distance \( y \) from Seattle will be 0.

Exercises 1.1.

1. Find the equation of the line through \((1,1)\) and \((-5,−3)\) in the form \( y = mx + b \). ⇒
2. Find the equation of the line through \((-1,2)\) with slope \(-2\) in the form \( y = mx + b \). ⇒
3. Find the equation of the line through \((-1,1)\) and \((5,−3)\) in the form \( y = mx + b \). ⇒
4. Change the equation \( y = 2x + 2 \) to the form \( y = mx + b \), graph the line, and find the \( y \)-intercept and \( x \)-intercept. ⇒
5. Change the equation \( x = y + 6 \) to the form \( y = mx + b \), graph the line, and find the \( y \)-intercept and \( x \)-intercept. ⇒
6. Change the equation \( x = 2y − 1 \) to the form \( y = mx + b \), graph the line, and find the \( y \)-intercept and \( x \)-intercept. ⇒
7. Change the equation \( 3x + 2y = 0 \) to the form \( y = mx + b \), graph the line, and find the \( y \)-intercept and \( x \)-intercept. ⇒
8. Change the equation \( 2x + 3y + 6 = 0 \) to the form \( y = mx + b \), graph the line, and find the \( y \)-intercept and \( x \)-intercept. ⇒
9. Determine whether the lines \( 3x + 4y = 7 \) and \( 2x + 4y = 5 \) are parallel. ⇒
10. Suppose a triangle in the \( z \)-plane has vertices \((-1,0,0)\), \((1,0,0)\), and \((0,2,0)\). Find the equations of the three lines that lie along the sides of the triangle in the \( y = mx + b \) form. ⇒
11. Suppose that you are driving to Seattle at constant speed. After you have been traveling for an hour you pass a sign saying it is 130 miles to Seattle, and after driving another 20 minutes you pass a sign saying it is 105 miles to Seattle. Using the horizontal axis for the time \( t \) and the vertical axis for the distance from your starting point, graph and find the equation \( y = mt + b \) for your distance from your starting point. How long does the trip to Seattle take? ⇒
12. Let \( x \) stand for temperature in degrees Celsius (centigrade), and let \( y \) stand for temperature in degrees Fahrenheit. A temperature of \( 0^\circ \)C corresponds to \( 32^\circ \)F, and a temperature of \( 100^\circ \)C corresponds to \( 212^\circ \)F. Find the equation of the line that relates temperature Fahrenheit \( y \) to temperature Celsius \( x \) in the form \( y = mx + b \). Graph the line, and find the point at which this line intersects \( x = y \). What is the practical meaning of this point? ⇒

1.2 Distance Between Two Points; Circles

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will be linear. Let \( x \) be the exam score, and let \( y \) be the corresponding grade. Find a formula of \( y = mx + b \) which applies to scores between 40 and 90. ⇒

1.2 Distance Between Two Points; Circles

Given two points \((x_1,y_1)\) and \((x_2,y_2)\), recall that their horizontal distance from one another is \( \Delta x = x_2 − x_1 \) and their vertical distance from one another is \( \Delta y = y_2 − y_1 \). (Actually, the word “distance” normally denotes “positive distance” \( \Delta x \) and \( \Delta y \) are signed distances, but this is clear from context.) The actual (positive) distance from one point to the other is the length of the hypotenuse of a right triangle with legs \( \Delta x \) and \( \Delta y \), as shown in figure 1.2.1. The Pythagorean theorem then says that the distance between the two points is the square root of the sum of the squares of the horizontal and vertical sides:

\[ \text{distance} = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{(x_2 − x_1)^2 + (y_2 − y_1)^2}. \]

For example, the distance between points \( A(2,1) \) and \( B(3,3) \) is \( \sqrt{(3 − 2)^2 + (3 − 1)^2} = \sqrt{5} \).

Figure 1.2.1

Distance between two points, \( \Delta x \) and \( \Delta y \) positive.

As a special case of the distance formula, suppose we want to know the distance of a point \((x,y)\) to the origin. According to the distance formula, this is \( \sqrt{(x − 0)^2 + (y − 0)^2} = \sqrt{x^2 + y^2} \).

A point \((x,y)\) is at a distance \( r \) from the origin if and only if \( \sqrt{x^2 + y^2} = r \), or, if we square both sides: \( x^2 + y^2 = r^2 \). This is the equation of the circle of radius \( r \) centered at the origin. The special case \( r = 1 \) is called the unit circle; its equation is \( x^2 + y^2 = 1 \).

Similarly, if \( C(h,k) \) is any fixed point, then a point \((x,y)\) is at a distance \( r \) from the point \( C \) if and only if \( \sqrt{(x − h)^2 + (y − k)^2} = r \), or, if and only if

\[ (x − h)^2 + (y − k)^2 = r^2. \]

This is the equation of the circle of radius \( r \) centered at the point \((h,k)\). For example, the circle of radius 5 centered at the point \((0,−6)\) has equation \( (x − 0)^2 + (y − (−6))^2 = 25 \), or \( x^2 + (y + 6)^2 = 25 \). If we expand this we get \( x^2 + y^2 + 12y + 36 = 25 \) or \( x^2 + y^2 + 12y + 11 = 0 \), but the original form is usually more useful.
is also true that if \( x \) is any number not 1 there is no \( y \) which corresponds to \( x \), but that is not a problem—only multiple \( y \) values is a problem.

In addition to lines, another familiar example of a function is the parabola \( y = f(x) = x^2 \). We can draw the graph of this function by taking various values of \( x \) (say, at regular intervals) and plotting the points \((x, f(x))=(x, x^2)\). Then connect the points with a smooth curve. (See figure 1.3.1.)

The two examples \( y = f(x) = 2x + 1 \) and \( y = f(x) = x^2 \) are both functions which can be evaluated at any value of \( x \) from negative infinity to positive infinity. For many functions, however, it only makes sense to talk of \( x \) as an interval or outside of some "forbidden" region. The interval of \( x \)-values at which we're allowed to evaluate the function is called the domain of the function.

For example, the square-root function \( y = f(x) = \sqrt{x} \) is the rule which says, given an \( x \)-value, take the nonnegative number whose square is \( x \). This rule only makes sense if \( x \) is positive or zero. We say that the domain of this function is \( x \geq 0 \), or more formally \( \{x \in \mathbb{R} | x \geq 0 \} \). Alternatively, we can use interval notation, and write that the domain is \([0, \infty)\). (In interval notation, square brackets mean that the endpoint is included, and a parenthesis means that the endpoint is not included.) The fact that the domain of \( y = \sqrt{x} \) is \([0, \infty)\) means that in the graph of this function (see figure 1.3.1) we have points \((x, y)\) only above \( x \)-values on the right side of the \( x \)-axis. Another example of a function whose domain is not the entire \( x \)-axis: \( y = f(x) = 1/x \), the reciprocal function. We cannot substitute \( x = 0 \) into this formula. The function makes sense, however, for any nonzero \( x \), so we take the domain to be \( \{x \in \mathbb{R} | x \neq 0 \} \).

The graph of this function does not have any point \((x, y)\) with \( x = 0 \). As \( x \) gets close to \( 0 \) from either side, the graph goes off toward infinity. We call the vertical line \( x = 0 \) an asymptote.

To summarize, two reasons why certain \( x \)-values are excluded from the domain of a function are that (i) we cannot divide by zero, and (ii) we cannot take the square root of a negative number. We will encounter some other ways in which functions might be undefined later.

Another reason why the domain of a function might be restricted is that in a given situation the \( x \)-values outside of some range might have no practical meaning. For example, if \( y \) is the area of a square of side \( x \), then we can write \( y = f(x) = x^2 \). In a purely mathematical context the domain of the function \( y = x^2 \) is all \( x \). But in the story-problem context of finding areas of squares, we restrict the domain to positive values of \( x \), because a square with negative or zero side makes no sense.

In a problem in pure mathematics, we usually take the domain to be all values of \( x \) at which the formulas can be evaluated. But in a story problem there might be further restrictions on the domain because only certain values of \( x \) are of interest or make practical sense.

In a story problem, often letters different from \( x \) and \( y \) are used. For example, the volume \( V \) of a sphere is a function of the radius \( r \), given by the formula \( V = f(r) = \frac{4}{3}\pi r^3 \). Also, letters different from \( f \) may be used. For example, if \( y \) is the velocity of something at time \( t \), we may write \( y = v(t) \) with the letter \( v \) (instead of \( f \)) standing for the velocity function (and \( t \) playing the role of \( x \)).

The letter playing the role of \( x \) is called the independent variable, and the letter playing the role of \( y \) is called the dependent variable (because its value "depends on" the value of the independent variable). In story problems, when one has to translate from English into mathematics, a crucial step is to determine what letters stand for variables. If only words and no letters are given, then we have to decide which letters to use. Some letters are traditional. For example, almost always, \( t \) stands for time.

EXAMPLE 1.3.1 An open-top box is made from an \( a \times b \) rectangular piece of cardboard by cutting out a square of side \( x \) from each of the four corners, and then folding the sides up and sealing them with duct tape. Find a formula for the volume \( V \) of the box as a function of \( x \), and find the domain of this function.

The box we get will have height \( x \) and rectangular base of dimensions \( a-2x \) by \( b-2x \). Thus,

\[
V = f(x) = x(a-2x)(b-2x)
\]

Here \( a \) and \( b \) are constants, and \( V \) is the variable that depends on \( x \), i.e., \( V \) is playing the role of \( y \).

This formula makes mathematical sense for any \( x \), but in the story problem the domain is much less. In the first place, \( x \) must be positive. In the second place, it must be less than half the length of either of the sides of the cardboard. Thus, the domain is \( \{x \in \mathbb{R} | 0 < x < \frac{1}{2} \text{(minimum of } a \text{ and } b)\} \).

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or else \( x < 0 \) and \( 4 - x < 0 \). The latter alternative is impossible, since if \( x \) is negative, then \( x + 4 \) is greater than 4, and so cannot be negative. As for the first alternative, the condition \( 4 - x > 0 \) can be rewritten (adding \( x \) to both sides) as \( 4 > 0 \), so we need \( x > 0 \) and \( 4 > x \) (this is sometimes combined in the form \( 4 > x > 0 \), or, equivalently, \( 0 < x < 4 \)).

Second method. Write \( 4x - x^2 = -(x^2 - 4x) \), and then complete the square, obtaining \( -(x - 2)^2 - 4 = -(x^2 - 4x) \). For this to be positive we need \( x^2 - 4x \) is less than 4, which means that \( x - 2 \) must be less than 2 and greater than \(-2\) or \(-2 < x < 2 \). Adding 2 to everything gives \( 0 < x < 2 \). Both of these methods are equally correct; you may use either in a problem of this type.

A function does not always have to be given by a single formula, as we have already seen (in the income tax problem, for example). Suppose that \( y = f(t) \) is the velocity function for a car which starts out from rest (zero velocity) at time \( t = 0 \); then increases its speed steadily to 20 m/sec, taking 10 seconds to do this; then travels at constant speed 20 m/sec for 15 seconds; and finally applies the brakes to decrease speed steadily to 0, taking 5 seconds to do this. The formula for \( y = f(t) \) is different in each of the three time intervals. First \( y = 2t \); then \( y = 20 \); then \( y = -4t + 120 \). The graph of this function is shown in figure 1.3.3.
1.4 Shifts and Dilations

Many functions in applications are built up from simple functions by inserting constants in various places. It is important to understand the effect such constants have on the appearance of the graph.

Horizontal shifts. If we replace $x$ by $x - C$ everywhere it occurs in the formula for $f(x)$, then the graph shifts over $C$ to the right. (If $C$ is negative, then this means that the graph shifts over $|C|$ to the left.) For example, the graph of $y = x^2$ is the same as the graph of $y = (x - 2)^2$, which is the graph of $y = x^2$ shifted 2 units to the right.

Vertical dilations. If we replace $y$ by $y/k$ in the formula for $f(x)$, then the graph moves up $k$ units. Note well: when replacing $y$ by $y - k$ we must pay attention to meaning, not merely appearance.

1.4.1 Circles

A basic example of the two expansion principles is given by the circle of radius $b$ centered at the origin. (As we saw, this is not a single function $y = f(x)$, but rather two functions $y = \pm \sqrt{b^2 - x^2}$ put together; in any case, the two shifting principles apply to equations like this one that are not in the form $y = f(x)$.) If we replace $x$ by $x - C$ and $y$ by $y - D$ — getting the expression $(x - C)^2 + (y - D)^2 = r^2$ — the effect on the circle is to move it $C$ to the right and $D$ up, thereby obtaining the circle of radius $r$ centered at the point $(C, D)$. This tells us how to write the equation of any circle, not necessarily centered at the origin.

1.4.2 Ellipses

A basic example of the two expansion principles is given by the ellipse of semimajor axis $a$ and semiminor axis $b$. We get such an ellipse by starting with the unit circle—the circle of radius 1 centered at the origin, the equation of which is $x^2 + y^2 = 1$—and dilating by a factor of $a$ horizontally and by a factor of $b$ vertically. To get the equation of the resulting ellipse, which crosses the $x$-axis at $-a$ and crosses the $y$-axis at $b$ in the equation $x/a$ and $y/b$ in the equation for the unit circle. This gives

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Finally, if we want to analyze a function that involves both shifts and dilations, it is usually simplest to work with the dilations first, and then the shifts. For instance, if we want to dilate a function by a factor of $A$ in the $x$-direction and then shift $C$ to the right, we do this by replacing $x$ first by $x/A$ and then by $(x - C)$ in the formula. As an example, suppose that, after dilating our unit circle by $a$ in the $x$-direction and by $b$ in the $y$-direction, we want to shift it to a distance $h$ to the right and a distance $k$ upward, so as to be centered at the point $(h, k)$. The new ellipse would have equation

$$\left(\frac{x - h}{a}\right)^2 + \left(\frac{y - k}{b}\right)^2 = 1$$

Note well that this is different than first doing shifts by $h$ and $k$ and then dilations by $a$ and $b$.

See figure 1.4.1.

Exercises 1.4.

1. $f(x) = \sqrt{x - 2}$
2. $f(x) = 1/(x + 2)$
3. $f(x) = 4x \sqrt{x^2 - 1}$
4. $y = f(x) = \sqrt{x^2 - 1}$
5. $f(x) = -4 + \sqrt{4-x^2}$
6. $f(x) = 2x/\sqrt{1-x^2}$
7. $f(x) = 4 + \sqrt{1-x^2}$
8. $f(x) = 2\sqrt{1-x^2}$
9. $f(x) = 1/(x + 1)$
10. $f(x) = 4x + 2/(x^2 - 1)$
11. $f(x) = 2/(x - 4)$
12. $f(x) = \sqrt{16 - 25(x - 2)^2}$

The graph of $f(x)$ is shown below. Sketch the graphs of the following functions.

13. $y = f(x)$
14. $y = 1 + f(x)$
15. $y = 2/\sqrt{x}$
16. $y = 2/(x - 2)$
17. $y = \sqrt{4/3} \cdot (x - 2)$
18. $y = (1/2)/(x - 3)$
19. $y = f(1 + x)/3 + 2$