Many physical phenomena can be modeled using the language of calculus. For example, observational evidence suggests that the temperature of a cup of tea (or some other liquid) in a room of constant temperature will cool over time at a rate proportional to the difference between the room temperature and the temperature of the tea.

In symbols, if $t$ is the time, $M$ is the room temperature, and $f(t)$ is the temperature of the tea at time $t$ then $F(t) = k(M - f(t))$ where $k > 0$ is a constant which will depend on the kind of tea (or more generally the kind of liquid) but not on the room temperature or the temperature of the tea. This is *Newton's law of cooling* and the equation that we just wrote down is an example of a differential equation. Ideally we would like to solve this equation, namely, find the function $f(t)$ that describes the temperature over time, though this often turns out to be impossible, in which case various approximation techniques must be used. The use and solution of differential equations is an important field of mathematics; here we see how to solve some simple but useful types of differential equation.

Informally, a differential equation is an equation in which one or more of the derivatives of a function appear. Typically, a scientific theory will produce a differential equation (or a system of differential equations) that describes or governs some physical process, but the theory will not produce the desired function or functions directly.

Recall from section 6.2 that when the variable is time the derivative of a function $y(t)$ is sometimes written as $\dot{y}$ instead of $y'$; this is quite common in the study of differential equations.

### Example 17.1.1

Solve the differential equation $\dot{y} = 2(25 - y)$ for $y(0) = 40$.

The general first order equation is rather too general, that is, we can't describe methods that will work on them all, or even a large portion of them. We can make progress with specific kinds of first order differential equations. For example, much can be said about equations of the form $\dot{y} = a(t, y)$ where $a$ is a function of the two variables $t$ and $y$. Under reasonable conditions on $a$, such an equation has a solution and the corresponding initial value problem has a unique solution. However, in general, these equations can be very difficult or impossible to solve explicitly.

### Example 17.1.6

Consider this specific example of an initial value problem for Newton's law of cooling: $\dot{y} = 2(25 - y), y(0) = 40$. We first note that if $y(t) = 25$, the right hand side of the differential equation is zero and so the constant function $y(t) = 25$ is a solution to the differential equation. It is not a solution to the initial value problem, since $y(0) \neq 40$. (The physical interpretation of this constant solution is that if a liquid is at the same temperature as its surroundings, then the liquid will stay at that temperature.)

### Example 17.1.7

Solve the differential equation $\dot{y} = 2(25 - y)$. This is almost identical to the previous example. As before, $y(t) = 25$ is a solution. If $y(0) \neq 25$,

$$
\int \frac{1}{25 - y} \, dy = \int 2 \, dt
$$

so

$$
\int \frac{1}{25 - y} \, dy = 2t + C_0
$$

and

$$
l(25 - y) = 2t + C_0
$$

for some non-zero constant $C_0$.

As before, we substitute and solve for A:

$$
40 = 25 + Ac_0
$$

so $A = 5$. Hence, the solution to this equation is

$$
y(t) = 25 + 5e^{-2t}
$$

and so $y = 25 + 5e^{-2t}$ is a solution to the initial value problem. Note that $y$ is never 25, so this makes sense for all values of $t$. However, if we allow $A = 0$ we get the solution $y = 25$ to the differential equation, which would be the solution to the initial value problem if we were to require $y(0) = 25$. Thus, $y = 25 + 5e^{-2t}$ describes all solutions to the differential equation $\dot{y} = 2(25 - y)$, and all solutions to the associated initial value problems.

### Example 17.1.8

A first order differential equation is called separable if it can be written in the form $\dot{y} = f(y)g(y)$.

As in the examples, we can attempt to solve a separable equation by converting to the form

$$
\int \frac{1}{f(y)} \, dy = \int f(t) \, dt
$$

We can do this if we can find an anti-derivative of $f(t)$.
Of course, there are a few places this ideal description could go wrong; we need to be able to find the antiderivatives $G$ and $F$, and we need to solve the final equation for $y$. The upshot is that the solutions to the original differential equation are the constant solutions, if any, and all functions $y$ that satisfy $G(y) = F(t) + C$. 

**EXAMPLE 17.1.9** Consider the differential equation $y = ky$. When $k > 0$, this describes certain simple cases of population growth: it says that the change in the population $y$ is proportional to the population. The underlying assumption is that each organism in the current population reproduces at a fixed rate, so that the larger the population the more new organisms are produced. While this is too simple to model most real populations, it is useful in some cases over a limited time. When $k < 0$, the differential equation describes a quantity that decreases in proportion to the current value; this can be used to model radioactive decay.

The constant solution is $y(t) = 0$; of course this will not be the solution to any interesting initial value problem. For the non-constant solutions, we proceed much as before:

\[ \int \frac{1}{y} \, dy = \int k \, dt \]

\[ \ln |y| = kt + C \]

\[ y = e^{kt+C} = Ae^{kt}. \]

Again, if we allow $A = 0$ this includes the constant solution, and we can simply say that $y = Ae^{kt}$ is the general solution. With an initial value we can easily solve for $A$ to get the solution of the initial value problem. In particular, if the initial value is given for time $t = 0$: $y(0) = y_0$, then $A = y_0$ and the solution is $y = y_0 e^{kt}$.

**Exercises 17.1.**

1. Which of the following equations are separable?
   a. $\dot{y} = \sin(y)$
   b. $\dot{y} = e^t$
   c. $\dot{y} = -t$
   d. $\dot{y} = (t^2 - t) \arcsin(y)$
   e. $\dot{y} = t \ln y + 4e^{3\ln y}$

2. Solve $\dot{y} = 1/(1 + t^2)$. ⇒

3. Solve the initial value problem $\dot{y} = t^2$ with $y(0) = 1$ and $n \geq 0$. ⇒

4. Solve $\dot{y} = \sin t$. ⇒

**17.2 First Order Homogeneous Linear Equations**

**EXAMPLE 17.2.2** The equation $\dot{y} = 2(2t - 2) - y$ can be written $\dot{y} + 2ty = 50t$. This is linear, but not homogeneous. The equation $\dot{y} = ky$, or $\dot{y} - ky = 0$ is linear and homogeneous, with a particularly simple $P(t) = -k$.

Because first order homogeneous linear equations are separable, we can solve them in the usual way:

\[ \int \frac{1}{y} \, dy = \int -p(t) \, dt \]

\[ \ln |y| = P(t) + C \]

\[ y = -e^{P(t)} \]

where $P(t)$ is an anti-derivative of $-p(t)$. As in previous examples, if we allow $A = 0$ we get the constant solution $y = 0$.

**EXAMPLE 17.2.3** Solve the initial value problems $\dot{y} + y\cos t = 0$, $y(0) = 1/2$ and $y(2) = 1/2$. We start with

\[ P(t) = -\cos t \, dt = -\sin t, \]

so the general solution to the differential equation is $y = Ae^{-\sin t}$.

To compute $A$ we substitute:

\[ \frac{1}{2} = A e^{-\sin \pi/2} = A, \]

so the solutions is

\[ y = \frac{1}{2} e^{-\sin t}. \]

For the second problem,

\[ \frac{1}{2} = A e^{-\sin 2} \]

\[ A = \frac{1}{2} e^{\sin 2} \]

so the solution is

\[ y = \frac{1}{2} e^{\sin 2} e^{-\sin t}. \]
As you might guess, a first order linear differential equation has the form \( \dot{y} + p(t)y = f(t) \). Not only is this closely related in form to the first order linear differential equation, we can use what we know about solving homogeneous equations to solve the general linear equation.

Suppose that \( y_1(t) \) and \( y_2(t) \) are solutions to \( \dot{y} + p(t)y = f(t) \). Let \( y(t) = y_1(t) - y_2(t) \). Then

\[
\dot{y}(t) + p(t)y(t) = y_1'(t) - y_2'(t) + p(t)(y_1(t) - y_2(t)) \]

\[
= (y_1' + p(t)y_1) - (y_2' + p(t)y_2) \]

\[
= f(t) - f(t) = 0.
\]

In other words, \( y(t) = y_1(t) - y_2(t) \) is a solution to the homogeneous equation \( \dot{y} + p(t)y = 0 \). Turning this around, any solution to the linear equation \( \dot{y} + p(t)y = f(t) \), call it \( y \), can be written as \( y = y_1 + y_2 \), for some particular \( y_2 \) and some solution \( y(t) \) of the homogeneous equation \( \dot{y} + p(t)y = 0 \). Since we already know how to find all solutions of the homogeneous equation, finding just one solution to the equation \( \dot{y} + p(t)y = f(t) \) will give us all of them.

How might we find that one particular solution to \( \dot{y} + p(t)y = f(t) \)? Again, it turns out that what we already know helps. We know that the general solution to the homogeneous equation \( \dot{y} + p(t)y = 0 \) looks like \( Ae^{\int p(t)dt} \). We now make an inspired guess: consider the function \( v(t) = \int p(t)dt \), in which we have replaced the constant parameter \( A \) with the function \( v(t) \). This technique is called variation of parameters. For convenience write this as \( s(t) = v(t)h(t) \) where \( h(t) = e^{\int p(t)dt} \) is a solution to the homogeneous equation. Now let’s compute a bit with \( s(t) \):

\[
s'(t) + p(t)s(t) = v'(t)h(t) + v(t)h'(t) + p(t)v(t)h(t) \]

\[
= v(h'(t) + p(t)h(t)) + v(t)(h'(t) + s(t))h(t) \]

The last equality is true because \( h'(t) + p(t)h(t) = 0 \), since \( h(t) \) is a solution to the homogeneous equation. We are hoping to find a function \( v(t) \) such that \( s'(t) + p(t)s(t) = f(t) \). We have a function \( s(t) \) that every solution to the equation looks like, in the constant parameter \( A \) with the function \( v(t) \). This technique is called variation of parameters. For convenience write this as \( s(t) = v(t)h(t) \) where \( h(t) = e^{\int p(t)dt} \) is a solution to the homogeneous equation. Now let’s compute a bit with \( s(t) \):

\[
s'(t) + p(t)s(t) = v'(t)h(t) + v(t)h'(t) + p(t)v(t)h(t) \]

\[
= v(h'(t) + p(t)h(t)) + v(t)(h'(t) + s(t))h(t) \]

The last equality is true because \( h'(t) + p(t)h(t) = 0 \), since \( h(t) \) is a solution to the homogeneous equation. We are hoping to find a function \( v(t) \) such that \( s'(t) + p(t)s(t) = f(t) \). We have a function \( s(t) \) that every solution to the equation looks like, in the constant parameter \( A \) with the function \( v(t) \).

**EXAMPLE 17.3.1** Find the solution of the initial value problem \( \dot{y} + 3y/t = t^2 \), \( y(1) = 1/2 \). First find the general solution, since we are interested in a solution with a given boundary condition.

Integrating both sides gives

\[
e^{\int p(t)dt}y = \int e^{\int p(t)dt}f(t)dt + \int C dt.
\]

If you look carefully, you will see that this is exactly the same solution we found by variation of parameters, because \( e^{\int p(t)dt}f(t) = f(t)/h(t) \).

Some people find it easier to remember how to use the integrating factor method than variation of parameters. Since ultimately they require the same calculation, you should use whichever of the two you find easier to recall. Using this method, the solution of the previous example would look just a bit different: Starting with \( \dot{y} + 3y/t = t^2 \), we recall that the integrating factor is \( e^{\int p(t)dt} = e^{\int 3dt} = t^3 \). Then we multiply through by the integrating factor and solve:

\[
t^3\dot{y} + t^33y/t = t^3t^2 \]

\[
d/dt(t^3y) = t^5 \]

\[
t^3y = \int t^5 dt \]

\[
= t^6/6 + C \]

\[
y = t^6/6 + C/t^3.
\]

This is the same answer, of course, and the problem is then finished just as before.

**Exercises 17.3.**

In problems 1–10, find the general solution of the equation.

1. \( \dot{y} + 3y = 5 \)
2. \( \dot{y} + 2y = 6 \)
3. \( \dot{y} + 4y = 5t \)
4. \( \dot{y} - 2y = -2e^{-t} \)
5. \( \dot{y} - y = e^{t} \)
6. \( 2\dot{y} + y = t \)
7. \( \dot{y} + 2y = 0 \), \( t > 0 \)
8. \( \dot{y} - y = \sqrt{2}, t > 0 \)
9. \( \dot{x} + \dot{y} + \dot{z} = 1, -\pi/2 < t < \pi/2 \)
10. \( \dot{y} + \dot{x} = \dot{t}, -\pi/2 < t < \pi/2 \)

**Chapter 17 Differential Equations**

**17.4 Approximation**

We have seen how to solve a restricted collection of differential equations, or more accurately, how to attempt to solve them—we may not be able to find the required antiderivatives. Not surprisingly, non-linear equations can be even more difficult to solve. Yet much is known about solutions to some more general equations.

Suppose \( \phi(x,y) \) is a function of two variables. A first general class of first order differential equations has the form \( \dot{y} = \phi(t,y) \). This is not necessarily a linear first order equation, since \( \phi \) may depend on \( y \) in some complicated way; note however that \( \dot{y} \) appears in a very simple form. Under suitable conditions on the function \( \phi \), it can be shown that every such differential equation has a solution, and moreover that for each initial condition the associated initial value problem has exactly one solution. In practical applications this is obviously a very desirable property.

**EXAMPLE 17.4.1** The equation \( \dot{y} = -t \cdot \dot{y} \) is a first order non-linear equation, because \( \dot{y} \) appears to the second power. We will not be able to solve this equation.

**EXAMPLE 17.4.2** The equation \( \dot{y} = y^2 \) is also non-linear, but it is separable and can be solved by separation of variables.

Not all differential equations that are important in practice can be solved exactly; so techniques have been developed to approximate solutions. We describe one such technique, Euler’s Method, which is simple though not particularly useful compared to some more sophisticated techniques.

Suppose we wish to approximate a solution to the initial value problem \( \dot{y} = \phi(t,y), \ y(0) = y_0 \) for \( t \geq t_0 \). Under reasonable conditions on \( \phi \), we know the solution exists, represented by a curve in the \( t,y \) plane; call this solution \( f(t) \). The point \( (t_0,y_0) \) is of course on this curve. We also know the slope of the curve at this point, namely \( \phi(t_0,y_0) \). If we follow the tangent line for a brief distance, we arrive at a point that should be almost on the graph of \( f(t) \), namely \( (t_0 + \Delta t, y_0 + \phi(t_0,y_0)\Delta t) \); call this point \( (t_1,y_1) \). Now we pretend, in effect, that this point really is on the graph of \( f(t) \), in which case we again know the slope of the curve through \( (t_1,y_1) \), namely \( \phi(t_1,y_1) \). So we can compute a new point, \( (t_2,y_2) = (t_1 + \Delta t, y_1 + \phi(t_1,y_1)\Delta t) \). We can continue in this way, doing a sequence of straightforward calculations, until we have an approximation \( (t_n,y_n) \) for whatever time \( t_n \) we need. At each step we do essentially the same calculation, namely

\[
(t_{n+1},y_{n+1}) = (t_n + \Delta t, y_n + \phi(t_n,y_n)\Delta t)
\]

We expect that smaller time steps \( \Delta t \) will give better approximations, and for course it will require more work to compute to a specified time. It is possible to compute a guaranteed approximation.
upper bound on how far off the approximation might be, that is, how far \( y_0 \) is from \( f(t_0) \).

Suffice it to say that the bound is not particularly good and that there are other more complicated approximation techniques that do better.

**Example 17.4.3** Let us compute an approximation to the solution for \( y' = t - y^2 \), \( y(0) = 0 \), when \( t = 1 \). We will use \( \Delta t = 0.2 \), which is easy to do even by hand, though we should not expect the resulting approximation to be very good. We get

\[
\begin{array}{|c|c|c|}
\hline
i & (t_i, y_i) & \Delta t \\
\hline
0 & (0, 0) & 0 \\
1 & (0.2, 0.02) & 0.2 \\
2 & (0.4, 0.004) & 0.4 \\
3 & (0.6, 0.0004) & 0.6 \\
4 & (0.8, 0.00004) & 0.8 \\
\hline
\end{array}
\]

So \( y(1) \approx 0.3856 \). As it turns out, this is not accurate to even one decimal place. Figure 17.4.1 shows these points connected by line segments (the lower curve) compared to a solution obtained by a much better approximation technique. Note that the shape is approximately correct even though the end points are quite far apart.

![Figure 17.4.1 Approximating a solution to \( y = t - y^2 \), \( y(0) = 0 \).](image1.png)

If you need to do Euler's method by hand, it is useful to construct a table to keep track of the work, as shown in figure 17.4.2. Each row holds the computation for a single step: the starting point \((t_0, y_0)\), the stepsize \(\Delta t\), the computed slope \(f(t, y)\), the change in \(y\), \(\Delta y = f(t, y)\Delta t\), and the new point. \((t_1, y_1) = (t_0 + \Delta t, y_0 + \Delta y)\). The starting point in each row is the newly computed point from the end of the previous row.

It is easy to write a short function in Sage to do Euler's method; see this Sage worksheet.

![Figure 17.4.2 A slope field for \( y = t - y^2 \).](image2.png)

**Exercises 17.4.**

In problems 1-4, compute the Euler approximations for the initial value problem for \( 0 \leq t \leq 1 \) and \( \Delta t = 0.2 \). If you have access to Sage, generate the slope field first and attempt to sketch the solution curve. Then use Sage to compute better approximations with smaller values of \( \Delta t \).

1. \( y' = y, y(0) = 1 \) ⇒
2. \( y' = t + y^2, y(0) = 1 \) ⇒
3. \( y = \cos(t) + y, y(0) = 1 \) ⇒
4. \( y = \ln(t), y(0) = 2 \) ⇒

**17.5 Second Order Homogeneous Equations**

A second order differential equation is one containing the second derivative. These are in general quite complicated, but one fairly simple type is useful: the second order linear equation with constant coefficients.

**Example 17.5.1** Consider the initial value problem \( y' - y - 2y = 0 \), \( y(0) = 5 \), \( y(0) = 0 \). We make an inspired guess: might there be a solution of the form \( e^{rt} \)? This seems at least plausible, since in this case \( \dot{y}, y, \) and \( y \) all involve \( e^{rt} \).

If such a function is a solution then

\[
\begin{align*}
A^2e^{rt} - Ae^{rt} - 2e^{rt} &= 0 \\
e^{rt}(r^2 - r - 2) &= 0 \\
(r^2 - r) &= 0 \\
solve \quad r &= 0, 2 \end{align*}
\]

so \( r = 0 \) or \( 2 \). Not only are \( f = e^r \) and \( g = e^{-r} \) solutions, but notice that \( y = Af + Bg \) is also, for any constants \( A \) and \( B \):

\[
(\{Af + Bg\}'' - \{Af + Bg\}' - 2(Af + Bg)) = Af'' + Bg'' - Af' - Bg' - 2Af - 2Bg = Af'' - 2Af' + Bg'' - 2Bg = 0.
\]

Can we find \( A \) and \( B \) so that this is a solution to the initial value problem? Let's substitute:

\[
5 = y(0) = Af(0) + Bg(0) = Ae^0 + B = A + B
\]

and

\[
0 = y(1) = Af(1) + Bg(1) = 2Ae^2 + B = 2A + B.
\]

So we need to find \( A \) and \( B \) that make both \( 5 = A + B \) and \( 0 = 2A - B \) true. This is a simple system of simultaneous equations: solve \( 2A - B = 0 \), substitute to get \( 5 = A + 2A = 3A \). Then \( A = 5/3 \) and \( B = 10/3 \), and the desired solution is \( 5(5/3)e^{t} + (10/3)e^{-t} \). You now see why the initial condition in this case included both \( y(0) \) and \( y(0) \); we needed two equations in the two unknowns \( A \) and \( B \).

You should of course wonder whether there might be other solutions: the answer is no. We will not prove this, but here is the theorem that tells us what we need to know:

**Theorem 17.5.2** Given the differential equation \( ay' + by + cy = 0 \), \( a \neq 0 \), consider the quadratic polynomial \( ax^2 + bx + c \), called the characteristic polynomial. Using the quadratic formula, this polynomial always has one or two roots, call them \( r \) and \( s \). The general solution of the differential equation is:

\[
\begin{align*}
\text{(a)} \quad y &= Ae^{rt} + Be^{st} \quad \text{if the roots \( r \) and \( s \) are real and \( r \neq s \).} \\
\text{(b)} \quad y &= Ae^{rt} + Be^{st} \quad \text{if \( r = s \) is real.} \\
\text{(c)} \quad y &= A \cos(\beta t)e^{rt} + B \sin(\beta t)e^{rt} \quad \text{if the roots \( r \) and \( s \) are complex numbers \( \alpha + \beta i \) and \( \alpha - \beta i \).}
\end{align*}
\]
EXAMPLE 17.5.3 Suppose a mass $m$ is hung on a spring with spring constant $k$. If the spring is compressed or stretched and then released, the mass will oscillate up and down. Because of friction, the oscillation will be damped; eventually the motion will cease. The damping will depend on the amount of friction; for example, if the system is suspended in oil, then the friction involved is greater than if the system is in air. Using some simple physics, it is not hard to see that the position of the mass is described by the differential equation: 

$$m\ddot{y} + b\dot{y} + ky = 0.$$ 

Using $m = 1$, $b = 4$, and $k = 5$ we find the motion of the mass. The characteristic polynomial is $s^2 + 4s + 5 = 0$, with roots $-2 \pm i\sqrt{3}$. Thus the general solution is $y = A\cos(2t) - B\sin(2t)$. Suppose we know that $y(0) = 1$ and $\dot{y}(0) = 2$. Then as before we form two simultaneous equations: from $y(0) = 1$ we get $1 = A\cos(0) - B\sin(0) = A$. For the second we compute

$$\dot{y}(0) = -2A\sin(t) + 2Ac\cos(t) + 2B\cos(t) - 2B\sin(t).$$

and then

$$2A = -2A\cos(0) - 2B\sin(0) - 2B\cos(0) - 2A + B = -2A + B.$$

So we get $A = 1, B = 4$, and $y = \cos(t)e^{-2t} + 3\sin(t)e^{-2t}$.

Here is a useful trick that makes this easier to understand: We have $y = (c\cos(2t) + d\sin(2t))$. The expression $\cos(t) + 4t\sin(2t)$ is a bit reminiscent of the trigonometric formula $\cos(a - \beta) = \cos(a)\cos(\beta) + \sin(a)\sin(\beta)$ with $a = 1$. Let's rewrite it as

$$\sqrt{17}(\sin(t) + 4t\cos(t)).$$

Note that $(1/\sqrt{17})^2 + (4/\sqrt{17})^2 = 1$, which means that there is an angle $\beta$ with $\cos \beta = 1/\sqrt{17}$ and $\sin \beta = 4/\sqrt{17}$ (of course, $\beta$ may not be a “nice” angle). Thus

$$\cos(t) + 4t\sin(2t) = \sqrt{17}(\cos(t) + 4\sin(\beta t)).$$

Finally, the solution may also be written $y = \sqrt{17}\cos(2t - \beta t)$. This is a cosine curve that has been shifted $\beta$ to the right; the $\sqrt{17}\cos(2t)$ has the effect of diminishing the amplitude of the cosine as $t$ increases; see figure 17.5.1. The oscillation is damped very quickly, so in the first graph it is not clear that this is an oscillation. The second graph shows a restricted range for $t$.

Other physical systems that oscillate can also be described by such differential equations. Some electric circuits, for example, generate oscillating current.

EXAMPLE 17.5.4 Find the solution to the initial value problem $\ddot{y} - 2\dot{y} + 4\dot{y} + 4y = 0$, $y(0) = -3$, $\dot{y}(0) = 1$. The characteristic polynomial is $s^2 - 4s + 4 = 0$, so there is a repeated root $s = 2$. The only solution is $y = c_1e^{2t} + c_2te^{2t}$.
To make this equal to \( \cos(4 \omega t) \). For the particular solution, we get

\[
(9Ct^2e^{3t} + 6Ce^{3t} + 2Cte^{3t}) - 6(3Ce^{3t} + 2Cte^{3t}) + 9Ct^2e^{3t} = e^{3t}(3Ce^{3t}).
\]

The solution is thus \( 3Ce^{3t} + Bte^{3t} + (1/3)t^2e^{3t} \).

It is common in various physical systems to encounter an \( f(t) \) of the form \( a \cos(\omega t) + b \sin(\omega t) \).

**EXAMPLE 17.6.6**

Find the general solution to \( y'' + 6y + 25y = -e^{3t} \). The roots of the characteristic equation are \(-3 \pm i\), so the solution to the homogeneous equation is \( e^{-3t}(Ae^{3t} + Be^{3t}) \). For a particular solution, we guess \( C \cos(\omega t) + D \sin(\omega t) \). Substituting as usual:

\[
(-16C \cos(4t) + 16D \sin(4t)) + (-6\omega^2 \cos(4t) + 4\omega^2 \sin(4t)) = (24D + 9C) \cos(4t) + (-24C + 9D) \sin(4t).
\]

To make this equal to \( -e^{3t} \) we need

\[
24D + 9C = 1 \quad \text{and} \quad 24C - 9D = 0
\]

which gives \( C = 1/73 \) and \( D = 8/219 \). The full solution is then \( e^{-3t}(Ae^{3t} + Be^{3t}) + (1/73)(C \cos(4t) + (8/219) \sin(4t)) \).
Now the particular solution we seek is
\[
we^{2t} + ve^{3t} = \frac{1}{5} \left( 2 \sin t + \cos t \right) e^{2t} - \frac{1}{10} (3 \sin t + \cos t) e^{3t} = \frac{1}{10} (\sin t + \cos t),
\]
and the solution to the differential equation is \( Ae^{2t} + Be^{3t} + (\sin t + \cos t)/10. \) For comparison (and practice) you might want to solve this using the method of undetermined coefficients.

**EXAMPLE 17.7.2** The differential equation \( \ddot{y} - 5\dot{y} + 6y = e^{\frac{t}{2}} \) can be solved using the method of undetermined coefficients, though we have not seen any examples of such a solution. Again, we will solve it by variation of parameters. The equations to be solved are
\[
\ddot{u}e^{2t} + \dot{v}e^{3t} = 0 \\
2\dot{u}e^{2t} + 3\dot{v}e^{3t} = e^{\frac{t}{2}}.
\]
If we multiply the first equation by 2 and subtract it from the second equation we get
\[
\dot{v}e^{3t} = e^{\frac{t}{2}} \sin t \\
\dot{v} = e^{3t} \sin t = e^{\frac{t}{2}} \sin t \\
v = -\frac{1}{3} (2 \sin t + \cos t) e^{\frac{t}{2}}.
\]
Then substituting we get
\[
\ddot{u} = -e^{2t} \dot{v} e^{3t} = -e^{\frac{t}{2}} e^{3t} \sin t e^{\frac{t}{2}} = -e^{t} \sin t \\
u = \frac{1}{3} (2 \sin t + \cos t).
\]
The particular solution is
\[
we^{2t} + ve^{3t} = \frac{1}{5} \left( 2 \sin t + \cos t \right) e^{2t} - \frac{1}{10} (3 \sin t + \cos t) e^{3t} = \frac{1}{10} (\sin t + \cos t) e^{\frac{t}{2}},
\]
and the solution to the differential equation is \( Ae^{2t} + Be^{3t} + \left( \frac{1}{10} \sin t + \frac{1}{10} \cos t \right) e^{\frac{t}{2}} \).