17
Differential Equations

Many physical phenomena can be modeled using the language of calculus. For example, observational evidence suggests that the temperature of a cup of tea (or some other liquid) in a room of constant temperature will cool over time at a rate proportional to the difference between the room temperature and the temperature of the tea.

In symbols, if \( t \) is the time, \( M \) is the room temperature, and \( f(t) \) is the temperature of the tea at time \( t \) then \( f'(t) = k(M - f(t)) \) where \( k > 0 \) is a constant which will depend on the kind of tea (or more generally the kind of liquid) but not on the room temperature or the temperature of the tea. This is Newton's law of cooling and the equation that we just wrote down is an example of a differential equation. Ideally we would like to solve this equation, namely, find the function \( f(t) \) that describes the temperature over time, though this often turns out to be impossible, in which case various approximation techniques must be used. The use and solution of differential equations is an important field of mathematics; here we see how to solve some simple but useful types of differential equations.

Informally, a differential equation is an equation in which one or more of the derivatives of some function appear. Typically, a scientific theory will produce a differential equation (or a system of differential equations) that describes or governs some physical process, but the theory will not produce the desired function or functions directly.

Recall from section 6.2 that when the variable is time the derivative of a function \( y(t) \) is sometimes written as \( \dot{y} \) instead of \( y' \); this is quite common in the study of differential equations.

17.1 First Order Differential Equations

So long as \( y \) is not 25, we can rewrite the differential equation as

\[
\frac{dy}{dt} = \frac{1}{25 - y} - 2
\]

and solve for \( \dot{y} \) to get the solution \( y = 25 - Ae^{-\frac{t}{2}} \), allowing \( A \) to be zero.

Thus \( y = 25 + 15e^{-\frac{t}{2}} \) is a solution to the initial value problem. Note that \( y \) is never 25, so this makes sense for all values of \( t \). However, if we allow \( A = 0 \) we get the solution \( y = 25 \) to the differential equation, which would be the solution to the initial value problem if we were to require \( y(0) = 25 \). Thus, \( y = 25 + 15e^{-\frac{t}{2}} \) describes all solutions to the differential equation \( y = 2(25 - y) \), and all solutions to the associated initial value problems.

Why could we solve this problem? Our solution depended on rewriting the equation so that all instances of \( y \) on one side of the equation and all instances of \( t \) on the other, of course in this case the equation was originally written, since we didn't write \( dy/dt \) in the original equation. This is not required, however.

17.1.1 A first order differential equation is an equation of the form \( F(t, y, y') = 0 \). A solution of a first order differential equation is a function \( f(t) \) that makes \( F(t, f(t), f'(t)) = 0 \) for every value of \( t \).

Here, \( F \) is a function of three variables which we label \( t, y, \) and \( y' \). It is understood that \( y \) will explicitly appear in the equation although \( t \) and \( y \) need not. The term “first order” means that the first derivative of \( y \) appears, but no higher order derivatives do.

EXAMPLE 17.1.2 The equation from Newton's law of cooling, \( \dot{y} = k(M - y) \) is a first order differential equation; \( F(t, y, \dot{y}) = k(M - y) - \dot{y} \).

EXAMPLE 17.1.3 \( y - t^2 + 1 \) is a first order differential equation; \( \dot{y} = -t^2 - 1 \). All solutions to this equation are of the form \( t^3/3 + t + C \).

DEFINITION 17.1.4 A first order initial value problem is a system of equations of the form \( F(t, y, y') = 0, y(t_0) = y_0 \). Here \( t_0 \) is a fixed time and \( y_0 \) is a number. A solution of an initial value problem is a solution \( f(t) \) of the differential equation that also satisfies the initial condition \( f(t_0) = y_0 \).

EXAMPLE 17.1.5 The initial value problem \( y = t^2 + 1, y(1) = 4 \) has solution \( f(t) = t^3/3 + t + 8/3 \).

The general first order equation is rather too general, that is, we can’t describe methods that will work on them all, or even a large portion of them. We can make progress with specific kinds of first order differential equations. For example, much can be said about equations of the form \( y = \phi(t, y) \) where \( \phi \) is a function of the two variables \( t \) and \( y \). Under reasonable conditions on \( \phi \), such an equation has a solution and the corresponding initial value problem has a unique solution. However, in general, these equations can be very difficult or impossible to solve explicitly.

EXAMPLE 17.1.6 Consider this specific example of an initial value problem for Newton's law of cooling: \( y = 2(25 - y), y(0) = 40 \). We first note that if \( y(0) = 25 \), the right hand side of the differential equation is zero, and so the constant function \( y(t) = 25 \) is a solution to the differential equation. It is not a solution to the initial value problem, since \( y(0) \neq 40 \). (The physical interpretation of this constant solution is that if a liquid is at the same temperature as its surroundings, then the liquid will stay at that temperature.)
Of course, there are a few places this ideal description could go wrong; we need to be able to find the antiderivatives of $G$ and $F$, and we need to solve the final equation for $y$. The upshot is that the solutions to the original differential equation are the constant solutions, if any, and all functions $y$ that satisfy $G(y) = F(t) + C$.

**EXAMPLE 17.1.9** Consider the differential equation $\dot{y} = ky$. When $k > 0$, this describes certain simple cases of population growth; it says that the change in the population $y$ is proportional to the population. The underlying assumption is that each organism in the current population reproduces at a fixed rate, so that the larger the population the more new organisms are produced. While this is too simple to model most real populations, it is useful in some cases over a limited time. When $k < 0$, the differential equation describes a quantity that decreases in proportion to the current value; this can be used to model radioactive decay.

The constant solution is $y(t) = 0$; of course this will not be the solution to any interesting initial value problem. For the non-constant solutions, we proceed much as before:

$$\int \frac{1}{y} \, dy = \int k \, dt$$
$$\ln |y| = kt + C$$
$$|y| = e^{kt+C}$$
$$y = \pm e^{kt}$$
$$y = Ae^{kt}$$

Again, if we allow $A = 0$ this includes the constant solution, and we can simply say that $y = Ae^{kt}$ is the general solution. With an initial value we can easily solve for $A$ to get the solution of the initial value problem. In particular, if the initial value is given for time $t = 0$, $y(0) = y_0$, then $A = y_0$ and the solution is $y = y_0 e^{kt}$.

**Exercises 17.1.**

1. Which of the following equations are separable?
   a. $y = \sin(y)$
   b. $e^y = t$
   c. $y = t^2$
   d. $y = (t^2 - t) \cos(y)$
   e. $y = t \ln y + 4t^3 \ln y$

2. Solve $\dot{y} = 1/(t + t^2)$. ⇒

3. Solve the initial value problem $\dot{y} = t^2$ with $y(0) = 1$ and $n \geq 0$. ⇒

4. Solve $\dot{y} = \ln t$. ⇒

### 17.2 First Order Homogeneous Linear Equations

**EXAMPLE 17.2.2** The equation $\dot{y} = 2(25 - y)$ can be written $\dot{y} + 2y = 50$. This is linear, but not homogeneous. The equation $\dot{y} = ky$, or $\dot{y} - ky = 0$ is linear and homogeneous, with a particularly simple form $\dot{y} = k$. ⇒

Because first order homogeneous linear equations are separable, we can solve them in the usual way:

$$\int \frac{1}{y} \, dy = \int (-p(t)) \, dt$$
$$\ln |y| = \int P(t) \, dt$$
$$y = \pm e^{\int P(t) \, dt}$$
$$y = Ae^{\int P(t) \, dt}$$

where $P(t)$ is an anti-derivative of $-p(t)$. As in previous examples, if we allow $A = 0$ we get the constant solution $y = 0$.

**EXAMPLE 17.2.3** Solve the initial value problems $\dot{y} + y \cos t = 0$, $y(0) = 1/2$ and $\dot{y}(2) = 1/2$. We start with

$$P(t) = -\cos t \, dt = -\sin t,$$

so the general solution to the differential equation is $y = Ae^{-\sin t}$.

To compute $A$ we substitute:

$$\frac{1}{2} = Ae^{-\sin 0} = A,$$

so the solutions is $y = \frac{1}{2}e^{-\sin t}$.

For the second problem,

$$\frac{1}{2} = Ae^{-\sin 2}$$

$A = \frac{1}{2}e^{\sin 2}$

so the solution is $y = \frac{1}{2}e^{\sin 2} e^{-\sin 2}$.
17.3 First Order Linear Equations

As you might guess, a first order linear differential equation has the form \( \dot{y} + p(t)y = f(t) \). Not only is this closely related in form to the first order linear differential equation, we can use what we know about solving homogeneous equations to solve the general linear equation.

Suppose that \( y_1(t) \) and \( y_2(t) \) are solutions to \( \dot{y} + p(t)y = f(t) \). Let \( g(t) = y_1 - y_2 \). Then

\[
\dot{g}(t) + p(t)g(t) = \dot{y}_1(t) - \dot{y}_2(t) + p(t)(y_1(t) - y_2(t)) = (\dot{y}_1 + p(t)y_1(t) - (\dot{y}_2 + p(t)y_2(t)) = f(t) - f(t) = 0.
\]

In other words, \( g(t) = y_1 - y_2 \) is a solution to the homogeneous equation \( \dot{y} + p(t)y = 0 \). Turning around this, any solution to the linear equation \( \dot{y} + p(t)y = f(t) \), call it \( y_3 \), can be written as \( y_3(t) = y_2(t) + g(t) \), for some particular \( y_2 \) and some solution \( y_3(t) \) of the homogeneous equation \( \dot{y} + p(t)y = 0 \). Since we already know how to find all solutions of the homogeneous equation, finding just one solution to the equation \( \dot{y} + p(t)y = f(t) \) will give us all of them.

How might we find that one particular solution to \( \dot{y} + p(t)y = f(t) \)? Again, it turns out that what we already know helps. We know that the general solution to the homogeneous equation \( \dot{y} + p(t)y = 0 \) looks like \( Ae^{P(t)} \). We now make an inspired guess: consider the function \( y(v) = e^{\int P(t) dt} \), where we have replaced the constant parameter \( A \) with the function \( v(t) \). This technique is called variation of parameters. For convenience write this as \( s(t) = e^{v(t)}h(t) \) where \( h(t) = e^{P(t)} \) is a solution to the homogeneous equation. Now let’s compute a bit with \( h(t) \):

\[
s'(t) + p(t)s(t) = v'(t)e^{v(t)}h(t) + v(t)e^{v(t)}h(t) + p(t)e^{v(t)}h(t) = v'(t)e^{v(t)}h(t) + v(t)e^{v(t)}h(t) + v(t)e^{v(t)}h(t) = v(t)e^{v(t)}h(t) + v(t)e^{v(t)}h(t) = v(t)e^{v(t)}h(t).
\]

The last equality is true because \( h(t) + p(t)h(t) = 0 \), since \( h(t) \) is a solution to the homogeneous equation. We hope to find a solution \( s(t) \) so that \( s'(t) + p(t)s(t) = f(t) \), we will have such a function if we can arrange to have \( v'(t)e^{v(t)}h(t) = f(t) \), that is, \( e^{v(t)}h(t) = f(t) \). But this is as easy (or hard) as finding an anti-derivative of \( f(t)/h(t) \). Putting this all together, the general solution to \( \dot{y} + p(t)y = f(t) \) is

\[
y(v(t))h(t) + Ae^{P(t)} = s(t)e^{P(t)} + Ae^{P(t)}.
\]

**EXAMPLE 17.3.1** Find the solution of the initial value problem \( \dot{y} + 3y/t = t^2 \), \( y(1) = 1/2 \). First find the general solution, since we are interested in a solution with a given condition at \( t = 1 \), we may assume \( t > 0 \). We start by solving the homogeneous equation as usual: call the solution \( y \):

\[
y = Ae^{\int P(t)} = Ae^{-3\ln t} = At^{-3}.
\]

Then as in the discussion, \( h(t) = t^{-3} \) and \( v(t) = t^{-3}/t^3 \), so \( v(t) = t^3/6 \). We know that every solution to the equation looks like

\[
y(t) = At^{-3} + Ae^{P(t)} = t^3/6 + At^{-3}.
\]

Finally we substitute to find \( A \):

\[
\frac{6}{A} = 2/3 - 1 + A.
\]

Now the solution is

\[
y = \frac{t^3}{6} + \frac{1}{1-t^3}.
\]

Here is an alternate method for finding a particular solution to the differential equation, using an integrating factor. In the differential equation \( \dot{y} + p(t)y = f(t) \), we note that if we multiply through by a function \( I(t) \) to get \( I(t)\dot{y} + I(t)p(t)y = I(t)f(t) \), the left hand side looks like it could be a derivative computed by the product rule:

\[
\frac{d}{dt}(I(t)y) = I(t)f(t).
\]

Now if we could choose \( I(t) \) so that \( I(t)f(t) = I(t)\dot{y} \), this would be exactly the left hand side of the differential equation. But this is just a first order homogeneous linear equation, and we know a solution is \( y = e^{\int P(t) dt} \), where \( Q(t) = -P(t) \), that is, \( Q(t) = -P(t) \), where \( P(t) \) appears in the variation of parameters method and \( P(t) + y = f(t) \). Now the modified differential equation is

\[
e^{-P(t)}\dot{y} + e^{-P(t)}p(t)y = e^{-P(t)}f(t)
\]

The equation is

\[
\frac{d}{dt}(e^{-P(t)}y) = e^{-P(t)}f(t).
\]

**EXERCISES 17.3.**

In problems 1–10, find the general solution of the equation.

1. \( \dot{y} + 3y = 8 \) ⇒
2. \( \dot{y} - 2y = 6 \) ⇒
3. \( \dot{y} + 5y = 5t \) ⇒
4. \( \dot{y} + 2y = -2t \) ⇒
5. \( \dot{y} - y = t \) ⇒
6. \( 2\dot{y} + 4y = t \) ⇒
7. \( \dot{y} - 2y = 1/t, t > 0 \) ⇒
8. \( \dot{y} + y = \sqrt{t}, t > 0 \) ⇒
9. \( y\cos t + y\sin t = 1, -\pi/2 < t < \pi/2 \) ⇒
10. \( \dot{y} + \dot{y} = t, -\pi/2 < t < \pi/2 \) ⇒

This is the same answer, of course, and the problem is then finished just as before.
upper bound on how far off the approximation might be, that is, how far \( y_0 \) is from \( f(t_0) \). Suffice it to say that the bound is not particularly good and that there are other more complicated approximation techniques that do better.

**EXAMPLE 17.4.3** Let us compute an approximation to the solution for \( y = t - y^2 \), \( y(0) = 0 \), when \( t = 1 \). We will use \( \Delta t = 0.2 \), which is easy to do by hand, though we should not expect the resulting approximation to be very good. We get

\[
(t_1, y_1) = (0.2, 0.2) \quad (t_2, y_2) = (0.4, 0.4) \quad (t_3, y_3) = (0.6, 0.6) \quad (t_4, y_4) = (0.8, 0.8)
\]

So \( y(1) \approx 0.856 \). As it turns out, this is not accurate to even one decimal place. Figure 17.4.1 shows these points connected by line segments (the lower curve) compared to a solution obtained by a much better approximation technique. Note that the shape is approximately correct even though the end points are quite far apart.

![Figure 17.4.1 Approximating a solution to \( y = t - y^2 \), \( y(0) = 0 \).](image)

If you need to do Euler’s method by hand, it is useful to construct a table to keep track of the work, as shown in figure 17.4.2. Each row holds the computation for a single step: the starting point \((t_i, y_i)\), the step size \(\Delta t\), the computed slope \(s(t_i, y_i)\), the change in \(y\), \(\Delta y = s(t_i, y_i)\Delta t\), and the new point, \((t_{i+1}, y_{i+1}) = (t_i + \Delta t, y_i + \Delta y)\). The starting point in each row is the newly computed point from the end of the previous row.

It is easy to write a short function in Sage to do Euler’s method, see this Sage worksheet.

![Figure 17.4.2 A slope field for \( y = t - y^2 \).](image)

**Exercises 17.4.**

In problems 1-4, compute the Euler approximations for the initial value problem for \( 0 \leq t \leq 1 \) and \( \Delta t = 0.2 \). If you have access to Sage, generate the slope field first and attempt to sketch the solution curve. Then use Sage to compute better approximations with smaller values of \( \Delta t \).

1. \( y = t^2/y \), \( y(0) = 1 \)
2. \( y = t + y^2 \), \( y(0) = 1 \)
3. \( y = \cos(t) + y \), \( y(0) = 1 \)
4. \( y = \sin(t), y(0) = 2 \)

**17.5 Second Order Homogeneous Equations**

A second order differential equation is one containing the second derivative. These are in general quite complicated, but one fairly simple type is useful: the second order linear equation with constant coefficients.

**EXAMPLE 17.5.1** Consider the initial value problem \( y - y = 2y \), \( y(0) = 5 \), \( y(0) = 0 \). We make an inspired guess: might there be a solution of the form \( e^{rt} \). This seems at least plausible, since in this case \( y, \dot{y}, \text{and} \ y \) all involve \( e^{rt} \).

...
EXAMPLE 17.5.3  Suppose a mass m is hung on a spring with spring constant k. If the spring is compressed or stretched and then released, the mass will oscillate up and down. Because of friction, the oscillation will be damped; eventually the motion will cease. The damping will depend on the amount of friction; for example, if the system is suspended in oil, the friction will cause the mass to slow down more quickly. Using some simple physics, it is not hard to see that the position of the mass is described by the differential equation: \[ m\ddot{y} + b\dot{y} + ky = 0. \]

Using m = 1, b = 4, and k = 5 we find the motion of the mass. The characteristic polynomial is \( x^2 + 4x + 5 = 0 \) with roots \( -2 + \sqrt{3}i \) and \( -2 - \sqrt{3}i \). Thus the general solution is \( y = A\cos(\sqrt{3}t) - B\sin(\sqrt{3}t) \). Suppose we know that \( y(0) = 1 \) and \( \dot{y}(0) = 2 \). Then as before we form two simultaneous equations: \( y(0) = 1 \) we get \( 1 = A\cos(0) - B\sin(0) = A \). For the second we compute

\[
\dot{y} = -2A\sqrt{3}\cos(t) + A\sqrt{3}\sin(t) - 2B\sqrt{3}\sin(t) + B\cos(t),
\]

and then

\[
2 = -2A\sqrt{3}\cos(0) - 2A\sin(0) - 2B\sin(0) + B\cos(0) = -2A + B.
\]

So we get \( A = 1, B = 4, \) and \( y = \cos(\sqrt{3}t) + 4\sin(\sqrt{3}t - 2\). Here is a useful trick that makes this easier to understand: We have \( y = (\cos(\sqrt{3}t) + 4\sin(\sqrt{3}t) \) is a solution to the differential equation

\[
\ddot{y} + 4\sqrt{3}\omega_0 \dot{y} + 4\omega_0^2 y = 0.
\]

Note that \((1/\sqrt{\omega_0^2})^2 + (4/\sqrt{\omega_0} t)^2 = 1, \) which means that there is an angle \( \beta \) with \( \cos \beta = 1/\sqrt{\omega_0} \) and \( \sin \beta = 4/\sqrt{\omega_0} \) (of course, \( \beta \) may not be a “nice” angle). Then

\[
\ddot{y} + 4\sqrt{3}\omega_0 \dot{y} + 4\omega_0^2 y = 0.
\]

Thus, the solution may also be written \( y = \sqrt{3}\omega_0 \cos(\beta t) - \cos(\beta t). \) This is a cosine curve that has been shifted \( \beta \) to the right: the \( \sqrt{3}\omega_0 \cos(\beta t) \) gives \( \cos(t) \) of the cosine as \( t \) increases; see figure 17.5.1. The oscillation is damped very quickly, so in the first graph it is not clear that this is an oscillation. The second graph shows a restricted range for \( t \).

Other physical systems that oscillate can also be described by such differential equations. Some electric circuits, for example, generate oscillating current.

EXAMPLE 17.5.4  Find the solution to the initial value problem \( \ddot{y} - 4\dot{y} + 4y = 0, \) with \( y(0) = -3, \dot{y}(0) = 1. \) The characteristic polynomial is \( x^2 - 4x + 4 = (x - 2)^2, \) so there is a unique solution, and we can write

\[
y(t) = Ae^{2t} + Bt e^{2t}.
\]

16. Solve the initial value problem \( \ddot{y} + 4\dot{y} + 13y = 0, \) with \( y(0) = 1, \dot{y}(0) = 1. \) Put your answer in the form developed at the end of example 17.5.3.

17. Solve the initial value problem \( \ddot{y} + 2\dot{y} + 3y = 0, \) with \( y(0) = 3, \dot{y}(0) = 0. \) Put your answer in the form developed at the end of example 17.5.3.

18. A mass-spring system \( m\ddot{y} + b\dot{y} + ky = 0, \) with \( m = 29, b = 4, \) and \( k = 1. \) At time \( t = 0 \) the position is \( y(0) = 2 \) and the velocity is \( \dot{y}(0) = 1. \) Find \( y(t). \)

19. A mass-spring system \( m\ddot{y} + b\dot{y} + ky = 0, \) with \( m = 24, b = 12, \) and \( k = 3. \) At time \( t = 0 \) the position is \( y(0) = 0 \) and the velocity is \( \dot{y}(0) = -1. \) Find \( y(t). \)

20. Consider the differential equation \( ay + by = 0, \) with \( a \) and \( b \) both non-zero. Find the general solution of the equation by the method of this section. Now let \( a = y = y \) : the equation may be written as \( ay + by = 0, \) a first order linear homogeneous equation. Solve this for \( y, \) then use the relationship \( y = y \) to find \( y. \)

21. Suppose that \( y(t) \) is a solution to the first order linear equation \( ay + by + cy = 0, \) with \( a = 0 \) and \( b = 0. \) Solve the initial value problem \( y(0) = y_0, \dot{y}(0) = y_0. \) Show that \( y(t) = 0. \)

17.6  Second Order Linear Equations

Now we consider second order equations of the form \( ay + by + cy = f(t), \) with \( a, b, \) and \( c \) constant. Of course, if \( a = 0, \) this is a first order equation, so we assume \( a \neq 0. \)

Also, as in exercise 20 of section 17.5, if \( c = 0 \) we can solve the related first order equation \( ah + bh = f(t), \) and then solve \( h = y \) for \( h. \) So we will only examine examples in which \( c \neq 0. \)

Suppose that \( y_1(t) \) and \( y_2(t) \) are solutions to \( ay + by + cy = f(t), \) and consider the function \( h = y_1 - y_2. \) We substitute this function into the left hand side of the differential equation and simplify:

\[
(a\dot{y}_1 - a\dot{y}_2) + (b\dot{y}_1 - b\dot{y}_2) + (c(y_1 - y_2)) = (a\dot{y}_1 + b\dot{y}_1 + cy_1) - (a\dot{y}_2 + b\dot{y}_2 + cy_2) = f(t) - f(t) = 0.
\]

So \( h(t) \) is a solution to the homogeneous equation \( ay + by + cy = 0. \) Since we know how to find all \( h, \) then with just one particular solution \( y_2 \) we can express all possible solutions \( y, \) namely, \( y = y_1 + y_2, \) where \( y_1 \) is the general solution to the homogeneous equation. Of course, this is exactly how we approached the first order linear equation.

To make use of this observation we need a method to find a single solution \( y_2. \) This turns out to be somewhat more difficult than the first order case, but if \( f(t) \) is of a certain simple form, we can find a solution using the method of undetermined coefficients, sometimes more whimsically called the method of judicious guessing.

EXAMPLE 17.6.1  Solve the differential equation \( \ddot{y} - 4\dot{y} - 6y = 18e^t. \) The general solution of the homogeneous equation is \( A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t) \). We guess that a solution to the non-homogeneous equation might look like \( f(t) \) itself, namely, a quadratic \( y = at^2 + bt + c. \)
To make this equal to \( \cos(4t) \), we guess it cannot work. Instead we guess \( \text{Cte}^{4t} \). Then
\[
(-2Ct^2 - 2Ct^2 + 4t(\text{Ct}^2) + 7(Ct^2) - 2(Ct^2) + 10Ct(-\text{Ct}^2) = e^{-2t}(-3C). \]
Thus \( C = -1/3 \) and the solution is \( \text{Ae}^{-3t} + \text{Bte}^{3t} - (1/3)t^2 \text{e}^{-3t}. \)

**EXAMPLE 17.6.4** Find the general solution to \( y + 7y + 10y = e^{3t} \). Following the last example we might guess \( \text{Cte}^{3t} \), but since this is a solution to the homogeneous equation it cannot work. Instead we guess \( \text{Cte}^{4t} \). Then
\[
(-2Ct^2 - 2Ct^2 + 4t(\text{Ct}^2) + 7(Ct^2) - 2(Ct^2) + 10Ct(-\text{Ct}^2) = e^{-2t}(-3C) \]
Then \( C = -1/3 \) and the solution is \( \text{Ae}^{-3t} + \text{Bte}^{3t} - (1/3)t^2 \text{e}^{-3t}. \)

**EXAMPLE 17.6.5** Find the general solution to \( y + 6y + 9y = e^{3t} \). The characteristic equation is \( r^2 - 6r + 9 - (r - 3)^2 \), so the general solution to the homogeneous equation is \( \text{Ae}^{3t} + \text{Bte}^{-3t} \). Guessing \( \text{Cte}^{4t} \) for the particular solution, we get
\[
(9t^2e^{4t}+6te^{4t}+6te^{4t}+2te^{4t}) - 6(3\text{Ct}e^{4t}+2\text{Cte}^{4t}) + 9\text{Ct}^2e^{4t} = e^{2t}2Ct. \]
The solution is thus \( \text{Ae}^{3t} + \text{Bte}^{3t} + (1/2)t^2 \text{e}^{-3t}. \)

It is common in various physical systems to encounter an \( f(t) \) of the form \( a\cos(\omega t) + b\sin(\omega t) \).

**EXAMPLE 17.6.6** Find the general solution to \( y + 6y + 29y = \cos(4t) \). The roots of the characteristic equation are \(-3 \pm 4i\), so the solution to the homogeneous equation is \( e^{-3t}(A\cos(4t) + B\sin(4t)) \). For a particular solution, we guess \( C\cos(4t) + D\sin(4t) \). Substituting as usual:

\[
(-16C\cos(4t) + 16D\sin(4t)) + (-4C\cos(4t) + 16D\sin(4t)) + 25(C\cos(4t) + D\sin(4t)) = (24D + 9C)\cos(4t) + (-24C + 9D)\sin(4t). \]

To make this equal to \( \cos(4t) \) we need
\[
24D + 9C = 1 \quad 24D - 9C = 0. \]
which gives \( C = 1/73 \) and \( D = 8/219 \). The full solution is then \( e^{-3t}(A\cos(4t) + B\sin(4t)) + (1/73)\cos(4t) + (8/219)\sin(4t) \).

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The function \( e^{-3t}(A\cos(4t) + B\sin(4t)) \) is a damped oscillation as in example 17.5.3, while \( (1/73)\cos(4t) + (8/219)\sin(4t) \) is a simple undamped oscillation. As \( t \) increases, the sum \( e^{-3t}(A\cos(4t) + B\sin(4t)) \) approaches zero, so the solution
\[
e^{-3t}(A\cos(4t) + B\sin(4t)) + (1/73)\cos(4t) + (8/219)\sin(4t) \]
becomes more and more like the simple oscillation \((1/73)\cos(4t) + (8/219)\sin(4t))\) notice that the initial conditions don't matter to this long-term behavior. The damped portion is called the transient part of the solution, and the simple oscillation is called the steady state part of the solution. A physical example is a mass-spring system. If the only force on the mass is due to the spring, then the behavior of the system is a damped oscillation. If in addition an external force is applied to the mass, and if the force varies according to a function of the form \( a\cos(\omega t) + b\sin(\omega t) \), then the long-term behavior will be a simple oscillation determined by the steady state part of the general solution, the initial position of the mass will not matter.

As with the exponential form, such a simple guess may not work.

**EXAMPLE 17.6.7** Find the general solution to \( y + 16y = -\sin(4t) \). The roots of the characteristic equation are \( -4 \pm 4i \), so the solution to the homogeneous equation is \( A\cos(4t) + B\sin(4t) \). Since both \( \cos(4t) \) and \( \sin(4t) \) are solutions to the homogeneous equation, \( C\cos(4t) + D\sin(4t) \) is also, so it cannot be a solution to the non-homogeneous equation. Instead, we guess \( C\cos(4t) + D\sin(4t) \). Then substituting:
\[
(-16C\cos(4t) - 16D\sin(4t)) + 8C(\cos(4t) - 8\sin(4t)) + 16(C\cos(4t) + D\sin(4t)) = 8D\cos(4t) - 8C\sin(4t). \]
Thus \( C = 1/8 \), \( D = 0 \), and the solution is \( C\cos(4t) + D\sin(4t) + (1/8)t\cos(4t) \).

In general, if \( f(t) = a\cos(\omega t) + b\sin(\omega t) \), and \( \omega \neq 2 \pi \) are the roots of the characteristic equation, then instead of \( C\cos(\omega t) + D\sin(\omega t) \) we guess \( C\cos(\omega t) + D\sin(\omega t) \).

**Exercises 17.6**

Find the general solution to the differential equation.

1. \( x - 10y + 29x = \cos t \)
2. \( x + 2\sqrt{7}y + 2y = 10 \)
3. \( x + 16y = \sin^2 t + 3 - 4 \)
4. \( x + 2y = \cos(5t) + \sin(5t) \)
5. \( x - 2y + 2x = e^t \)
6. \( y - 6y + 12y = 1 + 2t + e^t \)

**Chapter 17 Differential Equations**

Now substituting:
\[
y + 6y + 29y = a\sin(4t) + b\cos(4t), \text{ with } a, b \text{ and } k \text{ all positive and } k^2 < 2\omega; \text{ this equation is a model for a damped mass-spring system with external driving force } \cos(4t). \]
Show that the steady state part of the solution has amplitude
\[
\sqrt{a^2 + b^2} = e^{-x} \]
Now the particular solution we seek is
\[ y''(t) + y'(t) = \frac{1}{2}(2\sin t + \cos t)e^{-3t} + \frac{1}{10}(3\sin t + \cos t)e^{-3t}, \]
and the solution to the differential equation is
\[ Ae^{3t} + Be^{-3t} + (\sin t + \cos t)/10. \]
For comparison (and practice) you might want to solve this using the method of undetermined coefficients.

EXAMPLE 17.7.2
The differential equation \( y'' - 5y + 6y = e^t \sin t \) can be solved using the method of undetermined coefficients, though we have not seen any examples of such a solution. Again, we will solve it by variation of parameters. The equations to be solved are
\[ \dot{u}e^t + vte^t = 0 \]
\[ 2ue^t + 3vte^t = e^t \sin t. \]
If we multiply the first equation by 2 and subtract it from the second equation we get
\[ \dot{v}e^t + vte^t = e^t \sin t \]
\[ v = e^{-2t}e^t \sin t = e^{-t} \sin t \]
\[ v = \frac{1}{2}(2\sin t + \cos t)e^{-3t}. \]
Then substituting we get
\[ \dot{u} = -e^{-2t}e^t \sin t = -e^{-t} \sin t \]
\[ u = \frac{1}{2}(2\sin t + \cos t)e^{-t}. \]
The particular solution is
\[ y''(t) + y'(t) = \frac{1}{2}(2\sin t + \cos t)e^{-3t} + \frac{1}{10}(3\sin t + \cos t)e^{-3t}, \]
and the solution to the differential equation is
\[ Ae^{3t} + Be^{-3t} + (\sin t + \cos t)/10. \]

EXAMPLE 17.7.3
The differential equation \( y'' - 2y + y = e^t/5 \) is not of the form amenable to the method of undetermined coefficients. The solution to the homogeneous equation is \( Ae^{3t} + Be^{-3t} \) and so the simultaneous equations are
\[ \dot{u}e^t + vte^t = 0 \]
\[ \dot{u}e^t + vte^t = e^t/5. \]
Subtracting the equations gives
\[ \dot{v}e^t = e^t/5 \]
\[ v = \frac{1}{5} \]
\[ v = -\frac{1}{7}. \]
Then substituting we get
\[ \dot{u} = -\dot{v}e^t = \frac{1}{5}te^t \]
\[ u = \frac{1}{7} \]
\[ u = -\ln t. \]
The solution is \( Ae^{3t} + Be^{-3t} - e^t \ln t - e^t. \)

Exercises 17.7.
Find the general solution to the differential equation using variation of parameters.
1. \( y'' + y = \tan x \quad \Rightarrow \]
2. \( y'' + y = e^x \quad \Rightarrow \]
3. \( y'' + 4y = \sec x \quad \Rightarrow \]
4. \( y'' + 4y = \tan x \quad \Rightarrow \]
5. \( y'' + 2y = e^x \quad \Rightarrow \]
6. \( y'' + 2y = e^x \tan x \quad \Rightarrow \]
7. \( y'' - 2y + 2y = \sin(t) \cos(t) \) (This is rather messy when done by variation of parameters; compare to undetermined coefficients.) \( \Rightarrow \)