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# Single and Multivariable Calculus

*Early Transcendentals*

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This text was initially written by David Guichard. The single variable material in chapters 1–9 is a modification and expansion of notes written by Neal Koblitz at the University of Washington, who generously gave permission to use, modify, and distribute his work. New material has been added, and old material has been modified, so some portions now bear little resemblance to the original.

The book includes some exercises and examples from *Elementary Calculus: An Approach Using Infinitesimals*, by H. Jerome Keisler, available at <http://www.math.wisc.edu/~keisler/calc.html> under a Creative Commons license. In addition, the chapter on differential equations (in the multivariable version) and the section on numerical integration are largely derived from the corresponding portions of Keisler's book. Albert Schueller, Barry Balof, and Mike Wills have contributed additional material.

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I will be glad to receive corrections and suggestions for improvement at [guichard@whitman.edu](mailto:guichard@whitman.edu).

*For Kathleen,  
without whose encouragement  
this book would not have  
been written.*



# Contents

## 1

<b>Analytic Geometry</b>	<b>15</b>
1.1 Lines . . . . .	16
1.2 Distance Between Two Points; Circles . . . . .	21
1.3 Functions . . . . .	22
1.4 Shifts and Dilations . . . . .	27

## 2

<b>Instantaneous Rate of Change: The Derivative</b>	<b>31</b>
2.1 The slope of a function . . . . .	31
2.2 An example . . . . .	36
2.3 Limits . . . . .	38
2.4 The Derivative Function . . . . .	48
2.5 Adjectives For Functions . . . . .	53

### 3

---

#### Rules for Finding Derivatives 57

3.1	The Power Rule . . . . .	57
3.2	Linearity of the Derivative . . . . .	60
3.3	The Product Rule . . . . .	62
3.4	The Quotient Rule . . . . .	64
3.5	The Chain Rule . . . . .	67

### 4

---

#### Transcendental Functions 73

4.1	Trigonometric Functions . . . . .	73
4.2	The Derivative of $\sin x$ . . . . .	76
4.3	A hard limit . . . . .	77
4.4	The Derivative of $\sin x$ , continued . . . . .	80
4.5	Derivatives of the Trigonometric Functions . . . . .	81
4.6	Exponential and Logarithmic functions . . . . .	82
4.7	Derivatives of the exponential and logarithmic functions . . . . .	84
4.8	Implicit Differentiation . . . . .	89
4.9	Inverse Trigonometric Functions . . . . .	94
4.10	Limits revisited . . . . .	97
4.11	Hyperbolic Functions . . . . .	102

### 5

---

#### Curve Sketching 107

5.1	Maxima and Minima . . . . .	107
5.2	The first derivative test . . . . .	111
5.3	The second derivative test . . . . .	112
5.4	Concavity and inflection points . . . . .	113
5.5	Asymptotes and Other Things to Look For . . . . .	115

## 6

---

### Applications of the Derivative 119

<b>6.1</b>	Optimization . . . . .	119
<b>6.2</b>	Related Rates . . . . .	131
<b>6.3</b>	Newton's Method . . . . .	139
<b>6.4</b>	Linear Approximations . . . . .	143
<b>6.5</b>	The Mean Value Theorem . . . . .	145

## 7

---

### Integration 149

<b>7.1</b>	Two examples . . . . .	149
<b>7.2</b>	The Fundamental Theorem of Calculus . . . . .	153
<b>7.3</b>	Some Properties of Integrals . . . . .	160

## 8

---

### Techniques of Integration 165

<b>8.1</b>	Substitution . . . . .	166
<b>8.2</b>	Powers of sine and cosine . . . . .	171
<b>8.3</b>	Trigonometric Substitutions . . . . .	173
<b>8.4</b>	Integration by Parts . . . . .	176
<b>8.5</b>	Rational Functions . . . . .	180
<b>8.6</b>	Numerical Integration . . . . .	184
<b>8.7</b>	Additional exercises . . . . .	189

## 9

---

<b>Applications of Integration</b>	<b>191</b>
<b>9.1</b> Area between curves . . . . .	191
<b>9.2</b> Distance, Velocity, Acceleration . . . . .	196
<b>9.3</b> Volume . . . . .	199
<b>9.4</b> Average value of a function . . . . .	206
<b>9.5</b> Work . . . . .	209
<b>9.6</b> Center of Mass . . . . .	213
<b>9.7</b> Kinetic energy; improper integrals . . . . .	219
<b>9.8</b> Probability . . . . .	223
<b>9.9</b> Arc Length . . . . .	232
<b>9.10</b> Surface Area . . . . .	234

## 10

---

<b>Polar Coordinates, Parametric Equations</b>	<b>241</b>
<b>10.1</b> Polar Coordinates . . . . .	241
<b>10.2</b> Slopes in polar coordinates . . . . .	245
<b>10.3</b> Areas in polar coordinates . . . . .	247
<b>10.4</b> Parametric Equations . . . . .	250
<b>10.5</b> Calculus with Parametric Equations . . . . .	253



# 11

---

<b>Sequences and Series</b>	<b>257</b>
11.1 Sequences . . . . .	258
11.2 Series . . . . .	264
11.3 The Integral Test . . . . .	268
11.4 Alternating Series . . . . .	273
11.5 Comparison Tests . . . . .	275
11.6 Absolute Convergence . . . . .	278
11.7 The Ratio and Root Tests . . . . .	279
11.8 Power Series . . . . .	282
11.9 Calculus with Power Series . . . . .	285
11.10 Taylor Series . . . . .	287
11.11 Taylor's Theorem . . . . .	290
11.12 Additional exercises . . . . .	296

# 12

---

<b>Three Dimensions</b>	<b>299</b>
12.1 The Coordinate System . . . . .	299
12.2 Vectors . . . . .	302
12.3 The Dot Product . . . . .	307
12.4 The Cross Product . . . . .	313
12.5 Lines and Planes . . . . .	317
12.6 Other Coordinate Systems . . . . .	323

# 13

---

<b>Vector Functions</b>	<b>329</b>
13.1 Space Curves . . . . .	329
13.2 Calculus with vector functions . . . . .	331
13.3 Arc length and curvature . . . . .	339
13.4 Motion along a curve . . . . .	345

## 14

---

### Partial Differentiation 349

14.1	Functions of Several Variables . . . . .	349
14.2	Limits and Continuity . . . . .	353
14.3	Partial Differentiation . . . . .	357
14.4	The Chain Rule . . . . .	364
14.5	Directional Derivatives . . . . .	366
14.6	Higher order derivatives . . . . .	371
14.7	Maxima and minima . . . . .	373
14.8	Lagrange Multipliers . . . . .	378

## 15

---

### Multiple Integration 385

15.1	Volume and Average Height . . . . .	385
15.2	Double Integrals in Cylindrical Coordinates . . . . .	395
15.3	Moment and Center of Mass . . . . .	399
15.4	Surface Area . . . . .	402
15.5	Triple Integrals . . . . .	404
15.6	Cylindrical and Spherical Coordinates . . . . .	407
15.7	Change of Variables . . . . .	411

## 16

---

### Vector Calculus 419

16.1	Vector Fields . . . . .	419
16.2	Line Integrals . . . . .	421
16.3	The Fundamental Theorem of Line Integrals . . . . .	425
16.4	Green's Theorem . . . . .	428
16.5	Divergence and Curl . . . . .	433
16.6	Vector Functions for Surfaces . . . . .	436
16.7	Surface Integrals . . . . .	442
16.8	Stokes's Theorem . . . . .	446
16.9	The Divergence Theorem . . . . .	450

# 17

<b>Differential Equations</b>	<b>455</b>
17.1 First Order Differential Equations . . . . .	456
17.2 First Order Homogeneous Linear Equations . . . . .	460
17.3 First Order Linear Equations . . . . .	463
17.4 Approximation . . . . .	466
17.5 Second Order Homogeneous Equations . . . . .	469
17.6 Second Order Linear Equations . . . . .	473
17.7 Second Order Linear Equations, take two . . . . .	477

# A

<b>Selected Answers</b>	<b>481</b>
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# B

<b>Useful Formulas</b>	<b>507</b>
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<b>Index</b>	<b>511</b>
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# Introduction

The emphasis in this course is on problems—doing calculations and story problems. To master problem solving one needs a tremendous amount of practice doing problems. The more problems you do the better you will be at doing them, as patterns will start to emerge in both the problems and in successful approaches to them. You will learn fastest and best if you devote some time to doing problems every day.

Typically the most difficult problems are story problems, since they require some effort before you can begin calculating. Here are some pointers for doing story problems:

1. Carefully read each problem twice before writing anything.
2. Assign letters to quantities that are described only in words; draw a diagram if appropriate.
3. Decide which letters are constants and which are variables. A letter stands for a constant if its value remains the same throughout the problem.
4. Using mathematical notation, write down what you know and then write down what you want to find.
5. Decide what category of problem it is (this might be obvious if the problem comes at the end of a particular chapter, but will not necessarily be so obvious if it comes on an exam covering several chapters).
6. Double check each step as you go along; don't wait until the end to check your work.
7. Use common sense; if an answer is out of the range of practical possibilities, then check your work to see where you went wrong.

### Suggestions for Using This Text

1. Read the example problems carefully, filling in any steps that are left out (ask someone for help if you can't follow the solution to a worked example).
2. Later use the worked examples to study by covering the solutions, and seeing if you can solve the problems on your own.
3. Most exercises have answers in Appendix A; the availability of an answer is marked by " $\Rightarrow$ " at the end of the exercise. In the pdf version of the full text, clicking on the arrow will take you to the answer. The answers should be used only as a final check on your work, not as a crutch. Keep in mind that sometimes an answer could be expressed in various ways that are algebraically equivalent, so don't assume that your answer is wrong just because it doesn't have exactly the same form as the answer in the back.
4. A few figures in the pdf and print versions of the book are marked with "(AP)" at the end of the caption. Clicking on this should open a related interactive applet or Sage worksheet in your web browser. Occasionally another link will do the same thing, like this example. (Note to users of a printed text: the words "this example" in the pdf file are blue, and are a link to a Sage worksheet.)