1

Analytic Geometry

Much of the mathematics in this chapter will be review for you. However, the examples will be oriented toward applications and so will take some thought.

In the \((x,y)\) coordinate system we normally write the \(x\)-axis horizontally, with positive numbers to the right of the origin, and the \(y\)-axis vertically, with positive numbers above the origin. That is, unless stated otherwise, we take “rightward” to be the positive \(x\)-direction and “upward” to be the positive \(y\)-direction. In a purely mathematical situation, we normally choose the same scale for the \(x\) and \(y\)-axes. For example, the line joining the point to the \((a,a)\) makes an angle of \(45^\circ\) with the \(x\)-axis (and also with the \(y\)-axis).

In applications, often letters other than \(x\) and \(y\) are used, and often different scales are chosen in the horizontal and vertical directions. For example, suppose you drop something from a window, and you want to study how its height above the ground changes from second to second. It is natural to let the letter \(t\) denote the time (the number of seconds since the object was released) and to let the letter \(h\) denote the height. For each \(t\) (say, at one-second intervals) you have a corresponding height \(h\). This information can be tabulated, and then plotted on the \((t,h)\) coordinate plane, as shown in figure 1.0.1.

We use the word “quadrant” for each of the four regions into which the plane is divided by the axes: the first quadrant is where points have both coordinates positive, the second quadrant is where points have \(x\)-coordinate negative and \(y\)-coordinate positive, the third quadrant is where points have both coordinates negative, and the fourth quadrant is where points have \(x\)-coordinate positive and \(y\)-coordinate negative.

Suppose we have two points \(A\) and \(B\) in the \((x,y)\)-plane. We often want to know the change in \(x\)-coordinate (also called the “horizontal distance”) in going from \(A\) to \(B\). This is often written \(\Delta x\), where the meaning of \(\Delta\) (a capital delta in the Greek alphabet) is “change in.” (Thus, \(\Delta x\) can be read as “change in \(x\)” although it usually is read as “delta \(x\)”.) The point is that \(\Delta x\) denotes a single number, and should not be interpreted as “delta times \(x\)”.

For example, if \(A = (2,1)\) and \(B = (3,3)\), \(\Delta x = 3 - 2 = 1\). Similarly, the “change in \(y\)” is written \(\Delta y\). In our example, \(\Delta y = 3 - 1 = 2\), the difference between the \(y\)-coordinates of the two points. It is the vertical distance you have to move in going from \(A\) to \(B\). The general formulas for the change in \(x\) and the change in \(y\) between a point \((x_1,y_1)\) and a point \((x_2,y_2)\) are:

\[
\Delta x = x_2 - x_1, \quad \Delta y = y_2 - y_1.
\]

Figure 1.0.1 A data plot, height versus time.

Note that either or both of these might be negative.

1.1 Lines

If we have two points \(A(x_1,y_1)\) and \(B(x_2,y_2)\), then we can draw one and only one line through both points. By the slope of this line we mean the ratio of \(\Delta y\) to \(\Delta x\). The slope is often denoted \(m\): \(m = \Delta y/\Delta x = (y_2 - y_1)/(x_2 - x_1)\). For example, the line joining the points \((1,-2)\) and \((3,5)\) has slope \((5 + 2)/(3 - 1) = 7/2\).

EXAMPLE 1.1.1

According to the 1990 U.S. federal income tax schedules, a head of household paid 15% on taxable income up to $26,050. If taxable income was between $26,050 and $41,490, then, in addition, 28% was to be paid on the amount between $26,050 and $67,200, and 33% paid on the amount over $67,200 (if any). Interpret the tax bracket

\[
\begin{array}{c|c|c|c|c}
\text{tax bracket} & \text{tax}
\end{array}
\]

and any other point \((x,y)\) on the line. For example, if we want to find the equation of the line joining our earlier points \(A(2,1)\) and \(B(3,3)\), we can use this formula:

\[
y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1),
\]

where the meaning of \(\Delta\) (a capital delta in the Greek alphabet) is “change in”. (Thus, \(\Delta x\) can be read as “change in \(x\)” although it usually is read as “delta \(x\)”.) The point \((x_1,y_1)\) is the point you are using as the “starting point”, and \((x,y)\) is the point of the line you are using as the “ending point”.

Of course, this is really just the point-slope formula, except that we are not computing \(m\) in a separate step.

The slope \(m\) of a line in the form \(y = mx + b\) tells us the direction in which the line is pointing. If \(m\) is positive, the line goes into the 1st quadrant as you go from left to right. If \(m\) is large and positive, it has a steep incline, while if \(m\) is negative but near zero, then it points very little downward. These four possibilities are illustrated in figure 1.1.2.

\[
\begin{array}{c|c|c|c}
\text{graph of line} & \text{diagram}
\end{array}
\]

If \(m = 0\), then the line is horizontal: its equation is simply \(y = b\).

There is one type of line that cannot be written in the form \(y = mx + b\), namely, vertical lines. A vertical line has an equation of the form \(x = a\). Sometimes one says that a vertical line has an “infinite” slope.

Sometimes it is useful to find the \(x\)-intercept of a line of the form \(y = mx + b\). This is the \(x\)-value when \(y = 0\). Setting \(mx + b = 0\) and solving for \(x\), we get:

\[
x = -\frac{b}{m}.
\]

For example, the line \(y = 3x - 3\) passes through the points \(A(2,1)\) and \(B(3,3)\) has-x-intercept 3/2.

EXAMPLE 1.1.2

Suppose that you are driving to Seattle at constant speed, and notice that after you have been traveling for 1 hour (i.e., \(t = 1\)), you pass a sign saying it is 110 miles to Seattle, and after driving another half-hour you pass a sign saying it is 85 miles to Seattle. Using the horizontal axis for the time \(t\) and the vertical axis for the distance \(y\) from Seattle, graph and find the equation \(y = mt + b\) for your distance from Seattle. Find the slope, \(y\)-intercept, and \(t\)-intercept, and describe the practical meaning of each.

The graph of \(y\) versus \(t\) is a straight line because you are traveling at constant speed. The line passes through the two points \((1,110)\) and \((1.5,85)\), so its slope is \(m = (85 - 110)/1.5 = -16/1.5 = -32/3\). The \(y\)-intercept is \(b = 85\) and the \(t\)-intercept is \(t = 85/32\) hours.
1.1 Distance Between Two Points; Circles

1.2 Distance Between Two Points; Circles

Given two points $(x_1, y_1)$ and $(x_2, y_2)$, recall that their horizontal distance from one another is $\Delta x = x_2 - x_1$ and their vertical distance from one another is $\Delta y = y_2 - y_1$. (Actually, the word “distance” normally denoted $\Delta x$ and $\Delta y$ are signed distances, but this is clear from context.) The actual (positive) distance from one point to the other is the length of the hypotenuse of a right triangle with legs $|\Delta x|$ and $|\Delta y|$, as shown in figure 1.2.1. The Pythagorean theorem then says that the distance between the two points is the square root of the sum of the squares of the horizontal and vertical sides:

$$distance = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$  

For example, the distance between points $A(2,1)$ and $B(3,3)$ is $\sqrt{(3 - 2)^2 + (3 - 1)^2} = 2$.  

![Figure 1.2.1](image-url)  

Figure 1.2.1: Distance between two points, $\Delta x$ and $\Delta y$ positive.

As a special case of the distance formula, suppose we want to know the distance of a point $(x, y)$ to the origin. According to the distance formula, this is $\sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}$.

A point $(x, y)$ is at a distance $r$ from the origin if and only if $r = \sqrt{x^2 + y^2}$, or, if we square both sides: $x^2 + y^2 = r^2$. This is the equation of the circle of radius $r$ centered at the origin. The special case $r = 1$ is called the unit circle; its equation is $x^2 + y^2 = 1$.

Similarly, if $C(h, k)$ is any fixed point, then a point $(x, y)$ is at a distance $r$ from the point $C$ if and only if $r = \sqrt{(x - h)^2 + (y - k)^2}$, or, if we square both sides:

$$(x - h)^2 + (y - k)^2 = r^2.$$  

This is the equation of the circle of radius $r$ centered at the point $(h, k)$. For example, the circle of radius 5 centered at the point $(0, -6)$ has equation $(x - 0)^2 + (y - (-6))^2 = 25$, or $x^2 + (y + 6)^2 = 25$. If we expand this we get $x^2 + y^2 + 12y + 36 = 25$ or $x^2 + y^2 + 12y + 11 = 0$, but the original form is usually more useful.
is also true that if $x$ is any number not 1 there is no $y$ which corresponds to $x$, but that is not a problem—only multiple $y$ values is a problem.

In addition to lines, another familiar example of a function is the parabola $y = f(x) = x^2$. We can draw the graph of this function by taking various values of $x$ (say, at regular intervals) and plotting the points $(x, f(x)) = (x, x^2)$. Then connect the points with a smooth curve. (See figure 1.3.1.)

The two examples $y = f(x) = 2x + 1$ and $y = f(x) = x^2$ are both functions which can be evaluated at any value of $x$ from negative infinity to positive infinity. For many functions, however, it only makes sense to take values in some interval or outside of some “forbidden” region. The interval of $x$-values at which we’re allowed to evaluate the function is called the domain of the function.

![Graph of functions](image)

**Figure 1.3.1** Some graphs.

For example, the square-root function $y = f(x) = \sqrt{x}$ is the rule which says, given an $x$-value, take the nonnegative number whose square is $x$. This rule only makes sense if $x$ is positive or zero. We say that the domain of this function is $x \geq 0$, or more formally $x \in \mathbb{R} \; | \; x \geq 0$. Alternatively, we can use interval notation, and write that the domain is $[0, \infty)$. (In interval notation, square brackets mean that the endpoint is included, and a parenthesis means that the endpoint is not included.) The fact that the domain of $y = \sqrt{x}$ is $[0, \infty)$ means that in the graph of this function (see figure 1.3.1) we have points $(x, y)$ only above $x$-values on the right side of the $x$-axis.

Another example of a function whose domain is not the entire $x$-axis is: $y = f(x) = \frac{1}{x}$, the reciprocal function. We cannot substitute $x = 0$ in this formula. The function makes sense, however, for any nonzero $x$, so we take the domain to be: $x \in \mathbb{R} \; | \; x \neq 0$. The graph of this function does not have any point $(x, y)$ with $x = 0$. As $x$ gets close to 0 from either side, the graph goes off toward infinity. We call the vertical line $x = 0$ an asymptote.

To summarize, two reasons why certain $x$-values are excluded from the domain of a function are that (i) we cannot divide by zero, and (ii) we cannot take the square root of a negative number. We will encounter other ways in which functions might be undefined later.

Another reason why the domain of a function might be restricted is that in a given situation the $x$-values outside of some range might have no practical meaning. For example, if $y$ is the area of a square of side $x$, then we can write $y = f(x) = x^2$. In a purely mathematical context the domain of the function $y = x^2$ is all of $\mathbb{R}$, but in the story-problem context of finding areas of squares, we restrict the domain to positive values of $x$, because a square with negative or zero side makes no sense.

In a problem in pure mathematics, we usually take the domain to be all values of $x$ at which the formulas can be evaluated. But in a story problem there might be further restrictions on the domain because only certain values of $x$ are of interest or make practical sense.

In a story problem, often letters different from $x$ and $y$ are used. For example, the volume $V$ of a sphere is a function of the radius $r$, given by the formula $V = f(r) = \frac{4}{3} \pi r^3$. Also, letters different from $f$ may be used. For example, if $y$ is the velocity of something at time $t$, we may write $y = v(t)$ with the letter $v$ (instead of $f$) standing for the velocity function (and $t$ playing the role of $x$).

The letter playing the role of $x$ is called the independent variable, and the letter playing the role of $y$ is called the dependent variable (because its value “depends on” the value of the independent variable). In story problems, when one has to translate from English into mathematics, a crucial step is to determine what letters stand for variables. If only words and no letters are given, then we have to decide which letters to use. Some letters are traditional. For example, almost always, $t$ stands for time.

**EXAMPLE 1.3.1** An open-top box is made from an $a \times b$ rectangular piece of cardboard by cutting out a square of side $x$ from each of the four corners, and then folding the sides up and sealing them with duct tape. Find a formula for the volume $V$ of the box as a function of $x$, and find the domain of this function.

The box we get will have height $x$ and rectangular base of dimensions $a - 2x$ by $b - 2x$. Thus,

$$V = f(x) = x(a - 2x)(b - 2x).$$

Here $a$ and $b$ are constants, and $V$ is the variable that depends on $x$, i.e., $V$ is playing the role of $y$.

This formula makes mathematical sense for any $x$, but in the story problem the domain is much less. In the first place, $x$ must be positive. In the second place, it must be less than half the length of either of the sides of the cardboard. Thus, the domain is

$$\{x \in \mathbb{R} \; | \; 0 < x < \frac{1}{2} \text{(minimum of } a \text{ and } b)\}.$$
Exercises 1.3.
Find the domain of each of the following functions:
1. \( y = f(x) = \sqrt{\frac{x}{3} - 1} \Rightarrow \)
2. \( y = f(x) = \frac{1}{(x+1)} \Rightarrow \)
3. \( y = f(x) = 1/\sqrt{2-x} \Rightarrow \)
4. \( y = f(x) = \sqrt[3]{3x} \Rightarrow \)
5. \( y = f(x) = \frac{1}{x} \Rightarrow \)
6. \( y = f(x) = \sqrt[3]{x} \Rightarrow \)
7. \( y = f(x) = \sqrt{3}\left(x - \frac{3}{2}\right) \), where \( r \) is a positive constant. \( \Rightarrow \)
8. \( y = f(x) = \sqrt{1 - (2x)\frac{1}{3}} \Rightarrow \)
9. \( y = f(x) = 1/\sqrt{3x} \Rightarrow \)
10. \( y = f(x) = \sqrt{x} + 1(x - 1) \Rightarrow \)
11. \( y = f(x) = 1/(\sqrt{x} - 1) \Rightarrow \)

12. Find the domain of \( b(x) = \left(\sqrt{x^2 - 9}(x-3) \right) x \neq 3 \Rightarrow \)
13. Suppose \( f(x) = 3x - 9 \) and \( g(x) = \sqrt{x} \). What is the domain of the composition \( g(f(x)) \)? (Recall that composition is defined as \( (g \circ f)(x) = g(f(x)) \).) What is the domain of \( f \circ g(x) \)?

14. A farmer wants to build a fence along a river. He has 500 feet of fencing and wants to enclose a rectangular pen on three sides (with the river providing the fourth side). If \( x \) is the length of the side perpendicular to the river, determine the area of the pen as a function of \( x \). What is the domain of this function?

15. A can in the shape of a cylinder is to be made to hold a volume of one liter (1000 cubic centimeters). The manufacturer wants to use the least possible material for the can. Write the volume as a function of the radius \( r \) and the height \( h \) of the can. Find the domain of the function.

1-4 Shifting and Dilations

Many functions in applications are built up from simple functions by inserting constants in various places. It is important to understand the effect such constants have on the appearance of the graph.

**Horizontal shifts.** If we replace \( x \) by \( x - C \) everywhere it occurs in the formula for \( f(x) \), then the graph shifts over \( C \) to the right. If \( C \) is negative, then this means that the graph shifts over \( |C| \) to the left. For example, suppose that, after dilating our unit circle by a factor of 3, we move it 4 units to the left and then dilate it by a factor of 2/3. The new ellipse would have equation

\[
\left(\frac{x+4}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1.
\]

Finally, if we want to analyze a function that involves both shifts and dilations, it is usually simplest to work with the dilations first, and then the shifts. For instance, if we want to dilate a function by a factor of \( A \) in the x-direction and then shift \( C \) to the right, we do this by replacing \( x \) first by \( x/A \) and then by \( x - C \) in the formula. As an example, suppose that, after dilating our unit circle by a factor of 2 in the x-direction and by \( A \) in the y-direction to get the ellipse in the last paragraph, we then wanted to shift it a distance \( h \) to the right and a distance \( k \) upward, so as to be centered at the point \((h, k)\). The new ellipse would have equation

\[
\left(\frac{x-h}{A}\right)^2 + \left(\frac{y-k}{2}\right)^2 = 1.
\]

Note well that this is different than first doing shifts by \( h \) and \( k \) and then dilations by \( A \) and \( 2 \).

See figure 1.4.1.

**Exercises 1.4.**

Start with the graph of \( y = \sqrt{x} \), the graph of \( y = 1/x \), and the graph of \( y = \sqrt{1-x^2} \) (the upper unit semicircle), sketch the graph of each of the following functions:

1. \( f(x) = \sqrt{x - 2} \)
2. \( f(x) = -1/(x+2) \)
3. \( f(x) = 1/(2-x) \)
4. \( f(x) = \sqrt{1-x^2} \)
5. \( f(x) = -\sqrt{1-x^2} \)
6. \( f(x) = \sqrt{1-x^2} \)
7. \( f(x) = \sqrt{x-2} \)
8. \( f(x) = \sqrt{4-x^2} \)
9. \( f(x) = 1/(x+1) \)
10. \( f(x) = 1/(x-1) \)
11. \( f(x) = \sqrt{1-x} \)
12. \( f(x) = \sqrt{1+x} \)

The graph of \( f(x) \) is shown below. Sketch the graphs of the following functions:

13. \( y = f(x) + 2 \)
14. \( y = f(x) - 2 \)
15. \( y = f(x+2) \)
16. \( y = f(x)/2 \)
17. \( y = f(x)/2 + 2 \)
18. \( y = f(x)/2 - 2 \)
19. \( y = f(x)/2 + 1 \)
20. \( y = f(x)/2 - 1 \)