7
Integration

7.1 Two Examples

Up to now we have been concerned with extracting information about how a function changes from the function itself. Given knowledge about an object’s position, for example, we want to know the object’s speed. Given information about the height of a curve we want to know its slope. We now consider problems that are, whether obviously or not, the reverse of such problems.

Example 7.1.1 An object moves in a straight line so that its speed at time \( t \) is given by \( v(t) = 3t \) in, say, cm/sec. If the object is at position 10 on the straight line when \( t = 0 \), where is the object at any time \( t \)?

There are two reasonable ways to approach this problem. If \( s(t) \) is the position of the object at time \( t \), we know that \( s(t) = \int v(t) \, dt \). Because of our knowledge of derivatives, we know therefore that \( s(t) = \frac{3t^2}{2} + k \). And because \( s(0) = 10 \), we easily discover that \( k = 10 \), so \( s(t) = \frac{3t^2}{2} + 10 \). For example, at \( t = 1 \) the object is at position \( \frac{3}{2} + 10 = 11.5 \). This is certainly the easiest way to deal with this problem. Not all similar problems are so easy, as we will see; the second approach to the problem is more difficult but also more general.

We start by considering how we might approximate a solution. We know that at \( t = 0 \) the object is at position 10. How might we approximate its position at, say, \( t = 1 \)? We know that the speed of the object at time \( t = 0 \) is 0; if its speed were constant then in the first second the object would not move and its position would still be 10 when \( t = 1 \). In fact, the object will not be far from 10 at \( t = 1 \), but certainly we can do better. Let’s look at the times 0.1, 0.2, 0.3, 1.0, and try approximating the location of the object at each, by supposing that during each tenth of a second the object is going at a constant speed. Since the object initially has speed 0, we again suppose it maintains this speed, but only for a tenth of a second; during that time the object would not move. During the tenth of a second from \( t = 0.1 \) to \( t = 0.2 \), we suppose that the object is traveling at 0.3 cm/sec; namely, its actual speed at \( t = 0.1 \). In this case the object would travel \( 0.3(0.1) = 0.03 \) centimeters: 0.3 cm/sec times 0.1 seconds. Similarly, between \( t = 0.2 \) and \( t = 0.3 \) the object would travel \((0.0)(0.1) = 0.006 \) centimeters. Continuing, we get as an approximation that the object travels

\[(0.0)(0.1) + (0.3)(0.1) + (0.6)(0.1) + \cdots + (2.7)(0.1) = 1.35\]

centimeters, ending up at position 11.35. This is a better approximation than 10, certainly, but is still just an approximation. (We know in fact that the object ends up at position 11.5, because we’ve already done the problem using the first approach.) Presumably, we will get a better approximation if we divide the time into one hundred intervals of a hundredth of a second each, and repeat the process:

\[(0.0)(0.01) + (0.03)(0.01) + (0.06)(0.01) + \cdots + (2.97)(0.01) = 1.495\]

We thus approximate the position as 11.495. Since we know the exact answer, we can see that this is much closer, but if we did not already know the answer, we wouldn’t really know how close.

We can keep this up, but we’ll never really know the exact answer if we simply compute more and more examples. Let’s instead look at a “typical” approximation. Suppose we divide the time into \( n \) equal intervals, and imagine that on each of these the object travels at a constant speed. Over the first time interval we approximate the distance traveled as \( 0.0(1/n) = 0 \), as before. During the second time interval, from \( t = 1/n \) to \( t = 2/n \), the object travels approximately \( 0.1(1/n) \) centimeters. During time interval \( i \), the object travels approximately \( 0.0(1/n) \) centimeters, distance \( i \) of \( 0(1/n) \), \( \cdots \), time the length of time interval \( i \), \( 1/n \). Adding these up as before, we approximate the distance traveled as

\[(1/n) + (2/n) + (3/n) + \cdots + (n-1)/n \]

centimeters. What can we say about this? At first it looks rather less useful than the concrete calculations we’ve already done. But in fact a lot of algebra reveals it to be much

more useful. We can factor out a 3 and \( 1/n^2 \) to get

\[\frac{3}{n^2} \left[ 0 + 1 + 2 + 3 + \cdots + (n-1) \right] \]

that is, \( 3/n^2 \) times the sum of the first \( n-1 \) positive integers. Now we make use of a fact you may have run across before:

\[1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \]

In our case we’re interested in \( k = n - 1 \), so

\[1 + 2 + 3 + \cdots + (n-1) = \frac{(n-1)n}{2} \]

This simplifies the approximate distance traveled to

\[3 \frac{n^3 - n}{2n^2} - \frac{3n^2}{2n} - \frac{3n^2}{n^2} - \frac{3}{2} \left( \frac{n^3}{n^2} \right) \]

Now this is quite easy to understand: as \( n \) gets larger and larger this approximation gets closer and closer to \( (3/2)(1-0) = 3/2 \), so that \( 3/2 \) is the exact distance traveled during one second, and the final position is 11.5.

So for \( t = 1 \), at least, this rather cumbersome approach gives the same answer as the first approach. But really there’s nothing special about \( t = 1 \); let’s just call it \( t \) instead. In this case the approximate distance traveled during time interval \( i \) is \( 3(i-1)(t/n)/(t/n) = 3(i-1)t^2/n^2 \). This is speed \( (i-1)t/n \) times \( t/n \), and the total distance traveled is approximately

\[0 + \frac{3}{n^2} \left[ 0 + 1 + 2 + 3 + \cdots + (n-1) \right] + \frac{3(3)^2}{n^2} + \cdots + \frac{3(n-1)^2}{n^2} \]

As before we can simplify this to

\[3 \frac{n^2}{n^2} \left[ 0 + 1 + 2 + \cdots + (n-1) \right] = 3 \frac{n^2}{n^2} \frac{n(n-1)}{2} \]

In the limit, as \( n \) gets larger, this gets closer and closer to \( (3/2)t^2 \) and the approximated position of the object gets closer and closer to \( (3/2)t^2 + 10 \), so the actual position is \( (3/2)t^2 + 10 \), exactly the answer given by the first approach to the problem.

Example 7.1.2 Find the area under the curve \( y = 3x \) between \( x = 0 \) and any positive value \( x \). There is here no obvious analogue to the first approach in the previous example, but the second approach works fine. (Because the function \( y = 3x \) is so simple, there is another approach that works here, but it is even more limited in potential application than is approach number one.) How might we approximate the desired area? We know how to compute areas of rectangles, so we approximate the area by rectangles. Jumping straight to the general case, suppose we divide the interval between 0 and \( x \) into \( n \) equal subintervals, and use a rectangle above each subinterval to approximate the area under the curve. There are many ways we might do this, but let’s use the height of the curve at the left endpoint of the subinterval as the height of the rectangle, as in figure 7.1.1. The height of rectangle number \( i \) is then \( 3(x - i)/n \), the width is \( x/n \), and the area is \( 3/(n^2) \). The total area of the rectangles is

\[\frac{3}{n^2} \left[ 0 + 1 + 2 + \cdots + (n-1) \right] = \frac{3}{n^2} \frac{n(n-1)}{2} \]

By factoring out \( 3x^2/n^2 \) this simplifies to

\[3x^2 \frac{(0 + 1 + 2 + \cdots + (n-1))}{n^2} - \frac{3x^2}{2} \left( \frac{1}{n} \right) \]

As \( n \) gets larger this gets closer and closer to \( 3x^2/2 \), which must therefore be the true area under the curve.

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Figure 7.1.1 Approximating the area under \( y = 3x \) with rectangles. Drag the slider to change the number of rectangles.

What you will have noticed, of course, is that while the problem in the second example appears to be much different than the problem in the first example, and while the easy approach to problem one does not appear to apply to problem two, the “approximation” approach works in both, and moreover the calculations are identical. As we will see, there
are many, many problems that appear much different on the surface but that turn out to be
the same as those problems, in the sense that when we try to approximate solutions we
end up with mathematics that looks like the two examples, though of course the function
involved will not always be so simple.

Even better, we now see that while the second problem did not appear to be amenable
to approach one, it can in fact be solved in the same way. The reasoning is this: we know
that problem one can be solved easily by finding a function whose derivative is \( f \).

Therefore, we don’t really need to compute the limit of either sum because we know that we
will get the same answer by computing a function with the derivative \( f \) or, which is the same thing, \( 3x \).

It’s true that the first problem had the added complication of the “\( 10^9 \),” and we certainly
need to be able to deal with such minor variations, but that turns out to be quite simple.
The lesson then is this: whenever we can solve a problem by taking the limit of a sum of
a certain form, we can instead of computing the (often nasty) limit find a new function
with a certain derivative.

Exercises 7.1.
1. Suppose an object moves in a straight line so that its speed at time \( t \) is given by \( v(t) = 2t + 2 \),
and that at \( t = 1 \) the object is at position 5. Find the position of the object at \( t = 2 \).
2. Suppose an object moves in a straight line so that its speed at time \( t \) is given by \( v(t) = x^2 + 2 \),
and that at \( t = 0 \) the object is at position 5. Find the position of the object at \( t = 2 \).
3. By a method similar to that in example 7.1.2, find the area under the curve \( y = 2x \) between \( x = 0 \)
and any positive value for \( x \).
4. By a method similar to that in example 7.1.2, find the area under the curve \( y = 4x \) between \( x = 0 \)
and any positive value for \( x \).
5. By a method similar to that in example 7.1.2, find the area under the curve \( y = 4x \) between \( x = 2 \)
and any positive value for \( x \) bigger than 2.
6. By a method similar to that in example 7.1.2, find the area under the curve \( y = 4x \) between any two
positive values for \( x \), say \( x < 0 \).
7. Let \( f(x) = x^2 + 3x + 2 \). Approximate the area under the curve between \( x = 0 \) and \( x = 2 \)
using 4 rectangles and also using 8 rectangles.

7.2 The Fundamental Theorem of Calculus

Let’s recall the first example from the previous section. Suppose that the speed of the
object is \( 3x \) at time \( t \). How far does the object travel between time \( t = a \) and time \( t = b ? \)
We are no longer assuming that we know where the object is at time \( t = 0 \) or at any other
time. It is certainly true that it is somewhere, so let’s suppose that at \( t = 0 \) the position is \( k \).

Then just as in the example, we know that the position of the object at any time is \( 3x^2/2 + k \).
This means that at time \( t = a \) the position is \( 3a^2/2 + k \) and at time \( t = b \) the position is
\( 3b^2/2 + k \). Therefore the change in position is \( 3b^2/2 + k − (3a^2/2 + k) = 3b^2/2 − 3a^2/2 \).
Notice that the \( k \) drops out; this means that it doesn’t matter that we don’t know \( k \)!

It doesn’t even matter if we use the wrong \( k \), we get the correct answer. In other words, to
find the change in position between time \( a \) and time \( b \) we can use any antiderivative of the
speed function \( 3x^2/2 \); it need not be the one antiderivative that actually gives the location
of the object.

What about the second approach to this problem, in the new form? We now want to
approximate the change in position between time \( a \) and time \( b \). We take the interval of
time between \( a \) and \( b \), divide it into \( n \) subintervals, and approximate the distance traveled
during each. The starting time of subinterval number \( i \) is now \( a + (i − 1)(b − a)/n \), which we
abbreviate as \( t_i \), so that \( t_0 = a, t_1 = a + (b − a)/n \), and so on. The speed of the object is
\( f(t) = 3x^2/2 \), and each subinterval is \( (b − a)/n \) \( \Delta t \) seconds long. The distance
taveled during subinterval number \( i \) is approximately \( f(t_i) \Delta t \), and the total change in
distance is approximately

\[
\sum_{i=0}^{n-1} f(t_i) \Delta t = \sum_{i=0}^{n-1} f(t_i) \Delta t + f(t_n) - f(t_0).
\]

The exact change in position is in the limit of this sum as \( n \) goes to infinity. We abbreviate this
sum using sigma notation:

\[
\sum_{i=0}^{n} f(t_i) \Delta t = \sum_{i=0}^{n-1} f(t_i) \Delta t + f(t_n) - f(t_0).
\]

The notation on the left side of the equal sign uses a large capital sigma, a Greek letter,
and the left side is an abbreviation for the right side. The answer we seek is

\[
\lim_{n \to \infty} \sum_{i=0}^{n-1} f(t_i) \Delta t.
\]

Since this must be the same as the answer we have already obtained, we know that

\[
\lim_{n \to \infty} \sum_{i=0}^{n-1} f(t_i) \Delta t = \int_a^b f(x) \, dx.
\]

The significance of \( 3b^2/2 \), into which we substitute \( t = b \) and \( t = a \), is of course that it is
a function whose derivative is \( f(x) \). As we have discussed, by the time we know that we
want to compute

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(t_i) \Delta t.
\]

it no longer matters what \( f(t) \) stands for—it could be a speed, or the height of a curve,
or something else entirely. We know that the limit can be computed by finding any
function with derivative \( f(x) \), substituting \( a \) and \( b \), and subtracting. We summarize this
in a theorem. First, we introduce some new notation and terms.

We write

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(t_i) \Delta t
\]

if the limit exists. That is, the left hand side means, or is an abbreviation for, the right
hand side. The symbol \( \int \) is called an integral sign, and the whole expression is read as
“the integral of \( f(x) \) from \( a \) to \( b \).” What we have learned is that this integral can be
computed by finding a function, say \( F(x) \), with the property that \( F'(x) = f(x) \), and then
computing \( F(b) − F(a) \). The function \( F(x) \) is called an antiderivative of \( f(x) \). Now the theorem:

THEOREM 7.2.1 Fundamental Theorem of Calculus

Suppose that \( f(x) \) is continuous on the interval \([a, b]\). If \( F(x) \) is any antiderivative of \( f(x) \), then

\[
\int_a^b f(x) \, dx = F(b) − F(a).
\]

Let’s rewrite this slightly:

\[
\int_a^b f(x) \, dx = F(x) \bigg|_a^b = F(b) − F(a).
\]

We’ve replaced the variable \( x \) by \( t \) and \( b \) by \( x \). Those are just different names for quantities,
so the substitution doesn’t change the meaning. It does make it easier to think of the two
sides of the equation as functions. The expression

\[
\int_a^b f(x) \, dx
\]

is a function: plug in a value for \( x \), get out some other value. The expression \( F(x) − F(a) \)
is of course also a function, and it has a nice property:

\[
\frac{d}{dx} (F(x) − F(a)) = f(x) − f(a).
\]
The vertical line with subscript and superscript is used to indicate the operation “substitute and subtract” that is needed to finish the evaluation.
7.3 Some Properties of Integrals

right to left, so that \( t_1 = b \) and \( t_n = a \). Then \( \Delta t = t_{n+1} - t_i \) is negative and in

\[
\int_0^5 v(t) \, dt = \frac{125}{6} - \frac{25}{6} = \frac{100}{6} = \frac{50}{3}
\]

the values \( v(t) \) are negative but also \( \Delta t \) is negative, so all terms are positive again. On the other hand, in

\[
\int_5^0 v(t) \, dt = \frac{17}{6} - \frac{4}{6} = \frac{13}{6}
\]

the values \( v(t) \) are positive but \( \Delta t \) is negative, and we get a negative result:

\[
\int_0^5 v(t) \, dt - \int_5^0 v(t) \, dt = \frac{125}{6} - \frac{25}{6} - \frac{50}{3} + \frac{13}{6} = \frac{50}{3} - \frac{50}{3} + \frac{13}{6} = \frac{13}{6}
\]

Finally we note one simple property of integrals:

\[
\int_a^b f(x) + g(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx.
\]

This is easy to understand once you recall that \( (F(x) + G(x))' = F'(x) + G'(x) \). Hence, if \( F(x) = f(x) \) and \( G(x) = g(x) \), then

\[
\int_a^b f(x) + g(x) \, dx = (F(b) + G(b)) - (F(a) + G(a)) = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx.
\]

In summary, we will frequently use these properties of integrals:

1. \( \int_a^b f(x) \, dx = \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \)
2. \( \int_a^b f(x) + g(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \)
3. \( \int_a^b f(x) \, dx = \int_a^b f(x) \, dx \)

Exercises 7.3.

1. An object moves so that its velocity at time \( t \) is \( v(t) = -9.8t + 20 \) m/s. Describe the motion of the object between \( t = 0 \) and \( t = 2 \), find the total distance traveled by the object during that time, and find the net distance traveled. \( \Rightarrow \)
2. An object moves so that its velocity at time \( t \) is \( v(t) = \sin t \). Set up and evaluate a single definite integral to compute the net distance traveled between \( t = 0 \) and \( t = 2\pi \). \( \Rightarrow \)
3. An object moves so that its velocity at time \( t \) is \( v(t) = 1 + 2\sin t \) m/s. Find the net distance traveled by the object between \( t = 0 \) and \( t = 2\pi \), and find the total distance traveled during the same period. \( \Rightarrow \)
4. Consider the function \( f(x) = (x + 2)(x + 1)(x - 1)(x - 2) \) on \([-2, 2]\). Find the total area between the curve and the x-axis (measuring all area as positive). \( \Rightarrow \)
5. Consider the function \( f(x) = x^3 - 3x^2 + 2 \) on \([0, 4]\). Find the total area between the curve and the x-axis (measuring all area as positive). \( \Rightarrow \)
6. Evaluate the three integrals:

\[
A = \int_0^2 (x^2 + 9) \, dx \quad B = \int_1^4 (x^2 + 9) \, dx \quad C = \int_0^2 (x^2 + 9) \, dx,
\]

and verify that \( A = B + C \). \( \Rightarrow \)