8

Techniques of Integration

Over the next few sections we examine some techniques that are frequently successful when seeking antiderivatives of functions. Sometimes this is a simple problem, since it will be apparent that the function you wish to integrate is a derivative in some straightforward way. For example, faced with

$$\int x^{10}\,dx$$

we realize immediately that the derivative of x^{11} will supply an x^{10}: $(x^{11})' = 11x^{10}$. We don't want the "11", but constants are easy to alter, because differentiation "ignores" them in certain circumstances, so

$$\frac{d}{dx}\frac{1}{11}x^{11} = \frac{1}{11}11x^{10} = x^{10}.$$

From our knowledge of derivatives, we can immediately write down a number of antiderivatives. Here is a list of those most often used:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad \text{if } n \neq -1$$

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

8.1 SUBSTITUTION

Needless to say, most problems we encounter will not be so simple. Here's a slightly more complicated example: find

$$\int 2x\cos(x^2)\,dx.$$

This is not a "simple" derivative, but a little thought reveals that it must have come from an application of the chain rule. Multiplied on the "outside" is $2x$, which is the derivative of the "inside" function x^2. Checking:

$$\frac{d}{dx}\sin(x^2) = \cos(x^2)\frac{d}{dx}x^2 = 2x\cos(x^2),$$

so

$$\int 2x\cos(x^2)\,dx = \sin(x^2) + C.$$

Even when the chain rule has "produced" a certain derivative, it is not always easy to see. Consider this problem:

$$\int x^3\sqrt{1-x^2}\,dx.$$

There are two factors in this expression, x^3 and $\sqrt{1-x^2}$, but it is not apparent that the chain rule is involved. Some clever rearrangement reveals that it is:

$$\int x^3\sqrt{1-x^2}\,dx = \int (-2x)\left(-\frac{1}{2}\right)(1-(1-x^2))\sqrt{1-x^2}\,dx.$$

This looks messy, but we do now have something that looks like the result of the chain rule: the function $1-x^2$ has been substituted into $-(1/2)(1-x)\sqrt{x}$, and the derivative of $1-x^2$, $-2x$, multiplied on the outside. If we can find a function $F(x)$ whose derivative is $-(1/2)(1-x)\sqrt{x}$ we'll be done, since then

$$\begin{aligned}\frac{d}{dx}F(1-x^2) &= -2xF'(1-x^2) = (-2x)\left(-\frac{1}{2}\right)(1-(1-x^2))\sqrt{1-x^2}\\ &= x^3\sqrt{1-x^2}\end{aligned}$$

But this isn't hard:

$$\begin{aligned}\int -\frac{1}{2}(1-x)\sqrt{x}\,dx &= \int -\frac{1}{2}(x^{1/2}-x^{3/2})\,dx \\ &= -\frac{1}{2}\left(\frac{2}{3}x^{3/2}-\frac{2}{5}x^{5/2}\right)+C \\ &= \left(\frac{1}{5}x-\frac{1}{3}\right)x^{3/2}+C.\end{aligned}\tag{8.1}$$

So finally we have

$$\int x^3\sqrt{1-x^2}\,dx = \left(\frac{1}{5}(1-x^2)-\frac{1}{3}\right)(1-x^2)^{3/2}+C.$$

So we succeeded, but it required a clever first step, rewriting the original function so that it looked like the result of using the chain rule. Fortunately, there is a technique that makes such problems simpler, without requiring cleverness to rewrite a function in just the right way. It does sometimes not work, or may require more than one attempt, but the idea is simple: guess at the most likely candidate for the "inside function", then do some algebra to see what this requires the rest of the function to look like.

One frequently good guess is any complicated expression inside a square root, so we start by trying $u=1-x^2$, using a new variable, u, for convenience in the manipulations that follow. Now we know that the chain rule will multiply by the derivative of this inner function:

$$\frac{du}{dx} = -2x,$$

so we need to rewrite the original function to include this:

$$\int x^3\sqrt{1-x^2} = \int x^3\sqrt{u}\frac{-2x}{-2x}\,dx = \int \frac{x^2}{-2}\sqrt{u}\,\frac{du}{dx}\,dx.$$

Recall that one benefit of the Leibniz notation is that it often turns out that what looks like ordinary arithmetic gives the correct answer, even if something more complicated is going on. For example, in Leibniz notation the chain rule is

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}.$$

The same is true of our current expression:

$$\int \frac{x^2}{-2}\sqrt{u}\,\frac{du}{dx}\,dx = \int \frac{x^2}{-2}\sqrt{u}\,du.$$

Now we're almost there: since $u=1-x^2$, $x^2=1-u$ and the integral is

$$\int -\frac{1}{2}(1-u)\sqrt{u}\,du.$$

It's no coincidence that this is exactly the integral we computed in (8.1), we have simply renamed the variable u to make the calculations less confusing. Just as before:

$$\int -\frac{1}{2}(1-u)\sqrt{u}\,du = \left(\frac{1}{5}u-\frac{1}{3}\right)u^{3/2}+C.$$

Then since $u=1-x^2$:

$$\int x^3\sqrt{1-x^2}\,dx = \left(\frac{1}{5}(1-x^2)-\frac{1}{3}\right)(1-x^2)^{3/2}+C.$$

To summarize: if we suspect that a given function is the derivative of another via the chain rule, we let u denote a likely candidate for the inner function, then translate the given function so that it is written entirely in terms of u, with no x remaining in the expression. If we can integrate this new function of u, then the antiderivative of the original function is obtained by replacing u by the equivalent expression in x.

Even in simple cases you may prefer to use this mechanical procedure, since it often helps to avoid silly mistakes. For example, consider again this simple problem:

$$\int 2x\cos(x^2)\,dx.$$

Let $u=x^2$, then $du/dx=2x$ or $du=2x\,dx$. Since we have exactly $2x\,dx$ in the original integral, we can replace it by du:

$$\int 2x\cos(x^2)\,dx = \int \cos u\,du = \sin u + C = \sin(x^2)+C.$$

This is not the only way to do the algebra, and typically there are many paths to the correct answer. Another possibility, for example, is: Since $du/dx=2x$, $dx=du/2x$, and then the integral becomes

$$\int 2x\cos(x^2)\,dx = \int 2x\cos u\,\frac{du}{2x} = \int \cos u\,du.$$

The important thing to remember is that you must eliminate all instances of the original variable x.

EXAMPLE 8.1 Evaluate $\int (ax+b)^n\,dx$, assuming that a and b are constants, $a \neq 0$, and n is a positive integer. We let $u = ax+b$ so $du = a\,dx$ or $dx = du/a$. Then

$$\int (ax+b)^n\,dx = \int \frac{1}{a}u^n\,du = \frac{1}{a(n+1)}u^{n+1} + C = \frac{1}{a(n+1)}(ax+b)^{n+1} + C.$$

□

EXAMPLE 8.2 Evaluate $\int \sin(ax+b)\,dx$, assuming that a and b are constants and $a \neq 0$. Again we let $u = ax+b$ so $du = a\,dx$ or $dx = du/a$. Then

$$\int \sin(ax+b)\,dx = \int \frac{1}{a}\sin u\,du = \frac{1}{a}(-\cos u) + C = -\frac{1}{a}\cos(ax+b) + C.$$

□

EXAMPLE 8.3 Evaluate $\int_2^4 x\sin(x^2)\,dx$. First we compute the antiderivative, then evaluate the definite integral. Let $u = x^2$ so $du = 2x\,dx$ or $x\,dx = du/2$. Then

$$\int x\sin(x^2)\,dx = \int \frac{1}{2}\sin u\,du = \frac{1}{2}(-\cos u) + C = -\frac{1}{2}\cos(x^2) + C.$$

Now

$$\int_2^4 x\sin(x^2)\,dx = -\frac{1}{2}\cos(x^2)\Big|_2^4 = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

A somewhat neater alternative to this method is to change the original limits to match the variable u. Since $u = x^2$, when $x = 2$, $u = 4$, and when $x = 4$, $u = 16$. So we can do this:

$$\int_2^4 x\sin(x^2)\,dx = \int_4^{16} \frac{1}{2}\sin u\,du = -\frac{1}{2}(\cos u)\Big|_4^{16} = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

An incorrect, and dangerous, alternative is something like this:

$$\int_2^4 x\sin(x^2)\,dx = \int_2^4 \frac{1}{2}\sin u\,du = -\frac{1}{2}\cos(u)\Big|_2^4 = -\frac{1}{2}\cos(x^2)\Big|_2^4 = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

This is incorrect because $\int_2^4 \frac{1}{2}\sin u\,du$ means that u takes on values between 2 and 4, which is wrong. It is dangerous, because it is very easy to get to the point $-\frac{1}{2}\cos(u)\Big|_2^4$ and forget to substitute x^2 back in for u, thus getting the incorrect answer $-\frac{1}{2}\cos(4) + \frac{1}{2}\cos(2)$. A somewhat clumsy, but acceptable, alternative is something like this:

$$\int_2^4 x\sin(x^2)\,dx = \int_{x=2}^{x=4} \frac{1}{2}\sin u\,du = -\frac{1}{2}\cos(u)\Big|_{x=2}^{x=4} = -\frac{1}{2}\cos(x^2)\Big|_2^4 = -\frac{\cos(16)}{2} + \frac{\cos(4)}{2}.$$

□

EXAMPLE 8.4 Evaluate $\int_{1/4}^{1/2} \frac{\cos(\pi t)}{\sin^2(\pi t)}\,dt$. Let $u = \sin(\pi t)$ so $du = \pi\cos(\pi t)\,dt$ or $du/\pi = \cos(\pi t)\,dt$. We change the limits to $\sin(\pi/4) = \sqrt{2}/2$ and $\sin(\pi/2) = 1$. Then

$$\int_{1/4}^{1/2} \frac{\cos(\pi t)}{\sin^2(\pi t)}\,dt = \int_{\sqrt{2}/2}^{1} \frac{1}{\pi}\frac{1}{u^2}\,du = \int_{\sqrt{2}/2}^{1} \frac{1}{\pi}u^{-2}\,du = \frac{1}{\pi}\frac{u^{-1}}{-1}\Big|_{\sqrt{2}/2}^{1} = -\frac{1}{\pi} + \frac{\sqrt{2}}{\pi}.$$

□

Exercises 8.1.

Find the antiderivatives.

1. $\int (1-t)^9\,dt \Rightarrow$

2. $\int (x^2+1)^2\,dx \Rightarrow$

3. $\int x(x^2+1)^{100}\,dx \Rightarrow$

4. $\int \frac{1}{\sqrt[3]{1-5t}}\,dt \Rightarrow$

5. $\int \sin^3 x\cos x\,dx \Rightarrow$

6. $\int x\sqrt{100-x^2}\,dx \Rightarrow$

7. $\int \frac{x^2}{\sqrt{1-x^3}}\,dx \Rightarrow$

8. $\int \cos(\pi t)\cos\big(\sin(\pi t)\big)\,dt \Rightarrow$

9. $\int \frac{\sin x}{\cos^3 x}\,dx \Rightarrow$

10. $\int \tan x\,dx \Rightarrow$

11. $\int_0^{\pi} \sin^5(3x)\cos(3x)\,dx \Rightarrow$

12. $\int \sec^2 x\tan x\,dx \Rightarrow$

13. $\int_0^{\sqrt{\pi}/2} x\sec^2(x^2)\tan(x^2)\,dx \Rightarrow$

14. $\int \frac{\sin(\tan x)}{\cos^2 x}\,dx \Rightarrow$

15. $\int_3^4 \frac{1}{(3x-7)^2}\,dx \Rightarrow$

16. $\int_0^{\pi/6} (\cos^2 x - \sin^2 x)\,dx \Rightarrow$

17. $\int \frac{6x}{(x^2-7)^{1/9}}\,dx \Rightarrow$

18. $\int_{-1}^{1} (2x^3-1)(x^4-2x)^6\,dx \Rightarrow$

19. $\int_{-1}^{1} \sin^7 x\,dx \Rightarrow$

20. $\int f(x)f'(x)\,dx \Rightarrow$

8.2 POWERS OF SINE AND COSINE

Functions consisting of products of the sine and cosine can be integrated by using substitution and trigonometric identities. These can sometimes be tedious, but the technique is straightforward. Some examples will suffice to explain the approach.

EXAMPLE 8.5 Evaluate $\int \sin^5 x\,dx$. Rewrite the function:

$$\int \sin^5 x\,dx = \int \sin x \sin^4 x\,dx = \int \sin x(\sin^2 x)^2\,dx = \int \sin x(1-\cos^2 x)^2\,dx.$$

Now use $u = \cos x$, $du = -\sin x\,dx$:

$$\begin{aligned}\int \sin x(1-\cos^2 x)^2\,dx &= \int -(1-u^2)^2\,du\\ &= \int -(1-2u^2+u^4)\,du\\ &= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C\\ &= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C.\end{aligned}$$

□

EXAMPLE 8.6 Evaluate $\int \sin^6 x\,dx$. Use $\sin^2 x = (1-\cos(2x))/2$ to rewrite the function:

$$\begin{aligned}\int \sin^6 x\,dx = \int (\sin^2 x)^3\,dx &= \int \frac{(1-\cos 2x)^3}{8}\,dx\\ &= \frac{1}{8}\int 1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x\,dx.\end{aligned}$$

Now we have four integrals to evaluate:

$$\int 1\,dx = x$$

and

$$\int -3\cos 2x\,dx = -\frac{3}{2}\sin 2x$$

are easy. The $\cos^3 2x$ integral is like the previous example:

$$\begin{aligned}\int -\cos^3 2x\,dx &= \int -\cos 2x\cos^2 2x\,dx\\ &= \int -\cos 2x(1-\sin^2 2x)\,dx\\ &= \int -\frac{1}{2}(1-u^2)\,du\\ &= -\frac{1}{2}\left(u - \frac{u^3}{3}\right)\\ &= -\frac{1}{2}\left(\sin 2x - \frac{\sin^3 2x}{3}\right).\end{aligned}$$

And finally we use another trigonometric identity, $\cos^2 x = (1+\cos(2x))/2$:

$$\int 3\cos^2 2x\,dx = 3\int \frac{1+\cos 4x}{2}\,dx = \frac{3}{2}\left(x + \frac{\sin 4x}{4}\right).$$

So at long last we get

$$\int \sin^6 x\,dx = \frac{x}{8} - \frac{3}{16}\sin 2x - \frac{1}{16}\left(\sin 2x - \frac{\sin^3 2x}{3}\right) + \frac{3}{16}\left(x + \frac{\sin 4x}{4}\right) + C.$$

□

EXAMPLE 8.7 Evaluate $\int \sin^2 x\cos^2 x\,dx$. Use the formulas $\sin^2 x = (1-\cos(2x))/2$ and $\cos^2 x = (1+\cos(2x))/2$ to get:

$$\int \sin^2 x\cos^2 x\,dx = \int \frac{1-\cos(2x)}{2}\cdot\frac{1+\cos(2x)}{2}\,dx.$$

The remainder is left as an exercise. □

Exercises 8.2.

Find the antiderivatives.

1. $\int \sin^2 x\,dx \Rightarrow$

2. $\int \sin^3 x\,dx \Rightarrow$

3. $\int \sin^4 x\,dx \Rightarrow$

4. $\int \cos^2 x\sin^3 x\,dx \Rightarrow$

5. $\int \cos^3 x\,dx \Rightarrow$

6. $\int \sin^2 x\cos^2 x\,dx \Rightarrow$

7. $\int \cos^3 x\sin^2 x\,dx \Rightarrow$

8. $\int \sin x(\cos x)^{3/2}\,dx \Rightarrow$

9. $\int \sec^2 x\csc^2 x\,dx \Rightarrow$

10. $\int \tan^3 x\sec x\,dx \Rightarrow$

8.3 TRIGONOMETRIC SUBSTITUTIONS

So far we have seen that it sometimes helps to replace a subexpression of a function by a single variable. Occasionally it can help to replace the original variable by something more complicated. This seems like a "reverse" substitution, but it is really no different in principle than ordinary substitution.

EXAMPLE 8.8 Evaluate $\int \sqrt{1-x^2}\,dx$. Let $x=\sin u$ so $dx=\cos u\,du$. Then

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2 u}\cos u\,du = \int \sqrt{\cos^2 u}\cos u\,du.$$

We would like to replace $\sqrt{\cos^2 u}$ by $\cos u$, but this is valid only if $\cos u$ is positive, since $\sqrt{\cos^2 u}$ is positive. Consider again the substitution $x=\sin u$. We could just as well think of this as $u=\arcsin x$. If we do, then by the definition of the arcsine, $-\pi/2\le u\le \pi/2$, so $\cos u\ge 0$. Then we continue:

$$\begin{aligned}\int \sqrt{\cos^2 u}\cos u\,du &= \int \cos^2 u\,du = \int \frac{1+\cos 2u}{2}\,du = \frac{u}{2}+\frac{\sin 2u}{4}+C\\ &= \frac{\arcsin x}{2}+\frac{\sin(2\arcsin x)}{4}+C.\end{aligned}$$

This is a perfectly good answer, though the term $\sin(2\arcsin x)$ is a bit unpleasant. It is possible to simplify this. Using the identity $\sin 2x = 2\sin x\cos x$, we can write $\sin 2u = 2\sin u\cos u = 2\sin(\arcsin x)\sqrt{1-\sin^2 u} = 2x\sqrt{1-\sin^2(\arcsin x)} = 2x\sqrt{1-x^2}$. Then the full antiderivative is

$$\frac{\arcsin x}{2}+\frac{2x\sqrt{1-x^2}}{4} = \frac{\arcsin x}{2}+\frac{x\sqrt{1-x^2}}{2}+C.$$

□

This type of substitution is usually indicated when the function you wish to integrate contains a polynomial expression that might allow you to use the fundamental identity $\sin^2 x+\cos^2 x=1$ in one of three forms:

$$\cos^2 x = 1-\sin^2 x \qquad \sec^2 x = 1+\tan^2 x \qquad \tan^2 x = \sec^2 x - 1.$$

If your function contains $1-x^2$, as in the example above, try $x=\sin u$; if it contains $1+x^2$ try $x=\tan u$; and if it contains x^2-1, try $x=\sec u$. Sometimes you will need to try something a bit different to handle constants other than one.

EXAMPLE 8.9 Evaluate $\int \sqrt{4-9x^2}\,dx$. We start by rewriting this so that it looks more like the previous example:

$$\int \sqrt{4-9x^2}\,dx = \int \sqrt{4(1-(3x/2)^2)}\,dx = \int 2\sqrt{1-(3x/2)^2}\,dx.$$

Now let $3x/2=\sin u$ so $(3/2)\,dx=\cos u\,du$ or $dx=(2/3)\cos u\,du$. Then

$$\begin{aligned}\int 2\sqrt{1-(3x/2)^2}\,dx &= \int 2\sqrt{1-\sin^2 u}\,(2/3)\cos u\,du = \frac{4}{3}\int \cos^2 u\,du\\ &= \frac{4u}{6}+\frac{4\sin 2u}{12}+C\\ &= \frac{2\arcsin(3x/2)}{3}+\frac{2\sin u\cos u}{3}+C\\ &= \frac{2\arcsin(3x/2)}{3}+\frac{2\sin(\arcsin(3x/2))\cos(\arcsin(3x/2))}{3}+C\\ &= \frac{2\arcsin(3x/2)}{3}+\frac{2(3x/2)\sqrt{1-(3x/2)^2}}{3}+C\\ &= \frac{2\arcsin(3x/2)}{3}+\frac{x\sqrt{4-9x^2}}{2}+C,\end{aligned}$$

using some of the work from example 8.8. □

EXAMPLE 8.10 Evaluate $\int \sqrt{1+x^2}\,dx$. Let $x=\tan u$, $dx=\sec^2 u\,du$, so

$$\int \sqrt{1+x^2}\,dx = \int \sqrt{1+\tan^2 u}\sec^2 u\,du = \int \sqrt{\sec^2 u}\sec^2 u\,du.$$

Since $u=\arctan(x)$, $-\pi/2\le u\le\pi/2$ and $\sec u\ge 0$, so $\sqrt{\sec^2 u}=\sec u$. Then

$$\int \sqrt{\sec^2 u}\sec^2 u\,du = \int \sec^3 u\,du.$$

In problems of this type, two integrals come up frequently: $\int \sec^3 u\,du$ and $\int \sec u\,du$. Both have relatively nice expressions but they are a bit tricky to discover.

First we do $\int \sec u\,du$, which we will need to compute $\displaystyle\int \sec^3 u\,du$:

$$\begin{aligned}\int \sec u\,du &= \int \sec u\,\frac{\sec u+\tan u}{\sec u+\tan u}\,du\\ &= \int \frac{\sec^2 u+\sec u\tan u}{\sec u+\tan u}\,du.\end{aligned}$$

Now let $w=\sec u+\tan u$, $dw=\sec u\tan u+\sec^2 u\,du$, exactly the numerator of the function we are integrating. Thus

$$\begin{aligned}\int \sec u\,du = \int \frac{\sec^2 u+\sec u\tan u}{\sec u+\tan u}\,du = \int \frac{1}{w}\,dw &= \ln|w|+C\\ &= \ln|\sec u+\tan u|+C.\end{aligned}$$

Now for $\displaystyle\int \sec^3 u\,du$:

$$\begin{aligned}\sec^3 u &= \frac{\sec^3 u}{2}+\frac{\sec^3 u}{2} = \frac{\sec^3 u}{2}+\frac{(\tan^2 u+1)\sec u}{2}\\ &= \frac{\sec^3 u}{2}+\frac{\sec u\tan^2 u}{2}+\frac{\sec u}{2} = \frac{\sec^3 u+\sec u\tan^2 u}{2}+\frac{\sec u}{2}.\end{aligned}$$

We already know how to integrate $\sec u$, so we just need the first quotient. This is "simply" a matter of recognizing the product rule in action:

$$\int \sec^3 u+\sec u\tan^2 u\,du = \sec u\tan u.$$

So putting these together we get

$$\int \sec^3 u\,du = \frac{\sec u\tan u}{2}+\frac{\ln|\sec u+\tan u|}{2}+C,$$

and reverting to the original variable x:

$$\begin{aligned}\int \sqrt{1+x^2}\,dx &= \frac{\sec u\tan u}{2}+\frac{\ln|\sec u+\tan u|}{2}+C\\ &= \frac{\sec(\arctan x)\tan(\arctan x)}{2}+\frac{\ln|\sec(\arctan x)+\tan(\arctan x)|}{2}+C\\ &= \frac{x\sqrt{1+x^2}}{2}+\frac{\ln|\sqrt{1+x^2}+x|}{2}+C,\end{aligned}$$

using $\tan(\arctan x)=x$ and $\sec(\arctan x)=\sqrt{1+\tan^2(\arctan x)}=\sqrt{1+x^2}$. □

Exercises 8.3.

Find the antiderivatives.

1. $\displaystyle\int \csc x\,dx \Rightarrow$

2. $\displaystyle\int \csc^3 x\,dx \Rightarrow$

3. $\displaystyle\int \sqrt{x^2-1}\,dx \Rightarrow$

4. $\displaystyle\int \sqrt{9+4x^2}\,dx \Rightarrow$

5. $\displaystyle\int x\sqrt{1-x^2}\,dx \Rightarrow$

6. $\displaystyle\int x^2\sqrt{1-x^2}\,dx \Rightarrow$

7. $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx \Rightarrow$

8. $\displaystyle\int \sqrt{x^2+2x}\,dx \Rightarrow$

9. $\displaystyle\int \frac{1}{x^2(1+x^2)}\,dx \Rightarrow$

10. $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx \Rightarrow$

11. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx \Rightarrow$

12. $\displaystyle\int \frac{x^3}{\sqrt{4x^2-1}}\,dx \Rightarrow$

8.4 INTEGRATION BY PARTS

We have already seen that recognizing the product rule can be useful, when we noticed that

$$\int \sec^3 u+\sec u\tan^2 u\,du = \sec u\tan u.$$

As with substitution, we do not have to rely on insight or cleverness to discover such antiderivatives; there is a technique that will often help to uncover the product rule.

Start with the product rule:

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x)+f(x)g'(x).$$

We can rewrite this as

$$f(x)g(x) = \int f'(x)g(x)\,dx+\int f(x)g'(x)\,dx,$$

and then

$$\int f(x)g'(x)\,dx = f(x)g(x)-\int f'(x)g(x)\,dx.$$

This may not seem particularly useful at first glance, but it turns out that in many cases we have an integral of the form

$$\int f(x)g'(x)\,dx$$

but that

$$\int f'(x)g(x)\,dx$$

is easier. This technique for turning one integral into another is called **integration by parts**, and is usually written in more compact form. If we let $u=f(x)$ and $v=g(x)$ then

$du = f'(x)\,dx$ and $dv = g'(x)\,dx$ and

$$\int u\,dv = uv - \int v\,du.$$

To use this technique we need to identify likely candidates for $u = f(x)$ and $dv = g'(x)\,dx$.

EXAMPLE 8.11 Evaluate $\int x\ln x\,dx$. Let $u = \ln x$ so $du = 1/x\,dx$. Then we must let $dv = x\,dx$ so $v = x^2/2$ and

$$\int x\ln x\,dx = \frac{x^2\ln x}{2} - \int \frac{x^2}{2}\frac{1}{x}\,dx = \frac{x^2\ln x}{2} - \int \frac{x}{2}\,dx = \frac{x^2\ln x}{2} - \frac{x^2}{4} + C.$$

□

EXAMPLE 8.12 Evaluate $\int x\sin x\,dx$. Let $u = x$ so $du = dx$. Then we must let $dv = \sin x\,dx$ so $v = -\cos x$ and

$$\int x\sin x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

□

EXAMPLE 8.13 Evaluate $\int \sec^3 x\,dx$. Of course we already know the answer to this, but we needed to be clever to discover it. Here we'll use the new technique to discover the antiderivative. Let $u = \sec x$ and $dv = \sec^2 x\,dx$. Then $du = \sec x\tan x$ and $v = \tan x$ and

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x\tan x - \int \tan^2 x\sec x\,dx\\ &= \sec x\tan x - \int (\sec^2 x - 1)\sec x\,dx\\ &= \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx.\end{aligned}$$

At first this looks useless—we're right back to $\int \sec^3 x\,dx$. But looking more closely:

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx\\ \int \sec^3 x\,dx + \int \sec^3 x\,dx &= \sec x\tan x + \int \sec x\,dx\\ 2\int \sec^3 x\,dx &= \sec x\tan x + \int \sec x\,dx\\ \int \sec^3 x\,dx &= \frac{\sec x\tan x}{2} + \frac{1}{2}\int \sec x\,dx\\ &= \frac{\sec x\tan x}{2} + \frac{\ln|\sec x + \tan x|}{2} + C.\end{aligned}$$

□

EXAMPLE 8.14 Evaluate $\int x^2\sin x\,dx$. Let $u = x^2$, $dv = \sin x\,dx$; then $du = 2x\,dx$ and $v = -\cos x$. Now $\int x^2\sin x\,dx = -x^2\cos x + \int 2x\cos x\,dx$. This is better than the original integral, but we need to do integration by parts again. Let $u = 2x$, $dv = \cos x\,dx$; then $du = 2$ and $v = \sin x$, and

$$\begin{aligned}\int x^2\sin x\,dx &= -x^2\cos x + \int 2x\cos x\,dx\\ &= -x^2\cos x + 2x\sin x - \int 2\sin x\,dx\\ &= -x^2\cos x + 2x\sin x + 2\cos x + C.\end{aligned}$$

□

Such repeated use of integration by parts is fairly common, but it can be a bit tedious to accomplish, and it is easy to make errors, especially sign errors involving the subtraction in the formula. There is a nice tabular method to accomplish the calculation that minimizes the chance for error and speeds up the whole process. We illustrate with the previous example. Here is the table:

sign	u	dv
	x^2	$\sin x$
$-$	$2x$	$-\cos x$
	2	$-\sin x$
$-$	0	$\cos x$

or

u	dv
x^2	$\sin x$
$-2x$	$-\cos x$
2	$-\sin x$
0	$\cos x$

To form the first table, we start with u at the top of the second column and repeatedly compute the derivative; starting with dv at the top of the third column, we repeatedly compute the antiderivative. In the first column, we place a "$-$" in every second row. To form the second table we combine the first and second columns by ignoring the boundary; if you do this by hand, you may simply start with two columns and add a "$-$" to every second row.

To compute with this second table we begin at the top. Multiply the first entry in column u by the second entry in column dv to get $-x^2\cos x$, and add this to the integral of the product of the second entry in column u and second entry in column dv. This gives:

$$-x^2\cos x + \int 2x\cos x\,dx,$$

or exactly the result of the first application of integration by parts. Since this integral is not yet easy, we return to the table. Now we multiply twice on the diagonal, $(x^2)(-\cos x)$

and $(-2x)(-\sin x)$ and then once straight across, $(2)(-\sin x)$, and combine these as

$$-x^2\cos x + 2x\sin x - \int 2\sin x\,dx,$$

giving the same result as the second application of integration by parts. While this integral is easy, we may return yet once more to the table. Now multiply three times on the diagonal to get $(x^2)(-\cos x)$, $(-2x)(-\sin x)$, and $(2)(\cos x)$, and once straight across, $(0)(\cos x)$. We combine these as before to get

$$-x^2\cos x + 2x\sin x + 2\cos x + \int 0\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

Typically we would fill in the table one line at a time, until the "straight across" multiplication gives an easy integral. If we can see that the u column will eventually become zero, we can instead fill in the whole table; computing the products as indicated will then give the entire integral, including the "$+C$", as above.

Exercises 8.4.

Find the antiderivatives.

1. $\int x\cos x\,dx \Rightarrow$

2. $\int x^2\cos x\,dx \Rightarrow$

3. $\int xe^x\,dx \Rightarrow$

4. $\int xe^{x^2}\,dx \Rightarrow$

5. $\int \sin^2 x\,dx \Rightarrow$

6. $\int \ln x\,dx \Rightarrow$

7. $\int x\arctan x\,dx \Rightarrow$

8. $\int x^2\sin x\,dx \Rightarrow$

9. $\int x\sin^2 x\,dx \Rightarrow$

10. $\int x\sin x\cos x\,dx \Rightarrow$

11. $\int \arctan(\sqrt{x})\,dx \Rightarrow$

12. $\int \sin(\sqrt{x})\,dx \Rightarrow$

13. $\int \sec^2 x\csc^2 x\,dx \Rightarrow$

8.5 RATIONAL FUNCTIONS

A **rational function** is a fraction with polynomials in the numerator and denominator. For example,

$$\frac{x^3}{x^2+x-6}, \qquad \frac{1}{(x-3)^2}, \qquad \frac{x^2+1}{x^2-1},$$

are all rational functions of x. There is a general technique called "partial fractions" that, in principle, allows us to integrate any rational function. The algebraic steps in the technique are rather cumbersome if the polynomial in the denominator has degree more than 2, and the technique requires that we factor the denominator, something that is not always possible. However, in practice one does not often run across rational functions with high degree polynomials in the denominator for which one has to find the antiderivative function. So we shall explain how to find the antiderivative of a rational function only when the denominator is a quadratic polynomial ax^2+bx+c.

We should mention a special type of rational function that we already know how to integrate: If the denominator has the form $(ax+b)^n$, the substitution $u = ax+b$ will always work. The denominator becomes u^n, and each x in the numerator is replaced by $(u-b)/a$, and $dx = du/a$. While it may be tedious to complete the integration if the numerator has high degree, it is merely a matter of algebra.

EXAMPLE 8.15 Find $\int \frac{x^3}{(3-2x)^5}\,dx$. Using the substitution $u = 3-2x$ we get

$$\begin{aligned}
\int \frac{x^3}{(3-2x)^5}\,dx &= \frac{1}{-2}\int \frac{\left(\frac{u-3}{-2}\right)^3}{u^5}\,du = \frac{1}{16}\int \frac{u^3-9u^2+27u-27}{u^5}\,du \\
&= \frac{1}{16}\int u^{-2}-9u^{-3}+27u^{-4}-27u^{-5}\,du \\
&= \frac{1}{16}\left(\frac{u^{-1}}{-1}-\frac{9u^{-2}}{-2}+\frac{27u^{-3}}{-3}-\frac{27u^{-4}}{-4}\right)+C \\
&= \frac{1}{16}\left(\frac{(3-2x)^{-1}}{-1}-\frac{9(3-2x)^{-2}}{-2}+\frac{27(3-2x)^{-3}}{-3}-\frac{27(3-2x)^{-4}}{-4}\right)+C \\
&= -\frac{1}{16(3-2x)}+\frac{9}{32(3-2x)^2}-\frac{9}{16(3-2x)^3}+\frac{27}{64(3-2x)^4}+C
\end{aligned}$$

□

We now proceed to the case in which the denominator is a quadratic polynomial. We can always factor out the coefficient of x^2 and put it outside the integral, so we can assume that the denominator has the form x^2+bx+c. There are three possible cases, depending on how the quadratic factors: either $x^2+bx+c = (x-r)(x-s)$, $x^2+bx+c = (x-r)^2$, or it doesn't factor. We can use the quadratic formula to decide which of these we have, and to factor the quadratic if it is possible.

EXAMPLE 8.16 Determine whether x^2+x+1 factors, and factor it if possible. The quadratic formula tells us that $x^2+x+1=0$ when

$$x = \frac{-1\pm\sqrt{1-4}}{2}.$$

Since there is no square root of -3, this quadratic does not factor. □

EXAMPLE 8.17 Determine whether $x^2 - x - 1$ factors, and factor it if possible. The quadratic formula tells us that $x^2 - x - 1 = 0$ when

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Therefore

$$x^2 - x - 1 = \left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right).$$

□

If $x^2 + bx + c = (x-r)^2$ then we have the special case we have already seen, that can be handled with a substitution. The other two cases require different approaches.

If $x^2 + bx + c = (x-r)(x-s)$, we have an integral of the form

$$\int \frac{p(x)}{(x-r)(x-s)}\,dx$$

where $p(x)$ is a polynomial. The first step is to make sure that $p(x)$ has degree less than 2.

EXAMPLE 8.18 Rewrite $\int \frac{x^3}{(x-2)(x+3)}\,dx$ in terms of an integral with a numerator that has degree less than 2. To do this we use long division of polynomials to discover that

$$\frac{x^3}{(x-2)(x+3)} = \frac{x^3}{x^2+x-6} = x - 1 + \frac{7x-6}{x^2+x-6} = x - 1 + \frac{7x-6}{(x-2)(x+3)},$$

so

$$\int \frac{x^3}{(x-2)(x+3)}\,dx = \int x - 1\,dx + \int \frac{7x-6}{(x-2)(x+3)}\,dx.$$

The first integral is easy, so only the second requires some work. □

Now consider the following simple algebra of fractions:

$$\frac{A}{x-r} + \frac{B}{x-s} = \frac{A(x-s) + B(x-r)}{(x-r)(x-s)} = \frac{(A+B)x - As - Br}{(x-r)(x-s)}.$$

That is, adding two fractions with constant numerator and denominators $(x-r)$ and $(x-s)$ produces a fraction with denominator $(x-r)(x-s)$ and a polynomial of degree less than 2 for the numerator. We want to reverse this process: starting with a single fraction, we want to write it as a sum of two simpler fractions. An example should make it clear how to proceed.

EXAMPLE 8.19 Evaluate $\int \frac{x^3}{(x-2)(x+3)}\,dx$. We start by writing $\frac{7x-6}{(x-2)(x+3)}$ as the sum of two fractions. We want to end up with

$$\frac{7x-6}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}.$$

If we go ahead and add the fractions on the right hand side we get

$$\frac{7x-6}{(x-2)(x+3)} = \frac{(A+B)x + 3A - 2B}{(x-2)(x+3)}.$$

So all we need to do is find A and B so that $7x - 6 = (A+B)x + 3A - 2B$, which is to say, we need $7 = A + B$ and $-6 = 3A - 2B$. This is a problem you've seen before: solve a system of two equations in two unknowns. There are many ways to proceed; here's one: If $7 = A+B$ then $B = 7-A$ and so $-6 = 3A-2B = 3A-2(7-A) = 3A-14+2A = 5A-14$. This is easy to solve for A: $A = 8/5$, and then $B = 7 - A = 7 - 8/5 = 27/5$. Thus

$$\int \frac{7x-6}{(x-2)(x+3)}\,dx = \int \frac{8}{5}\frac{1}{x-2} + \frac{27}{5}\frac{1}{x+3}\,dx = \frac{8}{5}\ln|x-2| + \frac{27}{5}\ln|x+3| + C.$$

The answer to the original problem is now

$$\begin{aligned}\int \frac{x^3}{(x-2)(x+3)}\,dx &= \int x - 1\,dx + \int \frac{7x-6}{(x-2)(x+3)}\,dx \\ &= \frac{x^2}{2} - x + \frac{8}{5}\ln|x-2| + \frac{27}{5}\ln|x+3| + C.\end{aligned}$$

□

Now suppose that $x^2 + bx + c$ doesn't factor. Again we can use long division to ensure that the numerator has degree less than 2. Now we can complete the square to turn the integral into a trigonometric substitution problem.

EXAMPLE 8.20 Evaluate $\int \frac{x}{x^2+x+1}\,dx$. We have seen that this quadratic does not factor. We complete the square:

$$x^2 + x + 1 = x^2 + x + \frac{1}{4} + 1 - \frac{1}{4} = x^2 + x + \frac{1}{4} + \frac{3}{4} = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

Now factor out 3/4:

$$\frac{3}{4}\left(\frac{4}{3}\left(x + \frac{1}{2}\right)^2 + 1\right) = \frac{3}{4}\left(\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2 + 1\right).$$

Now let

$$\tan u = \frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}$$

$$\sec^2 u\,du = \frac{2}{\sqrt{3}}\,dx$$

$$dx = \frac{\sqrt{3}}{2}\sec^2 u\,du.$$

Now we can substitute in the original integral, using $x = (\sqrt{3}/2)(\tan u - 1/\sqrt{3})$ in the numerator.

$$\begin{aligned}
\int \frac{x}{x^2+x+1}\,dx &= \frac{4}{3}\int \frac{(\sqrt{3}/2)(\tan u - 1/\sqrt{3})}{\tan^2 u + 1}\,\frac{\sqrt{3}}{2}\sec^2 u\,du \\
&= \frac{4}{3}\int \frac{(\sqrt{3}/2)(\tan u - 1/\sqrt{3})}{\sec^2 u}\,\frac{\sqrt{3}}{2}\sec^2 u\,du \\
&= \frac{4}{3}\int \frac{\sqrt{3}}{2}\left(\tan u - \frac{1}{\sqrt{3}}\right)\frac{\sqrt{3}}{2}\,du \\
&= \int \tan u - \frac{1}{\sqrt{3}}\,du \\
&= -\ln|\cos u| - \frac{u}{\sqrt{3}} + C. \\
&= \ln|\sec u| - \frac{u}{\sqrt{3}} + C.
\end{aligned}$$

Finally, we can substitute

$$u = \arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)$$

and

$$\sec u = \sqrt{\tan^2 u + 1} = \sqrt{\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2 + 1}$$

to get

$$\ln\left|\sqrt{\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2 + 1}\right| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + C.$$

The details here are admittedly a bit unpleasant, but the whole process is fairly mechanical and "easy" in principle. □

Exercises 8.5.

Find the antiderivatives.

1. $\displaystyle\int \frac{1}{4-x^2}\,dx \Rightarrow$

2. $\displaystyle\int \frac{x^4}{4-x^2}\,dx \Rightarrow$

3. $\displaystyle\int \frac{1}{x^2+10x+25}\,dx \Rightarrow$

4. $\displaystyle\int \frac{x^2}{4-x^2}\,dx \Rightarrow$

5. $\displaystyle\int \frac{x^4}{4+x^2}\,dx \Rightarrow$

6. $\displaystyle\int \frac{1}{x^2+10x+29}\,dx \Rightarrow$

7. $\displaystyle\int \frac{x^3}{4+x^2}\,dx \Rightarrow$

8. $\displaystyle\int \frac{1}{x^2+10x+21}\,dx \Rightarrow$

9. $\displaystyle\int \frac{1}{2x^2-x-3}\,dx \Rightarrow$

10. $\displaystyle\int \frac{1}{x^2+3x}\,dx \Rightarrow$

8.6 Additional exercises

These problems require the techniques of this chapter, and are in no particular order. Some problems may be done in more than one way.

1. $\displaystyle\int (t+4)^3\,dt \Rightarrow$

2. $\displaystyle\int t(t^2-9)^{3/2}\,dt \Rightarrow$

3. $\displaystyle\int (e^{t^2}+16)te^{t^2}\,dt \Rightarrow$

4. $\displaystyle\int \sin t\cos 2t\,dt \Rightarrow$

5. $\displaystyle\int \tan t\sec^2 t\,dt \Rightarrow$

6. $\displaystyle\int \frac{2t+1}{t^2+t+3}\,dt \Rightarrow$

7. $\displaystyle\int \frac{1}{t(t^2-4)}\,dt \Rightarrow$

8. $\displaystyle\int \frac{1}{(25-t^2)^{3/2}}\,dt \Rightarrow$

9. $\displaystyle\int \frac{\cos 3t}{\sqrt{\sin 3t}}\,dt \Rightarrow$

10. $\displaystyle\int t\sec^2 t\,dt \Rightarrow$

11. $\displaystyle\int \frac{e^t}{\sqrt{e^t+1}}\,dt \Rightarrow$

12. $\displaystyle\int \cos^4 t\,dt \Rightarrow$

13. $\displaystyle\int \frac{1}{t^2+3t}\,dt \Rightarrow$

14. $\displaystyle\int \frac{1}{t^2\sqrt{1+t^2}}\,dt \Rightarrow$

15. $\displaystyle\int \frac{\sec^2 t}{(1+\tan t)^3}\,dt \Rightarrow$

16. $\displaystyle\int t^3\sqrt{t^2+1}\,dt \Rightarrow$

17. $\displaystyle\int e^t\sin t\,dt \Rightarrow$

18. $\displaystyle\int (t^{3/2}+47)^3\sqrt{t}\,dt \Rightarrow$

19. $\displaystyle\int \frac{t^3}{(2-t^2)^{5/2}}\,dt \Rightarrow$

20. $\displaystyle\int \frac{1}{t(9+4t^2)}\,dt \Rightarrow$

21. $\displaystyle\int \frac{\arctan 2t}{1+4t^2}\,dt \Rightarrow$

22. $\displaystyle\int \frac{t}{t^2+2t-3}\,dt \Rightarrow$

23. $\displaystyle\int \sin^3 t\cos^4 t\,dt \Rightarrow$

24. $\displaystyle\int \frac{1}{t^2-6t+9}\,dt \Rightarrow$

25. $\displaystyle\int \frac{1}{t(\ln t)^2}\,dt \Rightarrow$

26. $\displaystyle\int t(\ln t)^2\,dt \Rightarrow$

27. $\displaystyle\int t^3e^t\,dt \Rightarrow$

28. $\displaystyle\int \frac{t+1}{t^2+t-1}\,dt \Rightarrow$