4

Transcendental Functions

So far we have used only algebraic functions as examples when finding derivatives, that is, functions that can be built up by the usual algebraic operations of addition, subtraction, multiplication, division, and raising to constant powers. Both in theory and practice there are other functions, called transcendental, that are very useful. Most important among these are the trigonometric functions, the inverse trigonometric functions, exponential functions, and logarithms.

4.1 Trigonometric Functions

When you first encountered the trigonometric functions it was probably in the context of “triangle trigonometry,” defining, for example, the sine of an angle as the “side opposite over the hypotenuse.” While this will still be useful in an informal way, we need to use a more expansive definition of the trigonometric functions. First an important note: while degree measure of angles is sometimes convenient because it is so familiar, it turns out to be ill-suited to mathematical calculation, so (almost) everything we do will be in terms of radian measure of angles.

to any angles, as indicated in this figure:

The angle $x$ is subtended by the heavy arc in the figure, that is, $x \approx \pi/6$. Both coordinates of point $A$ in this figure are negative, so the sine and cosine of $\pi/6$ are both negative.

The remaining trigonometric functions can be most easily defined in terms of the sine and cosine, as usual:

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

and they can also be defined as the corresponding ratios of coordinates.

Although the trigonometric functions are defined in terms of the unit circle, the unit circle diagram is not what we normally consider the graph of a trigonometric function. (The unit circle is the graph of, well, the circle.) We can easily get a qualitatively correct idea of the graphs of the trigonometric functions from the unit circle diagram. Consider the sine function, $y = \sin x$. As $x$ increases from 0 in the unit circle diagram, the second coordinate of the point $A$ goes from 0 to a maximum of 1, then back to 0, then to a minimum of -1, then back to 0, and then it obviously repeats itself. So the graph of $y = \sin x$ must look something like this:

An angle, $x$, at the center of the circle is associated with an arc of the circle which is said to subtend the angle. In the figure, this arc is the portion of the circle from point (1,0) to point $A$. The length of this arc is the radian measure of the angle $x$; the fact that the radian measure is an actual geometric length is largely responsible for the usefulness of radian measure. The circumference of the unit circle is $2\pi$, so the radian measure of the full circular angle (that is, of the 360 degree angle) is $2\pi$.

While an angle with a particular measure can appear anywhere around the circle, we need a fixed, conventional location so that we can use the coordinate system to define properties of the angle. The standard convention is to place the starting radius for the angle on the positive x-axis, and to measure positive angles counterclockwise around the circle. In the figure, $x$ is the standard location of the angle $\pi/6$, that is, the length of the arc from (1,0) to $A$ is $\pi/6$. The angle $y$ in the picture is $-x/6$, because the distance from (1,0) to $B$ along the circle is also $\pi/6$, but in a clockwise direction.

Now the fundamental trigonometric definitions are: the cosine of $x$ and the sine of $x$ are the first and second coordinates of the point $A$, as indicated in the figure. The angle $x$ shown can be viewed as an angle of a right triangle, meaning the usual triangle definitions of the sine and cosine also make sense. Since the hypotenuse of the triangle is 1, the “side opposite over hypotenuse” definition of the sine is the second coordinate of point A over 1, which is just the second coordinate; in other words, both methods give the same value for the sine.

The simple triangle definitions work only for angles that can “fit” in a right triangle, namely, angles between 0 and $\pi/2$. The coordinate definitions, on the other hand, apply to any angles, as indicated in this figure:

Exercises 4.1.

1. Find all values of $\theta$ such that $\sin(\theta) = -1$; give your answer in radians. ⇒
2. Find all values of $\theta$ such that $\cos(\theta) = 1/2$; give your answer in radians. ⇒
3. Use an angle sum identity to compute $\cos(\pi/12)$. ⇒
4. Use an angle sum identity to compute $\tan(5\pi/12)$. ⇒
5. Verify the identity $\cos^2(\theta)/\cos(\theta) = 1 + \sin(\theta)$. ⇒
6. Verify the identity $2\cos(2\theta) = \cos(\theta)\cos(\theta)$. ⇒
7. Verify the identity $\sin(3\theta) - \sin(\theta) = 2\cos(2\theta)\sin(\theta)$. ⇒
8. Sketch $y = 2\sin(x)$. ⇒
9. Sketch $y = \sin(3x)$. ⇒
10. Sketch $y = \sin(-x)$. ⇒
11. Find all of the solutions of $2\sin(\theta) - \sin^2(\theta) = 0$ in the interval $[0,2\pi]$. ⇒

4.2 The Derivative of $\sin x$

What about the derivative of the sine function? The rules for derivatives that we have are no help, since $\sin x$ is not an algebraic function. We need to return to the definition of the derivative, set up a limit, and try to compute it. Here’s the definition:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Using some trigonometric identities, we can make a little progress on the quotient:

$$\frac{\sin(x + \Delta x) - \sin x}{\Delta x} = \sin x \cos \Delta x + \sin \Delta x \cos x - \sin x = \sin x \frac{\cos \Delta x - 1}{\Delta x} + \sin \Delta x \frac{\cos x}{\Delta x}.$$
This isolates the difficult parts in the two limits
\[
\lim_{\Delta x \to 0} \cos \Delta x - 1 \quad \text{and} \quad \lim_{\Delta x \to 0} \sin \Delta x.
\]
Here we get a little lucky: it turns out that once we know the second limit the first is quite easy. The second is quite tricky, however. Indeed, it is the hardest limit we will actually compute, and we devote a section to it.

### 4.3 A HARD LIMIT

We want to compute this limit:
\[
\lim_{x \to 0} \frac{\sin x}{x}.
\]
Equivalently, to make the notation a bit simpler, we can compute
\[
\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}.
\]
In the original context we need to keep \(x\) and \(\Delta x\) separate, but here it doesn't hurt to rename \(\Delta x\) to something more convenient.

To do this we need to be quite clever, and to employ some indirect reasoning. The indirect reasoning is embodied in a theorem, frequently called the squeeze theorem.

**THEOREM 4.3.1 Squeeze Theorem.** Suppose that \(g(x) \leq f(x) \leq h(x)\) for all \(x\) close to \(a\) but not equal to \(a\). If \(\lim_{x \to a} g(x) = L = \lim_{x \to a} h(x)\), then \(\lim_{x \to a} f(x) = L\).

This theorem can be proved using the official definition of limit. We won't prove it here, but point out that it is easy to understand and believe graphically. The condition says that \(f(x)\) is trapped between \(g(x)\) below and \(h(x)\) above, and that as \(x \to a\), both \(g\) and \(h\) approach the same value. This means the situation looks something like figure 4.3.1.

The wiggly curve is \(x^2\sin(1/x)\), the upper and lower curves are \(x^2\) and \(-x^2\). Since the sine function is always between \(-1\) and \(1\), \(-x^2 \leq x^2\sin(1/x) \leq x^2\), and it is easy to see that \(\lim_{x \to 0} -x^2 = 0 = \lim_{x \to 0} x^2\). It is not so easy to see directly, but that is algebraically, that \(\lim_{x \to 0} x^2\sin(1/x) = 0\), because the \(1/x\) prevents us from simply plugging in \(x = 0\). The squeeze theorem makes this "hard limit" as easy as the trivial limits involving \(x^2\).

To do the hard limit that we want, \(\lim_{\Delta x \to 0} \sin(\Delta x)/(\Delta x)\), we will find two simpler functions \(g\) and \(h\) so that \(g(\Delta x) \leq \sin(\Delta x)/(\Delta x) \leq h(\Delta x)\), and so that \(\lim_{\Delta x \to 0} g(\Delta x) = \lim_{\Delta x \to 0} h(\Delta x) = 0\). Not too surprisingly, this will require some trigonometry and geometry. Referring to figure 4.3.2, \(x\) is the measure of the angle in radians. Since the circle has radius 1, the coordinates of point \(A\) are \((\cos x, \sin x)\), and the area of the small triangle is \((\cos x \sin x)/2\). This triangle is completely contained within the circular wedge-shaped region bounded by two lines and the circle from \((1,0)\) to point \(A\). Comparing the areas of the triangle and the wedge we see \((\cos x \sin x)/2 \leq x/2\), since the area of a circular region with angle \(\theta\) and radius \(r\) is \(\theta r^2/2\). With a little algebra this turns into \((\sin x)/x \leq 1/\cos x\) giving us the \(h\) we seek.

\[
\cos x \leq \frac{\sin x}{x} \leq \frac{1}{\cos x}
\]

**Figure 4.3.1** The squeeze theorem.

To find \(g\), we note that the circular wedge is completely contained inside the larger triangle. The height of the triangle, from \((1,0)\) to point \(B\) on \(x\), so comparing areas we get \(x/2 \leq (\tan x)/2 = \sin x/(2\cos x)\). With a little algebra this becomes \(\cos x \leq (\sin x)/x\). So now we have

\[
\cos x \leq \frac{\sin x}{x} \leq \frac{1}{\cos x}
\]

**Figure 4.3.2** Visualizing \(\sin x/x\).

### Exercises 4.3

1. Compute \(\lim_{x \to 0} \sin(5x)/x\) \(\Rightarrow\)
2. Compute \(\lim_{x \to 0} \sin(7x)/x\) \(\Rightarrow\)
3. Compute \(\lim_{x \to 0} \cos(4x)\) \(\Rightarrow\)
4. Compute \(\lim_{x \to 0} \sin x / x\) \(\Rightarrow\)
5. Compute \(\lim_{x \to 0} \cos x - x - x^2\) \(\Rightarrow\)
6. For all \(x \geq 0\), \(4x - 9 \leq f(x) \leq x^2 - 4x + 7\). Find \(\lim_{x \to 0} f(x)\) \(\Rightarrow\)
7. For all \(x, 2x \leq g(x) \leq x^2 - x + 2\). Find \(\lim_{x \to 0} g(x)\) \(\Rightarrow\)
8. Use the Squeeze Theorem to show that \(\lim_{x \to 0} x^4 \cos(1/x) = 0\).
4.5 Derivatives of the Trigonometric Functions

All of the other trigonometric functions can be expressed in terms of the sine, and so their derivatives can easily be calculated using the rules we already have. For instance we need to use two identities,
\[
\cos x = \sin(x + \frac{\pi}{2}), \quad \sin x = \cos(x - \frac{\pi}{2}).
\]

Now,
\[
\frac{d}{dx} \cos x = \frac{d}{dx} \sin(x + \frac{\pi}{2}) = \cos x \cdot 1 = \cos x
\]
\[
\frac{d}{dx} \sin x = \frac{d}{dx} \cos(x - \frac{\pi}{2}) = -\sin x \cdot 1 = -\sin x
\]
\[
\frac{d}{dx} \sec x = \frac{d}{dx} \cos(x - \frac{\pi}{2}) = \sec x \tan x
\]

The derivatives of the cotangent and cosecant are similar and left as exercises.

Exercises 4.5.

Find the derivatives of the following functions.
1. \(\sin x \cos x \Rightarrow\)
2. \(\sin(3x) \Rightarrow\)
3. \(\sqrt[3]{\tan x} \Rightarrow\)
4. \(\cos x \Rightarrow\)
5. \(\sec x \Rightarrow\)

4.6 Exponential and Logarithmic Functions

The logarithm is the exponent. In other words, \(\log_a b = c\) means \(a^c = b\). The next easiest number to understand is 0: \(2^0 = 1\). After the positive integers, the next easiest number is \(-1\): \(2^{-1} = \frac{1}{2}\). It is clear that \(2^1 = 2\), it is clear that \(a^n \cdot a^m = a^{n+m}\). \(\log_a b \cdot \log_a c = \log_a (bc)\) and again this gives us a fact about logarithms. If \(a^m = b^n\), then \(A^m \cdot B^n = 2^{1/3} \cdot 2^{-1/3} = 2^{1/3-1/3} = 2^0 = 1\). Again it's clear that more generally \((a^n)^m = a^{nm}\), and again this gives us a fact about logarithms. If \(A^m = b^n\), then \(X = a^m\), \(y = b^n\), and \(\log_a (A^m) = xy = \log_a (b^n)\) the exponent can be "pulled out in front of the logarithm." We've cheated a bit in the previous two paragraphs. It is obvious that \(a^0 = 1\), \(a^m = a \cdot a \cdots a\) (where the number of terms is \(m\)). The answer to the first question is "yes." We've evaded the full answer to the hard question, but we have to know something about exponential functions. You need first to understand that since it's not "obvious" what \(2^0\) should mean, we are really free to make it mean whatever we want, so long as we keep the behavior that is obvious, namely, when \(x = 0\) is a positive integer. Which else do we want to be true about \(\mathbb{R}^\mathbb{N}\)? We want the properties of the previous two paragraphs to be true for all exponents, \(2^0 = 1\) and \(2^1 = 2\). After the positive integers, the next easiest number to understand is \(2^0 = 1\). You have probably learned this fact in the past, why is it true? It is true precisely because we want \(2^1 = 2\) to be true about the function \(2\). We need it to be true that \(2^a = 2^a\) and this only works if \(a = 1\). The same argument implies that \(a^0 = 1\) for any \(a\). The next easiest set of numbers to understand is the negative integers: for example, \(2^{-1} = \frac{1}{2}\). We know that whatever \(2^{-2}\) means it must be that \(2^{-2} = \frac{1}{2}\). Which means that \(2^2\) must be \(1/2^2\). Once we know that \(2^0\) means for some value of \(z\), \(2^0 = 1\) and more generally \(2^{-z} = \frac{1}{2^z}\). Next, consider an exponent \(1/q\), where \(q\) is a positive integer. We want it to be true that \(2^{1/q} = \sqrt[3]{2}\). This allows \(2^{1/4}\) to be a real root of \(2\). What do we want to be true about the function \(2^x\)? We need it to be true that \(2^{x+y} = 2^x \cdot 2^y\) and \(2^{-x} = \frac{1}{2^x}\). What's left is the hard part: what does \(2^x\) mean when \(x\) cannot be written as a fraction, like \(x = \sqrt{2}\) or \(x = \pi/2\) or \(x = \pi?\) What we know so far is how to assign meaning to 2^x whenever \(x = p/q\) if we were to graph this we'd see something like this:

But this is a poor picture, because you can't see that the "curve" is really a whole lot of individual points, above the rational numbers on the x-axis. There are really a lot of "holes" in the curve, above \(x = \pi\), for example. But (this is the hard part) it is possible to prove that the holes can be "filled in" and that the resulting function, called \(2^x\), really does have the properties we want, namely that \(2^z = 2^{z+y}\) and \(2^{z+y} = 2^z \cdot 2^y\).

Exercises 4.6.
1. Expand \(\log_3((x + 4)^2)\) for \(x > -2\).
2. Expand \(\log_{10}(x^2 - 2xy + y^2)\).
3. Write \(\log_{\sqrt{3}}(17 + \log_{\sqrt{2}}(x^2 - 2x + 1))\) as a single logarithm.
4. Solve \(\log_2(1 + \sqrt{5}) = 6\) for \(x\).
5. Solve \(2^x = 8\) for \(x\).
6. Solve \(\log_2(\log_3(x)) = 1\) for \(x\).

4.7 Derivatives of the Exponential and Logarithmic Functions

As with the sine, we don't know anything about derivatives that allows us to compute the derivatives of the exponential and logarithmic functions without going back to basics. Let's do a little work with the definition again:
\[
\frac{d}{dx} e^x = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}
\]
\[
= \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}
\]
\[
= \lim_{\Delta x \to 0} \frac{e^{x+\Delta x}}{\Delta x}
\]
\[
= \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}
\]
\[
= \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x}
\]
There are two interesting things to note here. As in the case of the sine function, we are left with a limit that involves $\Delta x$ but not $x$, which means that whatever $\lim_{\Delta x \to 0} (a^{\Delta x} - 1)/\Delta x$ is, we know that it is a number, that is, a constant. This means that $a^x$ has a remarkable property: its derivative is a constant times itself.

We earlier remarked that the hardest limit we would compute is $\lim_{\Delta x \to 0} \sin(x)/x = 1$; we now have a limit that is just a bit too hard to include here. In fact the hard part is to see that $\lim_{\Delta x \to 0} (a^{\Delta x} - 1)/\Delta x$ even exists—does this function really get closer and closer to some fixed value? Yes it does, but we will not prove this fact.

We can look at some examples. Consider $(2^x - 1)/x$ for some small values of $x$: 1, 0.828247124, 0.756582469, 0.724061864, 0.70839651, 0.708705777 when $x$ is 1, 1/2, 1/4, 1/8, 1/16, 1/32, respectively. It looks like this is settling in around 0.7, which turns out to be true (but the limit is not exactly 0.7). Consider next $(x^2 - 1)/x$: 2.1, 0.64104916, 0.26249069, 0.11776258, 0.011769854, at the same values of $x$. It turns out to be true that in the limit this is about 1.1. Two examples don’t establish a pattern, but if you do more examples you will find that the limit varies directly with the value of $a$: larger $a$, bigger limit; smaller $a$, smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between $a = 2$ and $a = 3$ the limit will be exactly 1; the value at which this happens is called $e$, so that

$$\lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1.$$

As you might guess from our two examples, $e$ is closer to 3 than to 2, and in fact $e \approx 2.718$.

Now we see that the function $e^x$ has a truly remarkable property:

$$\frac{d}{dx} e^x = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{x} e^{\Delta x} - e^x}{\Delta x} = e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = e^x.$$

That is, $e^x$ is its own derivative, or in other words the slope of $e^x$ is the same as its height, or the same as its second coordinate. The function $f(x) = e^x$ goes through the point $(x, e^x)$ and has slope $e^x$ there, no matter what $x$ is. It is sometimes convenient to express the function $e^x$ without an exponent, since complicated exponents can be hard to read. In such cases we use $\exp(x)$, e.g., $\exp(1 + x^2)$ instead of $e^{1+x^2}$.

4.7 Derivatives of the exponential and logarithmic functions

We have discussed this from the point of view of the graphs, which is easy to understand but is not normally considered a rigorous proof—it is too easy to be led astray by pictures that seem reasonable but that miss some hard point. It is possible to do this derivation without resorting to pictures, and indeed we will see an alternate approach soon.

In the previous example.

EXAMPLE 4.7.3

Compute the derivative of $\log_a x$.

$$\frac{d}{dx} \log_a x = \frac{d}{dx} \left( \frac{1}{\ln a} \log_e x \right) = \frac{1}{\ln a} \frac{d}{dx} \log_e x = \frac{1}{\ln a} \frac{1}{x} = \frac{1}{x \ln a}.$$
EXAMPLE 4.7.4 Recall that we have not justified the power rule except when the exponent is a positive or negative integer. We can use the exponential function to take care of other exponents.

\[
\frac{d}{dx} x^r = \frac{d}{dx} e^{r \ln x} = \left( \frac{d}{dx} \ln x \right) e^{r \ln x} = \left( \frac{1}{x} \right) e^{r \ln x} = r x^{r-1}
\]

Exercises 4.7.
In 1–19, find the derivatives of the functions.

1. \(2^x \Rightarrow\)
2. \(\ln x \Rightarrow\)
3. \((e^x)^r \Rightarrow\)
4. \(\sin(e^x) \Rightarrow\)
5. \(e^{rx} \Rightarrow\)
6. \(x^{2^x} \Rightarrow\)
7. \((\ln x)^r \Rightarrow\)
8. \(x^2 \Rightarrow\)
9. \((1/y)^r \Rightarrow\)
10. \(y^{x^r} \Rightarrow\)

11. \(\ln(x^2 + 3x) \Rightarrow\)
12. \(\ln(x \cos(x)) \Rightarrow\)
13. \(\sqrt{\ln(x^2)} \Rightarrow\)
14. \(\ln(x + \tan(x)) \Rightarrow\)
15. \(e^{x+1} \Rightarrow\)
16. \(x \ln x \Rightarrow\)
17. \(\ln(x^3) \Rightarrow\)
18. \(1 + \ln(4x) \Rightarrow\)
19. \(\frac{x^2(x - 2)^{1/2}}{27x^2(4x - 6)^2} \Rightarrow\)

20. Find the value of \(n\) so that the tangent line to \(y = \ln(x)\) at \(x = a\) is a line through the origin. Sketch the resulting situation. \(\Rightarrow\)

21. If \(f(x) = \ln(e^x + 2)\) compute \(f'(x)\).

4.8 Implicit Differentiation

As we have seen, there is a close relationship between the derivatives of \(e^x\) and \(\ln x\) because these functions are inverses. Rather than relying on pictures for our understanding, we would like to be able to exploit this relationship computationally. In fact this technique can help us find derivatives in many situations, not just when we seek the derivative of an inverse function.

4.8 Implicit Differentiation

chain rule where \(y\) appears.

\[
\frac{d}{dx} x^2 + y^2 = \frac{d}{dx} (x^2 + y^2) = \frac{d}{dx} x^2 + \frac{d}{dx} y^2 = 2x + 2yy' = 2x - 2y y' = x - y
\]

Now we have an expression for \(y'\), but it contains \(y\) as well as \(x\). This means that if we want to compute \(y'\) for some particular value of \(x\) we'll have to know or compute \(y\) at that value of \(x\) as well. It is at this point that we will need to know whether \(y = U(x)\) or \(L(x)\). Occasionally it will turn out that we can avoid explicit use of \(U(x)\) or \(L(x)\) by the nature of the problem.

EXAMPLE 4.8.1 Find the slope of the circle \(x^2 + y^2 = 1\) at the point \((1, -\sqrt{5})\). Since we know both the \(x\) and \(y\) coordinates of the point of interest, we do not need to explicitly recognize that this point is on \(L(x)\), and we do not need to use \(L(x)\) to compute \(y'\) but we could. Using the calculation of \(y'\) from above,

\[
y' = y = \frac{x}{\sqrt{5} - 1} = \frac{1}{\sqrt{5} - 1}
\]

It is instructive to compare this approach to others.

We might have recognized at the start that \((1, -\sqrt{5})\) is on the function \(y = L(x) = -\sqrt{5} - 1\). We could then take the derivative of \(L(x)\), using the power rule and the chain rule, to get

\[
L'(x) = \frac{d}{dx} \left( 1 - x^2 \right)^{1/2} \cdot (-2x) = \frac{x}{\sqrt{5} - 1}
\]

Then we could compute \(L'(1) = 1/\sqrt{5} - 1\) by substituting \(x = 1\).

Alternatively, we could realize that the point is on \(L(x)\), but use the fact that \(y' = -x/y\). Since the point is on \(L(x)\) we can replace \(y\) by \(L(x)\) to get

\[
y' = x \quad \frac{x}{L(x)} = \frac{x}{\sqrt{5} - 1}
\]

without computing the derivative of \(L(x)\) explicitly. Then we substitute \(x = 1\) and get the same answer as before.

In the case of the circle it is possible to find the functions \(U(x)\) and \(L(x)\) explicitly, but there are potential advantages to using implicit differentiation anyway. In some cases it is more difficult or impossible to find an explicit formula for \(y\) and implicit differentiation is the only way to find the derivative.

EXAMPLE 4.8.2 Find the derivative of any function defined implicitly by \(yx^2 + \cos y = x\). We treat \(y\) as an unspecified function and use the chain rule:

\[
\frac{d}{dx} \left( yx^2 + \cos y \right) = \frac{d}{dx} x
\]

\[
yx^2 + y'x^2 + y'\cos y = 1 - 2xy y'x^2 + y'\cos y = 1 - 2xy y'(x^2 + \cos y) = 1 - 2xy y' = \frac{1 - 2xy}{x^2 + \cos y}
\]

You might think that the step in which we solve for \(y'\) could sometimes be difficult—after all, we're using implicit differentiation here because we can't solve the equation \(yx^2 + \cos y = x\) for \(y\), so maybe after taking the derivative we get something that is hard to solve for \(y'\). In fact, this never happens. All occurrences \(y'\) come from applying the chain rule, and whenever the chain rule is used it deposits a single \(y'\) multiplied by some other expression. So it will always be possible to group the terms containing \(y'\) together and factor out the \(y'\) just as in the previous example. If you ever get anything more difficult you have made a mistake and should fix it before trying to continue.

It is sometimes the case that a situation leads naturally to an equation that defines a function implicitly.

EXAMPLE 4.8.3 Consider all the points \((x, y)\) that have the property that the distance from \((x, y)\) to \((1, 1, y)\) plus the distance from \((x, y)\) to \((x, 1, y)\) is 20. (\(a\) is some constant.) These points form an ellipse, which like a circle is not a function but can viewed as two functions pasted together. Because we know how to write down the distance between two points, we can write down an implicit equation for the ellipse:

\[
\sqrt{\left( x - 1 \right)^2 + \left( y - 1 \right)^2 + \left( x - x \right)^2 + \left( y - y \right)^2} = 20.
\]

Then we can use implicit differentiation to find the slope of the ellipse at any point, though the computation is rather messy.

EXAMPLE 4.8.4 We have already justified the power rule by using the exponential function, but we could also do it for rational exponents by using implicit differentiation. Suppose that \(y = x^{m/n}\), where \(m\) and \(n\) are positive integers. We can write this implicitly as \(y^m = x^n\), then because we justified the power rule for integers, we can take the derivative

\[
\frac{d}{dx} y^m = \frac{d}{dx} x^n \Rightarrow m y^{m-1} y' = n x^{n-1} \Rightarrow y' = \frac{n \left( x^{n-1} \right) }{m \left( y^{m-1} \right) }
\]

Then using the chain rule on the right hand side:

\[
1 - \left( \frac{d}{dx} y^m \right) y' = y^n \Rightarrow
\]

Then we can solve for \(y'\):

\[
y' = \frac{1}{y} \Rightarrow \frac{1}{y} = \frac{1}{y}
\]

There is one little difficulty here. To use the chain rule to compute \(d/dx(e^x) = e^x\) we need to know that the function \(y = e^x\) has a derivative. All we have shown is that if it has a derivative then that derivative must be 1/x. When using this method we will always have to assume that the desired derivative exists, but fortunately this is a safe assumption for most such problems.

The example \(y = \ln x\) involved an inverse function defined implicitly, but other functions can be defined implicitly, and sometimes a single equation can be used to implicitly define more than one function. Here’s a familiar example. The equation \(x^2 + y^2 = 1\) describes a circle of radius 1. The circle is not a function \(y = f(x)\) because for some values of \(x\) there are two corresponding values of \(y\). If we want to work with a function, we can break the circle into two pieces, the upper and lower semicircles, each of which is a function. Let’s call these \(y = U(x)\) and \(y = L(x)\); in fact this is a fairly simple example, and it’s possible to give explicit expressions for these: \(U(x) = \sqrt{1 - x^2}\) and \(L(x) = -\sqrt{1 - x^2}\). But it’s somewhat easier, and quite useful, to view both functions as given implicitly by \(x^2 + y^2 = 1\), both \(y^2 + y^2 = x^2 + L(x)^2\) and \(y^2 + y^2 = x^2 + U(x)^2\) are true, and we can think of \(x^2 + y^2 = x^2 + L(x)^2\) as defining both \(U(x)\) and \(L(x)\).

Now we can take the derivative of both sides as before, remembering that \(y\) is not simply a variable but a function—in this case, \(y\) is either \(U(x)\) or \(L(x)\) but we’re not yet specifying which one. When we take the derivative we just have to remember to apply the
Now taking the derivative of both sides, we get

\[ y'(x^2 + y^2) = 2x \]

As we expect when using implicit differentiation, \( y \) appears on the right hand side here. We would certainly prefer to have \( y \) written in terms of \( x \), and as in the case of \( h(x) \) we can actually do that here. Since \( \sin(y) = \cos(x) = 1 \), we \( \cos(y) = 1 - \sin^2(x) \). So \( \cos(y) = 1 - x^2 \), but which is it—plus or minus? It could in general be either, but this isn’t “in general”: since \( y = \arcsin(x) \) we know that \(-\pi/2 \leq y \leq \pi/2 \), and the cosine of an angle in this interval is always positive. Thus \( \cos(y) = \sqrt{1 - x^2} \)

\[ \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} \]

Note that this agrees with figure 4.9.1: the graph of the arcsine has positive slope everywhere.

We can do something similar for the cosine. As with the sine, we must first truncate the cosine so that it can be inverted, as shown in figure 4.9.2. Then we use implicit differentiation to find that

\[ \frac{dy}{dx} = \frac{-1}{\sqrt{1 - y^2}} \]

Note that the truncated cosine uses a different interval than the truncated sine, so that if \( y = \arccos(x) \) we know that \( 0 \leq y \leq \pi \). The computation of the derivative of the arccosine is left as an exercise.

Finally we look at the tangent; the other trigonometric functions also have “partial inverses” but the sine, cosine and tangent are enough for most purposes. The tangent, truncated tangent and inverse tangent are shown in figure 4.9.3; the derivative of the arctangent is left as an exercise.

\[ \frac{dy}{dx} = \frac{1}{x^2 + 1} \]

Exercises 4.9.

1. Show that the derivative of \( \arccos(x) \) is \( \frac{-1}{\sqrt{1 - x^2}} \).

2. Show that the derivative of \( \arctan(x) \) is \( \frac{1}{x^2 + 1} \).
3. The inverse of cot is usually defined so that the range of its cot is [0, π]. Sketch the graph of y = cot x. In the process you will make it clear what the domain of its cot is. Find the derivative of the arccotangent.

4. Show that arccot x + arccot y = π/2.

5. Find the derivative of arccot(x).

6. Find the derivative of arccot(x^2).

7. Find the derivative of arccos(ax).

8. Find the derivative of arccos(ax^2).

9. Find the derivative of arccos(x).

10. Find the derivative of arccos(πx).

11. Find the derivative of log(arccot(x^2)) =

4.10 LIMITS REVISITED

We have defined and used the concept of limit, primarily in our development of the derivative. Recall that lim x→a f(x) = L is true if, in a precise sense, f(x) gets closer and closer to L as x gets closer and closer to a. While some limits are easy to see, others take some ingenuity; in particular, the limits that are always difficult on their face, since in

\[ \lim_{x \to a} f(x + a_0) - f(x) \]

both the numerator and denominator approach zero. Typically this difficulty can be resolved when f is a “nice” function and we are trying to compute a derivative. Occasionally such limits are interesting for other reasons, and the limit of a fraction of which both numerator and denominator approach zero can be difficult to analyze. Now that we have the derivative available, there is another technique that can sometimes be helpful in such circumstances.

Before we introduce the technique, we will also expand our concept of limit, in two ways. When the limit of f(x) as x approaches a does not exist, it may be useful to note in what way it does not exist. We have already talked about one such case: one-sided limits. Another case is when “f goes to infinity”!

We will also occasionally want to know what happens to f when x “goes to infinity”.

EXAMPLE 4.10.1 What happens to 1/x as x goes to 0? From the right, 1/x gets bigger and bigger, or goes to infinity. From the left it goes to negative infinity.

EXAMPLE 4.10.2 What happens to the function cos(1/x) as x goes to infinity? It seems clear that as x gets larger and larger, 1/x gets closer and closer to zero, so cos(1/x) should be getting closer and closer to cos(0) = 1.

4.10 LIMITS REVISITED

First we use L'Hôpital's Rule. Since the numerator and denominator both approach zero,

\[ \lim_{x \to 0} \frac{x^2 - x^3}{\sin x} = \lim_{x \to 0} \frac{2x - 3x^2}{\cos x} \]

provided the latter exists. But in fact this is an easy limit, since the denominator now approaches −1, so

\[ \lim_{x \to 0} \frac{x^2 - x^3}{\sin x} = \frac{2}{-1} = -2. \]

We don't really need L'Hôpital's Rule to do this limit. Rewrite it as

\[ \lim_{x \to 0} \left( x + \frac{x - \pi}{\sin x} \right) \]

and note that

\[ \lim_{x \to 0} \frac{\pi}{\sin x} = \frac{\pi}{\sin(\pi/2)} = \frac{\pi}{1} = \pi \]

since x − π approaches zero as x approaches π. Now

\[ \lim_{x \to 0} (x + \pi) = \lim_{x \to 0} (x + \pi) \lim_{x \to 0} \frac{x}{\sin x} = 2\pi(-1) = -2\pi \]

as before.

EXAMPLE 4.10.7 Compute \( \lim_{x \to \infty} \frac{2x^2 - 3x + 7}{x^2 + 4x + 1} \) in two ways.

As x goes to infinity both the numerator and denominator go to infinity, so we may apply L'Hôpital's Rule:

\[ \lim_{x \to \infty} \frac{2x^2 - 3x + 7}{x^2 + 4x + 1} = \lim_{x \to \infty} \frac{4x - 3}{4x + 1} = \frac{4}{4} = 1 \]

In the second quotient, it is still the case that the numerator and denominator both go to infinity, so we are allowed to use L'Hôpital's Rule again:

\[ \lim_{x \to \infty} \frac{4x - 3}{4x + 1} = \frac{4}{4} = 2 \]

So the original limit is 2 as well.

Again, we don't really need L'Hôpital's Rule, and in fact a more elementary approach is easier—we divide the numerator and denominator by x^2:

\[ \lim_{x \to \infty} \frac{2x^2 - 3x + 7}{x^2 + 4x + 1} = \lim_{x \to \infty} \frac{2x - 3 + \frac{7}{x^2}}{1 + \frac{4}{x} + \frac{1}{x^2}} = \lim_{x \to \infty} \frac{2 - \frac{3}{x^2}}{1 + \frac{4}{x^2} + \frac{1}{x^2}} = 2. \]

Now as x approaches infinity, all the quotients with some power of x in the denominator approach zero, leaving 2 in the numerator and 1 in the denominator, so the limit again is 2.

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EXAMPLE 4.10.8 Compute \( \lim_{x \to 0} \frac{\sec x - 1}{\sin x} \) in two ways.

Both the numerator and denominator approach zero, so applying L'Hôpital's Rule:

\[ \lim_{x \to 0} \frac{\sec x - 1}{\sin x} = \lim_{x \to 0} \frac{\sec x \tan x}{\cos x} = \frac{1}{1} = 1. \]

EXAMPLE 4.10.9 Compute \( \lim_{x \to 0} x \ln x \).

This doesn't appear to be suitable for L'Hôpital's Rule, but it also is not "obvious". As x approaches zero, ln x goes to −∞, so the product looks like something very small (something very large and negative). But this could be anything: it depends on how small and how large. For example, consider (x^3)(1/x), (x^2)(1/x), and (x^3)(1/x^2). As x approaches zero, each of these is (something very small) (something very large), yet the limits are respectively zero, 1, and 0.

We can in fact turn this into a L'Hôpital's Rule problem:

\[ x \ln x = \lim_{x \to 0} \frac{x}{1/x} \]

Now as x approaches zero, both the numerator and denominator approach infinity (one −∞ and one +∞), but only the size is important. Using L'Hôpital's Rule:

\[ \lim_{x \to 0} \frac{x}{1/x} = \lim_{x \to 0} \frac{1/x - x}{1/x^2} = \lim_{x \to 0} x - x = 0. \]

One way to interpret this is that since \( \lim_{x \to 0} x \ln x = 0 \), the x approaches zero much faster than the ln x approaches −∞.

EXERCISES 4.10

Compute the limits.

1. \( \lim_{x \to 0} \frac{\cos x - 1}{\sin x} \)

2. \( \lim_{x \to 0} x^2 \)

3. \( \lim_{x \to 0} \sqrt{x^2 + x^3} - \sqrt{x^2} = \frac{x^3}{\sqrt{x^2 + x^3} + \sqrt{x^2}} \)

4. \( \lim_{x \to 0} \frac{\sin 4x}{x} \)

5. \( \lim_{x \to 0} \frac{\ln x}{\cot x} \)

6. \( \lim_{x \to 0} \frac{\sin x}{4x} \)

7. \( \lim_{x \to 0} \frac{\sqrt{x^2 + 1} - 1}{x} \)

8. \( \lim_{x \to 0} \frac{x}{x - 1} - \frac{1}{x} \)

9. \( \lim_{x \to 0} \frac{1}{1 - x} - \frac{1}{x} \)

10. \( \lim_{x \to 0} \frac{1}{x} - \frac{1}{1 - x} \)

11. \( \lim_{x \to 0} \frac{x^2 + 2x + 3}{x} \)

12. \( \lim_{x \to 0} \frac{x^2 + 2x + 3}{x} \)
The proof is a straightforward computation:

\[ \lim_{x \to \infty} \frac{1}{x^2} (x^2 - 1) = 1 \]

This immediately gives two additional identities:

1. \( 1 - \tanh^2 x = \text{sech}^2 x \)
2. \( \cosh^2 x - 1 = \text{csch}^2 x \)

The identity of the theorem also helps to provide a geometric motivation. Recall that the graph of \( x^2 - y^2 = 1 \) is a hyperbola with asymptotes \( x = \pm y \), whose \( x \)-intercepts are \( \pm \sqrt{2} \) and \( \pm \sqrt{2} \). The hyperbolas \( x^2 - y^2 = k \) have \( x \)-intercepts \( \pm \sqrt{k} \).

4.11 Hyperbolic Functions

**DEFINITION 4.11.1** The hyperbolic cosine is the function

\[ \cosh x = \frac{e^x + e^{-x}}{2} \]

and the hyperbolic sine is the function

\[ \sinh x = \frac{e^x - e^{-x}}{2} \]

Notice that \( \cosh \) is even (that is, \( \cosh(-x) = \cosh(x) \)) while \( \sinh \) is odd (\( \sinh(-x) = -\sinh(x) \)), and \( \cosh x + \sinh x = e^x \). Also, for all \( x \), \( \cosh x > 0 \), while \( \sinh x = 0 \) if and only if \( e^x - e^{-x} = 0 \), which is true precisely when \( x = 0 \).

**LEMMA 4.11.2** The range of \( \cosh x \) is \([1, \infty)\).

**Proof.** Let \( y = \cosh x \). We solve for \( x \):

\[ y = \frac{e^x + e^{-x}}{2} \]

\[ 2y = e^x + e^{-x} \]

\[ 2y = e^x - 2e^x + 1 \]

\[ e^x = \sqrt{y^2 - 1} \]

From the last equation, we see \( y^2 \geq 1 \), and since \( y \geq 0 \), it follows that \( y \geq 1 \). Now suppose \( y \geq 1 \), so \( y = \sqrt{y^2 - 1} > 0 \). Then \( x = \ln(y \pm \sqrt{y^2 - 1}) \) is a real number, and \( y = \cosh x \), so \( y \) is in the range of \( \cosh(x) \).

---

**THEOREM 4.11.5**

\[ \frac{d}{dx} \cosh x = \sinh x \]

**Proof.**

\[ \frac{d}{dx} \cosh x = \frac{d}{dx} \frac{e^x + e^{-x}}{2} = e^x - e^{-x} = \sinh x \]

Since \( \cosh x > 0 \), \( \sinh x \) is increasing and hence injective, so \( \sinh x \) has an inverse, \( \text{arsinh} x \). Also, \( \sinh x > 0 \) when \( x > 0 \), so \( \cosh x \) is injective on \([0, \infty)\), and has a (partial) inverse, \( \text{arcosh} x \). The other hyperbolic functions have inverses as well, though \( \text{arcosh} x \) is only a partial inverse. We may compute the derivatives of these functions as we have other inverse functions.

**THEOREM 4.11.6**

\[ \frac{d}{dx} \text{arsinh} x = \frac{1}{\sqrt{1 + x^2}} \]

**Proof.** Let \( y = \text{arsinh} x \), so \( \sinh y = x \). Then \( \frac{d}{dy} \sinh y = \cosh(y) \cdot y' = 1 \), and so

\[ y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}} \]
4.11 Hyperbolic Functions

The other derivatives are left to the exercises.

**Exercises 4.11.**

1. Show that the range of sinh \( x \) is all real numbers. (Hint: show that if \( y = \sinh x \) then \( x = \ln(y + \sqrt{y^2 + 1}) \).)

2. Compute the following limits:
   a. \( \lim_{x \to \infty} \cosh x \)
   b. \( \lim_{x \to \infty} \sinh x \)
   c. \( \lim_{x \to \infty} \tanh x \)
   d. \( \lim_{x \to \infty} (\cosh x - \sinh x) \)

3. Show that the range of \( \tanh x \) is \((-1, 1)\). What are the ranges of \( \coth, \text{sech}, \) and \( \text{csch} \)? (Use the fact that they are reciprocal functions.)

4. Prove that for every \( x, y \in \mathbb{R} \), \( \sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y \). Obtain a similar identity for \( \sinh(x - y) \).

5. Prove that for every \( x, y \in \mathbb{R} \), \( \cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y \). Obtain a similar identity for \( \cosh(x - y) \).

6. Use exercises 4 and 5 to show that \( \sinh(2x) = 2 \sinh x \cosh x \) and \( \cosh(2x) = \cosh^2 x + \sinh^2 x \) for every \( x \). Conclude also that \( (\cosh(2x) - 1)/2 = \sinh^2 x \).

7. Show that \( \frac{d}{dx}(\tanh x) = \text{sech}^2 x \). Compute the derivatives of the remaining hyperbolic functions as well.

8. What are the domains of the six inverse hyperbolic functions?

9. Sketch the graphs of all six inverse hyperbolic functions.