9

Applications of Integration

9.1 Area between Curves

We have seen how integration can be used to find an area between a curve and the x-axis. With very little change we can find some areas between curves; indeed, the area between a curve and the x-axis may be interpreted as the area between the curve and a second “curve” with equation $y = 0$. In the simplest of cases, the idea is quite easy to understand.

**Example 9.1.1** Find the area below $f(x) = -x^2 + 4x + 3$ and above $g(x) = -x^2 + 7x^2 - 10x + 5$ over the interval $1 \leq x \leq 2$. In figure 9.1.1 we show the two curves together, with the desired area shaded, then $f$ alone with the area under $f$ shaded, and then $g$ alone with the area under $g$ shaded.

![Figure 9.1.1 Area between curves as a difference of areas.](image)

**Example 9.1.2** Find the area below $f(x) = -x^2 + 4x + 1$ and above $g(x) = -x^2 + 7x^2 - 10x + 3$ over the interval $1 \leq x \leq 2$. These are the same curves as before but lowered by 2. In figure 9.1.2 we show the two curves together. Note that the lower curve now dips below the $x$-axis. This makes it somewhat tricky to view the desired area as a big area minus a smaller area, which leads naturally to the difference between two integrals. But it is instructive to consider how we might find the desired area directly. We can approximate the area by dividing the area into thin sections and approximating the area of each section by a rectangle, as indicated in figure 9.1.2. The area of a typical rectangle is $\Delta x(f(x) - g(x))$, so the total area is approximately

$$
\sum_{i=0}^{n-1} [f(x_i) - g(x_i)]\Delta x.
$$

This is exactly the sort of sum that turns into an integral in the limit, namely the integral

$$
\int_a^b [f(x) - g(x)] dx.
$$

Of course, this is the integral we actually computed above, but we have now arrived at it directly rather than as a modification of the difference between two other integrals. In that example it really doesn’t matter which approach we take, but in some cases this second approach is better.

**Example 9.1.3** Find the area between $f(x) = -x^2 + 4x$ and $g(x) = x^2 - 6x + 5$ over the interval $0 \leq x \leq 1$, the curves are shown in figure 9.1.4. Generally we should interpret

$$
\text{“area” in the usual sense, as a necessarily positive quantity. Since the two curves cross,}
$$

we need to compute two areas and add them. First we find the intersection point of the curves:

$$
-2x^2 + 6x - 10 = 0 \Rightarrow x = \frac{10 \pm \sqrt{100 - 40}}{4} = \frac{5 \pm \sqrt{15}}{2}.
$$

The intersection point we want is $x = a = (5 - \sqrt{15})/2$. Then the total area is

$$
\int_0^a [-2x^2 + 6x - 10] dx + \int_a^1 [2x^2 - 6x + 5] dx
$$

after a bit of simplification.

**Example 9.1.4** Find the area between $f(x) = -x^2 + 4x$ and $g(x) = x^2 - 6x + 5$, the curves are shown in figure 9.1.5. Here we are not given a specific interval, so it must...
be the case that there is a “natural” region involved. Since the curves are both parabolas, the only reasonable interpretation is the region between the two intersection points, which we found in the previous example:

\[ y = \frac{2x^3}{3} + \frac{3}{2} \]

If we let \( a = (5 - \sqrt{15})/2 \) and \( b = (5 + \sqrt{15})/2 \), the total area is

\[
\int_a^b \left[ \frac{3}{2} - 2x^2 \right] \, dx = \frac{5 + \sqrt{15}}{2} - \frac{5 - \sqrt{15}}{2} = 5\sqrt{15}.
\]

after a bit of simplification.

![Figure 9.1.5 Area bounded by two curves.](image)

**Exercises 9.1.**

Find the area bounded by the curves.

1. \( y = x^2 - x^2 \) and \( y = x^2 \) (the parabola to the right of the y-axis) ⇒
2. \( x = y^2 - y^2 \) and \( x = y^2 \) ⇒
3. \( x = 1 - y^2 \) and \( y = -x - 1 \) ⇒
4. \( y = 1 - y^2 \) and \( x = 3 \) ⇒
5. \( y = \cos(\pi/2) \) and \( y = -x^2 \) (in the first quadrant) ⇒
6. \( y = \sin(\pi/2) \) and \( y = x \) (in the first quadrant) ⇒
7. \( y = \sqrt{y} \) and \( x = x^2 \) ⇒
8. \( y = -\sqrt{y} \) and \( y = \sqrt{x^2 - 1} \), \( 0 \leq x \leq 4 \) ⇒
9. \( x = 0 \) and \( x = 25 - y^2 \) ⇒
10. \( y = \sin x \cos x \) and \( y = \sin x \), \( 0 \leq x \leq \pi \) ⇒

### 9.2 Distance, Velocity, Acceleration

**Example 9.2.1** Suppose an object is acted upon by a constant force \( F \). Find \( v(t) \) and \( a(t) \). By Newton’s law \( F = ma \), so the acceleration is \( F/m \), where \( m \) is the mass of the object. Then we first have

\[
v(t) = v(0) + \int_0^t F/m \, du = v_0 + F/m (t - t_0).
\]

using the usual convention \( v_0 = \dot{v}(t_0) \). Then

\[
a(t) = a(0) + \int_0^t \left( \ddot{v}(0) + \frac{F}{m} (t - t_0) \right) \, du = \ddot{v}(0) + \frac{F}{m} (t - t_0),
\]

For example, if an object is thrown straight upward at 19.6 m/sec, its velocity function is

\[
v(t) = t - 9.8t + 9.8 \sqrt{7/6} t^2/2.
\]

Recall that the integral of the velocity function gives the net distance traveled. If you want to know the total distance traveled, you must find out where the velocity function crosses the x-axis, integrate separately over the time intervals when \( v(t) \) is positive and when \( v(t) \) is negative, and add up the absolute values of the different intervals. For example, if an object is thrown straight up at 19.6 m/sec, its velocity is

\[
v(t) = -9.8t + 9.8 \sqrt{7/6} t^2/2.
\]

To find the total distance traveled, we need to know when \( 0.5 (\sin(t)) \) is positive and when it is negative. This function is 0 when \( \sin(t) = 0 \), i.e., when \( t = n\pi/2 \). Then \( 0 \leq t \leq 4 \) is the only case in which \( 0 \leq t \leq 1.5 \). Since \( v(t) > 0 \) for \( t < 7/6 \) and \( v(t) < 0 \) for \( t > 7/6 \), the total distance traveled is

\[
\int_0^{7/6} (0.5 + \sin(t)) \, dt + \int_{7/6}^2 (0.5 - \sin(t)) \, dt
\]

Similarly, since the velocity is an anti-derivative of the acceleration function \( a(t) \), we have

\[
v(t) = v(0) + \int_0^t a(u) \, du.
\]

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11. \( y = x^{1/3} \) and \( y = x^{1/3} \) ⇒
12. \( y = x^2 - 2x \) and \( y = x - 2 \) ⇒

The following three exercises expand on the geometric interpretation of the hyperbolic functions. Refer to section 4.11 and particularly to figure 4.11.2 and exercise 6 in section 4.11.

13. Compute \( \int \sqrt{x^2 - 1} \) using the substitution \( u = \cosh t \), \( t = \cosh u \); use exercise 6 in section 4.11.
14. For \( t > 0 \). Sketch the region \( R \) in the right half plane bounded by the curves \( y = x \tan t \), \( y = -x \tan t \), and \( x^2 - y^2 = 1 \). Note well: \( t \) is fixed, the plane is the x-y plane.
15. Prove that the area of \( R \) is \( t \).

### 9.3 Applications of Integration

**Example 9.2.2** The acceleration of an object is given by \( a(t) = \cos(\pi t) \), and its velocity at time \( t = 0 \) is 1/2(π). Find both the net and the total distance traveled in the first 5 seconds.

We compute

\[
v(t) = v(0) + \int_0^t \cos(\pi t) \, du = \frac{1}{2} \sin(\pi t) - \frac{1}{2} \frac{1}{\pi} \sin(\pi t).
\]

The net distance traveled is then

\[
s(3/2) - s(0) = \int_0^{3/2} \frac{1}{2} \left( \frac{1}{\pi} \sin(\pi t) \right) \, dt
\]

To find the total distance traveled, we need to know when \( (0.5 + \sin(t)) \) is positive and when it is negative. This function is 0 when \( \sin(t) = -0.5 \), i.e., when \( t = \pi - \pi/6 \). Then the value \( t = \pi - \pi/6 \), is the only one in the range \( 0 \leq t \leq 1.5 \). Since \( v(t) > 0 \) for \( t < 7/6 \) and \( v(t) < 0 \) for \( t > 7/6 \), the total distance traveled is

\[
\int_0^{7/6} \left( \frac{1}{2} + \sin(t) \right) \, dt + \int_{7/6}^2 \left( \frac{1}{2} - \sin(t) \right) \, dt
\]

For each velocity function find both the net distance and the total distance traveled during the indicated time interval (graph \( v(t) \) to determine when it’s positive and when it’s negative):

1. \( v = \cos(\pi t) \), \( 0 \leq t \leq 2.5 \) ⇒
2. \( v = -9.8t + 49 \), \( 0 \leq t \leq 10 \) ⇒
3. \( v = (t - 3)(1 - t) \), \( 0 \leq t \leq 5 \) ⇒
4. \( v = \sin(t)(\pi/2) - t \), \( 0 \leq t \leq 1 \) ⇒

An object is shot upwards from ground level with an initial velocity of 2 meters per second; it is subject only to the force of gravity (no air resistance). Find its maximum altitude and the time at which it hits the ground.
7. An object is shot upwards from ground level with an initial velocity of 100 meters per second; it is subject only to the force of gravity (no air resistance). Find its maximum altitude and the time at which it hits the ground. 

8. An object moves along a straight line with acceleration given by \( a(t) = -\cos(t) \), and \( a(0) = 1 \) and \( v(0) = 0 \). Find the maximum distance the object travels from zero, and find its maximum speed. Describe the motion of the object. 

9. An object moves along a straight line with acceleration given by \( a(t) = \sin(\pi t) \). Assume that when \( t = 0 \), \( v(t) = v_0 \) and \( x(t) = x_0 \). Find \( x(t) \) when \( t = 0 \). Assume that when \( t = 0 \), \( a(t) = -v_0 \). Find \( v(t) \). 

10. An object moves along a straight line with acceleration given by \( a(t) = 1 + \sin(\pi t) \). Assume that when \( t = 0 \), \( a(t) = v(t) = 0 \). Find \( x(t) \) and \( v(t) \). 

11. An object moves along a straight line with acceleration given by \( a(t) = 1 - \sin(\pi t) \). Assume that when \( t = 0 \), \( a(t) = v(t) = 0 \). Find \( x(t) \) and \( v(t) \). 

9.3 VOLUME

We have seen how to compute certain areas by using integration; some volumes may also be computed by evaluating an integral. Generally, the volumes that we can compute this way have cross-sections that are easy to describe.

EXAMPLE 9.3.1 Find the volume of a pyramid with a square base that is 20 meters tall and 20 meters on a side at the base. As with most of our applications of integration, we begin by asking how we might approximate the volume. Since we can easily compute the volume of a rectangular prism (that is, a "box"), we will use some boxes to approximate the volume.

As you may know, the volume of a pyramid is \( \frac{1}{3} \) base \( \times \) height. Hence, as shown in figure 9.3.1, on the left is a cross-sectional view, on the right is a 3D view of part of the pyramid with some of the boxes used to approximate the volume.

Each box has volume of form \( (2x_i)(2x_i)\Delta y \). Unfortunately, there are two variables here; fortunately, we can write \( x \) in terms of \( y \), \( x = 10 - y/2 \) or \( x = 10 - y_i/2 \). Then the total volume is approximately

\[
\sum_{i=0}^{n-1} \int_{-20}^{20} (20 - y_i/2) \Delta y.
\]

In the limit we get the volume as the value of an integral:

\[
\int_{-20}^{20} (20 - y)^2 dy = \int_{-20}^{20} 400 - 40y + y^2 dy = \frac{400^3}{3} - \frac{400^3}{3} = 8000
\]

Thus the total volume is \( \frac{8000}{3} \) cubic meters.

As you may know, the volume of a pyramid is \( \frac{1}{3} \) (height)(area of base). This agreement with our answer.

EXAMPLE 9.3.4 The base of a solid is the region between \( f(x) = x^2 - 1 \) and \( g(x) = -x^2 + 1 \), and its cross-sections perpendicular to the \( x \)-axis are equilateral triangles, as indicated in figure 9.3.2. The solid has been truncated to show a triangular cross-section above \( x = 1/2 \). Find the volume of the solid.

The volume of the pyramid, as shown in figure 9.3.1, on the left is a cross-sectional view, on the right is a 3D view of part of the pyramid with some of the boxes used to approximate the volume.

Each box has volume of form \( (2x_i)(2x_i)\Delta y \). Unfortunately, there are two variables here; fortunately, we can write \( x \) in terms of \( y \), \( x = 10 - y/2 \) or \( x = 10 - y_i/2 \). Then the total volume is approximately

\[
\sum_{i=0}^{n-1} \int_{-20}^{20} (10 - y_i/2) \Delta y
\]

and in the limit we get the volume as the value of an integral:

\[
\int_{-20}^{20} (10 - y)^2 dy = \int_{-20}^{20} (100 - 20y + y^2) dy = \frac{2000}{3} - \frac{2000}{3} = 8000
\]

As you may know, the volume of a pyramid is \( \frac{1}{3} \) (height)(area of base) = \( \frac{1}{3} (20 \times 400) \), which agrees with our answer.

EXAMPLE 9.3.2 The base of a solid is the region between \( f(x) = x^2 - 1 \) and \( g(x) = -x^2 + 1 \), and its cross-sections perpendicular to the \( x \)-axis are equilateral triangles, as indicated in figure 9.3.2. The solid has been truncated to show a triangular cross-section above \( x = 1/2 \). Find the volume of the solid.

The volume of the pyramid, as shown in figure 9.3.1, on the left is a cross-sectional view, on the right is a 3D view of part of the pyramid with some of the boxes used to approximate the volume.

Each box has volume of form \( (2x_i)(2x_i)\Delta y \). Unfortunately, there are two variables here; fortunately, we can write \( x \) in terms of \( y \), \( x = 10 - y/2 \) or \( x = 10 - y_i/2 \). Then the total volume is approximately

\[
\sum_{i=0}^{n-1} \int_{-20}^{20} (10 - y_i/2) \Delta y
\]

and in the limit we get the volume as the value of an integral:

\[
\int_{-20}^{20} (10 - y)^2 dy = \int_{-20}^{20} (100 - 20y + y^2) dy = \frac{2000}{3} - \frac{2000}{3} = 8000
\]

As you may know, the volume of a pyramid is \( \frac{1}{3} \) (height)(area of base). This agrees with our answer.

EXAMPLE 9.3.4 The volume of the object generated when the area between \( y = x^2 \) and \( y = x \) is rotated around the \( x \)-axis. This solid has a "hole" in the middle; we can compute the volume by subtracting the volume of the hole from the volume enclosed by the outer surface of the solid. In figure 9.3.5 we show the region that is rotated, the resulting solid with the front half cut away, the cone that forms the outer surface, the horn-shaped hole, and a cross-section perpendicular to the \( x \)-axis.

We have already computed the volume of a cone; in this case it is \( \pi/3 \). At a particular value of \( x \), say \( x_i \), the cross-section of the horn is a circle with radius \( x_i^2 \), so the volume of the horn is

\[
\int_{x_i}^{x} \frac{1}{2} \pi x^2 \Delta x = \int_{x_i}^{x} \frac{1}{2} \pi x^2 dx = \frac{1}{2} \pi x^3.
\]

so the desired volume is \( \pi/3 - \pi/5 = 2\pi/15 \).

As with the area between curves, there is an alternate approach that computes the desired volume “all at once” by approximating the volume of the actual solid. We can approximate the volume of a slice of the solid with a washer-shaped volume, as indicated in figure 9.3.5.

The volume of such a washer is the area of the face times the thickness. The thickness, as usual, is \( \Delta x \), while the area of the face is the area of the outer circle minus the area of...
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9.3 Volume

The base of a solid is the region between \( f(x) = \cos x \) and \( g(x) = -\cos x \), \(-\pi/2 \leq x \leq \pi/2\), and its cross-sections perpendicular to the \( x \)-axis are equilateral triangles. Express its volume \( V \) as an integral, and find a formula for \( V \) in terms of \( h \) and \( s \). Verify that your answer is \((1/3)(\text{area of base})(\text{height})\).

10. Use integration to compute the volume of a sphere of radius \( r \). You should of course get the well-known formula \( 4\pi r^3/3 \).

11. A hemispheric bowl of radius \( r \) contains water to a depth \( h \). Find the volume of water in the bowl.

12. The base of a tetrahedron (a triangular pyramid) of height \( h \) is an equilateral triangle of side \( s \). Its cross-sections perpendicular to an altitude are equilateral triangles. Express its volume \( V \) as an integral, and find a formula for \( V \) in terms of \( h \) and \( s \). Verify that your answer is \((1/3)(\text{area of base})(\text{height})\).

9.4 Average Value of a Function

The average of some finite set of values is a familiar concept. If, for example, the class scores on a quiz are 10, 9, 10, 8, 7, 5, 7, 6, 3, 2, 7, 8, then the average score is the sum of these numbers divided by the size of the class:

\[
\text{average score} = \frac{10 + 9 + 10 + 8 + 7 + 5 + 7 + 6 + 3 + 2 + 7 + 8}{12} = \frac{63}{12} = 5.25.
\]

Suppose that \( t = 0 \) and \( t = 1 \) the speed of an object is \( \sin(\pi t) \). What is the average speed of the object over that time? The question sounds as if it must make sense, yet we can't merely add up some number of speeds and divide, since the speed is changing continuously over the time interval.

To make sense of “average” in this context, we fall back on the idea of approximation. Consider the speed of the object at tenth of a second intervals: \( \sin(0.1\pi), \sin(0.2\pi), \sin(0.3\pi), \ldots, \sin(0.9\pi) \). The average speed “should” be fairly close to the average of these ten speeds:

\[
\frac{1}{10} \sum_{i=1}^{10} \sin(\pi i/10) \approx 0.583 = 0.63.
\]

Of course, if we compute more speeds at more times, the average of these speeds should be closer to the “real” average. If we take the average of \( n \) speeds at evenly spaced times, we get:

\[
\frac{1}{n} \sum_{i=1}^{n-1} \sin(\pi i/n).
\]

Here the individual times are \( t_i = i/n \), so rewriting slightly we have

\[
\frac{1}{n} \sum_{i=1}^{n-1} \sin(\pi i/n).
\]

This is almost the sort of sum that we know turns into an integral; what’s apparently missing is \( \Delta t \) — but in fact, \( \Delta t = 1/n \), the length of each subinterval. So rewriting again:

\[
\frac{1}{n} \sum_{i=1}^{n-1} \sin(\pi i/n) = \frac{1}{n} \sum_{i=1}^{n-1} \sin(\pi i/n) \Delta t.
\]

Now this has exactly the right form, so that in the limit we get

\[
\text{average speed} = \int_0^1 \sin(\pi t) \, dt = \frac{\cos(\pi t)}{\pi} \bigg|_0^1 = \frac{\cos(\pi)}{\pi} - \frac{\cos(0)}{\pi} = -\frac{2}{\pi} \approx 0.6366 \approx 0.64.
\]

It’s not entirely obvious from this one simple example how to compute such an average in general. Let’s look at a somewhat more complicated case. Suppose that the velocity breaks the problem into two parts and compute two integrals:

\[
\pi \int_0^1 (1 + \sqrt{y})^2 - (1 - \sqrt{y})^2 \, dy + \pi \int_0^1 (1 + \sqrt{y})^2 - (y - 1)^2 \, dy = \frac{8}{\pi} + \frac{65}{\pi} = \frac{77}{\pi}.
\]
of an object is $10^2 + 5$ feet per second. What is the average velocity between $t = 1$ and $t = 3$? Again we set up an approximation to the average:

$$\frac{1}{3-1} \sum_{i=0}^{n-1} (16i^2 + 5),$$

where the values $t_i$ are evenly spaced times between 1 and 3. Once again we are “missing” $\Delta t_i$ and this time $1/n$ is not the correct value. What is $\Delta t_i$ in general? It is the length of a subinterval; in this case we take the interval $[1, 3]$ and divide it into $n$ subintervals, so each has length $(3-1)/n = 2/n = \Delta t_i$. Now with the usual “multiply and divide by the same thing” trick we can rewrite this:

$$\frac{1}{3-1} \sum_{i=0}^{n-1} \left(16i^2 + 5\right) = \frac{1}{2} \sum_{i=0}^{n-1} (16i^2 + 5) \Delta t_i.$$

In the limit this becomes

$$\frac{1}{2} \int_1^3 16x^2 + 5 \, dx = \frac{1446}{3} \approx 482.$$

Does this seem reasonable? Let’s picture it: in figure 9.4.1 is the velocity function together with the horizontal line $y = 223/3 \approx 74.3$. Certainly the height of the horizontal line looks at least plausible for the average height of the curve.

![Figure 9.4.1 Average velocity.](image)

Here’s another way to interpret “average” that may make our computation appear even more reasonable. The object of our example goes a certain distance between $t = 1$ and $t = 3$. If instead the object were to travel at the average speed over the same time, it should go the same distance. At an average speed of 223/3 feet per second for two seconds the object would go 446/3 feet. How far does it actually go? We know how to compute this:

$$\int_1^3 v(t) \, dt = \int_1^3 16x^2 + 5 \, dx = \frac{1446}{3}.$$

So now we see that another interpretation of the calculation is: total distance traveled divided by the time in transit, namely, the usual interpretation of average speed.

In the case of speed, or more properly velocity, we can always interpret “average” as total (net) distance divided by time. But in the case of a different sort of quantity this interpretation does not obviously apply, while the approximation approach always does. We might interpret the same problem geometrically: what is the average height of $16x^2 + 5$ on the interval $[1, 3]$? We approximate this in exactly the same way, by adding up many sample heights and dividing by the number of samples. In the limit we get the same result:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} (16i^2 + 5) = \frac{1}{2} \int_1^3 16x^2 + 5 \, dx = \frac{1446}{3} \approx 482.$$

We can interpret this result in a slightly different way. The area under $y = 16x^2 + 5$ above $[1, 3]$ is

$$\int_1^3 16x^2 + 5 \, dx = \frac{1446}{3}.$$

The area under $y = 223/3$ over the same interval $[1, 3]$ is simply the area of a rectangle that is 2 by $223/3$ with area 446/3. So the average height of a function is the height of the horizontal line that produces the same area over the given interval.

**Exercises 9.4.**

4. Find the average height of cos $x$ over the intervals $[0, \pi/2]$, $[-\pi/2, \pi/2]$, and $[0, 2\pi]$. ⇒

2. Find the average height of $e^x$ over the interval $[-2, 2]$. ⇒

3. Find the average height of $1/x^2$ over the interval $[1, A]$. ⇒

4. Find the average height of $\sqrt{x + 1}$ over the interval $[-1, 1]$. ⇒

5. An object moves with velocity $v(t) = t^2 + 1$ feet per second between $t = 0$ and $t = 2$. Find the average velocity and the average speed of the object between $t = 0$ and $t = 2$. ⇒

**9.5 Work**

A fundamental concept in classical physics is **work**: If an object is moved in a straight line against a force $F$ for a distance $s$ the work done is $W = Fs$.

**Example 9.5.1** How much work is done in lifting a 10 pound weight vertically a distance of 5 feet? The force due to gravity on a 10 pound weight is 10 pounds at the surface of the earth, and it does not change appreciably over 5 feet. The work done is

$$W = 10 \times 5 = 50 \text{ foot-pounds}.$$

In reality few situations are so simple. The force might not be constant over the range of motion, as in the next example.

**Example 9.5.2** How much work is done in lifting a 10 pound weight from the surface of the earth to an orbit 100 miles above the surface? Over 100 miles the force due to gravity does change significantly, so we need to take this into account. The force exerted on a 10 pound weight at a distance $r$ from the center of the earth is $F = k/r^2$ and by definition it is 10 when $r$ is the radius of the earth (we assume the earth is a sphere). How can we approximate the work done? We divide the path from the surface to orbit into $n$ small subpaths. On each subpath the force due to gravity is roughly constant, with value $k/r_1^2$ at distance $r_1$. The work to raise the object from $r_1$ to $r_2$ is thus approximately $k/r_1^2 \Delta r$ and the total work is approximately

$$\sum_{i=1}^{n-1} \frac{k}{r_i^2} \Delta r_i,$$

or in the limit

$$W = \int_{r_1}^{r_2} \frac{k}{r^2} \, dr,$$

where $r_1$ is the radius of the earth and $r_1$ is $r_1$ plus 100 miles. The work is

$$W = \int_{r_1}^{r_2} \frac{k}{r^2} \, dr = -k \frac{1}{r_2} - k \frac{1}{r_1}.$$

Using $r_1 = 6378100$ feet we have $r_1 = 2145325$. The force on the 10 pound weight at the surface of the earth is 10 pounds, so $10 = k/209932525^2$, giving $k = 4378775965256250$.

**Example 9.5.3** Suppose that a water tank is shaped like a right circular cone with the tip at the bottom, and has height 10 meters and radius 2 meters at the top. If the tank is full, how much work is required to pump all the water out over the top? Here we have a large number of atoms of water that must be lifted different distances to get to the top of the tank. Fortunately, we don’t really have to deal with individual atoms—we can consider all the atoms at a given depth together.
To approximate the work, we can divide the water in the tank into horizontal sections, approximate the volume of water in a section by a thin disk, and compute the amount of work required to lift each disk to the top of the tank. As usual, we take the limit as the sections get thinner and thinner to get the total work.

At depth \( h \) the circular cross-section through the tank has radius \( r = (10 - h)/5 \), by similar triangles, and area \( \pi (10 - h)^2/25 \). A section of the tank at depth \( h \) thus has volume approximately \( \pi (10 - h)^2/25 \Delta h \) and so contains \( \pi (10 - h)^2/25 \Delta h \) kilograms of water, where \( \sigma \) is the density of water in kilograms per cubic meter, \( \pi \approx 1000 \) The force due to gravity on this much water is \( 9.8 \pi (10 - h)^2/25 \Delta h \) and, finally, this section of water must be lifted a distance \( h \), which requires \( 8.98 \pi (10 - h)^2/25 \Delta h \) Newton-meters of work. The total work is therefore

\[
W = \int_0^{10} \frac{8.98 \pi (10 - h)^2}{25} \Delta h = \frac{898000}{25} \approx 359254 \text{ Newton-meters.}
\]

A spring has a “natural length,” its length if nothing is stretching or compressing it. If the spring is either stretched or compressed the spring provides an opposing force; according to Hooke’s Law the magnitude of this force is proportional to the distance the spring has been stretched or compressed: \( F = kx \). The constant of proportionality, \( k \), of course depends on the spring. Note that \( x \) here represents the change in length from the natural length.

**Example 9.5.5** Suppose \( k = 5 \) for a given spring that has a natural length of 0.1 meters. Suppose a force is applied that compresses the spring to length 0.8 meters. What is the magnitude of the force? Assuming that the constant \( k \) has appropriate dimensions (namely, \( \text{kg s}^2/\text{m} \)), the force is \( 5(0.1 - 0.8) = 5(0.02) = 0.1 \text{ Newtons.} \)

5. A water tank has the shape of a half sphere with radius \( r = 1 \) meter. If the tank is full, how much work is required to pump all the water out of the top of the tank? \( \Rightarrow \)

6. A spring has constant \( k = 10 \text{ kg s}^2/\text{m} \). How much work is done in compressing it \( 1 / 10 \) meter from its natural length? \( \Rightarrow \)

7. A force of 2 Newtons will compress a spring from 1 meter (its natural length) to 0.8 meters. How much work is required to stretch the spring from 1.1 meters to 1.5 meters? \( \Rightarrow \)

8. A 20 meter long steel cable has density \( 2 \text{ kg/m} \) and is hanging straight down. What is the weight of the bottom half of the cable? \( \Rightarrow \)

9. The cable in the previous problem has a 100 kilogram bucket of concrete attached to its lower end. How much work is required to lift the entire cable and bucket to the height of its top end? \( \Rightarrow \)

10. Consider again the cable and bucket of the previous problem. How much work is required to lift the bucket 10 meters by raising the cable 10 meters? (The top half of the cable ends up at the height of the top end of the cable, while the bottom half of the cable is lifted 10 meters.) \( \Rightarrow \)

### 9.6 Center of Mass

Suppose a beam is 10 meters long, and that there are three weights on the beam: a 10 kilogram weight 5 meters from the left end, a 5 kilogram weight 6 meters from the left end, and a 4 kilogram weight 8 meters from the left end. Where should a fulcrum be placed so that the beam balances? Let’s assign a scale to the beam, from 0 at the left end to 10 at the right, so that we can denote locations on the beam simply as \( x \) coordinates; the weights are at \( x = 3, 6, \) and \( 8 \), as in figure 9.6.1.

![Figure 9.6.1](image)

**Figure 9.6.1** A beam with three masses.

Suppose to begin with that the fulcrum is placed at \( x = 5 \). What will happen? Each weight applies a force to the beam that tends to rotate it around the fulcrum; this effect is measured by a quantity called torque, proportional to the mass times the distance from the fulcrum. Of course, weights on different sides of the fulcrum rotate the beam in opposite directions. We can distinguish this by using a signed distance in the formula for torque. So with the fulcrum at 5, the torques induced by the three weights will be proportional to \( (3 - 5)10 = -20, (6 - 5)5 = 5, \) and \( (8 - 5)4 = 12 \). For the beam to balance, the sum of the torques must be zero, so the sum is \(-20 + 5 + 12 = -3\). The beam rotates counter-clockwise, and to get the beam to balance we need to move the fulcrum to the left. To calculate exactly where the fulcrum should be, we let \( \bar{x} \) denote the location of the fulcrum when the beam is in balance. The total torque on the beam is then

\[
\sum_{i=1}^{n} (x_i\mu_i - \bar{x}\mu_i) \Delta x = 0
\]

where \( x_i \) and \( \mu_i \) denote the location and weight of the \( i \)-th mass, respectively.

**Exercise 9.5.5**

1. How much work is done in lifting a 100 kilogram weight from the surface of the earth to an orbit 35,786 kilometers above the surface of the earth? \( \Rightarrow \)

2. How much work is done in lifting a 100 kilogram weight from an orbit 1000 kilometers above the surface of the earth to the surface of the earth? \( \Rightarrow \)

3. A water tank has the shape of an upright cylinder with radius \( r = 1 \) meter and height 10 meters. If the depth of the water is 5 meters, how much work is required to pump all the water out the top of the tank? \( \Rightarrow \)

4. Suppose the tank of the previous problem is lying on its side, so that the circular ends are vertical, and that it has the same amount of water as before. How much work is required to pump the water out the top of the tank (which is now 2 meters above the bottom of the tank)? \( \Rightarrow \)

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**Example 9.5.6** How much work is done in compressing the spring in the previous example from its natural length to 0.08 meters? From 0.08 meters to 0.05 meters? How much work is done in stretching the spring from 0.1 meters to 0.15 meters? We can approximate the work by dividing the distance that the spring is compressed (or stretched) into small subintervals. Then the force exerted by the spring is approximately constant over the subinterval, so the work required to compress the spring from \( x_i \) to \( x_{i+1} \) is approximately

\[
\int_{0.1}^{0.08} (5x_i - 0.1) \Delta x
\]

and in the limit

\[
W = \int_{0.1}^{0.08} (5x - 0.1) \, dx = \frac{5}{2} (0.08 - 0.1) = \frac{5}{2} (0.08 - 0.1) = 0.1 \text{ N-m.}
\]

The other values we seek simply use different limits. To compress the spring from 0.08 meters to 0.05 meters takes

\[
W = \int_{0.08}^{0.05} (5x - 0.1) \, dx = \frac{5}{2} (0.05 - 0.08) = \frac{5}{2} (0.05 - 0.08) = 0.1 \text{ N-m.}
\]

and to stretch the spring from 0.1 meters to 0.15 meters requires

\[
W = \int_{0.1}^{0.15} (5x - 0.1) \, dx = \frac{5}{2} (0.15 - 0.1) = \frac{5}{2} (0.15 - 0.1) = 0.1 \text{ N-m.}
\]

### 9.6.1 The Torque of a Beam

Suppose the beam is 10 meters long and that the density is \( 1 + x \) kilograms per meter at location \( x \) on the beam. To approximate the work, we can think of the beam as a sequence of weights “on” a beam. For example, we can think of the portion of the beam between \( x = 0 \) and \( x = 1 \) as a weight sitting at \( x = 0 \), the portion between \( x = 1 \) and \( x = 2 \) as a weight sitting at \( x = 1 \), and so on, as indicated in figure 9.6.2.

Now suppose that we have a beam with varying density—some portions of the beam contain more mass than other portions of the same size. We want to figure out where to put the fulcrum so that the beam balances.

**Example 9.6.1** Suppose the beam is 10 meters long and that the density is \( 1 + x \) kilograms per meter at location \( x \) on the beam. Approximate the mass by dividing the beam into \( n \) portions, the mass of weight number \( i \) will be \( m_i = (1 + x_i)(x_i - x_{i-1}) = (x_i - x_{i-1}) \Delta x \) and the torque induced by weight number \( i \) will be \( (x_i - x_{i-1})(x_i - x) \Delta x \). The total torque is then

\[
T = \sum_{i=1}^{n} (x_i - x)(x_i - x) \Delta x + \sum_{i=1}^{n} (x_i - x)(x_i - x) \Delta x + \cdots + \sum_{i=1}^{n} (x_i - x)(x_i - x) \Delta x
\]

If we set this to zero and solve for \( x \) we get \( x = \bar{x} \). In general, if we divide the beam into \( n \) portions, the mass of weight number \( i \) will be \( m_i = (x_i - x_{i-1})(x_i - x_{i-1}) \Delta x \) and the torque induced by weight number \( i \) will be \( (x_i - x_{i-1})(x_i - x) \Delta x \). The total torque is then

\[
T = \sum_{i=1}^{n} (x_i - x)(x_i - x) \Delta x + \sum_{i=1}^{n} (x_i - x)(x_i - x) \Delta x + \cdots + \sum_{i=1}^{n} (x_i - x)(x_i - x) \Delta x
\]

Figure 9.6.2 A solid beam.
If we set this equal to zero and solve for \( x \) we get an approximation to the balance point of the beam:

\[
0 = \sum_{i=0}^{n-1} x_i (1 + x_i) \Delta x - e \sum_{i=0}^{n-1} (1 + x_i) \Delta x
\]

\[
\bar{x} = \frac{\sum_{i=0}^{n-1} x_i (1 + x_i) \Delta x}{\sum_{i=0}^{n-1} (1 + x_i) \Delta x}
\]

The denominator of this fraction has a very familiar interpretation. Consider one term of the sum in the numerator:

\[
i (1 + x_i) \Delta x = \text{the mass of the beam between } x_i \text{ and } x_{i+1}.
\]

When we add these up we get approximately the mass of the beam.

Now each of the sums in the fraction has the right form to turn into an integral, which in turn gives us the exact value of \( \bar{x} \):

\[
\bar{x} = \frac{\int_0^1 x (1 + x) \, dx}{\int_0^1 (1 + x) \, dx}.
\]

The numerator of this fraction is called the moment of the system around zero:

\[
\int_0^1 x (1 + x) \, dx = \int_0^1 x + x^2 \, dx = \frac{1150}{3}
\]

and the denominator is the mass of the beam:

\[
\int_0^1 (1 + x) \, dx = 60.
\]

and the balance point, officially called the center of mass, is

\[
\bar{x} = \frac{1150}{4 \cdot 60} = \frac{115}{18} \approx 6.39.
\]

Figure 9.6.3 Center of mass for a two dimensional plate.

of the “beam”, say between \( x \) and \( x_{i+1} \), is the mass of a strip of the plate between \( x \) and \( x_{i+1} \). See figure 9.6.3 showing the plate from above and as it appears edge on. Since the plate has uniform density we may as well assume that \( \sigma = 1 \). Then the mass of the plate between \( x_i \) and \( x_{i+1} \) is approximately \( m_i = \sigma (1 + x_i^2) \Delta x = (1 + x_i^2) \Delta x \). Now we can compute the moment around the \( y \)-axis:

\[
M_y = \int_0^1 x (1 + x^2) \, dx = \frac{1}{4}
\]

and the total mass

\[
M = \int_0^1 (1 + x^2) \, dx = \frac{2}{3}
\]

and finally

\[
\bar{x} = \frac{1}{\frac{3}{4}} = \frac{4}{3}.
\]

Next we do the same thing to find \( \bar{y} \). The mass of the plate between \( y \) and \( y_{i+1} \) is approximately \( n_i = \sqrt{2} \Delta y \), so

\[
M_y = \int_0^1 y \sqrt{2} \, dy = \frac{2}{5}
\]

and

\[
\bar{y} = \frac{2}{\frac{2}{5}} = 5.
\]

since the total mass \( M \) is the same. The center of mass is shown in figure 9.6.3.

EXAMPLE 9.6.4 Find the center of mass of a thin, uniform plate whose shape is the region between \( y = \cos x \) and the \( x \)-axis between \( x = -\pi/2 \) and \( x = \pi/2 \). It is clear that \( \bar{x} = 0 \), but for practice let’s compute it anyway. We will need the total mass, so we compute it first:

\[
M = \int_{-\pi/2}^{\pi/2} \cos x \, dx = \sin \frac{\pi}{2} - \sin \left(-\frac{\pi}{2}\right) = 2.
\]

and the moment around the \( y \)-axis is

\[
M_y = \int_{-\pi/2}^{\pi/2} x \cos x \, dx = \cos x + \sin x \sin \frac{\pi}{2} = 0
\]

and the moment around the \( x \)-axis is

\[
M_x = \int_{-\pi/2}^{\pi/2} y \, dx = y \cos x + \frac{\pi - \pi^2}{2} + \arcsin y \left( \frac{1}{2} - \frac{y^2}{2} \right)
\]

Thus

\[
\bar{x} = \frac{0}{2} = 0 \quad \bar{y} = \frac{\pi}{8} \approx 0.393.
\]

EXERCISES 9.6.4

1. A beam 10 meters long has density \( \sigma(x) = x^2 \) at distance \( x \) from the left end of the beam. Find the center of mass \( \bar{x} \).

2. A beam 10 meters long has density \( \sigma(x) = \sqrt{x}/10 \) at distance \( x \) from the left end of the beam. Find the center of mass \( \bar{x} \).

3. A beam 4 meters long has density \( \sigma(x) = x^3 \) at distance \( x \) from the left end of the beam. Find the center of mass \( \bar{x} \).

4. Verify that \( \int_{-\pi/2}^{\pi/2} \cos x \, dx = 0 \) and \( \int_{-\pi/2}^{\pi/2} \sin x \, dx = 0 \).

5. A thin plate lies in the region between \( y = \sin x \) and the \( x \)-axis between \( x = 1 \) and \( x = 2 \). Find the area.

6. A thin plate fills the upper half of the unit circle \( x^2 + y^2 = 1 \). Find the centroid.

7. A thin plate lies in the region contained by \( y = x \) and \( y = x^2 \). Find the centroid.

8. A thin plate lies in the region contained by \( y = 4 - x^2 \) and \( x = 2 \). Find the centroid.

9. A thin plate lies in the region contained by \( y = -x^2 \) and the \( x \)-axis between \( x = 0 \) and \( x = 1 \). Find the centroid.

10. A thin plate lies in the region contained by \( y = -x^2 + x \) and the \( x \)-axis in the first quadrant. Find the centroid.

11. A thin plate lies in the region between the circle \( x^2 + y^2 = 4 \) and the circle \( x^2 + y^2 = 9 \) above the \( x \)-axis. Find the centroid.

12. A thin plate lies in the region between the circle \( x^2 + y^2 = 2 \) and the circle \( x^2 + y^2 = 4 \) in the first quadrant. Find the centroid.

13. A thin plate lies in the region between the circle \( x^2 + y^2 = 25 \) and the circle \( x^2 + y^2 = 36 \) above the \( x \)-axis. Find the centroid.
9.7 Kinetic energy; improper integrals

Recall example 9.5.3 in which we computed the work required to lift an object from the surface of the earth to some large distance $D$ away. Since $F = k/x^2$ we computed

$$\int_0^D \frac{k}{x^2} \, dx = \frac{k}{D} - \frac{k}{x}$$

We noticed that as $D$ increases, $k/D$ decreases to zero so that the amount of work increases to $k/r_0$. More precisely,

$$\lim_{D \to \infty} \int_0^D \frac{k}{x^2} \, dx = \lim_{D \to \infty} \frac{k}{D} - \frac{k}{x} = \frac{k}{r_0}$$

We might reasonably describe this calculation as computing the amount of work required to lift the object “to infinity,” and abbreviate the limit as

$$\lim_{D \to \infty} \int_0^D \frac{k}{x^2} \, dx = \int_0^\infty \frac{k}{x^2} \, dx.$$ 

Such an integral, with a limit of infinity, is called an improper integral. This is a bit unfortunate, since it’s not really “improper” to do this, nor is it really “an integral”—it is an abbreviation for the limit of a particular sort of integral. Nevertheless, we’re stuck with the term, and the operation itself is perfectly legitimate. It may at first seem odd that a finite amount of work is sufficient to lift an object “to infinity,” but sometimes surprising things are nevertheless true, and this is such a case. If the value of an improper integral is a finite number, as in this example, we say that the integral converges, and if not we say that the integral diverges.

Here’s another way, perhaps even more surprising, to interpret this calculation. We know that one interpretation of

$$\int_0^D \frac{1}{x} \, dx$$

is the area under $y = 1/x^2$ from $x = 1$ to $x = D$. Of course, as $D$ increases this area increases. But since

$$\int_0^D \frac{1}{x^2} \, dx = \frac{1}{D} + \frac{1}{2},$$

while the area increases, it never exceeds 1, that is

$$\int_1^\infty \frac{1}{x^2} \, dx = 1.$$

The area of the infinite region under $y = 1/x^2$ from $x = 1$ to infinity is finite.

Consider a slightly different sort of improper integral: $\int_0^\infty x^{-\alpha} \, dx$. There are two ways we might try to compute this. First, we could break it up into two more familiar integrals:

$$\int_0^\infty x^{-\alpha} \, dx = \int_0^a x^{-\alpha} \, dx + \int_a^\infty x^{-\alpha} \, dx.$$ 

Now we do these as before:

$$\int_0^a x^{-\alpha} \, dx = \lim_{b \to \infty} \int_0^b x^{-\alpha} \, dx = \frac{1}{\alpha - 1} - \frac{1}{a^{\alpha - 1}}$$

and

$$\int_a^\infty x^{-\alpha} \, dx = \lim_{b \to \infty} \int_a^b x^{-\alpha} \, dx = \frac{1}{\alpha - 1} - \frac{1}{a^{\alpha - 1}},$$

so

$$\int_0^\infty x^{-\alpha} \, dx = \frac{1}{\alpha - 1} - \frac{1}{a^{\alpha - 1}} + \frac{1}{a^{\alpha - 1}} = \frac{1}{\alpha - 1}.$$ 

Alternatively, we might try

$$\int_0^\infty x^{-\alpha} \, dx = \lim_{b \to \infty} \int_0^b x^{-\alpha} \, dx = \frac{1}{\alpha - 1} - \frac{1}{a^{\alpha - 1}}$$

and

$$\int_a^\infty x^{-\alpha} \, dx = \lim_{b \to \infty} \int_a^b x^{-\alpha} \, dx = \frac{1}{\alpha - 1} - \frac{1}{b^{\alpha - 1}},$$

so

$$\int_0^\infty x^{-\alpha} \, dx = \frac{1}{\alpha - 1} - \frac{1}{a^{\alpha - 1}} + \frac{1}{b^{\alpha - 1}} = \frac{1}{\alpha - 1}.$$ 

So we get the same answer either way. This does not always happen; sometimes the second approach gives a finite number, while the first approach does not; the exercises provide examples. In general, we interpret the integral $\int_0^\infty f(x) \, dx$ according to the first method: both integrals $\int_0^a f(x) \, dx$ and $\int_a^\infty f(x) \, dx$ must converge for the original integral to converge. The second approach does turn out to be useful; when $\lim_{b \to \infty} \int_0^b f(x) \, dx = L$, and $L$ is finite, then $L$ is called the Cauchy Principal Value of $\int_0^\infty f(x) \, dx$.

Here’s a more concrete application of these ideas. We know that in general

$$W = \int_{v_0}^v F \, dx$$

is the work done against the force $F$ in moving from $x_0$ to $x_1$. In the case that $F$ is the force of gravity exerted by the earth, it is customary to make $F < 0$ since the force is

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“downward.” This makes the work $W$ negative when it should be positive, so typically the work in this case is defined as

$$W = -\int_{x_0}^x F \, dx.$$ 

Also, by Newton’s Law, $F = ma(t)$. This means that

$$W = \int_{v_0}^v ma(t) \, dt.$$ 

Unfortunately this integral is a bit problematic: $a(t)$ is in terms of $t$, while the limits and the “$dx$” are in terms of $x$. But $x$ and $t$ are certainly related here: $x = x(t)$ is the function that gives the position of the object at time $t$, so $v = v(t) = dx/dt = x(t)$ is its velocity and $a(t) = a(t) = dx/dt$. We can use $v = x(t)$ as a substitution to convert the integral from “$dx” to “$dv” in the usual way, with a bit of cleverness along the way:

$$\frac{dx}{dt} = x(t) \, dt = a(t) \, dt = a(t) \frac{dx}{dt} \, dx \, dt \, dt = \frac{dx}{dt}$$

Substituting in the integral:

$$W = \int_{x_0}^x ma(t) \, dt = \int_{v_0}^v mv \, dv = \frac{mv^2}{2} - \frac{mv_0^2}{2} = \frac{mv_0^2}{2} - \frac{mv^2}{2}.$$

You may recall seeing the expression $mv^2/2$ in a physics course—it is called the kinetic energy of the object. We have shown here that the work done in moving the object from one place to another is the same as the change in kinetic energy.

We know that the work required to move an object from the surface of the earth to infinity is

$$W = \int_{x_0}^\infty \frac{F}{x^2} \, dx = \frac{k}{x}.$$ 

At the surface of the earth the acceleration due to gravity is approximately 9.8 meters per second squared, so the force on an object of mass $m$ is $F = 9.8m$. The radius of the earth is approximately 6378.1 kilometers or 6378108 meters. Since the force due to gravity obeys an inverse square law, $F = k/r^2$ and $9.8m = k/6378108^2$, $k = 39860564178000m$ and $W = 62503880m$.

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Now suppose that the initial velocity of the object, $v_0$, is just enough to get it to infinity, that is, just enough so that the object never slows to a stop, but so that its speed decreases to zero, i.e., so that $v_f = 0$. Then

$$62503880m = \frac{mv_0^2}{2} - \frac{mv_f^2}{2},$$

so

$$v_0 = 11181 \text{ meters per second},$$

or about 40521 kilometers per hour. This speed is called the escape velocity. Notice that the mass of the object, $m$, canceled out at the last step; the escape velocity is the same for all objects. Of course, it takes considerably more energy to get a large object up to 40521 kph than a small one, so it is certainly more difficult to get a large object into deep space than a small one. Also, note that while we have computed the escape velocity for the earth, this speed would not in fact get an object “to infinity” because of the large mass in our neighborhood called the sun. Escape velocity for the sun starting at the distance of the earth from the sun is nearly 4 times the escape velocity we have calculated.

Exercises 9.7.

1. Is the area under $y = 1/x$ from 1 to infinity finite or infinite? If finite, compute the area. ⇒

2. Is the area under $y = 1/x^2$ from 1 to infinity finite or infinite? If finite, compute the area. ⇒

3. Does $\int_0^1 x^3 + 2x - 1 \, dx$ converge or diverge? If it converges, find the value. ⇒

4. Does $\int_0^1 1/\sqrt{3x} \, dx$ converge or diverge? If it converges, find the value. ⇒

5. Does $\int_0^1 x^{-\alpha} \, dx$ converge or diverge? If it converges, find the value. ⇒

6. $\int_{0.5}^{0.75} \frac{1}{(2x - 1)^2} \, dx$ is an improper integral of a slightly different sort. Express it as a limit and determine whether it converges or diverges, if it converges, find the value. ⇒

7. Does $\int_0^1 1/\sqrt{3x} \, dx$ converge or diverge? If it converges, find the value. ⇒

8. Does $\int_0^1 x e^{x^2} \, dx$ converge or diverge? If it converges, find the value. ⇒

9. Does $\int_0^1 x e^{-x^2} \, dx$ converge or diverge? If it converges, find the value. ⇒

10. Does $\int_0^1 x \sin x \, dx$ converge or diverge? If it converges, find the value. Also find the Cauchy Principal Value, if it exists. ⇒

11. Does $\int_0^1 x \cos x \, dx$ converge or diverge? If it converges, find the value. Also find the Cauchy Principal Value, if it exists. ⇒
12. Does \( \int_{-\infty}^{\infty} x \, dx \) converge or diverge? If it converges, find the value. Also find the Cundy Principal Value if it exists. \( \Rightarrow \)

13. Suppose the curve \( y = 1/e \) is rotated around the \( z \)-axis generating a sort of funnel or horn shape, called Gabriel’s horn or Torricelli’s trumpet. Is the volume of this funnel from \( z = 1 \) to infinity finite or infinite? If finite, compute the volume. \( \Rightarrow \)

14. An officially sanctioned baseball must be between 142 and 149 grams. How much work, in Newton-meters, does it take to throw a ball at 80 miles per hour? At 90 mph? (According to the Guinness Book of World Records, at http://www.baseball-almanac.com/reports/tn_gsw.htm, “The greatest reliably recorded speed at which a baseball has been pitched is 105.9 mph by Lynn Nolan Ryan (California Angels) at Anaheim Stadium in California on August 20, 1974.”) \( \Rightarrow \)

9.8 Probability

You perhaps have at least a rudimentary understanding of discrete probability, which measures the likelihood of an “event” when there are a finite number of possibilities. For example, when an ordinary six-sided die is rolled, the probability of getting any particular number is 1/6. In general, the probability of an event is the number of ways the event can happen divided by the number of ways that “anything” can happen.

For a slightly more complicated example, consider the case of two six-sided dice. The dice are physically distinct, which means that rolling a 2–5 is different than rolling a 5–2; the number is 1.

Most interesting events are not so simple. More interesting is the probability of rolling a 7, we should suspect that the dice are not fair.

DEFINITION 9.8.1 Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. If \( f(x) \geq 0 \) for every \( x \) and \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \) then \( f \) is a probability density function.

We associate a probability density function with a random variable \( X \) by stipulating that the probability that \( X \) is between \( a \) and \( b \) is \( \int_a^b f(x) \, dx \). Because of the requirement that the integral from \( -\infty \) to \( \infty \) be 1, all probabilities are less than or equal to 1, and the probabilities that \( X \) takes on some value between \( -\infty \) and \( \infty \) is 1, as it should be.

EXAMPLE 9.8.2 Consider again the two dice example; we can view it in a way that more resembles the probability density function approach. Consider a random variable \( X \) that takes on any real value with probabilities given by the probability density function in figure 9.8.1. The function \( f \) consists of just the top edges of the rectangles, with vertical sides drawn for clarity; the function is zero below 1.5 and above 12.5. The area of each rectangle is the probability of rolling the sum in the middle of the bottom of the rectangle, or \( P(n) = \frac{n^{1/2}}{9} \int_{2n-1}^{2n+1} f(x) \, dx \).

The probability of rolling a 4, 5, or 6 is \( P(n) = \frac{n^{1/2}}{9} \int_{2n-1}^{2n+1} f(x) \, dx \).

Of course, we could also compute probabilities that don’t make sense in the context of the dice, such as the probability that \( X \) is between 4 and 5.8.

Figure 9.8.1 A probability density function for two dice.

The study of probability is concerned with more difficult questions as well; for example, suppose the two dice are rolled many times. On the average, what sum will come up? In the language of probability, this average is called the expected value of the sum. This is at first a little misleading, as it does not tell us what to “expect” when the two dice are rolled, but what we expect the long term average will be.

Suppose that two dice are rolled 36 million times. Based on the probabilities, we would expect about 1 million rolls to be 2, about 2 million to be 3, and so on, with a roll of 7 topping the list at about 6 million. The sum of all rolls would be 1 million times 2 plus 2 million times 3, and so on, and dividing by 36 million we would get the average:

\[
\bar{x} = \frac{2 \cdot 1,000,000 + 3 \cdot 2,000,000 + \cdots + 6 \cdot 6,000,000 + \cdots + 12 \cdot 1,000,000}{36,000,000} = \frac{6}{36} \cdot 1,000,000 = 100.
\]

There is nothing special about the 36 million in this calculation. No matter what the number of rolls, once we simplify the average, we get the same \( \sum_{i=2}^{12} iP(i) \).

When the number of possible values for \( X \) is finite, we say that \( X \) is a discrete random variable. In many applications of probability, the number of possible values of a random variable is very large, perhaps even infinite. To deal with the infinite case we need a different approach, and since there is a sum involved, it should not be wholly surprising that integration turns out to be a useful tool. It then turns out that even when the number of possibilities is large but finite, it is frequently easier to pretend that the number is infinite. Suppose, for example, that a dart is thrown at a dart board. Since the dart board consists of a finite number of atoms, there are in some sense only a finite number of places for the dart to land, but it is easier to explore the probabilities involved by pretending that the dart can land on any point in the usual x-y plane.

DEFINITION 9.8.3 Suppose that \( a \leq b \) and \( f(x) = \begin{cases} 1 & \text{if } a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases} \)

Then \( f(x) \) is the uniform probability density function on \([a, b]\), and the corresponding distribution is the uniform distribution on \([a, b]\).

EXAMPLE 9.8.4 Consider the function \( f(x) = e^{-x^2/2} \). What can we say about \( \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \)?

We cannot find an antiderivative of \( f \), but we can see that this integral is some finite number. Notice that \( 0 < f(x) = e^{-x^2/2} \leq e^{-x^2} \) for \( |x| > 1 \). This implies that the area under \( e^{-x^2/2} \) is less than the area under \( e^{-x^2} \), over the interval \([1, \infty)\). It is easy to compute the latter area, namely \( \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = 2/\sqrt{2\pi} \). Since \( g(x) = f(x)/A \) is some finite number smaller than \( 2/\sqrt{7} \). Because \( f \) is symmetric around the \( y \)-axis, \( \int_{-\infty}^{-1} e^{-x^2/2} \, dx = \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \) which is zero. Therefore:

\[
\int_{1}^{\infty} e^{-x^2/2} \, dx = \int_{1}^{\infty} e^{-x^2/2} \, dx + \int_{-1}^{1} e^{-x^2/2} \, dx + \int_{-\infty}^{-1} e^{-x^2/2} \, dx = A
\]

for some finite positive number \( A \). Note that \( g(x) \) is a probability density function. It turns out to be very useful, and is called the standard normal probability density function or more informally the bell curve.
EXAMPLE 9.8.5 The exponential distribution has probability density function
\[
f(x) = \begin{cases} 
0 & x < 0 \\
ke^{-x} & x \geq 0
\end{cases}
\]
where \(k\) is a positive constant.

The mean or expected value of a random variable is quite useful, as hinted at in our discussion of dice. Recall that the mean for a discrete random variable is
\[
E(X) = \sum_{i=1}^{\infty} x_i P(X = x_i).
\]
In the more general context we use an integral in place of the sum.

DEFINITION 9.8.6 The mean of a random variable \(X\) with probability density function \(f\) is
\[
E(X) = \int_{-\infty}^{\infty} x f(x) \, dx,
\]
provided the integral converges. The mean does not always exist.

When the mean exists, it is unique, since it is the result of an explicit calculation. The mean might look familiar; it is essentially identical to the center of mass of a one-dimensional beam, as discussed in section 9.6. The probability density function \(f\) plays the role of the physical density function, but now the “beam” has infinite length. If we consider only a finite portion of the beam, say between \(a\) and \(b\), then the center of mass is
\[
\frac{\int_{a}^{b} x f(x) \, dx}{\int_{a}^{b} f(x) \, dx}.
\]

EXAMPLE 9.8.8 We compute the standard deviation of the normal distribution.

The variance is
\[
\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx,
\]
called the variance. The square root of the variance is the standard deviation, denoted \(\sigma\).

EXAMPLE 9.8.9 Here is a simple example showing how these ideas can be useful. Suppose it is known that, in the long run, 1 out of every 100 computer memory chips produced by a certain manufacturing plant is defective when the manufacturing process is running correctly. Suppose 1000 chips are selected at random and 15 of them are defective. This is more than the “expected” number (10), but is it so many that we should suspect that something has gone wrong in the manufacturing process? We are interested in the probability that the number of defective chips arise; the probability distribution is discrete: there can only be a whole number of defective chips. But (under reasonable assumptions) the distribution is very close to a normal distribution, namely this one:
\[
f(x) = \frac{1}{\sqrt{2\pi b_0x_0\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2b_0x_0\sigma^2}\right)
\]
which is pictured in figure 9.8.3 (recall that \(\exp(x) = e^x\)).

Figure 9.8.3 Normal density function for the defective chips example.

Now how do we measure how unlikely it is that under normal circumstances we would see 15 defective chips? We can’t compute the probability of exactly 15 defective chips, as this would be \(\int_{15}^{15.5} f(x) \, dx = 0.036\) (this means there is only a 3.6% chance that the number of defective chips is 15. (We cannot compute these integrals exactly; computer software has been used to approximate the integral values in this discussion.) But this is misleading: \(\int_{15}^{15.5} f(x) \, dx \approx 0.126\), which is larger, certainly, but still small, even for the “most likely” outcome. The most useful question, in most circumstances, is this: how likely is it that the number of defective chips is “far from” the mean? For example, how likely, or unlikely, is it that the number of defective chips is different by 5 or more from the expected value of 10? This is the probability that the number of defective chips is less than 5 or larger than 15, namely
\[
\int_{-\infty}^{5} f(x) \, dx + \int_{15}^{\infty} f(x) \, dx \approx 0.11.
\]
So there is an 11% chance that this happens—not large, but not tiny. Hence the 15 defective chips does not appear to be cause for alarm: about one time in nine we would
expect to see the number of defective chips 5 or more away from the expected 10. How about 20? Here we compute
\[ \int_{-\infty}^{10} f(x)\,dx + \int_{10}^{\infty} f(x)\,dx \approx 0.005. \]

So there is only a 0.15% chance that the number of defective chips is more than 10 away from the mean; this would typically be interpreted as too suspicious to ignore—it shouldn’t happen if the process is running normally.

The big question, of course, is what level of improbability should trigger concerns? It depends to some degree on the application, and in particular on the consequences of getting it wrong in one direction or the other. If we’re wrong, do we lose a little money? A lot of money? Do people die? In general, the standard choices are 5% and 1%. So what we should do is find the number of defective chips that has, let us say, a 1% chance of occurring under normal circumstances, and use that as the relevant number. In other words, we want to know when
\[ \int_{-\infty}^{a} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx < 0.01. \]

A bit of trial and error shows that with \( r = 8 \) the value is about 0.011, and with \( r = 9 \) it is about 0.004, so if the number of defective chips is 19 or more, or 1 or fewer, we should look for problems. If the number is high, we worry that the manufacturing process has a problem, or conversely that the process that tests for defective chips is not working correctly and is flagging good chips as defective. If the number is too low, we suspect that the testing procedure is broken, and is not detecting defective chips.

Exercises 9.8.

1. Verify that \( \int_{0}^{\infty} e^{-x^2/2}\,dx = \sqrt{\pi/2} \).
2. Show that the function in example 9.8.5 is a probability density function. Compute the mean and standard deviation.
3. Compute the mean and standard deviation of the uniform distribution on \([a, b]\). (See example 9.8.3.)
4. What is the expected value of one roll of a fair six-sided die?
5. What is the expected sum of one roll of three fair six-sided dice?
6. Let \( \mu \) and \( \sigma \) be real numbers with \( \sigma > 0 \). Show that

\[ N(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \]

is a probability density function. You will not be able to compute this integral directly, use a substitution to convert the integral into the one from example 9.8.4. The function \( N \) is the probability density function of the normal distribution with mean \( \mu \) and standard deviation \( \sigma \). Show that the mean of the normal distribution is \( \mu \) and the standard deviation is \( \sigma \).
7. Let

\[ f(x) = \begin{cases} \frac{1}{\sqrt{x}} & x \geq 1 \\ \frac{1}{\sqrt{x}} & x < 1 \end{cases} \]

Show that \( f \) is a probability density function, and that the distribution has no mean.
8. Let

\[ f(x) = \begin{cases} \frac{1}{\sqrt{1-x^2}} & -1 < x \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

Show that \( \int_{-\infty}^{\infty} f(x)\,dx = 1 \). Is \( f \) a probability density function? Justify your answer.
9. If you have access to appropriate software, find \( r \) so that

\[ \int_{-r}^{r} f(x)\,dx \approx 0.05, \]

using the function of example 9.8.9. Discuss the impact of using this new value of \( r \) to decide whether to investigate the chip manufacturing process.

9.9 Arc Length

Here is another geometric application of the integral: find the length of a portion of a curve. As usual, we need to think about how we might approximate the length, and turn the approximation into an integral.

We already know how to compute one simple arc length, that of a line segment. If the endpoints are \( P(x_0, y_0) \) and \( P(x_1, y_1) \) then the length of the segment is the distance between the points, \( \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \), from the Pythagorean theorem, as illustrated in figure 9.9.1.

Figure 9.9.2 approximating arc length with line segments.

Now if the graph of \( f \) is "nice" (say, differentiable) it appears that we can approximate the length of a portion of the curve with line segments, and that as the number of segments increases, and their lengths decrease, the sum of the lengths of the line segments will approach the true arc length, see figure 9.9.2.

Now we need to write a formula for the sum of the lengths of the line segments, in a form that we know becomes an integral in the limit. So we suppose we have divided the interval \([a, b]\) into \( n \) subintervals as usual, with each length \( \Delta x = (b-a)/n \), and endpoints \( a = x_0, x_1, x_2, \ldots, x_n = b \). The length of a typical line segment, joining \((x_i, f(x_i))\) to \((x_{i+1}, f(x_{i+1}))\), is \( \sqrt{\Delta x^2 + [(f(x_{i+1}) - f(x_i))^2]} \). By the Mean Value Theorem (6.5.2), there is a number \( t_i \) in \([x_i, x_{i+1}]\) such that \( f'(t_i)\Delta x = f(x_{i+1}) - f(x_i) \), so the length of the line segment can be written as

\[ \sqrt{\Delta x^2 + (f'(t_i))^2}\Delta x = \sqrt{1 + (f'(t_i))^2}\Delta x. \]

The arc length is then

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \sqrt{1 + (f'(t_i))^2}\Delta x = \int_{a}^{b} \sqrt{1 + (f'(x))^2}\,dx. \]

Note that the sum looks a bit different than others we have encountered, because the approximation contains a \( t_i \) instead of an \( x_i \). In the past we have always used left endpoints (namely, \( x_i \)) to get a representative value of \( f \) on \([x_i, x_{i+1}]\); now we are using a different point, but the principle is the same.

To summarize, to compute the length of a curve on the interval \([a, b]\), we compute the integral

\[ \int_{a}^{b} \sqrt{1 + (f'(x))^2}\,dx. \]

Unfortunately, integrals of this form are typically difficult or impossible to compute exactly, because usually none of our methods for finding antiderivatives will work. In practice this means that the integral will usually have to be approximated.

EXAMPLE 9.9.1 Let \( f(x) = \sqrt{x^2 - 1} \), the upper half circle of radius \( r \). The length of this curve is half the circumference, namely \( \pi r \). Let’s compute this with the arc length formula. The derivative \( f' = x/\sqrt{x^2 - 1} \), so the integral is

\[ \int_{0}^{r} \sqrt{1 + \frac{x^2}{x^2 - 1}}\,dx = \int_{0}^{\pi/2} \sqrt{1 + \csc^2 x}\,dx = \pi. \]

Using a trigonometric substitution, we find the antiderivative, namely \( \arcsin(x) \). Notice that the integral is improper at both endpoints, as the function \( \sqrt{1 + \csc^2 x} \) is undefined when \( x = \pm \sqrt{r} \).

So we need to compute

\[ \lim_{a \to 0^+} \int_{a}^{r} \sqrt{1 + \frac{x^2}{x^2 - 1}}\,dx + \lim_{b \to 0^+} \int_{0}^{b} \sqrt{1 + \frac{x^2}{x^2 - 1}}\,dx. \]

This is not difficult, and has value \( \pi \), so the original integral, with the extra \( \pi \) in front, has value \( 2\pi r \) as expected.

Exercises 9.9.

1. Find the arc length of \( f(x) = e^{x/2} \) on \([0, 2]\).
2. Find the arc length of \( f(x) = x^2/8 - \ln x \) on \([1, 2]\).
3. Find the arc length of \( f(x) = (1/3)(e^{x^2})^{1/2} \) on the interval \([0, a]\).
4. Find the arc length of \( f(x) = \ln(\sin x) \) on the interval \([\pi/4, \pi/3]\).
5. Let \( a > 0 \). Show that the length of \( y = \cos x \) on \([0, a]\) is equal to \( \int_{0}^{a} \cos x\,dx \).
6. Find the arc length of \( f(x) = \cosh x \) on \([0, 1]\).
7. Set up the integral to find the arc length of \( y = \sec x \) on the interval \([0, \pi/2]\); do not evaluate the integral. If you have access to appropriate software, approximate the value of the integral.
8. Set up the integral to find the arc length of \( y = \arctan x \) on the interval \([2, 3]\); do not evaluate the integral. If you have access to appropriate software, approximate the value of the integral.
9. Find the arc length of \( y = x^2 \) on the interval \([0, 1]\). (This can be done exactly; it is a bit tricky and a bit long.)

9.10 Surface Area

Another geometric question that arises naturally is: "What is the surface area of a volume?" For example, what is the surface area of a sphere? More advanced techniques are required to approach this question in general, but we can compute the areas of some volumes generated by revolution.
As usual, the question is: how might we approximate the surface area? For a surface obtained by rotating a curve around an axis, we can take a polygonal approximation to the curve, as in the last section, and rotate it around the same axis. This gives a surface composed of many “truncated cones,” a truncated cone is called a frustum of a cone. Figure 9.10.1 illustrates this approximation.

Figure 9.10.1 Approximating a surface (left) by portions of cones (right).

So we need to be able to compute the area of a frustum of a cone. Since the frustum can be formed by removing a small cone from the top of a larger one, we can compute the desired area if we know the surface area of a cone. Suppose a right circular cone has base radius \( r \) and slant height \( h \). If we cut the cone from the vertex to the base circle and flatten it out, we obtain a sector of a circle with radius \( h \) and arc length \( 2\pi r \), as in figure 9.10.2. The angle at the center, in radians, is then \( 2\pi r/h \).

This is not quite the sort of sum we have seen before, as it contains two different values \( r \) and \( h \) than we have available here, it turns out that

\[
\frac{A}{\pi r h} = 2\pi \frac{r}{h}
\]

Now suppose we have a frustum of a cone with slant height \( h \) and radii \( r_0 \) and \( r_1 \), as in figure 9.10.3. The area of the entire cone is \( \pi r_1 h_0 \), and the area of the small cone is \( \pi r_0 h_0 \); thus, the area of the frustum is \( \pi r_1 (h_0 - h) = \pi r_0 h_0 + \pi r_1 h \). By similar triangles, \( \frac{h_0}{r_0} = \frac{h}{r_1} \).

With a bit of algebra this becomes \( (r_1 - r_0)h_0 = r_0 h_1 \); substitution into the area gives

\[
\pi((r_1 - r_0)h_0 + r_1 h) = \pi h_0 (r_0 + r_1) = \pi h_0 (r_0 + r_1) = 2\pi rh.
\]

The final form is particularly easy to remember, with \( r \) equal to the average of \( r_0 \) and \( r_1 \), as it is also the formula for the area of a cylinder. (Think of a cylinder of radius \( r \) and height \( h \) as the frustum of a cone of infinite height.)

Figure 9.10.2 The area of a cone.

Figure 9.10.3 The area of a frustum.

Now we are ready to approximate the area of a surface of revolution. On one subinterval, the situation is as shown in figure 9.10.4. When the line joining two points on the curve is rotated around the \( x \)-axis, it forms a frustum of a cone. The area is

\[
2\pi rh = 2\pi f(x)\sqrt{1 + (f'(x))^2} \Delta x.
\]

Here \( \sqrt{1 + (f'(x))^2} \Delta x \) is the length of the line segment, as we found in the previous section. Assuming \( f \) is a continuous function, there must be some \( x_i \) in \([x_i, x_{i+1}]\) such that \( f(x_i) + f(x_{i+1})/2 = f(x_i') \), so the approximation for the surface area is

\[
\sum_{i=0}^{n-1} 2\pi f(x_i')\sqrt{1 + (f'(x_i'))^2} \Delta x.
\]

This is not quite the sort of sum we have seen before, as it contains two different values in the interval \([x_i, x_{i+1}]\), namely \( x_i' \) and \( t_i \). Nevertheless, using more advanced techniques than we have available here, it turns out that

\[
\lim_{\Delta x \to 0} \sum_{i=0}^{n-1} 2\pi f(x_i')\sqrt{1 + (f'(x_i'))^2} \Delta x = \int_{x_0}^{x_n} 2\pi f(x)\sqrt{1 + (f'(x))^2} dx
\]

is the surface area we seek. (Roughly speaking, this is because while \( x_i' \) and \( t_i \) are distinct values in \([x_i, x_{i+1}]\), they get closer and closer to each other as the length of the interval shrinks.)

Figure 9.10.4 One subinterval.

EXAMPLE 9.10.1 We compute the surface area of a sphere of radius \( r \). The sphere can be obtained by rotating the graph of \( f(x) = \sqrt{r^2 - x^2} \) about the \( x \)-axis. The derivative

\[
\frac{df}{dx} = \frac{-x}{\sqrt{r^2 - x^2}},
\]

so the surface area is given by

\[
A = 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2} dx
\]

\[
= 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2} \sqrt{\frac{r^2}{r^2 - x^2}} dx
\]

\[
= 2\pi \int_{-r}^{r} r dx = 4\pi r^2.
\]

If the curve is rotated around the \( y \)-axis, the formula is nearly identical, because the length of the line segment we use to approximate a portion of the curve doesn’t change.

EXAMPLE 9.10.2 Compute the area of the surface formed when \( f(x) = x^2 \) between 0 and 2 is rotated around the \( y \)-axis.

We compute \( f'(x) = 2x \), and then

\[
2\pi \int_{0}^{2} x \sqrt{1 + 4x^2} dx = \frac{\pi}{4} (17^{3/2} - 1),
\]

by a simple substitution.

Exercise 9.10.

1. Compute the area of the surface formed when \( f(x) = 2\sqrt{x} \) between \(-1 \) and \( 0 \) is rotated around the \( x \)-axis.

2. Compute the surface area of example 9.10.2 by rotating \( f(x) = \sqrt{x} \) around the \( z \)-axis.

3. Compute the area of the surface formed when \( f(x) = x^2 \) between 1 and 3 is rotated around the \( z \)-axis.

4. Compute the area of the surface formed when \( f(x) = 2 + \cosh(x) \) between 0 and 1 is rotated around the \( z \)-axis.

5. Consider the surface obtained by rotating the graph of \( f(x) = 1/2 \) \( x \geq 1 \), around the \( x \)-axis. This surface is called Gabriel’s horn or Torricelli’s trumpet. In exercise 13 in section 9.7 we saw that Gabriel’s horn has finite surface area.

6. Consider the circle \((x - 2)^2 + y^2 = 1\). Sketch the surface obtained by rotating this circle about the \( y \)-axis. (The surface is called a torus.) What is the surface area?
7. Consider the ellipse with equation \( x^2/4 + y^2 = 1 \). If the ellipse is rotated around the \( x \)-axis it forms an ellipsoid. Compute the surface area. ⇒

8. Generalize the preceding result: rotate the ellipse given by \( x^2/a^2 + y^2/b^2 = 1 \) about the \( x \)-axis and find the surface area of the resulting ellipsoid. You should consider two cases, when \( a > b \) and when \( a < b \). Compare to the area of a sphere. ⇒