6

Applications of the Derivative

6.1 Optimization

Many important applied problems involve finding the best way to accomplish some task. Often this involves finding the maximum or minimum value of some function: the minimum time to make a certain journey, the minimum cost for doing a task, the maximum power that can be generated by a device, and so on. Many of these problems can be solved by finding the appropriate function and then using techniques of calculus to find the maximum or the minimum value required.

Generally such a problem will have the following mathematical form: Find the largest (or smallest) value of $f(x)$ when $a \leq x \leq b$. Sometimes a or b are infinite, but frequently the real world imposes some constraint on the values that x may have.

Such a problem differs in two ways from the local maximum and minimum problems we encountered when graphing functions: We are interested only in the function between a and b, and we want to know the largest or smallest value that $f(x)$ takes on, not merely values that are the largest or smallest in a small interval. That is, we seek not a local maximum or minimum but a **global** maximum or minimum, sometimes also called an **absolute** maximum or minimum.

Any global maximum or minimum must of course be a local maximum or minimum. If we find all possible local extrema, then the global maximum, *if it exists*, must be the largest of the local maxima and the global minimum, *if it exists*, must be the smallest of the local minima. We already know where local extrema can occur: only at those points at which $f'(x)$ is zero or undefined. Actually, there are two additional points at which a maximum or minimum can occur if the endpoints a and b are not infinite, namely, at a

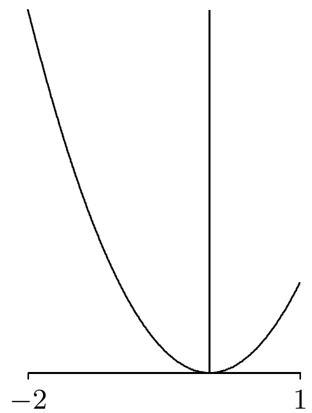

Figure 6.1.1 The function $f(x) = x^2$ restricted to $[-2, 1]$

and b. We have not previously considered such points because we have not been interested in limiting a function to a small interval. An example should make this clear.

EXAMPLE 6.1.1 Find the maximum and minimum values of $f(x) = x^2$ on the interval $[-2, 1]$, shown in figure 6.1.1. We compute $f'(x) = 2x$, which is zero at $x = 0$ and is always defined.

Since $f'(1) = 2$ we would not normally flag $x = 1$ as a point of interest, but it is clear from the graph that *when $f(x)$ is restricted to $[-2, 1]$ there is a local maximum at $x = 1$.* Likewise we would not normally pay attention to $x = -2$, but since we have truncated f at -2 we have introduced a new local maximum there as well. In a technical sense nothing new is going on here: When we truncate f we actually create a new function, let's call it g, that is defined only on the interval $[-2, 1]$. If we try to compute the derivative of this new function we actually find that it does not have a derivative at -2 or 1. Why? Because to compute the derivative at 1 we must compute the limit

$$\lim_{\Delta x \to 0} \frac{g(1 + \Delta x) - g(1)}{\Delta x}.$$

This limit does not exist because when $\Delta x > 0$, $g(1 + \Delta x)$ is not defined. It is simpler, however, simply to remember that we must always check the endpoints.

So the function g, that is, f restricted to $[-2, 1]$, has one critical value and two finite endpoints, any of which might be the global maximum or minimum. We could first determine which of these are local maximum or minimum points (or neither); then the largest local maximum must be the global maximum and the smallest local minimum must be the global minimum. It is usually easier, however, to compute the value of f at every point at which the global maximum or minimum might occur; the largest of these is the global maximum, the smallest is the global minimum.

So we compute $f(-2) = 4$, $f(0) = 0$, $f(1) = 1$. The global maximum is 4 at $x = -2$ and the global minimum is 0 at $x = 0$. □

It is possible that there is no global maximum or minimum. It is difficult, and not particularly useful, to express a complete procedure for determining whether this is the case. Generally, the best approach is to gain enough understanding of the shape of the graph to decide. Fortunately, only a rough idea of the shape is usually needed.

There are some particularly nice cases that are easy. A continuous function on a closed interval $[a, b]$ *always* has both a global maximum and a global minimum, so examining the critical values and the endpoints is enough:

THEOREM 6.1.2 Extreme value theorem If f is continuous on a closed interval $[a, b]$, then it has both a minimum and a maximum point. That is, there are real numbers c and d in $[a, b]$ so that for every x in $[a, b]$, $f(x) \leq f(c)$ and $f(x) \geq f(d)$. ∎

Another easy case: If a function is continuous and has a single critical value, then if there is a local maximum at the critical value it is a global maximum, and if it is a local minimum it is a global minimum. There may also be a global minimum in the first case, or a global maximum in the second case, but that will generally require more effort to determine.

EXAMPLE 6.1.3 Let $f(x) = -x^2 + 4x - 3$. Find the maximum value of $f(x)$ on the interval $[0, 4]$. First note that $f'(x) = -2x+4 = 0$ when $x = 2$, and $f(2) = 1$. Next observe that $f'(x)$ is defined for all x, so there are no other critical values. Finally, $f(0) = -3$ and $f(4) = -3$. The largest value of $f(x)$ on the interval $[0, 4]$ is $f(2) = 1$. □

EXAMPLE 6.1.4 Let $f(x) = -x^2 + 4x - 3$. Find the maximum value of $f(x)$ on the interval $[-1, 1]$.

First note that $f'(x) = -2x + 4 = 0$ when $x = 2$. But $x = 2$ is not in the interval, so we don't use it. Thus the only two points to be checked are the endpoints; $f(-1) = -8$ and $f(1) = 0$. So the largest value of $f(x)$ on $[-1, 1]$ is $f(1) = 0$. □

EXAMPLE 6.1.5 Find the maximum and minimum values of the function $f(x) = 7 + |x - 2|$ for x between 1 and 4 inclusive. The derivative $f'(x)$ is never zero, but $f'(x)$ is undefined at $x = 2$, so we compute $f(2) = 7$. Checking the end points we get $f(1) = 8$ and $f(4) = 9$. The smallest of these numbers is $f(2) = 7$, which is, therefore, the minimum value of $f(x)$ on the interval $1 \leq x \leq 4$, and the maximum is $f(4) = 9$. □

EXAMPLE 6.1.6 Find all local maxima and minima for $f(x) = x^3 - x$, and determine whether there is a global maximum or minimum on the open interval $(-2, 2)$. In example 5.1.2 we found a local maximum at $(-\sqrt{3}/3, 2\sqrt{3}/9)$ and a local minimum at $(\sqrt{3}/3, -2\sqrt{3}/9)$. Since the endpoints are not in the interval $(-2, 2)$ they cannot be con-

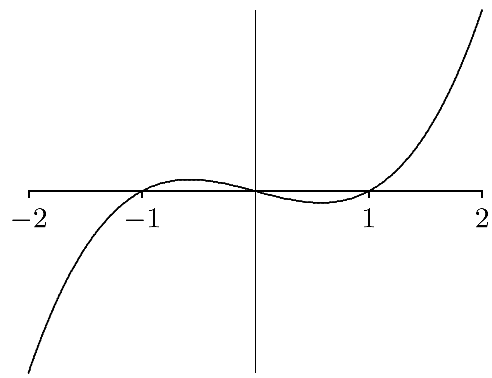

Figure 6.1.2 $f(x) = x^3 - x$

sidered. Is the lone local maximum a global maximum? Here we must look more closely at the graph. We know that on the closed interval $[-\sqrt{3}/3, \sqrt{3}/3]$ there is a global maximum at $x = -\sqrt{3}/3$ and a global minimum at $x = \sqrt{3}/3$. So the question becomes: what happens between -2 and $-\sqrt{3}/3$, and between $\sqrt{3}/3$ and 2? Since there is a local minimum at $x = \sqrt{3}/3$, the graph must continue up to the right, since there are no more critical values. This means no value of f will be less than $-2\sqrt{3}/9$ between $\sqrt{3}/3$ and 2, but it says nothing about whether we might find a value larger than the local maximum $2\sqrt{3}/9$. How can we tell? Since the function increases to the right of $\sqrt{3}/3$, we need to know what the function values do "close to" 2. Here the easiest test is to pick a number and do a computation to get some idea of what's going on. Since $f(1.9) = 4.959 > 2\sqrt{3}/9$, there is no global maximum at $-\sqrt{3}/3$, and hence no global maximum at all. (How can we tell that $4.959 > 2\sqrt{3}/9$? We can use a calculator to approximate the right hand side; if it is not even close to 4.959 we can take this as decisive. Since $2\sqrt{3}/9 \approx 0.3849$, there's really no question. Funny things can happen in the rounding done by computers and calculators, however, so we might be a little more careful, especially if the values come out quite close. In this case we can convert the relation $4.959 > 2\sqrt{3}/9$ into $(9/2)4.959 > \sqrt{3}$ and ask whether this is true. Since the left side is clearly larger than $4 \cdot 4$ which is clearly larger than $\sqrt{3}$, this settles the question.)

A similar analysis shows that there is also no global minimum. The graph of $f(x)$ on $(-2, 2)$ is shown in figure 6.1.2. □

EXAMPLE 6.1.7 Of all rectangles of area 100, which has the smallest perimeter?

First we must translate this into a purely mathematical problem in which we want to find the minimum value of a function. If x denotes one of the sides of the rectangle, then the adjacent side must be $100/x$ (in order that the area be 100). So the function we want to minimize is

$$f(x) = 2x + 2\frac{100}{x}$$

since the perimeter is twice the length plus twice the width of the rectangle. Not all values of x make sense in this problem: lengths of sides of rectangles must be positive, so $x > 0$. If $x > 0$ then so is $100/x$, so we need no second condition on x.

We next find $f'(x)$ and set it equal to zero: $0 = f'(x) = 2 - 200/x^2$. Solving $f'(x) = 0$ for x gives us $x = \pm 10$. We are interested only in $x > 0$, so only the value $x = 10$ is of interest. Since $f'(x)$ is defined everywhere on the interval $(0, \infty)$, there are no more critical values, and there are no endpoints. Is there a local maximum, minimum, or neither at $x = 10$? The second derivative is $f''(x) = 400/x^3$, and $f''(10) > 0$, so there is a local minimum. Since there is only one critical value, this is also the global minimum, so the rectangle with smallest perimeter is the 10×10 square. □

EXAMPLE 6.1.8 You want to sell a certain number n of items in order to maximize your profit. Market research tells you that if you set the price at \$1.50, you will be able to sell 5000 items, and for every 10 cents you lower the price below \$1.50 you will be able to sell another 1000 items. Suppose that your fixed costs ("start-up costs") total \$2000, and the per item cost of production ("marginal cost") is \$0.50. Find the price to set per item and the number of items sold in order to maximize profit, and also determine the maximum profit you can get.

The first step is to convert the problem into a function maximization problem. Since we want to maximize profit by setting the price per item, we should look for a function $P(x)$ representing the profit when the price per item is x. Profit is revenue minus costs, and revenue is number of items sold times the price per item, so we get $P = nx - 2000 - 0.50n$. The number of items sold is itself a function of x, $n = 5000 + 1000(1.5 - x)/0.10$, because $(1.5 - x)/0.10$ is the number of multiples of 10 cents that the price is below \$1.50. Now we substitute for n in the profit function:

$$\begin{aligned} P(x) &= (5000 + 1000(1.5 - x)/0.10)x - 2000 - 0.5(5000 + 1000(1.5 - x)/0.10) \\ &= -10000x^2 + 25000x - 12000 \end{aligned}$$

We want to know the maximum value of this function when x is between 0 and 1.5. The derivative is $P'(x) = -20000x + 25000$, which is zero when $x = 1.25$. Since $P''(x) = -20000 < 0$, there must be a local maximum at $x = 1.25$, and since this is the only critical value it must be a global maximum as well. (Alternately, we could compute $P(0) = -12000$, $P(1.25) = 3625$, and $P(1.5) = 3000$ and note that $P(1.25)$ is the maximum of these.) Thus the maximum profit is \$3625, attained when we set the price at \$1.25 and sell 7500 items. □

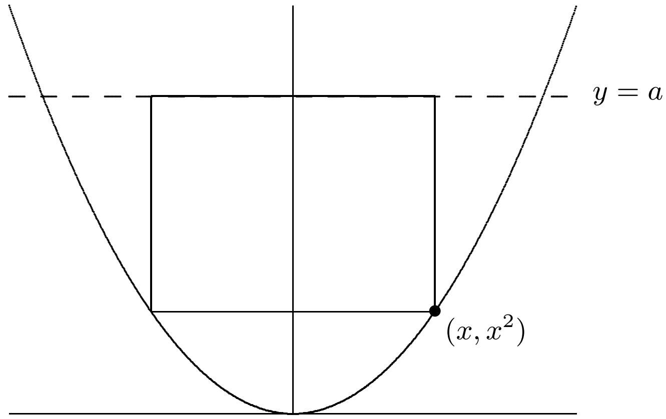

Figure 6.1.3 Rectangle in a parabola.

EXAMPLE 6.1.9 Find the largest rectangle (that is, the rectangle with largest area) that fits inside the graph of the parabola $y = x^2$ below the line $y = a$ (a is an unspecified constant value), with the top side of the rectangle on the horizontal line $y = a$; see figure 6.1.3.)

We want to find the maximum value of some function $A(x)$ representing area. Perhaps the hardest part of this problem is deciding what x should represent. The lower right corner of the rectangle is at (x, x^2), and once this is chosen the rectangle is completely determined. So we can let the x in $A(x)$ be the x of the parabola $f(x) = x^2$. Then the area is $A(x) = (2x)(a - x^2) = -2x^3 + 2ax$. We want the maximum value of $A(x)$ when x is in $[0, \sqrt{a}]$. (You might object to allowing $x = 0$ or $x = \sqrt{a}$, since then the "rectangle" has either no width or no height, so is not "really" a rectangle. But the problem is somewhat easier if we simply allow such rectangles, which have zero area.)

Setting $0 = A'(x) = -6x^2 + 2a$ we get $x = \sqrt{a/3}$ as the only critical value. Testing this and the two endpoints, we have $A(0) = A(\sqrt{a}) = 0$ and $A(\sqrt{a/3}) = (4/9)\sqrt{3}a^{3/2}$. The maximum area thus occurs when the rectangle has dimensions $2\sqrt{a/3} \times (2/3)a$. □

EXAMPLE 6.1.10 If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by "cone" we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.)

Let R be the radius of the sphere, and let r and h be the base radius and height of the cone inside the sphere. What we want to maximize is the volume of the cone: $\pi r^2 h/3$. Here R is a fixed value, but r and h can vary. Namely, we could choose r to be as large as possible—equal to R—by taking the height equal to R; or we could make the cone's height h larger at the expense of making r a little less than R. See the cross-section depicted in

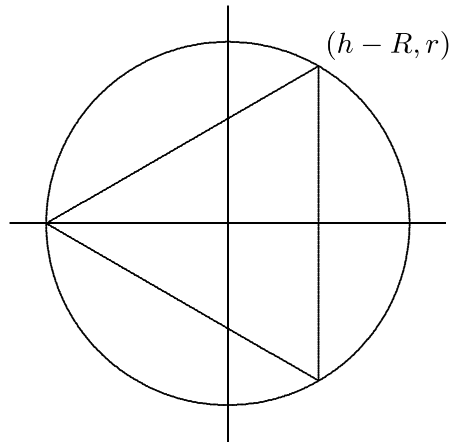

Figure 6.1.4 Cone in a sphere.

figure 6.1.4. We have situated the picture in a convenient way relative to the x and y axes, namely, with the center of the sphere at the origin and the vertex of the cone at the far left on the x-axis.

Notice that the function we want to maximize, $\pi r^2 h/3$, depends on *two* variables. This is frequently the case, but often the two variables are related in some way so that "really" there is only one variable. So our next step is to find the relationship and use it to solve for one of the variables in terms of the other, so as to have a function of only one variable to maximize. In this problem, the condition is apparent in the figure: the upper corner of the triangle, whose coordinates are $(h-R,r)$, must be on the circle of radius R. That is,

$$(h-R)^2 + r^2 = R^2.$$

We can solve for h in terms of r or for r in terms of h. Either involves taking a square root, but we notice that the volume function contains r^2, not r by itself, so it is easiest to solve for r^2 directly: $r^2 = R^2 - (h-R)^2$. Then we substitute the result into $\pi r^2 h/3$:

$$\begin{aligned} V(h) &= \pi(R^2 - (h-R)^2)h/3 \\ &= -\frac{\pi}{3}h^3 + \frac{2}{3}\pi h^2 R \end{aligned}$$

We want to maximize $V(h)$ when h is between 0 and $2R$. Now we solve $0 = f'(h) = -\pi h^2 + (4/3)\pi h R$, getting $h = 0$ or $h = 4R/3$. We compute $V(0) = V(2R) = 0$ and $V(4R/3) = (32/81)\pi R^3$. The maximum is the latter; since the volume of the sphere is $(4/3)\pi R^3$, the fraction of the sphere occupied by the cone is

$$\frac{(32/81)\pi R^3}{(4/3)\pi R^3} = \frac{8}{27} \approx 30\%.$$

□

EXAMPLE 6.1.11 You are making cylindrical containers to contain a given volume. Suppose that the top and bottom are made of a material that is N times as expensive (cost per unit area) as the material used for the lateral side of the cylinder. Find (in terms of N) the ratio of height to base radius of the cylinder that minimizes the cost of making the containers.

Let us first choose letters to represent various things: h for the height, r for the base radius, V for the volume of the cylinder, and c for the cost per unit area of the lateral side of the cylinder; V and c are constants, h and r are variables. Now we can write the cost of materials:

$$c(2\pi rh) + Nc(2\pi r^2).$$

Again we have two variables; the relationship is provided by the fixed volume of the cylinder: $V = \pi r^2 h$. We use this relationship to eliminate h (we could eliminate r, but it's a little easier if we eliminate h, which appears in only one place in the above formula for cost). The result is

$$f(r) = 2c\pi r\frac{V}{\pi r^2} + 2Nc\pi r^2 = \frac{2cV}{r} + 2Nc\pi r^2.$$

We want to know the minimum value of this function when r is in $(0, \infty)$. We now set $0 = f'(r) = -2cV/r^2 + 4Nc\pi r$, giving $r = \sqrt[3]{V/(2N\pi)}$. Since $f''(r) = 4cV/r^3 + 4Nc\pi$ is positive when r is positive, there is a local minimum at the critical value, and hence a global minimum since there is only one critical value.

Finally, since $h = V/(\pi r^2)$,

$$\frac{h}{r} = \frac{V}{\pi r^3} = \frac{V}{\pi(V/(2N\pi))} = 2N,$$

so the minimum cost occurs when the height h is $2N$ times the radius. If, for example, there is no difference in the cost of materials, the height is twice the radius (or the height is equal to the diameter). $\square$

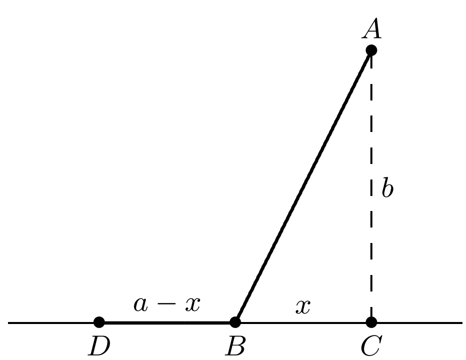

Figure 6.1.5 Minimizing travel time.

EXAMPLE 6.1.12 Suppose you want to reach a point A that is located across the sand from a nearby road (see figure 6.1.5). Suppose that the road is straight, and b is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from C. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A?

Let x be the distance short of C where you turn off, i.e., the distance from B to C. We want to minimize the total travel time. Recall that when traveling at constant velocity, time is distance divided by velocity.

You travel the distance $\overline{DB}$ at speed v, and then the distance $\overline{BA}$ at speed w. Since $\overline{DB} = a - x$ and, by the Pythagorean theorem, $\overline{BA} = \sqrt{x^2 + b^2}$, the total time for the trip is

$$f(x) = \frac{a-x}{v} + \frac{\sqrt{x^2+b^2}}{w}.$$

We want to find the minimum value of f when x is between 0 and a. As usual we set $f'(x) = 0$ and solve for x:

$$\begin{aligned}
0 = f'(x) &= -\frac{1}{v} + \frac{x}{w\sqrt{x^2+b^2}} \\
w\sqrt{x^2+b^2} &= vx \\
w^2(x^2+b^2) &= v^2x^2 \\
w^2b^2 &= (v^2-w^2)x^2 \\
x &= \frac{wb}{\sqrt{v^2-w^2}}
\end{aligned}$$

Notice that a does not appear in the last expression, but a is not irrelevant, since we are interested only in critical values that are in $[0, a]$, and $wb/\sqrt{v^2 - w^2}$ is either in this interval or not. If it is, we can use the second derivative to test it:

$$f''(x) = \frac{b^2}{(x^2+b^2)^{3/2}w}.$$

Since this is always positive there is a local minimum at the critical point, and so it is a global minimum as well.

If the critical value is not in $[0, a]$ it is larger than a. In this case the minimum must occur at one of the endpoints. We can compute

$$\begin{aligned}
f(0) &= \frac{a}{v} + \frac{b}{w} \\
f(a) &= \frac{\sqrt{a^2+b^2}}{w}
\end{aligned}$$

but it is difficult to determine which of these is smaller by direct comparison. If, as is likely in practice, we know the values of v, w, a, and b, then it is easy to determine this. With a little cleverness, however, we can determine the minimum in general. We have seen that $f''(x)$ is always positive, so the derivative $f'(x)$ is always increasing. We know that at $wb/\sqrt{v^2-w^2}$ the derivative is zero, so for values of x less than that critical value, the derivative is negative. This means that $f(0) > f(a)$, so the minimum occurs when $x = a$.

So the upshot is this: If you start farther away from C than $wb/\sqrt{v^2-w^2}$ then you always want to cut across the sand when you are a distance $wb/\sqrt{v^2-w^2}$ from point C. If you start closer than this to C, you should cut directly across the sand. □

Summary—Steps to solve an optimization problem.

1. Decide what the variables are and what the constants are, draw a diagram if appropriate, understand clearly what it is that is to be maximized or minimized.
2. Write a formula for the function for which you wish to find the maximum or minimum.
3. Express that formula in terms of only one variable, that is, in the form $f(x)$.
4. Set $f'(x) = 0$ and solve. Check all critical values and endpoints to determine the extreme value.

Exercises 6.1.

1. Let $f(x) = \begin{cases} 1+4x-x^2 & \text{for } x \le 3 \\ (x+5)/2 & \text{for } x > 3 \end{cases}$

Find the maximum value and minimum values of $f(x)$ for x in $[0,4]$. Graph $f(x)$ to check your answers. ⇒

2. Find the dimensions of the rectangle of largest area having fixed perimeter 100. ⇒

3. Find the dimensions of the rectangle of largest area having fixed perimeter P. ⇒

4. A box with square base and no top is to hold a volume 100. Find the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base. ⇒

5. A box with square base is to hold a volume 200. The bottom and top are formed by folding in flaps from all four sides, so that the bottom and top consist of two layers of cardboard. Find the dimensions of the box that requires the least material. Also find the ratio of height to side of the base. ⇒

6. A box with square base and no top is to hold a volume V. Find (in terms of V) the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base. (This ratio will not involve V.) ⇒

7. You have 100 feet of fence to make a rectangular play area alongside the wall of your house. The wall of the house bounds one side. What is the largest size possible (in square feet) for the play area? ⇒

8. You have l feet of fence to make a rectangular play area alongside the wall of your house. The wall of the house bounds one side. What is the largest size possible (in square feet) for the play area? ⇒

9. Marketing tells you that if you set the price of an item at \$10 then you will be unable to sell it, but that you can sell 500 items for each dollar below \$10 that you set the price. Suppose your fixed costs total \$3000, and your marginal cost is \$2 per item. What is the most profit you can make? ⇒

10. Find the area of the largest rectangle that fits inside a semicircle of radius 10 (one side of the rectangle is along the diameter of the semicircle). ⇒

11. Find the area of the largest rectangle that fits inside a semicircle of radius r (one side of the rectangle is along the diameter of the semicircle). ⇒

12. For a cylinder with surface area 50, including the top and the bottom, find the ratio of height to base radius that maximizes the volume. ⇒

13. For a cylinder with given surface area S, including the top and the bottom, find the ratio of height to base radius that maximizes the volume. ⇒

14. You want to make cylindrical containers to hold 1 liter using the least amount of construction material. The side is made from a rectangular piece of material, and this can be done with no material wasted. However, the top and bottom are cut from squares of side $2r$, so that $2(2r)^2 = 8r^2$ of material is needed (rather than $2\pi r^2$, which is the total area of the top and bottom). Find the dimensions of the container using the least amount of material, and also find the ratio of height to radius for this container. ⇒

15. You want to make cylindrical containers of a given volume V using the least amount of construction material. The side is made from a rectangular piece of material, and this can be done with no material wasted. However, the top and bottom are cut from squares of side $2r$, so that $2(2r)^2 = 8r^2$ of material is needed (rather than $2\pi r^2$, which is the total area of the top and bottom). Find the optimal ratio of height to radius. ⇒

16. Given a right circular cone, you put an upside-down cone inside it so that its vertex is at the center of the base of the larger cone and its base is parallel to the base of the larger cone. If you choose the upside-down cone to have the largest possible volume, what fraction of the volume of the larger cone does it occupy? (Let H and R be the height and base radius of the larger cone, and let h and r be the height and base radius of the smaller cone. Hint: Use similar triangles to get an equation relating h and r.) ⇒

17. In example 6.1.12, what happens if $w \geq v$ (i.e., your speed on sand is at least your speed on the road)? ⇒

18. A container holding a fixed volume is being made in the shape of a cylinder with a hemispherical top. (The hemispherical top has the same radius as the cylinder.) Find the ratio of height to radius of the cylinder which minimizes the cost of the container if (a) the cost per unit area of the top is twice as great as the cost per unit area of the side, and the container is made with no bottom; (b) the same as in (a), except that the container is made with a circular bottom, for which the cost per unit area is 1.5 times the cost per unit area of the side. ⇒

19. A piece of cardboard is 1 meter by 1/2 meter. A square is to be cut from each corner and the sides folded up to make an open-top box. What are the dimensions of the box with maximum possible volume? ⇒

20. (a) A square piece of cardboard of side a is used to make an open-top box by cutting out a small square from each corner and bending up the sides. How large a square should be cut from each corner in order that the box have maximum volume? (b) What if the piece of cardboard used to make the box is a rectangle of sides a and b? ⇒

21. A window consists of a rectangular piece of clear glass with a semicircular piece of colored glass on top; the colored glass transmits only 1/2 as much light per unit area as the the clear glass. If the distance from top to bottom (across both the rectangle and the semicircle) is 2 meters and the window may be no more than 1.5 meters wide, find the dimensions of the rectangular portion of the window that lets through the most light. ⇒

22. A window consists of a rectangular piece of clear glass with a semicircular piece of colored glass on top. Suppose that the colored glass transmits only k times as much light per unit area as the clear glass (k is between 0 and 1). If the distance from top to bottom (across both the rectangle and the semicircle) is a fixed distance H, find (in terms of k) the ratio of vertical side to horizontal side of the rectangle for which the window lets through the most light. ⇒

23. You are designing a poster to contain a fixed amount A of printing (measured in square centimeters) and have margins of a centimeters at the top and bottom and b centimeters at the sides. Find the ratio of vertical dimension to horizontal dimension of the printed area on the poster if you want to minimize the amount of posterboard needed. ⇒

24. The strength of a rectangular beam is proportional to the product of its width w times the square of its depth d. Find the dimensions of the strongest beam that can be cut from a cylindrical log of radius r. ⇒

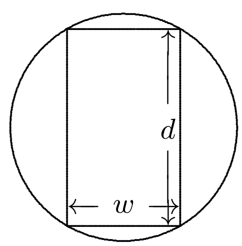

Figure 6.1.6 Cutting a beam.

25. What fraction of the volume of a sphere is taken up by the largest cylinder that can be fit inside the sphere? ⇒

26. The U.S. post office will accept a box for shipment only if the sum of the length and girth (distance around) is at most 108 in. Find the dimensions of the largest acceptable box with square front and back. ⇒

27. Find the dimensions of the lightest cylindrical can containing 0.25 liter (=250 cm^3) if the top and bottom are made of a material that is twice as heavy (per unit area) as the material used for the side. ⇒

28. A conical paper cup is to hold 1/4 of a liter. Find the height and radius of the cone which minimizes the amount of paper needed to make the cup. Use the formula $\pi r\sqrt{r^2+h^2}$ for the area of the side of a cone. ⇒

29. A conical paper cup is to hold a fixed volume of water. Find the ratio of height to base radius of the cone which minimizes the amount of paper needed to make the cup. Use the formula $\pi r\sqrt{r^2+h^2}$ for the area of the side of a cone, called the **lateral area** of the cone. ⇒

30. If you fit the cone with the largest possible surface area (lateral area plus area of base) into a sphere, what percent of the volume of the sphere is occupied by the cone? ⇒

31. Two electrical charges, one a positive charge A of magnitude a and the other a negative charge B of magnitude b, are located a distance c apart. A positively charged particle P is situated on the line between A and B. Find where P should be put so that the pull away from A towards B is minimal. Here assume that the force from each charge is proportional to the strength of the source and inversely proportional to the square of the distance from the source. ⇒

32. Find the fraction of the area of a triangle that is occupied by the largest rectangle that can be drawn in the triangle (with one of its sides along a side of the triangle). Show that this fraction does not depend on the dimensions of the given triangle. ⇒

33. How are your answers to Problem 9 affected if the cost per item for the x items, instead of being simply \$2, decreases below \$2 in proportion to x (because of economy of scale and volume discounts) by 1 cent for each 25 items produced? ⇒

34. You are standing near the side of a large wading pool of uniform depth when you see a child in trouble. You can run at a speed v_1 on land and at a slower speed v_2 in the water. Your perpendicular distance from the side of the pool is a, the child's perpendicular distance is b, and the distance along the side of the pool between the closest point to you and the closest point to the child is c (see the figure below). Without stopping to do any calculus, you instinctively choose the quickest route (shown in the figure) and save the child. Our purpose is to derive a relation between the angle θ_1 your path makes with the perpendicular to the side of the pool when you're on land, and the angle θ_2 your path makes with the perpendicular when you're in the water. To do this, let x be the distance between the closest point to you at the side of the pool and the point where you enter the water. Write the total time you run (on land and in the water) in terms of x (and also the constants a, b, c, v_1, v_2). Then set the derivative equal to zero. The result, called "Snell's law" or the "law of refraction," also governs the bending of light when it goes into water. ⇒

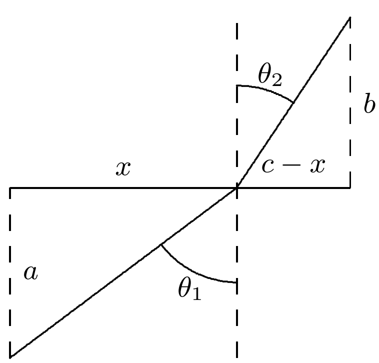

Figure 6.1.7 Wading pool rescue.

6.2 Related Rates

Suppose we have two variables x and y (in most problems the letters will be different, but for now let's use x and y) which are both changing with time. A "related rates" problem is a problem in which we know one of the rates of change at a given instant—say,

$\dot{x} = dx/dt$—and we want to find the other rate $\dot{y} = dy/dt$ at that instant. (The use of $\dot{x}$ to mean dx/dt goes back to Newton and is still used for this purpose, especially by physicists.)

If y is written in terms of x, i.e., $y = f(x)$, then this is easy to do using the chain rule:

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx}\dot{x}.$$

That is, find the derivative of $f(x)$, plug in the value of x at the instant in question, and multiply by the given value of $\dot{x} = dx/dt$ to get $\dot{y} = dy/dt$.

EXAMPLE 6.2.1 Suppose an object is moving along a path described by $y = x^2$, that is, it is moving on a parabolic path. At a particular time, say $t = 5$, the x coordinate is 6 and we measure the speed at which the x coordinate of the object is changing and find that $dx/dt = 3$. At the same time, how fast is the y coordinate changing?

Using the chain rule, $dy/dt = 2x \cdot dx/dt$. At $t = 5$ we know that $x = 6$ and $dx/dt = 3$, so $dy/dt = 2 \cdot 6 \cdot 3 = 36$. □

In many cases, particularly interesting ones, x and y will be related in some other way, for example $x = f(y)$, or $F(x, y) = k$, or perhaps $F(x, y) = G(x, y)$, where $F(x, y)$ and $G(x, y)$ are expressions involving both variables. In all cases, you can solve the related rates problem by taking the derivative of both sides, plugging in all the known values (namely, x, y, and $\dot{x}$), and then solving for $\dot{y}$.

To summarize, here are the steps in doing a related rates problem:

1. Decide what the two variables are.

2. Find an equation relating them.

3. Take d/dt of both sides.

4. Plug in all known values at the instant in question.

5. Solve for the unknown rate.

EXAMPLE 6.2.2 A plane is flying directly away from you at 500 mph at an altitude of 3 miles. How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

To see what's going on, we first draw a schematic representation of the situation, as in figure 6.2.1.

Because the plane is in level flight directly away from you, the rate at which x changes is the speed of the plane, $dx/dt = 500$. The distance between you and the plane is y; it is dy/dt that we wish to know. By the Pythagorean Theorem we know that $x^2 + 9 = y^2$.

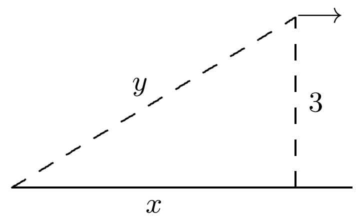

Figure 6.2.1 Receding airplane.

Taking the derivative:

$$2x\dot{x} = 2y\dot{y}.$$

We are interested in the time at which $x = 4$; at this time we know that $4^2 + 9 = y^2$, so $y = 5$. Putting together all the information we get

$$2(4)(500) = 2(5)\dot{y}.$$

Thus, $\dot{y} = 400$ mph. □

EXAMPLE 6.2.3 You are inflating a spherical balloon at the rate of 7 cm^3/sec. How fast is its radius increasing when the radius is 4 cm?

Here the variables are the radius r and the volume V. We know dV/dt, and we want dr/dt. The two variables are related by means of the equation $V = 4\pi r^3/3$. Taking the derivative of both sides gives $dV/dt = 4\pi r^2\dot{r}$. We now substitute the values we know at the instant in question: $7 = 4\pi 4^2\dot{r}$, so $\dot{r} = 7/(64\pi)$ cm/sec. □

EXAMPLE 6.2.4 Water is poured into a conical container at the rate of 10 cm^3/sec. The cone points directly down, and it has a height of 30 cm and a base radius of 10 cm; see figure 6.2.2. How fast is the water level rising when the water is 4 cm deep (at its deepest point)?

The water forms a conical shape within the big cone; its height and base radius and volume are all increasing as water is poured into the container. This means that we actually have three things varying with time: the water level h (the height of the cone of water), the radius r of the circular top surface of water (the base radius of the cone of water), and the volume of water V. The volume of a cone is given by $V = \pi r^2 h/3$. We know dV/dt, and we want dh/dt. At first something seems to be wrong: we have a third variable r whose rate we don't know.

But the dimensions of the cone of water must have the same proportions as those of the container. That is, because of similar triangles, $r/h = 10/30$ so $r = h/3$. Now we can eliminate r from the problem entirely: $V = \pi(h/3)^2 h/3 = \pi h^3/27$. We take the derivative of both sides and plug in $h = 4$ and $dV/dt = 10$, obtaining $10 = (3\pi \cdot 4^2/27)(dh/dt)$. Thus, $dh/dt = 90/(16\pi)$ cm/sec. □

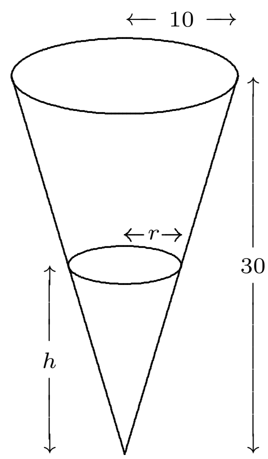

Figure 6.2.2 Conical water tank.

EXAMPLE 6.2.5 A swing consists of a board at the end of a 10 ft long rope. Think of the board as a point P at the end of the rope, and let Q be the point of attachment at the other end. Suppose that the swing is directly below Q at time $t = 0$, and is being pushed by someone who walks at 6 ft/sec from left to right. Find (a) how fast the swing is rising after 1 sec; (b) the angular speed of the rope in deg/sec after 1 sec.

We start out by asking: What is the geometric quantity whose rate of change we know, and what is the geometric quantity whose rate of change we're being asked about? Note that the person pushing the swing is moving horizontally at a rate we know. In other words, the horizontal coordinate of P is increasing at 6 ft/sec. In the xy-plane let us make the convenient choice of putting the origin at the location of P at time $t = 0$, i.e., a distance 10 directly below the point of attachment. Then the rate we know is dx/dt, and in part (a) the rate we want is dy/dt (the rate at which P is rising). In part (b) the rate we want is $\dot{\theta} = d\theta/dt$, where θ stands for the angle in radians through which the swing has swung from the vertical. (Actually, since we want our answer in deg/sec, at the end we must convert $d\theta/dt$ from rad/sec by multiplying by $180/\pi$.)

(a) From the diagram we see that we have a right triangle whose legs are x and $10 - y$, and whose hypotenuse is 10. Hence $x^2 + (10 - y)^2 = 100$. Taking the derivative of both sides we obtain: $2x\dot{x} + 2(10 - y)(0 - \dot{y}) = 0$. We now look at what we know after 1 second, namely $x = 6$ (because x started at 0 and has been increasing at the rate of 6 ft/sec for 1 sec), $y = 2$ (because we get $10 - y = 8$ from the Pythagorean theorem applied to the triangle with hypotenuse 10 and leg 6), and $\dot{x} = 6$. Putting in these values gives us $2 \cdot 6 \cdot 6 - 2 \cdot 8\dot{y} = 0$, from which we can easily solve for $\dot{y}$: $\dot{y} = 4.5$ ft/sec.

(b) Here our two variables are x and θ, so we want to use the same right triangle as in part (a), but this time relate θ to x. Since the hypotenuse is constant (equal to 10), the best way to do this is to use the sine: $\sin\theta = x/10$. Taking derivatives we obtain

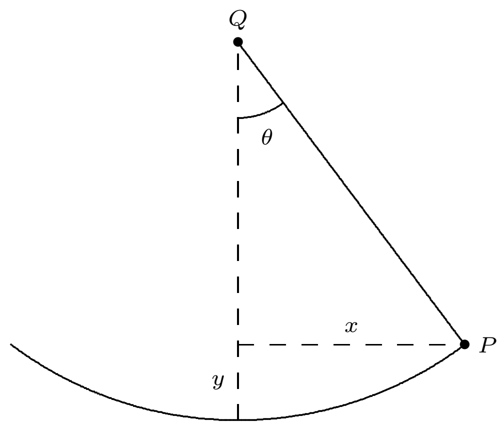

Figure 6.2.3 Swing.

$(\cos\theta)\dot\theta = 0.1\dot x$. At the instant in question ($t = 1$ sec), when we have a right triangle with sides 6–8–10, $\cos\theta = 8/10$ and $\dot x = 6$. Thus $(8/10)\dot\theta = 6/10$, i.e., $\dot\theta = 6/8 = 3/4$ rad/sec, or approximately 43 deg/sec. □

We have seen that sometimes there are apparently more than two variables that change with time, but in reality there are just two, as the others can be expressed in terms of just two. But sometimes there really are several variables that change with time; as long as you know the rates of change of all but one of them you can find the rate of change of the remaining one. As in the case when there are just two variables, take the derivative of both sides of the equation relating all of the variables, and then substitute all of the known values and solve for the unknown rate.

EXAMPLE 6.2.6 A road running north to south crosses a road going east to west at the point P. Car A is driving north along the first road, and car B is driving east along the second road. At a particular time car A is 10 kilometers to the north of P and traveling at 80 km/hr, while car B is 15 kilometers to the east of P and traveling at 100 km/hr. How fast is the distance between the two cars changing?

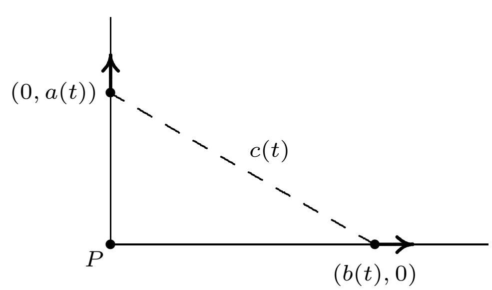

Figure 6.2.4 Cars moving apart.

Let $a(t)$ be the distance of car A north of P at time t, and $b(t)$ the distance of car B east of P at time t, and let $c(t)$ be the distance from car A to car B at time t. By the Pythagorean Theorem, $c(t)^2 = a(t)^2 + b(t)^2$. Taking derivatives we get $2c(t)c'(t) = 2a(t)a'(t) + 2b(t)b'(t)$, so

$$\dot{c} = \frac{a\dot{a} + b\dot{b}}{c} = \frac{a\dot{a} + b\dot{b}}{\sqrt{a^2 + b^2}}.$$

Substituting known values we get:

$$\dot{c} = \frac{10 \cdot 80 + 15 \cdot 100}{\sqrt{10^2 + 15^2}} = \frac{460}{\sqrt{13}} \approx 127.6\text{km/hr}$$

at the time of interest. □

Notice how this problem differs from example 6.2.2. In both cases we started with the Pythagorean Theorem and took derivatives on both sides. However, in example 6.2.2 one of the sides was a constant (the altitude of the plane), and so the derivative of the square of that side of the triangle was simply zero. In this example, on the other hand, all three sides of the right triangle are variables, even though we are interested in a specific value of each side of the triangle (namely, when the sides have lengths 10 and 15). Make sure that you understand at the start of the problem what are the variables and what are the constants.

Exercises 6.2.

1. A cylindrical tank standing upright (with one circular base on the ground) has radius 20 cm. How fast does the water level in the tank drop when the water is being drained at 25 cm^3/sec? ⇒
2. A cylindrical tank standing upright (with one circular base on the ground) has radius 1 meter. How fast does the water level in the tank drop when the water is being drained at 3 liters per second? ⇒
3. A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.6 m/sec. How fast is the top sliding down the wall when the foot of the ladder is 5 m from the wall? ⇒
4. A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The top of the ladder is being pulled up the wall at 0.1 meters per second. How fast is the foot of the ladder approaching the wall when the foot of the ladder is 5 m from the wall? ⇒
5. A rotating beacon is located 2 miles out in the water. Let A be the point on the shore that is closest to the beacon. As the beacon rotates at 10 rev/min, the beam of light sweeps down the shore once each time it revolves. Assume that the shore is straight. How fast is the point where the beam hits the shore moving at an instant when the beam is lighting up a point 2 miles along the shore from the point A? ⇒
6. A baseball diamond is a square 90 ft on a side. A player runs from first base to second base at 15 ft/sec. At what rate is the player's distance from third base decreasing when she is half way from first to second base? ⇒

7. Sand is poured onto a surface at 15 cm^3/sec, forming a conical pile whose base diameter is always equal to its altitude. How fast is the altitude of the pile increasing when the pile is 3 cm high? ⇒

8. A boat is pulled in to a dock by a rope with one end attached to the front of the boat and the other end passing through a ring attached to the dock at a point 5 ft higher than the front of the boat. The rope is being pulled through the ring at the rate of 0.6 ft/sec. How fast is the boat approaching the dock when 13 ft of rope are out? ⇒

9. A balloon is at a height of 50 meters, and is rising at the constant rate of 5 m/sec. A bicyclist passes beneath it, traveling in a straight line at the constant speed of 10 m/sec. How fast is the distance between the bicyclist and the balloon increasing 2 seconds later? ⇒

10. A pyramid-shaped vat has square cross-section and stands on its tip. The dimensions at the top are 2 m × 2 m, and the depth is 5 m. If water is flowing into the vat at 3 m^3/min, how fast is the water level rising when the depth of water (at the deepest point) is 4 m? Note: the volume of any "conical" shape (including pyramids) is (1/3)(height)(area of base). ⇒

11. The sun is rising at the rate of 1/4 deg/min, and appears to be climbing into the sky perpendicular to the horizon, as depicted in figure 6.2.5. How fast is the shadow of a 200 meter building shrinking at the moment when the shadow is 500 meters long? ⇒

12. The sun is setting at the rate of 1/4 deg/min, and appears to be dropping perpendicular to the horizon, as depicted in figure 6.2.5. How fast is the shadow of a 25 meter wall lengthening at the moment when the shadow is 50 meters long? ⇒

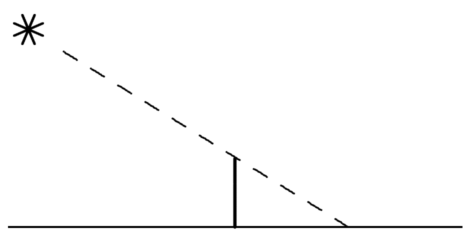

Figure 6.2.5 Sunrise or sunset.

13. The trough shown in figure 6.2.6 is constructed by fastening together three slabs of wood of dimensions 10 ft × 1 ft, and then attaching the construction to a wooden wall at each end. The angle θ was originally 30°, but because of poor construction the sides are collapsing. The trough is full of water. At what rate (in ft^3/sec) is the water spilling out over the top of the trough if the sides have each fallen to an angle of 45°, and are collapsing at the rate of 1° per second? ⇒

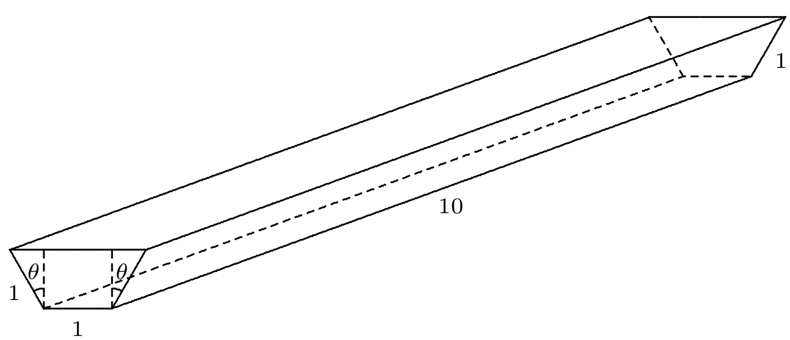

Figure 6.2.6 Trough.

14. A woman 5 ft tall walks at the rate of 3.5 ft/sec away from a streetlight that is 12 ft above the ground. At what rate is the tip of her shadow moving? At what rate is her shadow lengthening? ⇒

15. A man 1.8 meters tall walks at the rate of 1 meter per second toward a streetlight that is 4 meters above the ground. At what rate is the tip of his shadow moving? At what rate is his shadow shortening? ⇒

16. A police helicopter is flying at 150 mph at a constant altitude of 0.5 mile above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 1 mile from the helicopter, and that this distance is decreasing at 190 mph. Find the speed of the car. ⇒

17. A police helicopter is flying at 200 kilometers per hour at a constant altitude of 1 km above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 2 kilometers from the helicopter, and that this distance is decreasing at 250 kph. Find the speed of the car. ⇒

18. A light shines from the top of a pole 20 m high. A ball is falling 10 meters from the pole, casting a shadow on a building 30 meters away, as shown in figure 6.2.7. When the ball is 25 meters from the ground it is falling at 6 meters per second. How fast is its shadow moving? ⇒

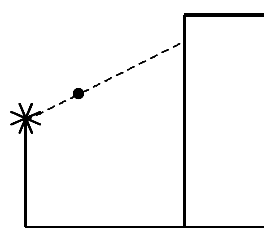

Figure 6.2.7 Falling ball.

19. Do example 6.2.6 assuming that the angle between the two roads is 120° instead of 90° (that is, the "north–south" road actually goes in a somewhat northwesterly direction from P). Recall the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\theta$. ⇒

20. Do example 6.2.6 assuming that car A is 300 meters north of P, car B is 400 meters east of P, both cars are going at constant speed toward P, and the two cars will collide in 10 seconds. ⇒

21. Do example 6.2.6 assuming that 8 seconds ago car A started from rest at P and has been picking up speed at the steady rate of 5 m/sec^2, and 6 seconds after car A started car B passed P moving east at constant speed 60 m/sec. ⇒

22. Referring again to example 6.2.6, suppose that instead of car B an airplane is flying at speed 200 km/hr to the east of P at an altitude of 2 km, as depicted in figure 6.2.8. How fast is the distance between car and airplane changing? ⇒

23. Referring again to example 6.2.6, suppose that instead of car B an airplane is flying at speed 200 km/hr to the east of P at an altitude of 2 km, and that it is gaining altitude at 10 km/hr. How fast is the distance between car and airplane changing? ⇒

24. A light shines from the top of a pole 20 m high. An object is dropped from the same height from a point 10 m away, so that its height at time t seconds is $h(t) = 20 - 9.8t^2/2$. How fast is the object's shadow moving on the ground one second later? ⇒

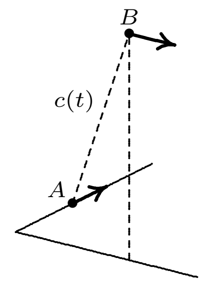

Figure 6.2.8 Car and airplane.

25. The two blades of a pair of scissors are fastened at the point A as shown in figure 6.2.9. Let a denote the distance from A to the tip of the blade (the point B). Let β denote the angle at the tip of the blade that is formed by the line $\overline{AB}$ and the bottom edge of the blade, line $\overline{BC}$, and let θ denote the angle between $\overline{AB}$ and the horizontal. Suppose that a piece of paper is cut in such a way that the center of the scissors at A is fixed, and the paper is also fixed. As the blades are closed (i.e., the angle θ in the diagram is decreased), the distance x between A and C increases, cutting the paper.

a. Express x in terms of a, θ, and β.

b. Express dx/dt in terms of a, θ, β, and $d\theta/dt$.

c. Suppose that the distance a is 20 cm, and the angle β is 5°. Further suppose that θ is decreasing at 50 deg/sec. At the instant when $\theta = 30°$, find the rate (in cm/sec) at which the paper is being cut. ⇒

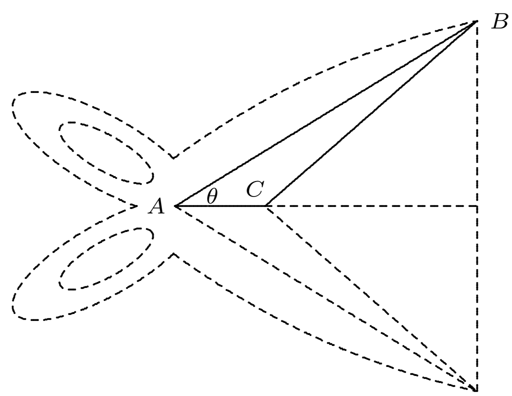

Figure 6.2.9 Scissors.

6.3 Newton's Method

Suppose you have a function $f(x)$, and you want to find as accurately as possible where it crosses the x-axis; in other words, you want to solve $f(x) = 0$. Suppose you know of no way to find an exact solution by any algebraic procedure, but you are able to use an approximation, provided it can be made quite close to the true value. Newton's method is a way to find a solution to the equation to as many decimal places as you want. It is what is called an "iterative procedure," meaning that it can be repeated again and again to get an answer of greater and greater accuracy. Iterative procedures like Newton's method are well suited to programming for a computer. Newton's method uses the fact that the tangent line to a curve is a good approximation to the curve near the point of tangency.

EXAMPLE 6.3.1 Approximate $\sqrt{3}$. Since $\sqrt{3}$ is a solution to $x^2 = 3$ or $x^2 - 3 = 0$, we use $f(x) = x^2 - 3$. We start by guessing something reasonably close to the true value; this is usually easy to do; let's use $\sqrt{3} \approx 2$. Now use the tangent line to the curve when $x = 2$ as an approximation to the curve, as shown in figure 6.3.1. Since $f'(x) = 2x$, the slope of this tangent line is 4 and its equation is $y = 4x - 7$. The tangent line is quite close to $f(x)$, so it crosses the x-axis near the point at which $f(x)$ crosses, that is, near $\sqrt{3}$. It is easy to find where the tangent line crosses the x-axis: solve $0 = 4x - 7$ to get $x = 7/4 = 1.75$. This is certainly a better approximation than 2, but let us say not close enough. We can improve it by doing the same thing again: find the tangent line at $x = 1.75$, find where this new tangent line crosses the x-axis, and use that value as a better approximation. We can continue this indefinitely, though it gets a bit tedious. Lets see if we can shortcut the process. Suppose the best approximation to the intercept we have so far is x_i. To find a better approximation we will always do the same thing: find the slope of the tangent line at x_i, find the equation of the tangent line, find the x-intercept. The slope is $2x_i$. The tangent line is $y = (2x_i)(x - x_i) + (x_i^2 - 3)$, using the point-slope formula for a line. Finally, the intercept is found by solving $0 = (2x_i)(x - x_i) + (x_i^2 - 3)$. With a little algebra this turns into $x = (x_i^2 + 3)/(2x_i)$; this is the next approximation, which we naturally call x_{i+1}. Instead of doing the whole tangent line computation every time we can simply use this formula to get as many approximations as we want. Starting with $x_0 = 2$, we get $x_1 = (x_0^2 + 3)/(2x_0) = (2^2 + 3)/4 = 7/4$ (the same approximation we got above, of course), $x_2 = (x_1^2 + 3)/(2x_1) = ((7/4)^2 + 3)/(7/2) = 97/56 \approx 1.73214$, $x_3 \approx 1.73205$, and so on. This is still a bit tedious by hand, but with a calculator or, even better, a good computer program, it is quite easy to get many, many approximations. We might guess already that 1.73205 is accurate to two decimal places, and in fact it turns out that it is accurate to 5 places. □

Let's think about this process in more general terms. We want to approximate a solution to $f(x) = 0$. We start with a rough guess, which we call x_0. We use the tangent line to $f(x)$ to get a new approximation that we hope will be closer to the true value. What is the equation of the tangent line when $x = x_0$? The slope is $f'(x_0)$ and the line goes through $(x_0, f(x_0))$, so the equation of the line is

$$y = f'(x_0)(x - x_0) + f(x_0).$$

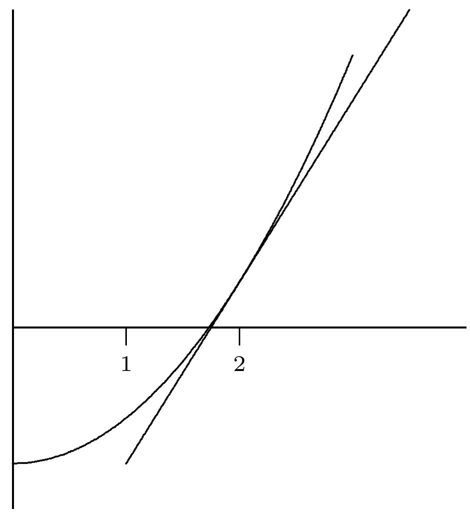

Figure 6.3.1 Newton's method. (AP)

Now we find where this crosses the x-axis by substituting $y = 0$ and solving for x:

$$x = \frac{x_0 f'(x_0) - f(x_0)}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

We will typically want to compute more than one of these improved approximations, so we number them consecutively; from x_0 we have computed x_1:

$$x_1 = \frac{x_0 f'(x_0) - f(x_0)}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)},$$

and in general from x_i we compute x_{i+1}:

$$x_{i+1} = \frac{x_i f'(x_i) - f(x_i)}{f'(x_i)} = x_i - \frac{f(x_i)}{f'(x_i)}.$$

EXAMPLE 6.3.2 Returning to the previous example, $f(x) = x^2 - 3$, $f'(x) = 2x$, and the formula becomes $x_{i+1} = x_i - (x_i^2 - 3)/(2x_i) = (x_i^2 + 3)/(2x_i)$, as before. □

In practice, which is to say, if you need to approximate a value in the course of designing a bridge or a building or an airframe, you will need to have some confidence that the approximation you settle on is accurate enough. As a rule of thumb, once a certain number of decimal places stop changing from one approximation to the next it is likely that those decimal places are correct. Still, this may not be enough assurance, in which case we can test the result for accuracy.

EXAMPLE 6.3.3 Find the x coordinate of the intersection of the curves $y = 2x$ and $y = \tan x$, accurate to three decimal places. To put this in the context of Newton's method, we note that we want to know where $2x = \tan x$ or $f(x) = \tan x - 2x = 0$. We compute $f'(x) = \sec^2 x - 2$ and set up the formula:

$$x_{i+1} = x_i - \frac{\tan x_i - 2x_i}{\sec^2 x_i - 2}.$$

From the graph in figure 6.3.2 we guess $x_0 = 1$ as a starting point, then using the formula we compute $x_1 = 1.310478030$, $x_2 = 1.223929096$, $x_3 = 1.176050900$, $x_4 = 1.165926508$, $x_5 = 1.165561636$. So we guess that the first three places are correct, but that is not the same as saying 1.165 is correct to three decimal places—1.166 might be the correct, rounded approximation. How can we tell? We can substitute 1.165, 1.1655 and 1.166 into $\tan x - 2x$; this gives -0.002483652, -0.000271247, 0.001948654. Since the first two are negative and the third is positive, $\tan x - 2x$ crosses the x axis between 1.1655 and 1.166, so the correct value to three places is 1.166. □

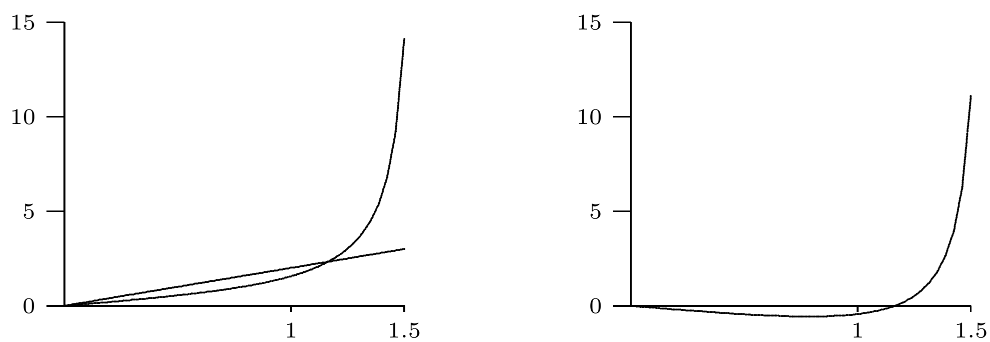

Figure 6.3.2 $y = \tan x$ and $y = 2x$ on the left, $y = \tan x - 2x$ on the right.

Exercises 6.3.

1. Approximate the fifth root of 7, using $x_0 = 1.5$ as a first guess. Use Newton's method to find x_3 as your approximation. ⇒
2. Use Newton's Method to approximate the cube root of 10 to two decimal places. ⇒
3. The function $f(x) = x^3 - 3x^2 - 3x + 6$ has a root between 3 and 4, because $f(3) = -3$ and $f(4) = 10$. Approximate the root to two decimal places. ⇒
4. A rectangular piece of cardboard of dimensions 8×17 is used to make an open-top box by cutting out a small square of side x from each corner and bending up the sides. (See exercise 20 in 6.1.) If $x = 2$, then the volume of the box is $2 \cdot 4 \cdot 13 = 104$. Use Newton's method to find a value of x for which the box has volume 100, accurate to 3 significant figures. ⇒

6.4 Linear Approximations

Newton's method is one example of the usefulness of the tangent line as an approximation to a curve. Here we explore another such application.

Recall that the tangent line to $f(x)$ at a point $x = a$ is given by $L(x) = f'(a)(x-a) + f(a)$. The tangent line in this context is also called the **linear approximation** to f at a.

If f is differentiable at a then L is a good approximation of f so long as x is "not too far" from a. Put another way, if f is differentiable at a then under a microscope f will look very much like a straight line. Figure 6.4.1 shows a tangent line to $y = x^2$ at three different magnifications.

If we want to approximate $f(b)$, because computing it exactly is difficult, we can approximate the value using a linear approximation, provided that we can compute the tangent line at some a close to b.

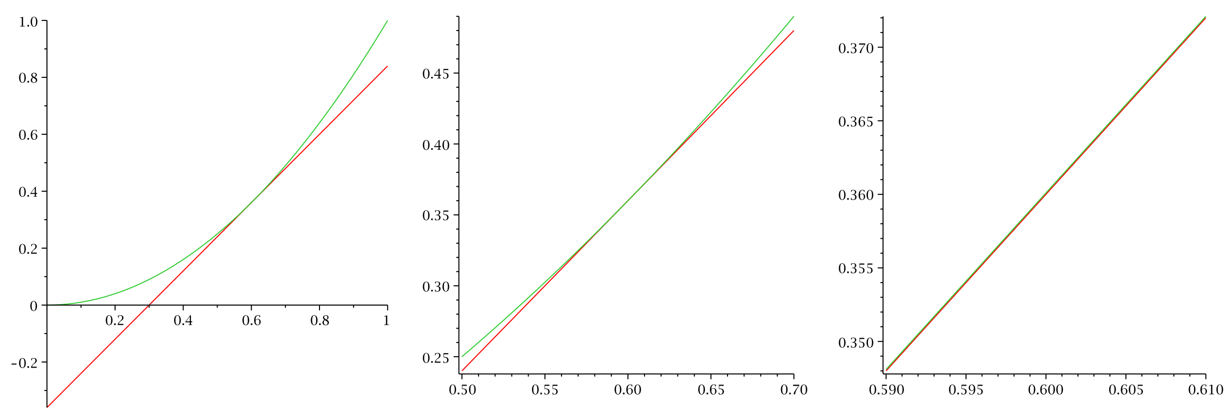

Figure 6.4.1 The linear approximation to $y = x^2$.

EXAMPLE 6.4.1 Let $f(x) = \sqrt{x+4}$. Then $f'(x) = 1/(2\sqrt{x+4})$. The linear approximation to f at $x = 5$ is $L(x) = 1/(2\sqrt{5+4})(x-5) + \sqrt{5+4} = (x-5)/6 + 3$. As an immediate application we can approximate square roots of numbers near 9 by hand. To estimate $\sqrt{10}$, we substitute 6 into the linear approximation instead of into $f(x)$, so $\sqrt{6+4} \approx (6-5)/6 + 3 = 19/6 \approx 3.1\overline{6}$. This rounds to 3.17 while the square root of 10 is actually 3.16 to two decimal places, so this estimate is only accurate to one decimal place. This is not too surprising, as 10 is really not very close to 9; on the other hand, for many calculations, 3.2 would be accurate enough. □

With modern calculators and computing software it may not appear necessary to use linear approximations. But in fact they are quite useful. In cases requiring an explicit numerical approximation, they allow us to get a quick rough estimate which can be used as a "reality check" on a more complex calculation. In some complex calculations involving functions, the linear approximation makes an otherwise intractable calculation possible, without serious loss of accuracy.

EXAMPLE 6.4.2 Consider the trigonometric function $\sin x$. Its linear approximation at $x = 0$ is simply $L(x) = x$. When x is small this is quite a good approximation and is used frequently by engineers and scientists to simplify some calculations. □

DEFINITION 6.4.3 Let $y = f(x)$ be a differentiable function. We define a new independent variable dx, and a new dependent variable $dy = f'(x)\,dx$. Notice that dy is a function both of x (since $f'(x)$ is a function of x) and of dx. We say that dx and dy are **differentials.** □

Let $\Delta x = x - a$ and $\Delta y = f(x) - f(a)$. If x is near a then Δx is small. If we set $dx = \Delta x$ then

$$dy = f'(a)\,dx \approx \frac{\Delta y}{\Delta x}\Delta x = \Delta y.$$

Thus, dy can be used to approximate Δy, the actual change in the function f between a and x. This is exactly the approximation given by the tangent line:

$$dy = f'(a)(x - a) = f'(a)(x - a) + f(a) - f(a) = L(x) - f(a).$$

While $L(x)$ approximates $f(x)$, dy approximates how $f(x)$ has changed from $f(a)$. Figure 6.4.2 illustrates the relationships.

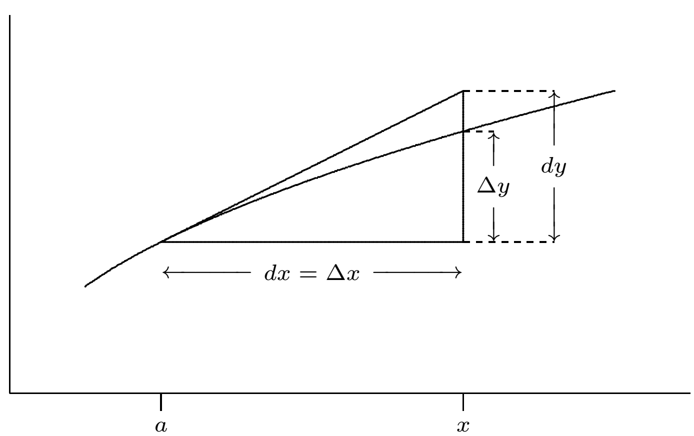

Figure 6.4.2 Differentials.

Exercises 6.4.

1. Let $f(x) = x^4$. If $a = 1$ and $dx = \Delta x = 1/2$, what are Δy and dy? $\Rightarrow$
2. Let $f(x) = \sqrt{x}$. If $a = 1$ and $dx = \Delta x = 1/10$, what are Δy and dy? $\Rightarrow$
3. Let $f(x) = \sin(2x)$. If $a = \pi$ and $dx = \Delta x = \pi/100$, what are Δy and dy? $\Rightarrow$
4. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.02 cm thick to a sphere with diameter 40 meters. (Recall that the volume of a sphere of radius r is $V = (4/3)\pi r^3$. Notice that you are given that $dr = 0.02$.) $\Rightarrow$
5. Show in detail that the linear approximation of $\sin x$ at $x = 0$ is $L(x) = x$ and the linear approximation of $\cos x$ at $x = 0$ is $L(x) = 1$.

6.5 The Mean Value Theorem

Here are two interesting questions involving derivatives:

1. Suppose two different functions have the same derivative; what can you say about the relationship between the two functions?
2. Suppose you drive a car from toll booth on a toll road to another toll booth at an average speed of 70 miles per hour. What can be concluded about your actual speed during the trip? In particular, did you exceed the 65 mile per hour speed limit?

While these sound very different, it turns out that the two problems are very closely related. We know that "speed" is really the derivative by a different name; let's start by translating the second question into something that may be easier to visualize. Suppose that the function $f(t)$ gives the position of your car on the toll road at time t. Your change in position between one toll booth and the next is given by $f(t_1) - f(t_0)$, assuming that at time t_0 you were at the first booth and at time t_1 you arrived at the second booth. Your average speed for the trip is $(f(t_1) - f(t_0))/(t_1 - t_0)$. If we think about the graph of $f(t)$, the average speed is the slope of the line that connects the two points $(t_0, f(t_0))$ and $(t_1, f(t_1))$. Your speed at any particular time t between t_0 and t_1 is $f'(t)$, the slope of the curve. Now question (2) becomes a question about slope. In particular, if the slope between endpoints is 70, what can be said of the slopes at points between the endpoints?

As a general rule, when faced with a new problem it is often a good idea to examine one or more simplified versions of the problem, in the hope that this will lead to an understanding of the original problem. In this case, the problem in its "slope" form is somewhat easier to simplify than the original, but equivalent, problem.

Here is a special instance of the problem. Suppose that $f(t_0) = f(t_1)$. Then the two endpoints have the same height and the slope of the line connecting the endpoints is zero. What can we say about the slope between the endpoints? It shouldn't take much experimentation before you are convinced of the truth of this statement: Somewhere between t_0 and t_1 the slope is exactly zero, that is, somewhere between t_0 and t_1 the slope is equal to the slope of the line between the endpoints. This suggests that perhaps the same is true even if the endpoints are at different heights, and again a bit of experimentation will probably convince you that this is so. But we can do better than "experimentation"—we can prove that this is so.

We start with the simplified version:

THEOREM 6.5.1 Rolle's Theorem Suppose that $f(x)$ has a derivative on the interval (a, b), is continuous on the interval $[a, b]$, and $f(a) = f(b)$. Then at some value $c \in (a, b)$, $f'(c) = 0$.

Proof. We know that $f(x)$ has a maximum and minimum value on $[a, b]$ (because it is continuous), and we also know that the maximum and minimum must occur at an endpoint, at a point at which the derivative is zero, or at a point where the derivative is undefined. Since the derivative is never undefined, that possibility is removed.

If the maximum or minimum occurs at a point c, other than an endpoint, where $f'(c) = 0$, then we have found the point we seek. Otherwise, the maximum and minimum both occur at an endpoint, and since the endpoints have the same height, the maximum and minimum are the same. This means that $f(x) = f(a) = f(b)$ at every $x \in [a, b]$, so the function is a horizontal line, and it has derivative zero everywhere in (a, b). Then we may choose any c at all to get $f'(c) = 0$. ■

Perhaps remarkably, this special case is all we need to prove the more general one as well.

THEOREM 6.5.2 Mean Value Theorem Suppose that $f(x)$ has a derivative on the interval (a, b) and is continuous on the interval $[a, b]$. Then at some value $c \in (a, b)$, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Proof. Let $m = \dfrac{f(b) - f(a)}{b - a}$, and consider a new function $g(x) = f(x) - m(x - a) - f(a)$. We know that $g(x)$ has a derivative everywhere, since $g'(x) = f'(x) - m$. We can compute $g(a) = f(a) - m(a - a) - f(a) = 0$ and

$$\begin{aligned} g(b) = f(b) - m(b - a) - f(a) &= f(b) - \frac{f(b) - f(a)}{b - a}(b - a) - f(a) \\ &= f(b) - (f(b) - f(a)) - f(a) = 0. \end{aligned}$$

So the height of $g(x)$ is the same at both endpoints. This means, by Rolle's Theorem, that at some c, $g'(c) = 0$. But we know that $g'(c) = f'(c) - m$, so

$$0 = f'(c) - m = f'(c) - \frac{f(b) - f(a)}{b - a},$$

which turns into

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

exactly what we want. ∎

Returning to the original formulation of question (2), we see that if $f(t)$ gives the position of your car at time t, then the Mean Value Theorem says that at some time c, $f'(c) = 70$, that is, at some time you must have been traveling at exactly your average speed for the trip, and that indeed you exceeded the speed limit.

Now let's return to question (1). Suppose, for example, that two functions are known to have derivative equal to 5 everywhere, $f'(x) = g'(x) = 5$. It is easy to find such functions: $5x$, $5x + 47$, $5x - 132$, etc. Are there other, more complicated, examples? No—the only functions that work are the "obvious" ones, namely, $5x$ plus some constant. How can we see that this is true?

Although "5" is a very simple derivative, let's look at an even simpler one. Suppose that $f'(x) = g'(x) = 0$. Again we can find examples: $f(x) = 0$, $f(x) = 47$, $f(x) = -511$ all have $f'(x) = 0$. Are there non-constant functions f with derivative 0? No, and here's why: Suppose that $f(x)$ is not a constant function. This means that there are two points on the function with different heights, say $f(a) \neq f(b)$. The Mean Value Theorem tells us that at some point c, $f'(c) = (f(b) - f(a))/(b - a) \neq 0$. So any non-constant function does not have a derivative that is zero everywhere; this is the same as saying that the only functions with zero derivative are the constant functions.

Let's go back to the slightly less easy example: suppose that $f'(x) = g'(x) = 5$. Then $(f(x) - g(x))' = f'(x) - g'(x) = 5 - 5 = 0$. So using what we discovered in the previous paragraph, we know that $f(x) - g(x) = k$, for some constant k. So any two functions with derivative 5 must differ by a constant; since $5x$ is known to work, the only other examples must look like $5x + k$.

Now we can extend this to more complicated functions, without any extra work. Suppose that $f'(x) = g'(x)$. Then as before $(f(x) - g(x))' = f'(x) - g'(x) = 0$, so $f(x) - g(x) = k$. Again this means that if we find just a single function $g(x)$ with a certain derivative, then every other function with the same derivative must be of the form $g(x) + k$.

EXAMPLE 6.5.3 Describe all functions that have derivative $5x - 3$. It's easy to find one: $g(x) = (5/2)x^2 - 3x$ has $g'(x) = 5x - 3$. The only other functions with the same derivative are therefore of the form $f(x) = (5/2)x^2 - 3x + k$.

Alternately, though not obviously, you might have first noticed that $g(x) = (5/2)x^2 - 3x + 47$ has $g'(x) = 5x - 3$. Then every other function with the same derivative must have the form $f(x) = (5/2)x^2 - 3x + 47 + k$. This looks different, but it really isn't. The functions of the form $f(x) = (5/2)x^2 - 3x + k$ are exactly the same as the ones of the form $f(x) = (5/2)x^2 - 3x + 47 + k$. For example, $(5/2)x^2 - 3x + 10$ is the same as $(5/2)x^2 - 3x + 47 + (-37)$, and the first is of the first form while the second has the second form. □

This is worth calling a theorem:

THEOREM 6.5.4 If $f'(x) = g'(x)$ for every $x \in (a, b)$, then for some constant k, $f(x) = g(x) + k$ on the interval (a, b). ■

EXAMPLE 6.5.5 Describe all functions with derivative $\sin x + e^x$. One such function is $-\cos x + e^x$, so all such functions have the form $-\cos x + e^x + k$. □

Exercises 6.5.

1. Let $f(x) = x^2$. Find a value $c \in (-1, 2)$ so that $f'(c)$ equals the slope between the endpoints of $f(x)$ on $[-1, 2]$. ⇒
2. Verify that $f(x) = x/(x + 2)$ satisfies the hypotheses of the Mean Value Theorem on the interval $[1, 4]$ and then find all of the values, c, that satisfy the conclusion of the theorem. ⇒
3. Verify that $f(x) = 3x/(x + 7)$ satisfies the hypotheses of the Mean Value Theorem on the interval $[-2, 6]$ and then find all of the values, c, that satisfy the conclusion of the theorem.
4. Let $f(x) = \tan x$. Show that $f(\pi) = f(2\pi) = 0$ but there is no number $c \in (\pi, 2\pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's theorem?
5. Let $f(x) = (x - 3)^{-2}$. Show that there is no value $c \in (1, 4)$ such that $f'(c) = (f(4) - f(1))/(4 - 1)$. Why is this not a contradiction of the Mean Value Theorem?
6. Describe all functions with derivative $x^2 + 47x - 5$. ⇒
7. Describe all functions with derivative $\dfrac{1}{1 + x^2}$. ⇒
8. Describe all functions with derivative $x^3 - \dfrac{1}{x}$. ⇒
9. Describe all functions with derivative $\sin(2x)$. ⇒
10. Show that the equation $6x^4 - 7x + 1 = 0$ does not have more than two distinct real roots.
11. Let f be differentiable on $\mathbb{R}$. Suppose that $f'(x) \neq 0$ for every x. Prove that f has at most one real root.
12. Prove that for all real x and y $|\cos x - \cos y| \leq |x - y|$. State and prove an analogous result involving sine.
13. Show that $\sqrt{1 + x} \leq 1 + (x/2)$ if $-1 < x < 1$.