Many physical phenomena can be modeled using the language of calculus. For example, observational evidence suggests that the temperature of a cup of tea (or some other liquid) in a room of constant temperature will cool over time at a rate proportional to the difference between the room temperature and the temperature of the tea.

In symbols, if \( t \) is the time, \( M \) is the room temperature, and \( f(t) \) is the temperature of the tea at time \( t \), then \( f'(t) = k(M - f(t)) \) where \( k > 0 \) is a constant which will depend on the kind of tea (or more generally the kind of liquid) but not on the room temperature or the temperature of the tea. This is Newton’s law of cooling and the equation that we just wrote down is an example of a differential equation. Ideally we would like to solve this equation, namely, find the function \( f(t) \) that describes the temperature over time, though this often turns out to be impossible, in which case various approximation techniques must be used. The use and solution of differential equations is an important field of mathematics; here we see how to solve some simple but useful types of differential equations.

Informally, a differential equation is an equation in which one or more of the derivatives of some function appear. Typically, a scientific theory will produce a differential equation (or a system of differential equations) that describes or governs some physical process, but the theory will not produce the desired function or functions directly.

Recall from section 6.2 that when the variable is time the derivative of a function so defined is usually written as \( y' \) instead of \( y' \); this is quite common in the study of differential equations.

**17.1 First Order Differential Equations**

So long as \( y \) is not 25, we can rewrite the differential equation as

\[
\frac{dy}{dt} = \frac{1}{25} - \frac{y}{25} = 2
\]

and so

\[
\int \frac{1}{25} \, dy = \int dt
\]

that is, the two antiderivatives must be the same except for a constant difference. We can calculate these antiderivatives and rearrange the results:

\[
\int \frac{1}{25} \, dy = \int dt \\
(1) \ln |25 - y| = \ln C
\]

\[
\ln |25 - y| = \ln C - \ln 25 + \ln e^k
\]

\[
y = 25 + Ce^k
\]

Here \( A = \pm e^k \) is some non-zero constant. Since we want \( y(0) = 40 \), we substitute and solve for \( A \):

\[
40 = 25 + Ae^0 \\
15 = A
\]

and so \( y = 25 + 15e^{-kt} \) is a solution to the initial value problem. Note that \( y \) is never 25, so this makes sense for all values of \( t \). However, if we allow \( A = 0 \) we get the solution \( y = 25 \) to the differential equation, which would be the solution to the initial value problem if we were to require \( y(0) = 25 \). Thus, \( y = 25 + Ae^{-kt} \) describes all solutions to the differential equation \( y' = 2(25 - y) \), and all solutions to the associated initial value problem.

Why could we solve this problem? Our solution depended on rewriting the equation so that all instances of \( y \) were on one side of the equation and all instances of \( t \) were on the other; of course, in this case the only \( t \) was originally hidden, since we didn’t write \( dy/dt \) in the original equation. This is not required, however.
17.1 First Order Differential Equations

5. Identify the constant solutions (if any) of $y = \tan y \Rightarrow$
6. Identify the constant solutions (if any) of $y = \sin y \Rightarrow$
7. Solve $y = e^y \Rightarrow$
8. Solve $y = e^y + 1 \Rightarrow$
9. Solve $y = t(y^2 - 5)$. You may leave your solution in implicit form: that is, you may stop once you have done the integration, without solving for $y \Rightarrow$
10. Find a non-constant solution of the initial value problem $y = y^2, y(0) = 0$, using separation of variables. Note that the constant function $y(t) = 0$ also solves the initial value problem. This shows that an initial value problem can have more than one solution. $\Rightarrow$
11. Solve the equation for Newton's law of cooling leaving $M$ and $k$ unknown. $\Rightarrow$
12. After 10 minutes in Jean-Luc's room, his tea has cooled to 40°Celsius from 80°Celsius. The room temperature is 25°Celsius. How much longer will it take to cool to 10°C? $\Rightarrow$
13. Solve the logistic equation $y = ky(M - y)$. (This is a somewhat more reasonable population model in most cases than the simpler $y = ky$.) Sketch the graph of the solution to this equation when $M = 1000, k = 0.002; y(0) = 1 \Rightarrow$
14. Suppose that $y = ky$, $y(0) = 2$, and $y(0) = 3$. What is $y(t) \Rightarrow$
15. A radioactive substance decays the equation $y = ky$ where $k < 0$ and $y$ is the mass of the substance at time $t$. Suppose that initially, the mass of the substance is $y(0) = M > 0$. At what time does half of the mass remain? (This is known as the half life. Note that the half life depends on $k$ but not on $M$.) $\Rightarrow$
16. Bismuth-210 has a half life of five days. If there is initially 600 milligrams, how much is left after 6 days? When will there be only 2 milligrams left? $\Rightarrow$
17. The half life of carbon-14 is 5730 years. If one starts with 100 milligrams of carbon-14, how much is left after 6000 years? How long do we have to wait before there is less than 2 milligrams? $\Rightarrow$
18. A certain species of bacteria doubles its population (or its mass) every hour in the lab. The differential equation that models this phenomenon is $y = ky, k > 0$ and $y$ is the population of bacteria at time $t$. What is $y(t) \Rightarrow$
19. If a certain microbe doubles its population every 4 hours and after 5 hours the total population has mass 500 grams, what was the initial mass? $\Rightarrow$

17.2 First Order Homogeneous Linear Equations

A simple, but important and useful, type of separable equation is the first order homogeneous linear equation.

DEFINITION 17.2.1 A first order homogeneous linear differential equation is one of the form $y + p(t)y = 0$ or equivalently $y = -p(t)y$. “Linear” in this definition indicates that both $y$ and $y$ occur to the first power. “Homogeneous” refers to the zero on the right hand side of the first form of the equation.

EXAMPLE 17.2.2 The equation $y = 2(5-y)$ can be written $y + 2y = 5t$. This is linear, but not homogeneous. The equation $y = ky$, or $y - ky = 0$ is linear and homogeneous, with a particularly simple $p(t) = -k$.

Because first order homogeneous linear equations are separable, we can solve them in the usual way:

$\int \frac{1}{y} dy = \int -p(t) dt$

$\ln |y| = \int -p(t) dt + C$

$y = \pm e^{\int -p(t) dt} + C$

where $P(t)$ is an anti-derivative of $-p(t)$. As in previous examples, if we allow $A = 0$ we get the constant solution $y = 0$.

EXAMPLE 17.2.3 Solve the initial value problems $\dot{y} + y \cos t = 0, y(0) = 1/2$ and $y(2) = 1/2$.

We start with

$P(t) = \int -\cos t dt = -\sin t$

so the general solution to the differential equation is

$y = A e^{-\sin t}$

To compute $A$ we substitute:

$\frac{A}{2} e^{-\sin 0} = A$

so the solutions is

$y = \left(\frac{1}{2}\right) e^{-\sin t}$

For the second problem,

$\frac{A}{2} e^{-\sin 2t} \Rightarrow A = 1/2$ so the solution is

$y = \left(\frac{1}{2}\right) e^{-\sin 2t}$

17.3 First Order Homogeneous Linear Equations

Exercises 17.2.

Find the general solution of each equation in 1-4.

1. $y + 2y = 0 \Rightarrow$
2. $\dot{y} + 2y = 0 \Rightarrow$
3. $y + 2y = 0 \Rightarrow$
4. $\dot{y} + 2y = 0 \Rightarrow$

In 5-14, solve the initial value problem.

5. $y + y = 0, y(0) = 4 \Rightarrow$
6. $y + 3y = 0, y(1) = -2 \Rightarrow$
7. $\dot{y} + y \sin t = 0, y(0) = 1 \Rightarrow$
8. $y + 2y = 0, y(0) = 0 \Rightarrow$
9. $\dot{y} + y \sqrt{1 + t} = 0, y(0) = 0 \Rightarrow$
10. $\dot{y} + y y \cos t = 0, y(0) = 0 \Rightarrow$
11. $\dot{y} + 2y = 0, y(1) = 4 \Rightarrow$
12. $\dot{y} + 2y = 0, y(0) = 2, t > 0 \Rightarrow$
13. $\dot{y} + 2y = 0, y(1) = 1, t > 0 \Rightarrow$
14. $\dot{y} + 2y = 0, y(0) = 0, t > 0 \Rightarrow$
15. A function $y(t)$ is as solution of $y + ky = 0$. Suppose that $y(0) = 100$ and $y(2) = 4$. Find $k$ and find $y(t)$.
16. Find $y(t)$, $y^{(2)} = 0$.
17. A bacterial culture grows at a rate proportional to its population. If the population is one million at $t = 0$ and 1.5 million at $t = 1$ hour, find the population as a function of time.
18. A radioactive element decays with a half-life of 6 years. If a block of the element has mass 10 kilograms at $t = 0$, find the amount of the element at time $t$.
17.3 First Order Linear Equations

As you might guess, a first order linear differential equation has the form \( y + p(t)y = f(t) \). Not only is this closely related in form to the first order linear homogeneous equation, we can use what we know about solving homogeneous equations to solve the general linear equation.

Suppose that \( y_1(t) \) and \( y_2(t) \) are solutions to \( \dot{y} + p(t)y = f(t) \). Let \( y(t) = y_1(t) - y_2(t) \). Then

\[
\begin{align*}
\dot{y}(t) + p(t)y(t) &= \dot{y}_1(t) + p(t)y_1(t) - \dot{y}_2(t) - p(t)y_2(t) \\
&= f(t) - f(t) = 0.
\end{align*}
\]

In other words, \( y(t) = y_1(t) - y_2(t) \) is a solution to the homogeneous equation \( \dot{y} + p(t)y = 0 \). Turning this around, any solution to the linear equation \( \dot{y} + p(t)y = f(t) \), call it \( y_1 \), can be written as \( y_1 = y_2 + \psi(t) \), for some particular \( y_2 \) and some solution \( \psi(t) \) of the homogeneous equation \( \dot{y} + p(t)y = 0 \). Since we already know how to find all solutions of the homogeneous equation, finding just one solution to the equation \( \dot{y} + p(t)y = f(t) \) will give us all of them.

How might we find that one particular solution to \( \dot{y} + p(t)y = f(t) \)? Again, it turns out that what we already know helps. We know that the general solution to the homogeneous equation \( \dot{y} + p(t)y = 0 \) looks like \( Ae^{\int p(t)dt} \), where we have replaced the constant parameter \( A \) with a function \( v(t)e^{\int p(t)dt} \).

This technique is called variation of parameters. For convenience write this as

\[
\dot{y}(t) + p(t)y(t) = v(t)e^{\int p(t)dt} + \psi(t)e^{\int p(t)dt}.
\]

The last equality is true because \( h(t) + p(t)h(t) = 0 \), since \( h(t) \) is a solution to the homogeneous equation. We are hoping to find a function \( v(t) \) so that \( \dot{y}(t) + p(t)v(t)e^{\int p(t)dt} = 0 \); we will have such a function if we can arrange to have \( v(t)e^{\int p(t)dt} = f(t) \), that is, \( \dot{v}(t) = f(t)/h(t) \). But this is as easy (or hard) as finding an anti-derivative of \( f(t)/h(t) \). Putting this all together, the general solution to \( \dot{y} + p(t)y = f(t) \) is

\[
y(t)e^{\int p(t)dt} = v(t)e^{\int p(t)dt} + Ae^{\int p(t)dt}.
\]

**EXAMPLE 17.3.1** Find the solution of the initial value problem \( \dot{y} + 3y/t = t^2 \), \( y(1) = 1/2 \). First we find the general solution; since we are interested in a solution with a given condition at \( t = 1 \), we may assume \( t > 0 \). We start by solving the homogeneous equation as usual; call the solution \( y \)

\[
y = Ae^{-\int 3/t dt} = Ae^{-3\log t} = Ae^{-3\log t}.
\]

Then as in the discussion, \( h(t) = t^{-3} \) and \( \dot{v}(t) = t^{-3}t^2 = t , so \( \psi(t) = t^2/6 \). We know that every solution to the equation looks like

\[
y(t) = t^2/6 + Ae^{-3t}.
\]

Finally we substitute to find \( A \)

\[
\frac{1}{6} = \frac{t^2}{6} + \frac{A}{1} - \frac{A}{3} - \frac{A}{3} + \frac{A}{3} = A.
\]

The solution is then

\[
y = \frac{t^2}{6} + \frac{1}{3} - \frac{t^3}{3}.
\]

Here is an alternate method for finding a particular solution to the differential equation, using an integrating factor. In the differential equation \( \dot{y} + p(t)y = f(t) \), we note that if we multiply through by a function \( I(t) \) to get \( I(t)\dot{y} + I(t)p(t)y = I(t)f(t) \), the left hand side looks like it could be a derivative computed by the product rule:

\[
\frac{d}{dt} \left( I(t)y(t) \right) = I(t)\dot{y} + \dot{I}(t)y(t).
\]

Now if we could choose \( I(t) \) so that \( I(t)\dot{y} = I(t)\dot{y} + \dot{I}(t)y(t) = -\dot{P}(t)y(t) \), where \( P(t) \) appears in the variation of parameters method and \( P'(t) = -p(t) \). Now the modified differential equation is

\[
y = \frac{d}{dt} \left( I(t)y(t) \right) = e^{-\int P(t) dt} \frac{d}{dt} (I(t)y(t)) = e^{-\int P(t) dt} I(t)\dot{y}.
\]

Integrating both sides gives

\[
e^{-\int P(t) dt} y = \int e^{-\int P(t) dt} f(t) dt + C.
\]

**Example 17.3.2** Suppose \( p(t, y) \) is a function of two variables. A more general class of first order differential equations has the form \( \dot{y} = f(t, y) \). This is not necessarily a linear first order equation, since \( \dot{y} \) may depend on \( y \) in some complicated way; note however that \( \dot{y} \) appears in a very simple form. Under suitable conditions on the function \( f \), it can be shown that every such differential equation has a solution, and moreover that for each initial condition the associated initial value problem has exactly one solution. In practical applications this is obviously a very desirable property.

**Exercise 17.3.**

In problems 1-10, find the general solution of the equation.

1. \( \dot{y} + 4y = 8 \)
2. \( \dot{y} - 2y = 6 \)
3. \( \dot{y} + 5y = 4 \)
4. \( \dot{y} + 5y = -2y \)
5. \( \dot{y} = t^2 \)
6. \( 2\dot{y} - y = t \)
7. \( \dot{y} + 2y = 1/t, t > 0 \)
8. \( \dot{y} + 2\sqrt{t}, t > 0 \)
9. \( \dot{y} = \text{constant}, t < a \)
10. \( \dot{y} + 2y = t, 0 < t < a \)

This is the same answer, of course, and the problem is then finished just as before.

17.4 Approximation

We have seen how to solve a restricted collection of differential equations, or more accurately, how to attempt to solve them—we may not be able to find the required anti-derivatives. Not surprisingly, non-linear equations can be even more difficult to solve. Yet much is known about solutions to some more general equations.

Suppose \( \dot{y}(t, y) \) is a function of two variables. A more general class of first order differential equations has the form \( \dot{y} = f(t, y) \). This is not necessarily a linear first order equation, since \( \dot{y} \) may depend on \( y \) in some complicated way; note however that \( \dot{y} \) appears in a very simple form. Under suitable conditions on the function \( f \), it can be shown that every such differential equation has a solution, and moreover that for each initial condition the associated initial value problem has exactly one solution. In practical applications this is obviously a very desirable property.

**Example 17.4.1** The equation \( \dot{y} = y^2 \) is a first order non-linear equation, because \( y \) appears to the second power. We will not be able to solve this equation.

**Example 17.4.2** The equation \( \dot{y} = y^2 \) is also non-linear, but it is separable and can be solved by separation of variables.

Not all differential equations that are important in practice can be solved exactly; some techniques have been developed to approximate solutions. We describe one such technique, Euler’s Method, which is simple though not particularly useful compared to some more sophisticated techniques.

Suppose we wish to approximate a solution to the initial value problem \( \dot{y} = \phi(t, y), \ y(0) = y_0 \) for \( t \geq 1 \). Under reasonable conditions on \( \phi \), we know the solution exists, represented by a curve in the \( t \)-\( y \) plane; call this solution \( f(t) \). The point \((t_0, y_0)\) is of course on this curve. We also know the slope of the curve at this point, namely \( \phi(t_0, y_0) \). If we follow the tangent line for a brief distance, we arrive at a point that should be almost on the graph of \( f(t) \), namely \((t_0 + \Delta t, y_0 + \phi(t_0, y_0)\Delta t)\). Call this point \((t_1, y_1)\). Now we pretend, in effect, that this point really is on the graph of \( f(t) \), in which case we again know the slope of the curve through \((t_1, y_1)\), namely \( \phi(t_1, y_1) \). So we can compute a new point, \((t_2, y_2) = (t_1 + \Delta t, y_1 + \phi(t_1, y_1)\Delta t) \) that is a little farther along, still close to the graph of \( f(t) \) but probably not quite so close as \((t_1, y_1)\). We can continue in this way, doing a sequence of straightforward calculations, until we have an approximation \((t_n, y_n)\) for whatever time \(t_n\) we need. At each step we do essentially the same calculation, namely

\[
(t_{n+1}, y_{n+1}) = (t_n + \Delta t, y_n + \phi(t_n, y_n)\Delta t).
\]

We expect that smaller time steps \( \Delta t \) will give better approximations, but of course it will require more work to compute to a specified time. It is possible to compute a guaranteed
EXAMPLE 17.4.3 Let us compute an approximation to the solution for $\dot{y} = t - y^2$, $y(0) = 0$, when $t = 1$. We will use $\Delta t = 0.2$, which is easy to do even by hand, though we should not expect the resulting approximation to be very good. We get

$(t_1, y_1) = (0, 0.2 + 0 + (0 - 0)^2) = (0.2, 0)$

$(t_2, y_2) = (0.2, 0.2 + 0 + (0.2 - 0)^2) = (0.4, 0.04)$

$(t_3, y_3) = (0.6, 0.4 + 0 - 0.04^2/2) = (0.6, 0.11968)$

$(t_4, y_4) = (0.8, 0.11968 - 0.6 - 0.11968^2) = (0.8, 0.2361533952)$

$(t_5, y_5) = (1.0, 0.2361533952 - 0.6 - 0.2361533952^2/2) = (1.0, 0.38520903513605)$

So $y(1) \approx 0.3856$. As it turns out, this is not accurate to even one decimal place. Figure 17.4.1 shows these points connected by line segments (the lower curve) compared to the upper bound on how far off the approximation might be, that is, how far $y_i$ is from $f(t_i)$. Suffice it to say that the bound is not particularly good and that there are other more complicated approximation techniques that do better.

EXAMPLE 17.4.4 Consider the initial value problem $\dot{y} - y - 2y = 0$, $y(0) = 5$, $\dot{y}(0) = 0$. We make an inspired guess: might there be a solution of the form $e^{rt}$? This seems at least plausible, since in this case $\ddot{y} = 0$. So we need to find $A$ and $B$ that make both $5 = A\phi(0) + B\phi(0)$ and $\dot{y}(0) = 0$.

\[
\begin{align*}
(\phi(0), \phi'(0)) & = (1, 1) \\
(0, -2) & = (A, B)
\end{align*}
\]

This system of linear equations has the solution $A = -2$ and $B = 1$, which leads to the general solution $y(t) = -2e^{t} + e^{t} = -e^{t}$. Is this a solution? No, because it does not satisfy the initial conditions.

The second order homogeneous equations are quite informative. It is apparent that if the initial population is smaller than $M$ it rises to $M$ over the long term, while if the initial population is greater than $M$ it decreases to $M$. It is quite easy to generate slope fields with Sage; follow the AP link in the figure caption.

**Exercises 17.4.**

In problems 1-4, compute the Euler approximations for the initial value problem for $0 \leq t \leq 1$ and $\Delta t = 0.2$. Using Sage, generate the slope field first and attempt to sketch the solution curve. Then use Sage to compute better approximations with smaller values of $\Delta t$.

1. $\dot{y} = t/y$, $y(0) = 1$ →
2. $\dot{y} = t + y^2$, $y(0) = 1$ →
3. $\dot{y} = \cos(t + y)$, $y(0) = 1$ →
4. $\dot{y} = \log(y)$, $y(0) = 2$ →

**17.5 Second Order Homogeneous Equations**

A second order differential equation is one containing the second derivative. These are in general quite complicated, but one fairly simple type is useful: the second order linear equation with constant coefficients.

**EXAMPLE 17.5.1** Consider the initial value problem $\dot{y} - y - 2y = 0$, $y(0) = 5$, $\dot{y}(0) = 0$. We make an inspired guess: might there be a solution of the form $e^{rt}$? This seems at least plausible, since in this case $\ddot{y} = 0$. So we need to find $A$ and $B$ that make both $5 = A\phi(0) + B\phi(0)$ and $\dot{y}(0) = 0$.

\[
\begin{align*}
(\phi(0), \phi'(0)) & = (1, 1) \\
(0, -2) & = (A, B)
\end{align*}
\]

This system of linear equations has the solution $A = -2$ and $B = 1$, which leads to the general solution $y(t) = -2e^{t} + e^{t} = -e^{t}$. Is this a solution? No, because it does not satisfy the initial conditions.

The second order homogeneous equations are quite informative. It is apparent that if the initial population is smaller than $M$ it rises to $M$ over the long term, while if the initial population is greater than $M$ it decreases to $M$. It is quite easy to generate slope fields with Sage; follow the AP link in the figure caption.

**Exercises 17.4.**

In problems 1-4, compute the Euler approximations for the initial value problem for $0 \leq t \leq 1$ and $\Delta t = 0.2$. Using Sage, generate the slope field first and attempt to sketch the solution curve. Then use Sage to compute better approximations with smaller values of $\Delta t$.

1. $\dot{y} = t/y$, $y(0) = 1$ →
2. $\dot{y} = t + y^2$, $y(0) = 1$ →
3. $\dot{y} = \cos(t + y)$, $y(0) = 1$ →
4. $\dot{y} = \log(y)$, $y(0) = 2$ →

**17.5 Second Order Homogeneous Equations**

A second order differential equation is one containing the second derivative. These are in general quite complicated, but one fairly simple type is useful: the second order linear equation with constant coefficients.

**EXAMPLE 17.5.1** Consider the initial value problem $\dot{y} - y - 2y = 0$, $y(0) = 5$, $\dot{y}(0) = 0$. We make an inspired guess: might there be a solution of the form $e^{rt}$? This seems at least plausible, since in this case $\ddot{y} = 0$. So we need to find $A$ and $B$ that make both $5 = A\phi(0) + B\phi(0)$ and $\dot{y}(0) = 0$.

\[
\begin{align*}
(\phi(0), \phi'(0)) & = (1, 1) \\
(0, -2) & = (A, B)
\end{align*}
\]

This system of linear equations has the solution $A = -2$ and $B = 1$, which leads to the general solution $y(t) = -2e^{t} + e^{t} = -e^{t}$. Is this a solution? No, because it does not satisfy the initial conditions.

The second order homogeneous equations are quite informative. It is apparent that if the initial population is smaller than $M$ it rises to $M$ over the long term, while if the initial population is greater than $M$ it decreases to $M$. It is quite easy to generate slope fields with Sage; follow the AP link in the figure caption.

**Exercises 17.4.**

In problems 1-4, compute the Euler approximations for the initial value problem for $0 \leq t \leq 1$ and $\Delta t = 0.2$. Using Sage, generate the slope field first and attempt to sketch the solution curve. Then use Sage to compute better approximations with smaller values of $\Delta t$.

1. $\dot{y} = t/y$, $y(0) = 1$ →
2. $\dot{y} = t + y^2$, $y(0) = 1$ →
3. $\dot{y} = \cos(t + y)$, $y(0) = 1$ →
4. $\dot{y} = \log(y)$, $y(0) = 2$ →

**17.5 Second Order Homogeneous Equations**

A second order differential equation is one containing the second derivative. These are in general quite complicated, but one fairly simple type is useful: the second order linear equation with constant coefficients.

**EXAMPLE 17.5.1** Consider the initial value problem $\dot{y} - y - 2y = 0$, $y(0) = 5$, $\dot{y}(0) = 0$. We make an inspired guess: might there be a solution of the form $e^{rt}$? This seems at least plausible, since in this case $\ddot{y} = 0$. So we need to find $A$ and $B$ that make both $5 = A\phi(0) + B\phi(0)$ and $\dot{y}(0) = 0$.

\[
\begin{align*}
(\phi(0), \phi'(0)) & = (1, 1) \\
(0, -2) & = (A, B)
\end{align*}
\]

This system of linear equations has the solution $A = -2$ and $B = 1$, which leads to the general solution $y(t) = -2e^{t} + e^{t} = -e^{t}$. Is this a solution? No, because it does not satisfy the initial conditions.

The second order homogeneous equations are quite informative. It is apparent that if the initial population is smaller than $M$ it rises to $M$ over the long term, while if the initial population is greater than $M$ it decreases to $M$. It is quite easy to generate slope fields with Sage; follow the AP link in the figure caption.
EXAMPLE 17.5.3 Suppose a mass $m$ is hung on a spring with spring constant $k$. If the spring is compressed or stretched and then released, the mass will oscillate up and down. Because of friction, the oscillation will be damped: eventually the motion will cease. The damping will depend on the amount of friction, for example, if the system is suspended in oil the motion will cease sooner than if the system is in air. Using some simple physics, it is not hard to see that the position of the mass is described by the differential equation: $m\ddot{y} + b\dot{y} + ky = 0$. Using $m = 1$, $b = 4$, and $k = 5$ we find the motion of the mass. The characteristic polynomial is $x^2 + 4x + 5$ with roots $(1 \pm \sqrt{16 - 20})/2 = -2 \pm 2i$. Thus the general solution is $y = \alpha e^{(-2\pm2i)t} = \alpha e^{-2t}(\cos(2t) + i\sin(2t))$. Suppose we know that $y(0) = 1$ and $y'(0) = 2$. Then as before we form two simultaneous equations: from $y(0) = 1$ we get $1 = \alpha e^{0} + B e^{0i} = A$.

For the second we compute

$$\dot{y} = -2\alpha e^{-2t}(\cos(2t) + i\sin(2t)) - 2Be^{-2t}(\sin(2t) + i\cos(2t))$$

and then

$$2 = -2\alpha e^{-2t}(0) - 2Be^{-2t}(0) + Be^{-2t}(0) = -2A + B.$$
To make this equal to \( \cos(4t) \). Substituting as usual:

\[
(−2C\cos^2 t − 2\cos t) + (7(−2\cos^2 t + 2\cos t)) + 10(\cos 2t − e^{−2t}) = e^{−2t}(−3\cos t).
\]

Then \( C = −1/3 \) and the solution is \( Ae^{−t} \cos 2t + (1/3)e^{−2t} \).

**Example 17.6.4** Find the general solution to \( y + 7y + 10y = e^{−2t} \). Following the last example we might guess \( Ce^{−2t} \), but since this is a solution to the homogeneous equation it cannot work. Instead we guess \( Cte^{−2t} \). Then

\[
−(2C\cos 2t + 4C \cos t) + 7(−2\cos 2t + 2\cos t) + 10(\cos 2t − 2e^{−2t}) = e^{−2t}(−3\cos t).
\]

In general, if \( f(t) = e^{it} \) and \( k \) is one of the roots of the characteristic equation, then we guess \( Cte^{kt} \) instead of \( Ce^{kt} \). If \( k \) is the only root of the characteristic equation, then \( Cte^{kt} \) will not work, and we must guess \( Ct^2e^{kt} \).

**Example 17.6.5** Find the general solution to \( y − 6y + 9y = e^{3t} \). The characteristic equation is \( r^2 − 6r + 9 = (r − 3)^2 \), so the general solution to the homogeneous equation is \( Ae^{3t} + Be^{3t} \). Guessing \( Ct^2e^{3t} \) for the particular solution, we get

\[
(9Ct^2e^{3t} + 6Ce^{3t} + 6Ce^{3t}) = 6(2Ct^2e^{3t} + 2Ce^{3t}) + 9Ce^3t = e^{3t}2Ct^2e^{3t}.
\]

The solution is then \( Ae^{3t} + Be^{3t} + (1/2)t^2e^{3t} \).

It is common in various physical systems to encounter an \( f(t) \) of the form \( a \cos(\omega t) + b \sin(\omega t) \).

**Example 17.6.6** Find the general solution to \( y + 6y + 25y = \cos(4t) \). The roots of the characteristic equation are \( −3 \pm 4i \), so the solution to the homogeneous equation is \( e^{−3t}(A \cos(4t) + B \sin(4t)) \). For a particular solution, we guess \( C \cos(4t) + D \sin(4t) \). Substituting as usual:

\[
−(16C \cos(4t) + 16D \sin(4t)) + 6(−4C \cos(4t) + 4D \sin(4t)) + 25(C \cos(4t) + D \sin(4t)) = (24D + 9C) \cos(4t) + (−24C + 9D) \sin(4t).
\]

To make this equal to \( \cos(4t) \) we need

\[
24D + 9C = 1 \quad \text{and} \quad −24C + 9D = 0
\]

which gives \( C = 1/73 \) and \( D = 8/219 \). The full solution is then \( e^{−3t}(A \cos(4t) + B \sin(4t)) + (1/73) \cos(4t) + (8/219) \sin(4t) \).

### 17.7 Second Order Linear Equations, Take Two

#### Exercises 17.6.6

1. \( y + 10y + 25y = \cos(4t) \)
2. \( y + 2v \sqrt{y} + 2y = 10 \)
3. \( y + 16y = 4 \)
4. \( y + 2y + \cos(4t) + \sin(5t) \)
5. \( y = 2y + 2y = e^{−t} \)
6. \( y = 6y + 13 = 1 + 2t + e^{−t} \)

This is a system of two equations in the two unknowns \( u \) and \( v \), but we can solve as usual to get

\( u = g(t) \) and \( v = h(t) \).

We can then find \( u \) and \( v \) by computing antiderivatives. This is of course the sticking point in the whole plan, since the antiderivatives may be impossible to find. Nevertheless, this sometimes works out and is worth a try.

**Example 17.7.1** Consider the equation \( y + 5y + 6y = \sin t \). We can solve this by the method of undetermined coefficients, but we will use variation of parameters. The solution to the homogeneous equation is \( Ae^{3t} + Be^{−2t} \), so the particular solutions to be solved are

\[
e^{3t} + e^{−2t} = 0 \quad \text{and} \quad 20e^{3t} + 30e^{−2t} = \sin t.
\]

If we multiply the first equation by 2 and subtract it from the second equation we get

\[
e^{3t} = \sin t \quad \text{and} \quad e^{−2t} = 0\]

using integration by parts. Then from the first equation:

\[
\sin t = e^{−2t}e^{3t} = e^{−2t}e^{3t} \sin t = e^{−2t} \sin t \quad \text{and} \quad \sin t = \frac{1}{3} \sin t + e^{−2t} \sin t,
\]

where \( \omega = \sqrt{3} \) is the resonant frequency of the system.
Now the particular solution we seek is
\[ u e^{2t} + v e^{3t} = \frac{1}{2} (2 \sin t + \cos t) e^{-2 t} - \frac{1}{2} (3 \sin t + \cos t) e^{-3 t} \]
and the solution to the differential equation is \( A u e^{2t} + B v e^{3t} + (\sin t + \cos t)/10 \). For comparison (and practice) you might want to solve this using the method of undetermined coefficients.

**EXAMPLE 17.7.2** The differential equation \( \ddot{y} - 5 \dot{y} + 6 y = e^t \sin t \) can be solved using the method of undetermined coefficients, though we have not seen any examples of such a solution. Again, we will solve it by variation of parameters. The equations to be solved are:
\[
\begin{align*}
\ddot{u} + \dot{v} &= 0 \\
2 \ddot{u} + 3 \dot{v} &= e^t \sin t.
\end{align*}
\]
If we multiply the first equation by 2 and subtract it from the second equation we get
\[
\ddot{v} = e^t \sin t
\]
and the solution to the differential equation is
\[ A e^{2t} + B e^{3t} + (\sin t + \cos t)/10. \]

**EXAMPLE 17.7.3** The differential equation \( \dot{y} - 2 \dot{y} + y = e^t / 2 \) is not of the form amenable to the method of undetermined coefficients. The solution to the homogeneous equation is \( A e^t + B t e^t \) and so the simultaneous equations are
\[
\begin{align*}
\ddot{u} &= \dot{v} \\
\ddot{v} &= e^t.
\end{align*}
\]
Subtracting the equations gives
\[
\ddot{u} = \ddot{v} - \dot{v} = \frac{1}{2} e^t.
\]
Then substituting we get
\[
\begin{align*}
u &= \int \frac{1}{2} e^t dt = \frac{1}{2} e^t \\
v &= \int \frac{1}{2} e^t dt = \frac{1}{2} e^t.
\end{align*}
\]
The solution is \( A e^t + B t e^t - e^t \ln t - e^t \).

**Exercises 17.7.**

Find the general solution to the differential equation using variation of parameters.
1. \( \ddot{y} + y = \tan x \)
2. \( \ddot{y} + y = e^x \)
3. \( \ddot{y} + 4y = \sec x \)
4. \( \ddot{y} + 4y = \tan x \)
5. \( \ddot{y} + y - 6y = e^x \)
6. \( \ddot{y} - 2 \dot{y} + 2y = e^x \tan t \):
7. \( \ddot{y} - 2 \dot{y} + 2y = \sin(t) \cos(t) \) (This is rather messy when done by variation of parameters; compare to undetermined coefficients)