5 Curve Sketching

Whether we are interested in a function as a purely mathematical object or in connection with some application to the real world, it is often useful to know what the graph of the function looks like. We can obtain a good picture of the graph using certain crucial information provided by derivatives of the function and certain limits.

5.1 Maxima and Minima

A local maximum point on a function is a point \((x, y)\) on the graph of the function whose \(y\)-coordinate is larger than all other \(y\)-coordinates on the graph at points "close to" \((x, y)\). More precisely, \((x, f(x))\) is a local maximum if there is an interval \((a, b)\) with \(x < b \) and \(f(x) \geq f(z)\) for every \(z\) in \((a, b)\). Similarly, \((x, y)\) is a local minimum point if it has locally the smallest \(y\)-coordinate. Again being more precise: \((x, f(x))\) is a local minimum if there is an interval \((a, b)\) with \(x < b \) and \(f(x) \leq f(z)\) for every \(z\) in \((a, b)\). A local extremum is either a local minimum or a local maximum.

Local maximum and minimum points are quite distinctive on the graph of a function, and are therefore useful in understanding the shape of the graph. In many applied problems we want to find the largest or smallest value that a function achieves (for example, we might want to find the minimum cost at which some task can be performed) and so identifying maximum and minimum points will be useful for applied problems as well. Some examples of local maximum and minimum points are shown in figure 5.1.1.

If \((x, f(x))\) is a point where \(f(x)\) reaches a local maximum or minimum, and if the derivative of \(f\) exists at \(x\), then the graph has a tangent line and the tangent line must be horizontal. This is important enough to state as a theorem, though we will not prove it.

![Testing for a maximum or minimum.](image)

Figure 5.1.3 Testing for a maximum or minimum.

It is not always easy to compute the value of a function at a particular point. The task is made easier by the availability of calculators and computers, but they have their own drawbacks—they do not always allow us to distinguish between values that are very close together. Nevertheless, because this method is conceptually simple and sometimes easy to perform, you should always consider it.

**EXAMPLE 5.1.2** Find all local maximum and minimum points for the function \(f(x) = x^3 - x\). The derivative is \(f'(x) = 3x^2 - 1\). This is defined everywhere and is zero at \(x = \pm \sqrt[3]{\frac{1}{3}}\). Looking first at \(x = \sqrt[3]{\frac{1}{3}}\), we see that \(f'(\sqrt[3]{\frac{1}{3}}) = -2\sqrt[3]{\frac{1}{9}}\). Now we test two points on either side of \(x = \sqrt[3]{\frac{1}{3}}\), making sure that neither is farther away than the nearest critical value; since \(\sqrt[3]{3} > \sqrt[3]{\frac{1}{3}} > 1\) and we can use \(x = 0\) and \(x = 1\). Since \(f(0) = 0 > -2\sqrt[3]{\frac{1}{9}}\) and \(f(1) = 0 > -2\sqrt[3]{\frac{1}{9}}\), there must be a local minimum at

**Exercises 5.1.**

In problems 1–12, find all local maximum and minimum points \((x, y)\) by the method of this section.

1. \(y = x^2 - x\) \(
\begin{align*}
&\Rightarrow 2 = 2 \times 3 - x^2 \\
&\Rightarrow 3 = 3^2 - 3 \\
&\Rightarrow 1 = \sqrt{3}x \\
&\Rightarrow x = \cos(2\pi) \\
&\Rightarrow x = 3 - 3 \\
&\Rightarrow x = 1 - 0 \\
&\Rightarrow x = x \\
&\Rightarrow x = 0
\end{align*}
\)

2. \(y = 2 + 3x - x^3\)

3. \(y = x^2 - 4x + 3\)

4. \(y = 3x - 3 \pm 3\)

5. \(y = (x^2 - 1)^{1/2}\)

6. \(y = (x^2 - 1)^{1/2}\)

7. \(y = 3x - 1/\sqrt{x}\)

8. \(y = x - x\)

9. \(f(x) = \begin{cases} x & \text{if } x < 1 \\ x & \text{if } x \geq 1 \end{cases}\)

10. \(f(x) = \frac{x^3 - 3x^2 + 2}{x^2 - 2x} \pm 1\)

11. \(f(x) = \frac{x^3 - 2x^2 + 4}{x^2 - 2x} \pm 1\)

12. \(f(x) = \frac{x^2 - 2x}{x^2 - 1} \pm 0\)

13. For any real number \(x\) there is a unique integer \(n\) such that \(n \leq x < n + 1\), and the greatest integer function is defined as \([x] = n\). Where are the critical values of the greatest integer function? Which are local maxima and which are local minima?

14. Explain why the function \(f(x) = 1/x\) has no local maxima or minima.

15. How many critical points can a quadratic polynomial function have?
5.2 The first derivative test

11. The sign of the derivative tells us whether a function is increasing or decreasing; for example, when \( f'(x) > 0 \), \( f \) is increasing, and when \( f'(x) < 0 \), \( f \) is decreasing. If \( f'(x) = 0 \), this means that near \( x = a \), \( f \) is neither increasing nor decreasing at a point then there is a local maximum at that point. Thus, we can get information from the sign of \( f' \) even when \( f' = 0 \). Suppose that \( f'(x) > 0 \). This means that near \( x = a \), \( f \) is increasing. If \( f'(a) > 0 \), this means that \( f \) slopes up and is getting steeper; if \( f'(a) < 0 \), this means that \( f \) slopes down and is getting concave up.

5.3 The second derivative test

The basis of the first derivative test is that if the derivative changes from positive to negative at a point at which the derivative is zero then there is a local maximum at the point, and similarly for a local minimum. If \( f' \) changes from positive to negative it is decreasing; this means that the derivative of \( f' \), \( f'' \), might be negative, and if in fact \( f'' \) is negative then \( f' \) is definitely decreasing, so there is a local maximum at the point in question. Note well that \( f'' \) might change from positive to negative while \( f'' = 0 \), in which case \( f'' \) gives us no information about the critical value. Similarly, if \( f'' \) changes from negative to positive there is a local minimum at the point, and \( f' \) is increasing. If \( f'' > 0 \) at the point, this tells us that \( f' \) is increasing, and so there is a local minimum.

5.4 Concavity and inflection points

5.4.1 Concavity and inflection points

We know that the sign of the derivative tells us whether a function is increasing or decreasing; for example, when \( f'(x) > 0 \), \( f \) is increasing. The sign of the second derivative \( f''(x) \) tells us whether \( f' \) is increasing or decreasing; we have seen that if \( f' \) is zero and increasing at a point then there is a local minimum at the point, and if \( f' \) is zero and decreasing at a point then there is a local maximum at the point. Thus, we can identify such points by first finding where \( f'' = 0 \), then checking to see whether \( f'' \) is positive or negative.

5.4.2 Concavity and inflection points

If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. Of particular interest are points at which the concavity changes from up to down or down to up; such points are called inflection points. If the concavity changes from up to down at \( a = x \), \( f'' \) changes from positive to the left of \( a \) to negative to the right of \( a \), and usually \( f''(a) = 0 \). We can identify such points by first finding where \( f'' \) is zero and then checking to see whether \( f'' \) does in fact go from positive to negative or vice versa at these points.

Note that it is possible that \( f''(a) = 0 \) but the concavity is the same on both sides; \( f''(x) > 0 \) at \( a = x = 0 \) is an example.

EXAMPLE 5.4.1 Describe the concavity of \( f(x) = x^2 - x \). \( f''(x) = 3x^2 - 1 \), \( f''(0) = 6 \). Since \( f''(0) > 0 \), there is potentially an inflection point at zero. Since \( f''(x) > 0 \) when \( x > 0 \) and \( f''(x) < 0 \) when \( x < 0 \) the concavity does change from down to up at zero, and the curve is concave down for all \( x < 0 \) and concave up for all \( x > 0 \).

Note that we need to compute and analyze the second derivative to understand concavity, so we may as well try to use the second derivative test for maxima and minima. If for some reason this fails we can then try one of the other tests.
Describe the concavity of the functions in 1–18.

5.4 Asymptotes and Other Things to Look For

Describe the concavity of the functions in 1–18.

A vertical asymptote is a place where the function becomes infinite, typically because the denominator of the function is zero. For example, the reciprocal function has the property that as $x$ approaches zero, the function has a zero denominator at $x = 0$, and the function $f(x) = 1/x$ has a vertical asymptote at $x = 0$ and also at $x = -0/2$, $x = -0/2$, etc. The vertical asymptote is a vertical line to which the function approaches infinity.

Horizontal asymptotes can be identified by computing the limits $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$. Since $\lim_{x \to \infty} 1/x = \lim_{x \to -\infty} 1/x = 0$, the line $y = 0$ (that is, the $x$-axis) is a horizontal asymptote in both directions.

5.5 Asymptotes and Other Things to Look For

For each of the following five functions, identify any vertical and horizontal asymptotes, and identify intervals on which the function is concave up and increasing; concave up and decreasing; concave down and increasing; concave down and decreasing.

1. $f(x) = x^3 - x - \pi$
2. $f(x) = 2x + 3x - x^3$
3. $f(x) = x^3 - 9x^2 + 24x$
4. $f(x) = x^3 - 2x^2 + 3$
5. $f(x) = x^3 - 4x^2 + 9x$
6. $f(x) = (x^2 - 1)/x$
7. $f(x) = 3x^2 - (1/x^2)$
8. $f(x) = \sin x + \cos x$
9. $f(x) = 4x + \sqrt{3 - x^2}$
10. $f(x) = (x + 1)/\sqrt{3x^2 + 35}$
11. $f(x) = x^3 - x$
12. $f(x) = 6x + \sin 3x$
13. $f(x) = x + 1/x$
14. $f(x) = x^3 + 1/x$
15. $f(x) = (x + 1)^2/35$
16. $f(x) = \tan x$
17. $f(x) = 3x^2 + 3x$
18. $f(x) = \sin x$
19. $f(x) = x + 1/x$
20. $f(x) = x^3 + 1/x$
21. $f(x) = x^3 + 6x^2 + 9x$
22. $f(x) = x^3/5x^3 + 9$
23. $f(x) = x^3/(x^2 + 9)$
24. $f(x) = 2x^3 - x$
25. $f(x) = 3\sin(x) - \sin^2(x)$, for $x \in [0, 2\pi]$
26. $f(x) = (x - 1)(x^2)$

Some functions have asymptotes that are neither horizontal nor vertical, but some other line. Such asymptotes are somewhat more difficult to identify and we will ignore them.

Finally, it is worthwhile to notice any symmetry. A function $f(x)$ that has the same value for $-x$ as for $x$, i.e., $f(-x) = f(x)$, is called an "even function." Its graph is symmetric with respect to the $y$-axis. Some examples of even functions are: $x^n$ when $n$ is an even number, $\cos x$, and $\sin^2 x$. On the other hand, a function that satisfies the property $f(-x) = -f(x)$ is called an "odd function." Its graph is symmetric with respect to the origin. Some examples of odd functions are: $x^n$ when $n$ is an odd number, sin $x$, and $\tan x$.

5.5 Asymptotes and Other Things to Look For

Sketch the curves. Identify clearly any interesting features, including local maximum and minimum points, inflection points, asymptotes, and intercepts. You can use this Sage worksheet to check your answers. Note that you may need to adjust the interval over which the function is graphed to capture all the details.

1. $y = x^3 - 5x^2 + 3x - 2$
2. $y = x^3 - 3x^2 - 9x + 5$
3. $y = (x - 1)^2/(x + 3)^2/3$
4. $y = x^3 + x^2 + x + 3$
5. $y = x^2 + x + 1$
6. $y = x^2 - x + 1$
7. $y = e^x - x^2$
8. $y = e^x - \sin x$
9. $y = e^x + x^2$
10. $y = x^2 + 1/x$
11. $y = (x + 1)/\sqrt{3x^2 + 35}$
12. $y = x^2 - x$
13. $y = 5x^2 - x^2$
14. $y = x + 1/x$
15. $y = x^3 - 1/x$
16. $y = (x^3 + 1)/x$
17. $y = \tan x$
18. $y = \cos x - x^2$
19. $y = \sin x$
20. $y = x(x^2 + 1)$
21. $y = x^3 + 6x^2 + 9x$
22. $y = x^3/(x^2 + 9)$
23. $y = x^3/(x^2 + 9)$
24. $y = 2x^3 - x$
25. $y = 3\sin(x) - \sin^2(x)$, for $x \in [0, 2\pi]$
26. $y = (x - 1)(x^2)$