8

Techniques of Integration

Over the next few sections we examine some techniques that are frequently successful when seeking antiderivatives of functions. Sometimes this is a simple problem, since it will be apparent that the function you wish to integrate is a derivative in some straightforward way. For example, faced with

\[ \int x^3 dx \]

we realize immediately that the derivative of \( x^4 \) will supply an \( x^3 \) (\( x^4' = 4x^3 \)). We don’t want the “11”, but constants are easy to alter, because differentiation “ignores” them in certain circumstances, so

\[ \frac{d}{dx} \left( \frac{1}{11} x^{11} \right) = \frac{1}{11} x^{11} \cdot 1 = x^3. \]

From our knowledge of derivatives, we can immediately write down a number of antiderivatives. Here is a list of those most often used:

\[
\begin{align*}
\int x^n \, dx &= \frac{x^{n+1}}{n+1} + C, & \text{if } n \neq -1 \\
\int x^{-1} \, dx &= \ln |x| + C \\
\int e^x \, dx &= e^x + C \\
\int \sin x \, dx &= -\cos x + C
\end{align*}
\]

8.1 Substitution

Needless to say, most problems we encounter will not be so simple. Here’s a slightly more complicated example: find

\[ \int 2x \cos(x^2) \, dx. \]

This is not a “simple” derivative, but a little thought reveals that it must have come from an application of the chain rule. Multiplied on the “outside” is \( 2x \), which is the derivative of the “inside” function \( x^2 \). Checking:

\[ \frac{d}{dx} \sin(x^2) \neq \cos(x^2) \cdot 2x. \]

so

\[ \int 2x \cos(x^2) \, dx = \sin(x^2) + C. \]

Even when the chain rule has “produced” a certain derivative, it is not always easy to see. Consider this problem:

\[ \int \sqrt{1 - x^2} \, dx. \]

There are two factors in this expression, \( x^2 \) and \( \sqrt{1 - x^2} \), but it is not apparent that the chain rule is involved. Some clever rearrangement reveals that it is:

\[ \int x^2 \sqrt{1 - x^2} \, dx = \int (-2x) \left( \frac{1}{2} \right) (1 - (1 - x^2)) \sqrt{1 - x^2} \, dx. \]

This looks messy, but we do now have something that looks like the result of the chain rule: the function \( 1 - x^2 \) has been substituted into \(- (1/2)(1 - x^2)^{3/2}\), and the derivative

\[ \sqrt{1 - x^2} \]

going on. For example, in Leibniz notation the chain rule is

\[ \frac{dy}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}. \]

The same is true of our current expression:

\[ \int \frac{d}{dx} x^2 \sqrt{1 - x^2} \, dx = \int \frac{d}{du} x^2 \sqrt{1 - x^2} \, du. \]

Now we’ve almost there: since \( u = 1 - x^2 \), \( x^2 = 1 - u \) and the integral is

\[ \int \frac{1}{2} (1 - u)^{1/2} \, du. \]

It’s no coincidence that this is exactly the integral we computed in (8.1), we have simply renamed the variable \( u \) to make the calculations less confusing. Just as before:

\[ \int \frac{1}{2} (1 - u)^{1/2} \, du = \left( \frac{u^{1/2}}{1/2} + C \right). \]

Then since \( u = 1 - x^2 \):

\[ \frac{1}{2} (1 - x^2)^{1/2} + C. \]

To summarize: if we suspect that a given function is the derivative of another via the chain rule, we let \( u \) denote a likely candidate for the inner function, then translate the given function so that it is written entirely in terms of \( u \), with no \( x \) remaining in the expression. If we can integrate this new function of \( u \), then the antiderivative of the original function is obtained by replacing \( u \) by the equivalent expression in \( x \).

Even in simple cases you may prefer to use this mechanical procedure, since it often helps to avoid silly mistakes. For example, consider again this simple problem:

\[ \int 2 \cos(x^2) \, dx. \]

Let \( u = x^2 \), then \( du/\sqrt{u} = 2x \, dx \). Since we have exactly \( 2x \) \( dx \) in the original integral, we can replace it by \( du \):

\[ \int 2x \cos(x^2) \, dx = \int \cos u \, du = \sin u + C. \]

This is not the only way to do the algebra, and typically there are many paths to the correct answer. Another possibility, for example, is:

\[ \text{Since } du/\sqrt{u} = 2x \, dx, \text{ then } \int 2x \cos(x^2) \, dx = \int \cos u \, du = \sin u + C = \sin(x^2) + C. \]
then the integral becomes
\[ \int 2x \cos(x^2) \, dx = \int 2x \cos u \, \frac{du}{2u} = \int \cos u \, du. \]
The important thing to remember is that you must eliminate all instances of the original variable \( x \).

**EXAMPLE 8.1** Evaluate \( \int (a + b)^n \, dx \), assuming that \( a \) and \( b \) are constants, \( a \neq 0 \), and \( n \) is a positive integer. We let \( u = ax + b \), then \( du/dx = a \). Then
\[ \int (a + b)^n \, dx = \int \frac{1}{a} \cos u \, \frac{du}{a} + C = \frac{1}{a(a + b)^n} + C. \]

**EXAMPLE 8.2** Evaluate \( \int \sin(ax + b) \, dx \), assuming that \( a \) and \( b \) are constants and \( a \neq 0 \). Again we let \( u = ax + b \), then \( du = a \, dx \). Then
\[ \int \sin(ax + b) \, dx = \int \frac{1}{a} \sin u \, du = \frac{1}{a} \cos u + C = \frac{1}{a} \cos(ax + b) + C. \]

**EXAMPLE 8.3** Evaluate \( \int x \sin(x^2) \, dx \). First we compute the antiderivative, then evaluate the definite integral. Let \( u = x^2 \), so \( du = 2x \, dx \) or \( dx = du/2 \). Then
\[ \int x \sin(x^2) \, dx = \int \frac{1}{2} \sin u \, du = \frac{1}{2} \cos u + C = \frac{1}{2} \cos(x^2) + C. \]
Now
\[ \int x \sin(x^2) \, dx = \frac{1}{2} \cos u + C = \frac{1}{2} \cos(x^2) + C. \]

A somewhat easier alternative to this method is to change the original limits to match the variable \( u \). Since \( u = x^2 \), when \( x = a \) or \( x = b \) we have
\[ u = a^2 \] and
\[ u = b^2, \]
respectively. So at long last we get
\[ \int \sin(2x) \, dx = \frac{1}{2} \cos(2x) + C. \]

**EXAMPLE 8.4** Evaluate \( \int \frac{\cos(x)}{\sin(x)} \, dx \). Let \( u = \sin(x) \), then \( du = \cos(x) \, dx \). We change the limits to \( \sin(\pi/4) = \sqrt{2}/2 \) and \( \sin(\pi/2) = 1 \). Then
\[ \int \frac{\cos(x)}{\sin(x)} \, dx = \int \frac{1}{u} \, du = \ln |u| + C = \ln |\sin(x)| + C. \]

**Exercises 8.1.**

Find the antiderivatives.
1. \( \int (1 + x)^3 \, dx \) \( \Rightarrow \)
2. \( \int (x^2 + 1)^3 \, dx \) \( \Rightarrow \)
3. \( \int \sqrt{1 + 2x} \, dx \) \( \Rightarrow \)
4. \( \int \sqrt{1 - x^2} \, dx \) \( \Rightarrow \)
5. \( \int \sqrt{\tan x} \, dx \) \( \Rightarrow \)
6. \( \int \sqrt{\sin x} \, dx \) \( \Rightarrow \)
7. \( \int \sqrt{\cos x} \, dx \) \( \Rightarrow \)
8. \( \int \frac{1}{\sqrt{1 - x^2}} \, dx \) \( \Rightarrow \)
9. \( \int \frac{\cos x}{\sin x} \, dx \) \( \Rightarrow \)
10. \( \int \tan x \, dx \) \( \Rightarrow \)
11. \( \int \frac{1}{x} \, dx \) \( \Rightarrow \)
12. \( \int \frac{1}{x^2 - 1} \, dx \) \( \Rightarrow \)
13. \( \int \frac{x}{\sqrt{2 + x^2}} \, dx \) \( \Rightarrow \)
14. \( \int \frac{1}{\sqrt{x^2 - 4}} \, dx \) \( \Rightarrow \)
15. \( \int \frac{1}{x^2 + 1} \, dx \) \( \Rightarrow \)
16. \( \int \frac{1}{\sqrt{x^2 - 1}} \, dx \) \( \Rightarrow \)
17. \( \int \frac{1}{x^2 - 1} \, dx \) \( \Rightarrow \)
18. \( \int \frac{1}{x^2 + 1} \, dx \) \( \Rightarrow \)
19. \( \int \frac{1}{x} \, dx \) \( \Rightarrow \)
20. \( \int (x^2 + 1)^2 \, dx \) \( \Rightarrow \)

**8.2 Powers of sine and cosine**

Functions consisting of products of the sine and cosine can be integrated by using substitution and trigonometric identities. These can sometimes be tedious, but the technique is straightforward. Some examples will suffice to explain the approach.

**EXAMPLE 8.5** Evaluate \( \int \sin^2 x \, dx \). Rewrite the function:
\[ \int \sin^2 x \, dx = \int \sin x \sin^2 x \, dx = \int -\sin x (1 - \cos^2 x) \, dx. \]
Now use \( u = \cos x \), then \( du = -\sin x \, dx \):
\[ \int \sin x (1 - \cos^2 x) \, dx = \int -u^2 \, du = \frac{1}{3} u^3 + C = \frac{1}{3} \cos^3 x + C. \]

**EXAMPLE 8.6** Evaluate \( \int \cos^2 x \, dx \). Use the identity \( \cos^2 x = (1 - \cos(2x))/2 \) to rewrite the function:
\[ \int \cos^2 x \, dx = \int \frac{1 - \cos(2x)}{2} \, dx = \frac{1}{2} x - \frac{1}{4} \sin 2x + C. \]

**Exercises 8.2.**

Find the antiderivatives:
1. \( \int \sin^2 x \, dx \) \( \Rightarrow \)
2. \( \int \cos^2 x \, dx \) \( \Rightarrow \)
3. \( \int \sin^3 x \, dx \) \( \Rightarrow \)
4. \( \int \cos^3 x \, dx \) \( \Rightarrow \)
5. \( \int \sin^3 x \cos x \, dx \) \( \Rightarrow \)
6. \( \int \cos^3 x \sin x \, dx \) \( \Rightarrow \)
7. \( \int \sin^2 x \cos^2 x \, dx \) \( \Rightarrow \)
8. \( \int \sin x \cos(\sin x) \, dx \) \( \Rightarrow \)
9. \( \int \cos x \sin^2 x \, dx \) \( \Rightarrow \)
10. \( \int \tan x \sec x \, dx \) \( \Rightarrow \)

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are easy. The \( \cos^2 x \) integral is like the previous example:
\[ \int -\cos^2 x \, dx = \int -\cos 2x \cos^2 2x \, dx = \int -\cos 2x (1 - \sin^2 2x) \, dx = \int -\frac{1}{2} (1 - u^2) \, du = \frac{1}{2} (u - \frac{u^3}{3}) + C = \frac{1}{2} \sin 2x - \frac{1}{2} \sin 2x + C. \]

And finally we use another trigonometric identity, \( \cos^2 x = (1 + \cos(2x))/2 \):
\[ \int 3 \cos^2 2x \, dx = 3 \frac{1 + \cos 4x}{2} \, dx = \frac{3}{2} x + \frac{3}{4} \sin 4x + C. \]

So at long last we get
\[ \int \sin^2 x \, dx = \frac{x}{2} - \frac{1}{8} \sin 2x = \frac{1}{8} \sin 2x + \frac{1}{16} (\sin 2x) + \frac{3}{16} (x + \sin 4x). \]

**EXAMPLE 8.7** Evaluate \( \int \sin x \cos x \, dx \). Use the identity \( \sin^2 x = (1 - \cos(2x))/2 \) and \( \cos^2 x = (1 + \cos(2x))/2 \) to get:
\[ \int \sin x \cos x \, dx = \frac{1 - \cos 2x}{2} \, dx. \]

The remainder is left as an exercise.
8.3 Trigonometric Substitutions

So far we have seen that it sometimes helps to replace a subexpression of a function by a single variable. Occasionally it can help to replace the original variable by something more complicated. This seems like a “reverse” substitution, but it really is no different in principle than ordinary substitution.

**EXAMPLE 8.8** Evaluate \( \int \sqrt{1 - x^2} \, dx \). Let \( x = \sin u \) so \( dx = \cos u \, du \). Then

\[
\int \sqrt{1 - x^2} \, dx = \int \sqrt{1 - \sin^2 u} \cos u \, du = \int \cos u \, du.
\]

We would like to replace \( \sqrt{\cos u} \) by \( \cos u \), but this is valid only if \( \cos u \) is positive, since \( \sqrt{\cos u} \) is positive. Consider again the substitution \( x = \sin u \). If we do then, by the definition of the arcsine, \(-\pi/2 \leq u \leq \pi/2\), so \( \cos u \geq 0 \). Then we continue:

\[
\int \sqrt{\cos u} \cos u \, du = \cos^2 u \, du = \frac{1 + \cos 2u}{2} \, du = \frac{u}{2} + \frac{\sin 2u}{4} + C.
\]

This is a perfectly good answer, though the term \( \sin(2\arcsin x) \) is a bit unpleasant. It is possible to simplify this. Using the identity \( \sin(2u) = 2\sin u \cos u \), we can write \( \sin 2u = 2\sin(u)\cos(u) = 2\sin(\arcsin x)\sqrt{1 - \sin^2 (\arcsin x)} = 2x\sqrt{1 - x^2} \). Then the full antiderivative is

\[
\frac{u}{2} + \frac{x}{2}\sqrt{1 - x^2} + C.
\]

This type of substitution is usually indicated when the function you wish to integrate contains a polynomial expression that might allow you to use the fundamental identity \( \sin^2 x + \cos^2 x = 1 \) in one of three forms:

\[
\cos^2 x = 1 - \sin^2 x \quad \sec^2 x = 1 + \tan^2 x \quad \tan^2 x = \sec^2 x - 1.
\]

If your function contains \( 1 - x^2 \), as in the example above, try \( x = \sin u \), if it contains \( 1 + x^2 \), try \( x = \tan u \), and if it contains \( x^2 - 1 \), try \( x = \sec u \). Sometimes you will need to try something a bit different to handle constants other than one.

### Exercises 8.3.

Find the antiderivatives.

1. \( \int \csc x \, dx \Rightarrow \)
2. \( \int \csc^2 x \, dx \Rightarrow \)
3. \( \int \sqrt{1 - x^2} \, dx \Rightarrow \)
4. \( \int \sqrt{1 + x^2} \, dx \Rightarrow \)
5. \( \int x\sqrt{1 - x^2} \, dx \Rightarrow \)
6. \( \int x\sqrt{1 + x^2} \, dx \Rightarrow \)
7. \( \int \sqrt{1 + 3x^2} \, dx \Rightarrow \)
8. \( \int \sqrt{1 - 3x^2} \, dx \Rightarrow \)
9. 10. \( \int \frac{1}{\sqrt{1 + x^2}} \, dx \Rightarrow \)
10. \( \int \frac{1}{\sqrt{1 - x^2}} \, dx \Rightarrow \)

### 8.4 Integration by Parts

We have already seen that recognizing the product rule can be useful, when we noticed that

\[
\int \sec^3 u \, du = \int \sec u \tan u \, du = \tan u.
\]

As with substitution, we do not have to rely on insight or cleverness to discover such antiderivatives; there is a technique that will often help to uncover the product rule.

Start with the product rule:

\[
\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x).
\]

We can write this as

\[
f(x)g(x) = \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx,
\]

and then

\[
f(x)g'(x) \, dx = f(x)g(x) - \int f(x)g'(x) \, dx.
\]

This may not seem particularly useful at first glance, but it turns out that in many cases we have an integral of the form

\[
\int f(x)g'(x) \, dx
\]

but that

\[
\int f'(x)g(x) \, dx
\]

is easier. This technique for turning one integral into another is called integration by parts, and is usually written in more compact form. If we let \( u = f(x) \) and \( v = g(x) \) then
\[
du = f'(x) \, dx \quad \text{and} \quad dv = g'(x) \, dx
\]
To use this technique we need to identify likely candidates for \( u = f(x) \) and \( dv = g'(x) \, dx \).

**EXAMPLE 8.11** Evaluate \( \int x \ln x \, dx \). Let \( u = \ln x \) so \( du = \frac{1}{x} \, dx \) and \( dv = x \, dx \) so \( v = \frac{x^2}{2} \) and
\[
\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C.
\]

**EXAMPLE 8.12** Evaluate \( \int \sin x \, dx \). Let \( u = -\cos x \) so \( du = \sin x \, dx \) and \( dv = dx \) so \( v = x \) and
\[
\int \sin x \, dx = -x \cos x - \int -x \cdot \sin x + x \cos x + C.
\]

**EXAMPLE 8.13** Evaluate \( \int \sec^3 x \, dx \). Of course we already know the answer to this, but we needed to be clever to discover it. Here we’ll use the new technique to discover the antiderivative. Let \( u = \sec x \) and \( dv = \sec^2 x \, dx \). Then \( du = \sec x \tan x \, dx \) and \( v = \tan x \) and
\[
\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan x \cdot \sec x \, dx = \sec x \tan x - \int \sec x \tan x \cdot \sec x \, dx + \int \sec x \, dx.
\]
At first this looks useless—we’re right back to \( \int \sec^3 x \, dx \). But looking more closely:
\[
\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan x \cdot \sec x \, dx + \int \sec x \, dx = \sec x \tan x - \frac{1}{2} \int \sec x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C,
\]
and \((-2x)(-\sin x)\) and then once straight across, \([2](x) \cdot (-\sin x)\), and combine these as
\[
-x^2 \cos x + 2x \sin x - 2 \sin x \, dx,
\]
giving the same result as the second application of integration by parts. While this integral is easy, we may return yet once more to the table. Now multiply three times on the diagonal to get \( (\sec x)(-\cos x)\), \((\sec x)(-\sin x)\), and \((2)(\cos x)\), and once straight across, \((0)(\cos x)\). We combine these as before to get
\[
-x^2 \cos x + 2x \sin x + 2 \cos x = \int \sec x \, dx - x^2 \cos x + 2x \sin x + 2 \cos x + C.
\]
Typically we would fill in the table one line at a time, until the “straight across” multiplication gives an easy integral. If we can see that the \( u \) column will eventually become zero, we can instead fill in the whole table; computing the products as indicated will then give the entire integral, including the “\(C\)” as above.

**Exercises 8.4.**
Find the antiderivatives.
1. \( \int x \cos x \, dx \) 
2. \( \int x^2 \cos x \, dx \) 
3. \( \int x^3 \, dx \) 
4. \( \int x^2 \, dx \) 
5. \( \int \sin^2 x \, dx \) 
6. \( \int e^x \, dx \) 
7. \( \int \sec x \tan x \, dx \) 
8. \( \int \sin x \, dx \) 
9. \( \int x^3 \cos x \, dx \) 
10. \( \int x \sin^2 x \, dx \) 
11. \( \int x \sin x \, dx \) 
12. \( \int \tan x + \sqrt{7} \, dx \) 
13. \( \int \sin \sqrt{x} \, dx \) 

**8.5 Rational Functions**
A rational function is a fraction with polynomials in the numerator and denominator. For example,
\[
x^3 \quad \text{and} \quad x^2 + 1
\]
are all rational functions of \( x \). There is a general technique called “partial fractions” that, in principle, allows us to integrate any rational function. The algebraic steps in the

**EXAMPLE 8.14** Evaluate \( \int x^2 \sin x \, dx \). Let \( u = x^2 \), \( dv = \sin x \, dx \), then \( du = 2x \, dx \) and \( v = -\cos x \). Now \( \int x^2 \sin x \, dx = x^2 \cos x + 2 \int x \cos x \, dx \). This is better than the original integral, but we need to do integration by parts again. Let \( u = 2x \), \( dv = \cos x \, dx \), then \( du = 2 \, dx \) and \( v = \sin x \) and
\[
\int x^2 \sin x \, dx = x^2 \cos x + 2 \int x \cos x \, dx = x^2 \cos x + 2 \sin x \, dx - 2 \cos x + x + C.
\]

Such repeated use of integration by parts is fairly common, but it can be a bit tedious to accomplish, and it is easy to make errors, especially sign errors involving the subtraction in the formula. There is a nice tabular method to accomplish the calculation that minimizes the chance for error and speeds up the whole process. We illustrate with the previous example. Here is the table:

<table>
<thead>
<tr>
<th>( u )</th>
<th>( dv )</th>
<th>( du )</th>
<th>( v )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 \sin x )</td>
<td>( x )</td>
<td>( 2x )</td>
<td>( \cos x )</td>
</tr>
<tr>
<td>( x \cos x )</td>
<td>( 1 )</td>
<td>( 0 )</td>
<td>( x^2 \sin x )</td>
</tr>
</tbody>
</table>

To form the first table, we start with \( u \) at the top of the second column and repeatedly compute the derivative; starting with \( dv \) at the top of the third column, we repeatedly compute the antiderivative. In the first column, we place a “-” in every second row. To form the second table we combine the first and second columns by ignoring the boundary; if you do this by hand, you may simply start with two columns and add a “-” to every second row.

To compute with the second table we begin at the top. Multiply the first entry in column \( u \) by the second entry in column \( dv \) to get \( -x^2 \cos x \), and add this to the integral of the product of the second entry in column \( u \) and second entry in column \( dv \). This gives:
\[
-x^2 \cos x + 2x \sin x + C
\]
or exactly the result of the first application of integration by parts. Since this integral is not yet easy, we return to the table. Now we multiply twice on the diagonal, \( (x^2)(-\cos x) \)

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**EXAMPLE 8.15** Find \( \int \frac{x^3}{(3 - 2x)^3} \, dx \). Using the substitution \( u = 3 - 2x \) we get
\[
\int \frac{x^3}{(3 - 2x)^3} \, dx = \frac{1}{2} \int \frac{u^2}{u^3} \, du = \frac{1}{2} \left[ \frac{u^2}{u^3} \right]_{u = 0}^{u = 3} + \frac{27u + 2\left(3 - 2x\right)^3}{u^2} \left(3 - 2x\right) \, du,
\]

\[
= \frac{1}{10} \left[ (3 - 2x)^3 \right]_{u = 0}^{u = 3} + \frac{9(3 - 2x)^4 + 27(3 - 2x)^5}{(3 - 2x)^2} \, du + C
\]

We now proceed to the case in which the denominator is a quadratic polynomial. We can always factor out the coefficient of \( x^2 \) and put it outside the integral, so we can assume that the denominator has the form \( x^2 + bx + c \). There are three possible cases, depending on how the quadratic factors: either \( x^2 + bx + c = (x - \alpha)(x - \beta) \), \( x^2 + bx + c = (x - \alpha)^2 \), or it doesn’t factor. We can use the quadratic formula to decide which of these we have, and to factor the quadratic if it is possible.

**EXAMPLE 8.16** Determine whether \( x^2 + x + 1 \) factors, and factor it if possible. The quadratic formula tells us that \( x^2 + x + 1 = 0 \) when
\[
x = -\frac{1 \pm \sqrt{3}}{2}
\]
Since there is no square root of \(-3\), this quadratic does not factor.
EXAMPLE 8.17 Determine whether \( x^2 - x - 1 \) factors, and factor it if possible. The quadratic formula tells us that \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \).

Therefore

\[
x^2 - x - 1 = \left( x - \frac{1 + \sqrt{5}}{2} \right) \left( x - \frac{1 - \sqrt{5}}{2} \right)
\]

If \( x^2 + bx + c = (x - r)^2 \) then we have the special case we have already seen, that can be handled with a substitution. The other two cases require different approaches.

**EXAMPLE 8.18** Using \( u \) substitution, we complete the square:

\[
\int \frac{x^3}{(x + 2)(x + 3)} \, dx = \int x - 1 \, dx + \int \frac{7x - 6}{(x + 2)(x + 3)} \, dx
\]

The first integral is easy, so only the second requires some work.

Now consider the following simple algebra of fractions:

\[
\frac{A}{x - 1} + \frac{B}{x - 2} = \frac{Ax - A + Bx - 2B}{(x - 1)(x - 2)} = \frac{(A + B)x - (A - 2B)}{(x - 1)(x - 2)}
\]

That is, adding two fractions with constant numerator and denominators \((x - 1)\) and \((x - 2)\) produce a fraction with denominator \((x - 1)(x - 2)\) and a polynomial of degree less than 2 for the numerator. We want to reverse this process: starting with a single fraction, we want to write it as a sum of two simpler fractions. An example should make it clear how to proceed.

For the second integral we complete the square:

\[
x^2 + 4x + 8 = (x + 2)^2 + 4 = 4 \left( \frac{x + 2}{2} \right)^2 + 1
\]

making the integral

\[
\frac{1}{7} \int \frac{1}{\left( \frac{x + 2}{2} \right)^2 + 1} \, dx
\]

Using \( u = \frac{x + 2}{2} \) we get

\[
\frac{1}{3} \int \frac{1}{u^2 + 1} \, du - \frac{1}{3} \int \frac{1}{\frac{u^2}{2} + 1} \, du = \frac{1}{2} \text{arctan} \left( \frac{u}{2} \right)
\]

The final answer is now

\[
\int \frac{x + 1}{x^2 + 4x + 8} \, dx = \frac{1}{2} \ln|x^2 + 4x + 8| - \frac{1}{2} \text{arctan} \left( \frac{x + 2}{2} \right) + C.
\]

**Exercises 8.5.**

Find the antiderivatives.

1. \( \int \frac{1}{x - 1} \, dx \)
2. \( \int \frac{1}{x + 1} \, dx \)
3. \( \int \frac{1}{x^2 + 2x + 2} \, dx \)
4. \( \int \frac{1}{x^2 - 1} \, dx \)
5. \( \int \frac{1}{x^2 + 2} \, dx \)
6. \( \int \frac{1}{x^2 + 4} \, dx \)
7. \( \int \frac{1}{x^2 + 3} \, dx \)
8. \( \int \frac{1}{x^2 + 6} \, dx \)
9. \( \int \frac{1}{x^2 - x - 1} \, dx \)

**8.6 Numerical Integration**

We have now seen some of the most generally useful methods for discovering antiderivatives, and there are others. Unfortunately, some functions have no simple antiderivatives; in such cases if the value of a definite integral is needed it will have to be approximated. We will see two methods that work reasonably well and yet are fairly simple; in some cases more sophisticated techniques will be needed.

**8.7 Chapter 8 Techniques of Integration**

EXAMPLE 8.19 Evaluate \( \int \frac{x^3}{(x - 2)(x + 3)} \, dx \). We start by writing 7\( x - 6 \) as the sum of two fractions. We want to end up with

\[
\frac{7x - 6}{(x - 2)(x + 3)} = \frac{A}{x - 2} + \frac{B}{x + 3}
\]

If we go ahead and add the fractions on the right hand side we get

\[
\frac{7x - 6}{(x - 2)(x + 3)} = \frac{(A + B)x + 3A + 2B}{(x - 2)(x + 3)}
\]

So all we need to do is find \( A \) and \( B \) so that \( 7x - 6 = (A + B)x + 3A + 2B \), which is to say, we need \( 7 = A + B \) and \( -6 = 3A + 2B \). This is easy to solve for \( A = 8 - 5B \), and then \( B = 7 - 8B/5 \). Thus

\[
\int \frac{7x - 6}{(x - 2)(x + 3)} \, dx = \int \frac{1}{5x/2} + \frac{1}{7x + 3} \, dx = \frac{1}{5} \ln|x - 2| + \frac{2}{7} \ln|x + 3| + C
\]

Now suppose that \( x^2 + 4x + c \) doesn’t factor. Again we can use long division to ensure that the numerator has degree less than 2, then we complete the square.

**EXAMPLE 8.20** Evaluate \( \int \frac{x + 1}{x^2 + 4x + 8} \, dx \). The quadratic denominator does not factor. We could complete the square and use a trigonometric substitution, but it is simpler to rearrange the integrand:

\[
\int \frac{x + 1}{x^2 + 4x + 8} \, dx = \int \frac{x + 2}{x^2 + 4x + 9} \, dx - \int \frac{1}{x^2 + 4} \, dx
\]

The first integral is an easy substitution problem, using \( u = x^2 + 4x + 8 \):

\[
\int \frac{x + 2}{x^2 + 4x + 8} \, dx = \frac{1}{2} \int \frac{du}{u} - \frac{1}{2} \ln|x^2 + 4x + 8|
\]

**Figure 8.1** Approximating area with rectangles and with trapezoids. (AP)

As with rectangles, we divide the interval into \( n \) equal subintervals of length \( \Delta x \). A typical trapezoid is pictured in figure 8.2. It has area \( \int (x_i) + f(x_{i+1}) \Delta x \). If we add up the areas of all trapezoids we get

\[
\int f(x_i) \, dx + f(x_1) \, dx + \int f(x_2) \, dx + \cdots + \int f(x_{n+1}) \, dx = \int f(x) \Delta x
\]

For a modest number of subintervals this is not too difficult to do with a calculator; a computer can easily do many subintervals.

**Figure 8.2** A single trapezoid.
In practice, an approximation is useful only if we know how accurate it is; for example, we might need a particular value accurate to three decimal places. When we compute a particular approximation to an integral, the error is the difference between the approximation and the true value of the integral. For any approximation technique, we need an error estimate, a value that is guaranteed to be larger than the actual error. If $A$ is an approximation and $E$ is the associated error estimate, then we know that the true value of the integral is between $A - E$ and $A + E$. In the case of our approximation of the integral, we want $E = |E(\Delta x)|$ to be a function of $\Delta x$ that gets small rapidly as $\Delta x$ gets small. Fortunately, for many functions, there is such an error estimate associated with the trapezoidal approximation.

**THEOREM 8.21** Suppose $f$ has a second derivative $f''$ everywhere on the interval $[a, b]$, and $|f''(x)| \leq M$ for all $x$ in the interval. With $\Delta x = (b - a)/n$, an error estimate for the trapezoidal approximation is

$$E(\Delta x) = \frac{b - a}{12n} f''(\xi)^2 \Delta x^3.$$  

Let's see how we can use this.

**EXAMPLE 8.22** Approximate $\int_0^1 e^{-x^2} \, dx$ to two decimal places. The second derivative of $f = e^{-x^2}$ is $(4x^2 - 2)e^{-x^2}$, and it is not hard to see that on $[0, 1]$, $|4x^2 - 2| e^{-x^2} \leq 2$.

We begin by estimating the number of subintervals we are likely to need. To get two decimal places of accuracy, we will certainly need $E(\Delta x) < 0.005$ or

$$\frac{1}{12n} < 0.005 \quad \text{or} \quad \frac{1}{600} < n.$$  

With $n = 6$, the error estimate is thus $1/600 < 0.00047$. We compute the trapezoidal approximation for six intervals:

$$\int_0^1 e^{-x^2} \, dx = \frac{1}{6} \left( f(0) + f(1/6) + f(2/6) + \cdots + f(5/6) + f(1) \right) = 0.74512.$$  

So the true value of the integral is between 0.74512 and 0.74042, or 0.74042 and 0.74752. Unfortunately, the first rounds to 0.74 and the second rounds to 0.75, so we can't be sure of the correct value in the second decimal place; we need to pick a larger $n$.

As it turns out, we need to go to 12 subintervals, and the approximation is

$$\int_0^1 e^{-x^2} \, dx = \frac{1}{12} \left( f(0) + f(1/12) + f(2/12) + \cdots + f(11/12) + f(1) \right) = 0.74982.$$  

For comparison, using 12 rectangles to approximate the integral gives 0.71715.

Not surprisingly, the solutions turn out to be quite messy. Nevertheless, Sage can easily compute and simplify the integral to get

$$\int_{x_{i-1}}^{x_{i+1}} e^{-x^2} \, dx = \frac{1}{2} \left( f(x_i) + f(x_{i+1}) \right) + \frac{1}{6} \left( f(x_i) + 4f(x_{i+1}) + f(x_{i+2}) \right).$$  

This is just slightly more complicated than the formula for trapezoids, so we need to remember the alternating 2 and 4 coefficients. This approximation technique is referred to as Simpson's Rule.

**THEOREM 8.23** Suppose $f$ has a fourth derivative $f^{(4)}$ everywhere on the interval $[a, b]$, and $|f^{(4)}(x)| \leq M$ for all $x$ in the interval. With $\Delta x = (b - a)/n$, an error estimate for Simpson’s approximation is

$$E(\Delta x) = \frac{b - a}{180n} f^{(4)}(\xi)^4 \Delta x^5 = \frac{b - a}{180n} \left( \frac{b - a}{n} \right)^4 \Delta x^5.$$  

With $n = 6$, the error estimate is thus $1/180 < 0.00047$. We compute the Simpson approximation for six intervals:

$$\int_0^1 e^{-x^2} \, dx = \frac{1}{8} \left( f(0) + 4f(1/8) + 2f(2/8) + \cdots + 4f(7/8) + f(1) \right) = 0.74855.$$  

8.6 Numerical Integration

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$$\int_{x_{i-1}}^{x_{i+1}} e^{-x^2} \, dx = \frac{1}{2} \left( f(x_i) + f(x_{i+1}) \right) + \frac{1}{6} \left( f(x_i) + 4f(x_{i+1}) + f(x_{i+2}) \right).$$  

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8.6 Numerical Integration

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$$\int_{x_{i-1}}^{x_{i+1}} e^{-x^2} \, dx = \frac{1}{2} \left( f(x_i) + f(x_{i+1}) \right) + \frac{1}{6} \left( f(x_i) + 4f(x_{i+1}) + f(x_{i+2}) \right).$$  

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$$E(\Delta x) = \frac{b - a}{180n} f^{(4)}(\xi)^4 \Delta x^5 = \frac{b - a}{180n} \left( \frac{b - a}{n} \right)^4 \Delta x^5.$$  

1. $\int_a^b x \, dx = \frac{b^2 - a^2}{2n} \Delta x^2$
2. $\int_a^b x^2 \, dx = \frac{b^3 - a^3}{3n} \Delta x^3$
3. $\int_a^b x^3 \, dx = \frac{b^4 - a^4}{4n} \Delta x^4$
4. $\int_a^b x^4 \, dx = \frac{b^5 - a^5}{5n} \Delta x^5$
5. $\int_a^b f(x) \, dx = \left( f(a) + f(b) \right) \Delta x / 2$
6. $\int_a^b f(x) \, dx = \left( f(a) + f(b) \right) \Delta x / 2$
7. $\int_a^b f(x) \, dx = \left( f(a) + f(b) \right) \Delta x / 2$
8. $\int_a^b f(x) \, dx = \left( f(a) + f(b) \right) \Delta x / 2$
9. $\int_a^b f(x) \, dx = \left( f(a) + f(b) \right) \Delta x / 2$
10. $\int_a^b f(x) \, dx = \left( f(a) + f(b) \right) \Delta x / 2$
11. Using Simpson’s rule on a parabola $f(x)$, even with just two subintervals, gives the exact value of the integral, because the parabolas used to approximate $f$ will be $f$ itself. Remarkably, Simpson’s rule also computes the integral of a cubic function $f(x)$ as $ax^3 + bx^2 + cx + d$ exactly. Show this is true by showing that

$$\int_a^b f(x) \, dx = \frac{b^3 - a^3}{3n} \Delta x^3 + \frac{b^2 - a^2}{2n} \Delta x^2 + \frac{b - a}{n} \Delta x.$$  

This does require a bit of messy algebra, so you may prefer to use Sage.
These problems require the techniques of this chapter, and are in no particular order. Some problems may be done in more than one way.

1. \[ \int (t + 4)^3 \, dt \Rightarrow \]
2. \[ 2t^2 - [y(t)]^{3/2} \, dt \Rightarrow \]
3. \[ \int e^{t^2} + 16t^3 \, dt \Rightarrow \]
4. \[ \int \sin t \cos 2t \, dt \Rightarrow \]
5. \[ \int \tan t \sec^2 t \, dt \Rightarrow \]
6. \[ \int 3t + 1 \, dt \Rightarrow \]
7. \[ \int \frac{1}{(t^2 - 1)^{3/2}} \, dt \Rightarrow \]
8. \[ \int \frac{1}{(x - y)^{3/2}} \, dt \Rightarrow \]
9. \[ \int \frac{\cos 3t}{\sin 3t} \, dt \Rightarrow \]
10. \[ \int t \sec^2 t \, dt \Rightarrow \]
11. \[ \int e^{t^2} \sqrt{e^{t^2} + 1} \, dt \Rightarrow \]
12. \[ \int \frac{1}{t^2 + 1} \, dt \Rightarrow \]
13. \[ \int e^{t^2} \sin t \, dt \Rightarrow \]
14. \[ \int \cos^4 t \, dt \Rightarrow \]
15. \[ \int \frac{1}{t^2 + 2t + 1} \, dt \Rightarrow \]
16. \[ \int \frac{1}{t^2 + 4t - 3} \, dt \Rightarrow \]
17. \[ \int \frac{1}{t^2 + 3t} \, dt \Rightarrow \]
18. \[ \int \frac{1}{t^2 + 6t + 9} \, dt \Rightarrow \]
19. \[ \int \frac{1}{\ln(t)^2} \, dt \Rightarrow \]
20. \[ \int \frac{1}{(t^2 + 4t + 9)^2} \, dt \Rightarrow \]
21. \[ \int \frac{1}{2 + \tan(t)} \, dt \Rightarrow \]
22. \[ \int \frac{1}{x^2 + 2t - 3} \, dt \Rightarrow \]
23. \[ \int \frac{1}{\ln(t) + 4t} \, dt \Rightarrow \]
24. \[ \frac{1}{x^2 - 6x + 6} \, dt \Rightarrow \]
25. \[ \int \frac{1}{\ln(t)^2} \, dt \Rightarrow \]
26. \[ \int \sqrt{t} \, dt \Rightarrow \]
27. \[ \int \frac{1}{\sqrt{t^2 + 1}} \, dt \Rightarrow \]
28. \[ \int \frac{1}{\sqrt{t^2 + 4}} \, dt \Rightarrow \]