16
Vector Calculus

16.1 Vector Fields

This chapter is concerned with applying calculus in the context of vector fields. A two-dimensional vector field is a function \( f \) that maps each point \((x, y)\) in \(\mathbb{R}^2\) to a two-dimensional vector \((u, v)\), and similarly a three-dimensional vector field maps \((x, y, z)\) to \((u, v, w)\). Since a vector has no position, we typically indicate a vector field in graphical form by placing the vector \( \mathbf{f}(x, y) \) with its tail at \((x, y)\). Figure 16.1.1 shows a representation of the vector field \( \mathbf{f}(x, y) = (−x/\sqrt{x^2 + y^2 + z^2})\mathbf{i} + (y/\sqrt{x^2 + y^2 + z^2})\mathbf{j} + (z/\sqrt{x^2 + y^2 + z^2})\mathbf{k} \). For such a graph to be readable, the vectors must be fairly short, which is accomplished by using a different scale for the vectors than for the axes. Such graphs are thus useful for understanding the sizes of the vectors relative to each other but not their absolute size.

Vector fields have many important applications, as they can be used to represent many physical quantities, such as wind speed or the velocity of some other fluid.

We have already seen a particularly important kind of vector field—the gradient. Given a function \( f(x, y) \), recall that the gradient is \( \langle f_x(x, y), f_y(x, y) \rangle \), a vector that depends on \((x, y)\). We usually picture the gradient vector with its tail at \((x, y)\), pointing in the direction of maximum increase. Vector fields that are gradients have some particularly nice properties, as we will see. An important example is

\[
\mathbf{F} = \left( \begin{array}{c} -x \\ -y \\ -z \\ \end{array} \right) / \left( (x^2 + y^2 + z^2)^{3/2} \right).
\]

16.2 Line Integrals

We have so far integrated “over” intervals, areas, and volumes with single, double, and triple integrals. We now investigate integration over or “along” a curve—“line integrals” are really “curve integrals”.

As with other integrals, a geometric example may be easiest to understand. Consider the function \( f = x + y \) and the parabola \( y = x^2 \) in the \(xy\)-plane, for \(0 \leq x \leq 2\). Imagine that we extend the parabola up to the surface \( f \), to form a curved wall or curtain, as in Figure 16.2.1. What is the area of the surface thus formed? We already know one way to compute surface area, but here we take a different approach that is more useful for the problems to come.

Typically the curve is in vector form, or can easily be put in vector form; in this example we have \( \mathbf{v}(t) = (t, t^2) \). Then as we have seen in section 13.3 on arc length, the length of one of the straight line segments in the approximation is approximately

\[
\Delta s \approx \sqrt{dx^2 + dy^2} = \sqrt{1 + 4t^2} \Delta t.
\]

As usual, we start by thinking about how to approximate the area. We pick some points along the part of the parabola we’re interested in, and connect adjacent points by straight lines, when the points are close together, the length of each line segment will be close to the length along the parabola. Using each line segment as the base of a rectangle, we choose the height to be the height of the surface \( f \) above the line segment. If we add up the areas of these rectangles, we get an approximation to the desired area, and in the limit this sum turns into an integral.

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We have already examined the idea of components of force, in example 12.3.4: the component of a force $F$ in the direction of a vector $v$ is

$$\frac{F \cdot v}{|v|},$$

the projection of $F$ onto $v$. The length of this vector, that is, the magnitude of the force in the direction of $v$, is

$$F \frac{v}{|v|},$$

the scalar projection of $F$ onto $v$. If an object moves subject to this (constant) force, in the direction of $v$, over a distance equal to the length of $v$, the work done is

$$\frac{F \cdot v}{|v|} = F \cdot v.$$

Thus, work in the vector setting is still “force times distance”, except that “times” means “dot product”.

If the force varies from point to point, it is represented by a vector field $F$, the displacement vector $v$ may also change, as an object may follow a curving path in two or three dimensions. Suppose that the path of an object is given by a vector function $r(t)$, at any point along the path, the (small) tangent vector $r'(t)$ gives an approximation to its motion over a short time $\Delta t$, so the work done during that time is approximately $F \cdot r'(t) \Delta t$; the total work over some time period is then

$$\int_{a}^{b} F \cdot r'(t) dt.$$

It is useful to rewrite this in various ways at different times. We start with

$$\int_{C} F \cdot r'(t) dt = \int_{C} F \cdot dr,$$

abbreviating $r'(t) dt$ by $dr$. Or we can write

$$\int_{C} F \cdot r'(t) dt = \int_{a}^{b} F \cdot r'(t) dt = \int_{a}^{b} F \cdot T \gamma(t) dt = \int_{C} F \cdot T ds,$$

using the unit tangent vector $T$, abbreviating $r'(t) dt$ as $ds$, and indicating the path of the object by $C$. In other words, work is computed using a particular line integral of the form

$$\int_{C} F \cdot r'(t) dt.$$

### 16.3 The Fundamental Theorem of Line Integrals

10. Compute $\int_{C} (1/y, 1/x + y) \cdot dr$ along the path from $(1, 1)$ to $(3, 3)$, using straight line segments. ⇒

11. Compute $\int_{C} (1/y, 1/(x + y)) \cdot dr$ along the curve $x^{2} + y^{2} = 4, 1 \leq t \leq 4$. ⇒

12. Compute $\int_{C} (1/y, 1/(x + y)) \cdot dr$ along the curve $(t, t^2)$, $1 \leq t \leq 4$. ⇒

13. Compute $\int_{C} y dx + x dy + x dy$ along the curve $(x, t^2, 0), 0 \leq t \leq 1$. ⇒

14. Compute $\int_{C} y dx + x dy + x dy$ along the curve $(\cos(t), 1, \sin(t)), 0 \leq t \leq \pi$. ⇒

15. An object moves from $(1, 1)$ to $(4, 8)$ along the path $r(t) = (t^2, t^3)$, subject to the force $F = (x^2, y^3)$. Find the work done. ⇒

16. An object moves along the line segment from $(1, 1)$ to $(2, 5)$, subject to the force $F = \langle x(x^2 + y^2), y(x^2 + y^2) \rangle$. Find the work done. ⇒

17. An object moves along the parabola $r(t) = (t^2, t^3), 0 \leq t \leq 1$, subject to the force $F = (x^2, y^3)$. Find the work done. ⇒

18. An object moves along the line segment from $(0, 0, 0)$ to $(3, 6, 10)$, subject to the force $F = \langle x, y, z \rangle$. Find the work done. ⇒

19. An object moves along the curve $r(t) = (\sqrt{t}, 1/\sqrt{t}, t) 1 \leq t \leq 4$, subject to the force $F = (y, z, x)$. Find the work done. ⇒

20. An object moves from $(1, 1, 2)$ to $(4, 8, 3)$ along the path $r(t) = (t^2, t^3, t^2)$, subject to the force $F = (x^2, y^3)$. Find the work done. ⇒

21. An object moves from $(1, 0, 0)$ to $(1, 1, 1)$ along the path $r(t) = (\cos(t), \sin(t), t), 1 \leq t \leq 1$. ⇒

22. Give an example of a non-trivial force field $F$ and non-trivial path $r(t)$ for which the total work done moving along the path is zero.

### 16.3 The Fundamental Theorem of Line Integrals

One way to write the Fundamental Theorem of Calculus (7.2.1) is:

$$\int_{a}^{b} f'(x) dx = f(b) - f(a).$$

That is, to compute the integral of a derivative $f'$ we need only compute the values of $f$ at the endpoints. Something similar is true for line integrals of a certain form.

**Theorem 16.3.1 Fundamental Theorem of Line Integrals** Suppose a curve $C$ is given by the vector function $r(t)$, with $a = r(a)$ and $b = r(b)$. Then

$$\int_{C} \nabla f \cdot dr = f(b) - f(a),$$

provided that $r$ is sufficiently nice.
Another immediate consequence of the Fundamental Theorem involves closed paths. A path \( C \) is closed if it forms a loop, so that traveling over the curve \( C \) brings you back to the starting point. If \( C \) is a closed path, we can integrate around it starting at any point \( a \); since the starting and ending points are the same,

\[
\int_C \nabla \cdot \mathbf{f} = \int_C df(a) - f(a) = 0.
\]

For example, in a gravitational field (an inverse square law field) the amount of work required to move an object around a closed path is zero. Of course, it’s only the net amount of work that is zero. It may well take a great deal of work to get from point \( a \) to point \( b \), but then the return trip will “produce” work. For example, it takes work to pump water from a lower to a higher elevation, but if you then let gravity pull the water back down, you can recover work by running a water wheel or generator. (In the real world you won’t recover all the work because of various losses along the way.)

To make use of the Fundamental Theorem of Line Integrals, we need to be able to spot conservative vector fields \( \mathbf{F} \) and to compute \( f \) so that \( \mathbf{F} = \nabla f \). Suppose that \( \mathbf{F} = (P, Q) = \nabla f \). Then \( P = f_x = Q_y \) and provided that \( f \) is sufficiently nice, we know from Clairaut’s Theorem (14.6.2) that \( P_x = f_{xx} = f_{yx} = Q_x \). If we compute \( P_x + Q_y \) and find that they are not equal, then \( \mathbf{F} \) is not conservative. If \( P_x = Q_y \), then, again provided that \( \mathbf{F} \) is sufficiently nice, we can be assured that \( \mathbf{F} \) is conservative. Ultimately, what’s important is that we be able to find \( f \); as this amounts to finding anti-derivatives, we may not always succeed.

**EXAMPLE 16.3.3** Find an \( f \) so that \( \mathbf{F} = (3 + 2xy, x^2 - 3y^2) = \nabla f \).

First, note that

\[
\frac{\partial}{\partial x}(3 + 2xy) = 2x \quad \text{and} \quad \frac{\partial}{\partial y}(x^2 - 3y^2) = 2y,
\]

so the desired \( f \) does exist. This means that \( f_x = 3 + 2xy \), so that \( f = 3x + x^2 y + g(y) \); the first two terms are needed to get \( 3 + 2xy \), and the \( g(y) \) could be any function of \( y \), as it would disappear upon taking a derivative with respect to \( x \). Likewise, since \( f_y = x^2 - 3y^2 \),

\[
f = x^2 y - y^3 + h(x).
\]

This question now becomes, is it possible to find \( g(y) \) and \( h(x) \) so that

\[
3x + x^2 y + g(y) = x^2 y - y^3 + h(x),
\]

and of course the answer is yes: \( g(y) = -y^3, h(x) = 3x \). Thus,

\[
f = 3x + x^2 y - y^3.
\]

We can test a vector field \( \mathbf{F} = (P, Q) \) in a similar way. Suppose that \( (P, Q, R) = (f_x, f_y, f_z) \). If we temporarily hold \( z \) constant, then \( f_x(x, y, z) \) is a function of \( x \) and \( y \).

16.4 Green’s Theorem

**THEOREM 16.4.1 Green’s Theorem** If the vector field \( \mathbf{F} = (P, Q) \), and the region \( D \) is sufficiently nice, and if \( C \) is the boundary of \( D \) (\( C \) is a closed curve), then

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA
\]

provided the integration on the right is done counter-clockwise around \( C \).

To indicate that an integral \( \int_C \) is being done over a closed curve in the counterclockwise direction, we usually write \( \oint_C \). We also use the notation \( \nabla \mathbf{F} \) to mean the boundary of \( D \) oriented in the counterclockwise direction. With this notation, \( \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \).

We already know one case, not particularly interesting, in which this theorem is true: if \( \mathbf{F} \) is conservative, we know that the integral \( \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \), because any integral of a conservative vector field around a closed curve is zero. We also know in this case that \( \partial P/\partial y - \partial Q/\partial x \) is a double integral in the theorem is simply the integral of the zero function, namely, \( 0 \). So in the case that \( \mathbf{F} \) is conservative, the theorem says simply that \( 0 = 0 \).

**EXAMPLE 16.4.2** We illustrate the theorem by computing both sides of,

\[
\int_C (x^2 + y^2) \, dx + y \, dy = \iint_D y \, -\, 0 \, dA,
\]

where \( D \) is the triangular region with corners \((0,0), (1,0), (0,1)\).

Starting with the double integral,

\[
\int_D y \, -\, 0 \, dA = \int_0^1 \int_0^{1-x} y \, dy \, dx = \int_0^1 \left( \frac{1}{2} x^2 - \frac{1}{2} \right) \, dx = \left( \frac{1}{3} x^3 \right) \bigg|_0^1 - \frac{1}{2}.
\]

There is no single formula to describe the boundary of \( D \), so we compute the left side directly we need to compute three separate integrals corresponding to the three sides of the triangle, and each of these integrals we break into two integrals, the “\( dx \)” part and the “\( dy \)” part. The three sides are described by \( y = 0, \quad y = 1 - x \), and \( x = 0 \). The integrals are then

\[
\int_0^1 x^2 \, dx + (1-x)y \, dy = \int_0^1 x^2 \, dx + 0 \, dy + \int_0^1 (1-x)y \, dy + \int_0^1 0 \, dx + \int_0^1 0 \, dy,
\]

Alternately, we could describe the three sides in vector form as \((1,0), (1-t, t), \) and \((0,1-t)\). Note that in each case, as \( t \) ranges from 0 to 1, we follow the corresponding side

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and by Clairaut’s Theorem \( P_x = f_{xx} = f_{yx} = Q_y \). Likewise, holding \( y \) constant implies \( P_y = f_{xy} = f_{yx} = R_x \). And with \( x \) constant we get \( Q_x = f_{xx} - f_{yx} - R_y \). Conversely, if we find that \( P_x = Q_y, \quad P_y = R_x, \) and \( Q_x = R_y \) then \( \mathbf{F} \) is conservative.

**Exercises 16.3.**

1. Find an \( f \) so that \( \mathbf{F} = (2x + y, 2y + z^2) \), or explain why there is no such \( f \).

2. Find an \( f \) so that \( \mathbf{F} = (z^2 - y^3, y^3 - z^2) \), or explain why there is no such \( f \).

3. Find an \( f \) so that \( \mathbf{F} = (x^2, ye^z, yz^2) \), or explain why there is no such \( f \).

4. Find an \( f \) so that \( \mathbf{F} = (xy, yz, xz) \), or explain why there is no such \( f \).

5. Find an \( f \) so that \( \mathbf{F} = (x^2 y^3, x^3 y^2, x^3 y) \), or explain why there is no such \( f \).

6. Evaluate \( \int_C (\mathbf{x} \times \mathbf{a}) \cdot ds \) where \( C \) is the part of the curve \( x^2 - 5xy = 7z^3 = 0 \) from \((3,2,0)\) to \((3,2,1)\).

7. Let \( \mathbf{F} = (x, y, z) \). Find the work done by this force field on an object that moves from \((1,0,0)\) to \((1,-1,3)\).

8. Evaluate \( \int_C x^2 + 2x \, dx + 3y^2 \, dy \) where \( C \) is the path of the curve \( x^3 - 5x^2 y^2 + 7z^2 = 0 \) from \((3,2,2)\) to \((3,2,0)\).

9. Let \( \mathbf{F} = (\mathbf{x}, \mathbf{y}, \mathbf{z}) \). Find the work done by this force field on an object that moves from \((0,0,0)\) to \((1,1,1)\).

10. Let \( \mathbf{F} = (\mathbf{x}, \mathbf{y}, \mathbf{z}) \). Find the work done by this force field on an object that moves from \((0,0,0)\) to \((1,1,1)\).

11. Let \( \mathbf{F} = (x^2 y^3 + y^2 z^3, x^3 y^2 + z^2 x^3 + y^2 z^3) \).

Find the work done by this force field on an object that moves from \((1,1,1)\) to \((4,4,4)\).
Using \( y = 0 \) gives
\[
\int_C y \, dx + 0 \, dy = \int_C \frac{\partial Q}{\partial y} \, dy = \int_C \frac{\partial Q}{\partial x} \, dx.
\]
Finally, using \( y = 2x \) and \( z \) gives
\[
\int_C \frac{\partial Q}{\partial y} \, dy = \int_C \frac{\partial Q}{\partial z} \, dz = \int_C \frac{\partial Q}{\partial z} \, dz = \int_C \frac{\partial Q}{\partial z} \, dz = z = \text{rub}.
\]
The first two integrals are not particularly difficult, but the third is very easy, though the choice of \( P \) and \( Q \) seems more complicated.

**Figure 16.4.1** A “standard” ellipse, \( x^2 + y^2 = 1 \).

**Proof of Green’s Theorem.** We cannot here prove Green’s Theorem in general, but we can do a special case. We seek to prove that
\[
\oint_C P \, dx + Q \, dy = \iint_D \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \, dA.
\]
It is sufficient to show that
\[
\oint_C P \, dx = \iint_D \frac{\partial P}{\partial y} \, dA \quad \text{and} \quad \oint_C Q \, dy = \iint_D \frac{\partial Q}{\partial x} \, dA,
\]
which we can do if we can compute the double integral in both possible ways, that is, using \( dA = dy \, dx \) and \( dA = dx \, dy \).

16.5 Divergence and Curl

**Theorem 16.5.1** \( \nabla \cdot (\nabla \times \mathbf{F}) = 0 \).

**Theorem 16.5.2** \( \nabla \times (\nabla \phi) = 0 \).

We defer the general proof of both theorems until a later chapter when we have the tools needed to prove them rigorously. The next two sections present simple results for divergence and curl, which are sufficient for our current applications.

**Exercises 16.4.**

1. Compute \( \iint_D 2y \, dx + 3x \, dy \) where \( D \) is described by \( 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \).
2. Compute \( \iint_D xy \, dx + y^2 \, dy \) where \( D \) is described by \( 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \).
3. Compute \( \iint_D x^2 \, dx + y^2 \, dy \) where \( D \) is described by \( 0 \leq x \leq 2, \quad 0 \leq y \leq 1 \).
4. Compute \( \iint_D \cos x \, dx + \sin x \, dy \) where \( D \) is described by \( 0 \leq x \leq \pi/2, \quad 1 \leq y \leq 2 \).
5. Compute \( \iint_D x^2 \, dy + xy^2 \, dx \) where \( D \) is described by \( 0 \leq x \leq 1, \quad 0 \leq y \leq \pi \).
6. Compute \( \iint_D x^2 \, dx + \sqrt{x^2 + y^2} \, dy \) where \( D \) is described by \( 1 \leq x \leq 2, \quad 1 \leq y \leq 4 \).
7. Compute \( \iint_D (x/y) \, dx + (2 + 3x) \, dy \) where \( D \) is described by \( 1 \leq x \leq 2, \quad 1 \leq y \leq e^x \).
8. Compute \( \iint_D \sin y \, dx + x^2 \, dy \) where \( D \) is described by \( 0 \leq x \leq \pi/2, \quad 0 \leq y \leq 1 \).
9. Compute \( \iint_D \sin y \, dx + x^2 \, dy \) where \( D \) is described by \( 1 \leq x \leq 2, \quad e^x \leq y \leq e^{2x} \).

16.4 Green’s Theorem

For the first equation, we start with
\[
\iint_D \frac{\partial P}{\partial y} \, dA - \iint_D \frac{\partial Q}{\partial x} \, dA = \int_C P(x, y(g(x))) - P(x, y(g(x))) \, dx.
\]
Here we have simply used the ordinary Fundamental Theorem of Calculus, since for the inner integral we are integrating a derivative with respect to \( y \), which is simply \( P(x, y) \).

**Exercises 16.4.**

1. Compute \( \iint_D 2xy \, dx + 3x \, dy \), where \( D \) is described by \( 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \).
2. Compute \( \iint_D xy \, dx + y^2 \, dy \), where \( D \) is described by \( 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \).
3. Compute \( \iint_D x^2 \, dx + y^2 \, dy \), where \( D \) is described by \( 0 \leq x \leq 2, \quad 0 \leq y \leq 1 \).
4. Compute \( \iint_D \cos x \, dx + \sin x \, dy \), where \( D \) is described by \( 0 \leq x \leq \pi/2, \quad 1 \leq y \leq 2 \).
5. Compute \( \iint_D x^2 \, dy + xy^2 \, dx \), where \( D \) is described by \( 0 \leq x \leq 1, \quad 0 \leq y \leq \pi \).
6. Compute \( \iint_D x^2 \, dx + \sqrt{x^2 + y^2} \, dy \), where \( D \) is described by \( 1 \leq x \leq 2, \quad 1 \leq y \leq 4 \).
7. Compute \( \iint_D (x/y) \, dx + (2 + 3x) \, dy \), where \( D \) is described by \( 1 \leq x \leq 2, \quad 1 \leq y \leq e^x \).
8. Compute \( \iint_D \sin y \, dx + x^2 \, dy \), where \( D \) is described by \( 0 \leq x \leq \pi/2, \quad 0 \leq y \leq 1 \).
9. Compute \( \iint_D \sin y \, dx + x^2 \, dy \), where \( D \) is described by \( 1 \leq x \leq 2, \quad e^x \leq y \leq e^{2x} \).

16.5 Divergence and Curl

**Theorem 16.5.1** \( \nabla \cdot (\nabla \times \mathbf{F}) = 0 \).

**Theorem 16.5.2** \( \nabla \times (\nabla \phi) = 0 \).

That is, the curl of a gradient is the zero vector. Recalling that gradients are conservative vector fields, this says that the curl of a conservative vector field is the zero vector field. Under suitable conditions, it is also true that if the curl of \( F \) is \( 0 \) then \( F \) is conservative. (Note that this is exactly the same test that we discussed on page 437.)

**Example 16.5.3** Let \( F = (e^x, 1, xe^z) \). Then \( \nabla \times F = (0, e^z - e^z) = 0 \). Thus, \( F \) is conservative, and we can exhibit this directly by finding the corresponding \( \phi \).

Since \( f_x = e^x \), \( F = xe^z + g(y,z) \). Since \( f_x = 1 \), it must be that \( g_y = 1 \), so \( g(y,z) = y + h(z) \). Thus \( F = xe^z + y + h(z) \). Let \( h'(z) = 0 \), i.e., \( h(z) = C \), and \( f = xe^z + y + C \).

We can rewrite Green’s Theorem using these new ideas; these rewritten versions in turn are closer to some later theorems we will see.

Suppose we write a two-dimensional vector field in the form \( \mathbf{F} = (P, Q, 0) \), where \( P \) and \( Q \) are functions of \( x \) and \( y \). Then
\[
\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \nabla \cdot (\nabla \phi) = 0,
\]
and so \( \nabla \times \mathbf{F} = (0, 0, 0) = \mathbf{0} \). So Green’s Theorem says
\[
\int_C F \cdot dr = \int_C P \, dx + Q \, dy = \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA - \int_D (\nabla \times \mathbf{F}) \cdot dA. \quad (16.5.1)
\]
16.5 Divergence and Curl

Roughly speaking, the rightmost integral adds up the curl (tendency to swirl) at each point in the region; the leftmost integral adds up the tangential components of the vector field around the entire boundary. Green’s Theorem says these are equal, or roughly, that the sum of the “microscopic” swirls is the same as the “macroscopic” swirl around the boundary.

Next, suppose that the boundary $\partial D$ has a vector form $r(t)$, so that $r'(t)$ is tangent to the boundary, and $\mathbf{T} = r'(t)/|r'(t)|$ is the usual unit tangent vector. Writing $r = (x(t), y(t))$, we get

$$T = \left(\frac{y'(t)}{|r'(t)|}, -\frac{x'(t)}{|r'(t)|}\right)$$

and then

$$\mathbf{N} = \left(\frac{-x'(t)}{|r'(t)|}, \frac{y'(t)}{|r'(t)|}\right)$$

is a unit vector perpendicular to $\mathbf{T}$, that is, a unit normal to the boundary. Now

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{N} \, ds = \int_{\partial D} (P, Q) \left(\frac{-x'(t)}{|r'(t)|}, \frac{y'(t)}{|r'(t)|}\right) |r'(t)| \, dt = \int_{\partial D} |r'(t)| P' \, dt - Q' \, dt$$

$$= \int_{\partial D} P \, dy - Q \, dx = \int_D -Q \, dx + P \, dy.$$

So far, we’ve just rewritten the original integral using alternate notation. The last integral looks just like the right side of Green’s Theorem (16.4.1) except that $P$ and $Q$ have traded places and $Q$ has acquired a negative sign. Then applying Green’s Theorem we get

$$\int_{\partial D} -Q \, dx + P \, dy = \int_D \left(\int_T P_x + Q_y \, dA\right) = \int_D \nabla \cdot \mathbf{F} \, dA.$$

Summarizing the long string of equalities,

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{N} \, ds = \int_D \nabla \cdot \mathbf{F} \, dA. \quad (16.5.2)$$

Roughly speaking, the first integral adds up the flow across the boundary of the region, from inside to out, and the second sums the divergence (tendency to spread) at each point in the interior. The theorem roughly says that the sum of the “macroscopic” spreads is the same as the total spread across the boundary and out of the region.

16.6 Vector Functions for Surfaces

value of $v$ and letting $r(u, v)$ sweep out a curve as it changes. Then $v$ can change a bit, and $r(u, v)$ sweeps out a new curve very close to the first. Put enough of these curves together and they form a surface.

EXAMPLE 16.6.1

Consider the function $r(u, v) = (v \cos u, v \sin u, v)$. For a fixed value of $v$, as $u$ varies from 0 to $2\pi$, this traces a circle of radius $v$ at height $v$ above the $x$-$y$ plane. Put lots and lots of these together, and they form a cone, as in figure 16.6.1.  

![Figure 16.6.1: Tracing a surface.](image)

EXAMPLE 16.6.2

Let $r = (v \cos u, v \sin u, u)$. If $u$ is constant, the resulting curve is a helix (as in figure 13.1.1). If $u$ is constant, the resulting curve is a straight line at height $u$ in the direction $x$ radians from the positive $x$ axis. Note in figure 16.6.2 how the helices and the lines both paint the same surface in a different way.

This technique allows us to represent many more surfaces than previously.

EXAMPLE 16.6.3

The curve given by

$$r = \left(\left(2 + \cos(3u/2)\right) \cos u, \left(2 + \cos(3u/2)\right) \sin u, \sin(3u/2)\right)$$

is called a trefoil knot. Recall that from the vector equation of the curve we can compute the unit tangent $\mathbf{T}$, the unit normal $\mathbf{N}$, and the binormal vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$; you may want to review section 13.3. The binormal is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$, one way to interpret this is that $\mathbf{N}$ and $\mathbf{B}$ define a plane perpendicular to $\mathbf{T}$, that is, perpendicular to the curve; since $\mathbf{N}$ and $\mathbf{B}$ are perpendicular to each other, they can function just as I
We have previously examined surfaces given in the form \( f(x, y) \). It is sometimes useful to represent such surfaces in the more general vector form, which is quite easy:

\[ \mathbf{r}(u, v) = (u, v, f(u, v)). \]

The names of the variables are not important of course; instead of designating \( x \) and \( y \), we could simply write \( \mathbf{r}(x, y) = (x, y, f(x, y)) \).

We have also previously dealt with surfaces that are not functions of \( x \) and \( y \); many of these are easy to represent in vector form. One common type of surface that cannot be represented as \( z = f(x, y) \) is a surface given by an equation involving only \( z \) and \( y \). For example, \( x + y = 1 \) and \( y = x^2 \) are “vertical” surfaces. For every point \((x, y)\) in the plane that satisfies the equation, the point \((x, y, z)\) is on the surface, for every value of \( z \). Thus, a corresponding vector form for the surface is something like \((f(x), y, v)\); for example, \( x + y = 1 \) becomes \((1, 1 - u, v)\) and \( y = x^2 \) becomes \((u, u^2, v)\).

Yet another sort of example is the sphere, say \( x^2 + y^2 + z^2 = 1 \). This cannot be written in the form \( z = f(x, y) \), but it is easy to write in vector form; indeed this particular surface is much like the cone, since it has circular cross-sections, or we can think of it as a tube around a portion of the \( z \)-axis, with a radius that varies depending on where along the axis we are. One vector expression for the sphere is

\[ \langle u, v, \sqrt{1 - u^2 - v^2} \rangle, \]

this emphasizes the tube structure, as it is naturally viewed as drawing a circle of radius \( \sqrt{1 - u^2 - v^2} \) around the \( z \)-axis at height \( v \). We could also take a cue from spherical coordinates, and write \((\sin u \cos v, \sin u \sin v, \cos u)\), where in effect \( u \) and \( v \) are \( \phi \) and \( \theta \) in disguise.

It is quite simple in Sage to plot any surface for which you have a vector representation. Using different vector functions sometimes gives different looking plots, because Sage in effect draws the surface by holding one variable constant and then the other. For example, you might have noticed in figure 16.6.2 that the curves in the two right-hand graphs are superimposed on the left-hand graph; the graph of the surface is just the combination of the two sets of curves, with the spaces filled in with color.

Here’s a simple but striking example: the plane \( x + y + z = 1 \) can be represented quite naturally as \((u, v, 1 - u - v)\). But we could also think of painting the same plane by choosing a particular point on the plane, say \((1, 0, 0)\), and then drawing circles or ellipses (or any of a number of other curves) as if that point were the origin in the plane. For example, \((1 - v \cos u - v \sin u, v \sin u, v \cos u)\) is one such vector function. Note that while it may not be obvious where this came from, it is quite easy to see that the sum of \( x, y, \) and \( z \) components of the vector is always 1. Computer renderings of the plane using these two functions are shown in figure 16.6.4.

Suppose we know that a plane contains a particular point \((x_0, y_0, z_0)\) and that two vectors \(u = (u_1, u_2, u_3)\) and \(v = (v_1, v_2, v_3)\) are parallel to the plane but not to each other. We know how to get an equation for the plane in the form \( ax + by + cz = d\), by first computing \( u \times v\). It’s easier to get a vector equation:

\[ \mathbf{r}(u, v) = (x_0, y_0, z_0) + u u_1 + v v_1. \]

**Example 16.6.4** We find the area of the surface \((\cos u \cos v, \sin u \cos v, \sin v)\) for \(0 \leq u \leq \pi\) and \(0 \leq v \leq \pi\); this is a portion of the helical surface in figure 16.6.2. We compute \(\mathbf{r}_u = (-\sin u \cos v, \cos u \cos v, 0)\) and \(\mathbf{r}_v = (\sin u \sin v, \cos u \sin v, \cos v)\). The cross product of these two vectors is \((\sin u \cos v, \cos u \sin v, \sin v)\) with length \(\sqrt{1 + \cos^2 v}\), and the surface area is

\[ \int_0^\pi \int_0^\pi \sqrt{1 + \cos^2 v} \, dv \, du. \]

**Exercises 16.6.**

1. Describe or sketch the surface with the given vector function.
   a. \(\mathbf{r}(u, v) = (u + v, 3 - v, 1 + 4u + 5v)\)
   b. \(\mathbf{r}(u, v) = (2u \sin u, 3 \cos u, v)\)
   c. \(\mathbf{r}(u, v) = (u, v, u^2 - v^2)\)
   d. \(\mathbf{r}(u, v) = (u \sin 2v, v, \cos 2v)\)

2. Find a vector function \(\mathbf{r}(u, v)\) for the surface.
   a. The plane that passes through the point \((1, 2, 3)\) and is parallel to the vectors \((1, 1, -1)\) and \((1, -1, 1)\).
   b. The lower half of the ellipsoid \(2x^2 + 4y^2 + z^2 = 1 \)
   c. The part of the sphere of radius 4 centered at the origin that lies between the planes \(z = -2\) and \(z = 2\).

3. Find the area of the portion of \(x^2 + y^2 = 10\) in the first octant.
4. Find the area of the portion of \(x^2 + 4y^2 + z^2 = 1\) in the first octant.
5. Find the area of \(x^2 + y^2 \leq 1\) that lies below \(z = 1\).
6. Find the area of \(x^2 + y^2 \geq 2\) that lies below \(z = 2\).
7. Find the area of \(x^2 + y^2 = z^2\) that lies in the first octant.
8. Find the area of \(x^2 + y^2 + z^2 = 1\) that lies above \(z^2 \leq 1\), \(x \leq 0\).
9. Find the area of \(x^2 + y^2 \leq 1\) that lies inside \(z^2 = x^2 + y^2\).
10. Find the area of \(x^2 + y^2 = z^2\) that lies inside \(x^2 + y^2 = 1\).
11. Find the area of \(x^2 + y^2 = z^2 = \theta^2\) that lies above the interior of the circle given in polar coordinates by \(r = \theta\).
12. Find the area of the cone \(z = \sqrt{x^2 + y^2}\) that lies above the interior of the circle given in polar coordinates by \(r = \theta\).
13. Find the area of the plane \(x = z\) that lies over a region \(D\) with area \(A\).
14. Find the area of the cone \(z = \sqrt{x^2 + y^2}\) that lies over a region \(D\) with area \(A\).
15. Find the area of the cylinder \(x^2 + y^2 = a^2\) that lies inside the cylinder \(x^2 + z^2 = a^2\).
16. The surface \(f(x, y)\) can be represented with the vector function \((x, y, f(x, y))\). Set up the surface area integral using this vector function and compute the integral of section 15.4.
We write the hemisphere as \( r(\theta, \phi) = (\cos \theta \sin \phi, \cos \theta \sin \phi, \cos \theta) \), \( 0 \leq \phi \leq \pi/2 \) and \( 0 \leq \theta \leq 2\pi \). So \( r_\phi = (-\sin \theta \sin \phi, \cos \theta \sin \phi, 0) \) and \( r_\theta = (-\sin \theta \cos \phi, \cos \theta \cos \phi, -\cos \theta) \). Then

\[
or \times r_\phi = (-\cos \theta \sin^2 \phi, -\cos \theta \sin \phi \cos \phi, -\cos \phi) \]

and

\[
or \times r_\theta = (\sin \theta \sin \phi, \sin \theta \cos \phi, 0) \]

since we are interested only in \( 0 \leq \phi \leq \pi/2 \). Finally, the density is \( z = \cos \phi \) and the integral for mass is

\[
\int_0^{2\pi} \int_0^{\pi/2} \cos \phi \sin \phi \, d\phi \, d\theta = \pi.
\]

By symmetry, the center of mass is on the \( z \)-axis, so we only need to find the \( z \)-coordinate of the center of mass. The moment around the \( x-y \) plane is

\[
\int_0^{2\pi} \int_0^{\pi/2} z \cos \phi \sin \phi \, d\phi \, d\theta = \frac{2\pi}{3},
\]

so the center of mass is at \((0, 0, 2/3)\).}

Now suppose that \( F \) is a vector field; imagine that it represents the velocity of some fluid at each point in space. We would like to know how much fluid is passing through a surface \( D \), the flux across \( D \). As usual, we imagine computing the flux across a very small section of the surface, with area \( dS \), and then adding up all such small fluxes across \( D \) with an integral. Suppose that vector \( N \) is a unit normal to the surface at a point; \( F \cdot N \) is the scalar projection of \( F \) onto the direction of \( N \), so it measures how fast the fluid is moving across the surface. In one unit of time the fluid moving across the surface will fill a volume of \( F \cdot N \, dS \), which is therefore the rate at which the fluid is moving across a small patch of the surface. Thus, the total flux across \( D \) is

\[
\iint_D F \cdot N \, dS = \iint_D F \cdot dS,
\]

defining \( dS = N \, dS \). As usual, certain conditions must be met for this to work out; chief among them is the nature of the surface. We can integrate over the surface, and we must choose the normal vectors \( N \) in such a way that they point “the same way” through the surface. For example, if the surface is roughly horizontal in orientation, we might want to choose the flux in the “upwards” direction, or if the surface is closed, like a sphere, we might want to measure the flux “outwards” across the surface. In the first case we would choose \( N \) to have positive \( z \)-component, in the second we would make sure that \( N \) points away from the origin. Unfortunately, there are surfaces that are not orientable; they have only one side, so that it is not possible to choose the normal vectors to point in the “same way” through the surface. The most famous such surface is the Möbius strip shown in figure 16.7.1. It is quite easy to make such a strip with a piece of paper and some tape. If you have never done this, it is quite instructive; in particular, you should draw a line down the center of the strip until you return to your starting point. No matter how unit normal vectors are assigned to the points of the Möbius strip, there will be normal vectors very close to each other pointing in opposite directions.

\[\text{Figure 16.7.1 A Möbius strip. (AP)}\]

Assuming that the quantities involved are well behaved, however, the flux of the vector field across the surface \( (x, y, z) \) is

\[
\iint_D F \cdot N \, dS = \iint_D F \cdot dS = \int_D \int (\nabla \times F) \cdot k \, dA.
\]

In practice, we may have to use \( r_x \times r_y \) or even something a bit more complicated to make sure that the normal vector points in the desired direction.

\[
\text{EXAMPLE 16.8.2 Compute the flux of } F = (x, y, z) \text{ across the cone } z = \sqrt{x^2 + y^2}, 0 \leq z \leq 1, \text{ in the downward direction.}
\]

We write the cone as a vector function: \( r = (\cos u, \sin u, v) \), \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq 1 \). Then \( r_u = (-\sin u, \cos u, 0) \) and \( r_v = (\cos u, \sin u, 1) \) and \( r_u \times r_v = \)

\[
\begin{bmatrix}
2 & 0 & -1 \\
0 & 1 & 0 \\
-1 & 0 & 2
\end{bmatrix}.
\]

12. Gauss’s Law says that the net charge, \( Q \), enclosed by a closed surface, \( S \), is

\[
Q = \varepsilon_0 \int_S \mathbf{E} \cdot dS,
\]

where \( \mathbf{E} \) is an electric field and \( \varepsilon_0 \) (the permittivity of free space) is a known constant; \( N \) is oriented outward. Use Gauss’s Law to find the charge contained in the cube with vertices \((\pm 1, \pm 1, \pm 1)\) if the electric field is \( \mathbf{E} = (x, y, z) \). \( \triangleright \)

16.8 Stokes’s Theorem

Recall that one version of Green’s Theorem (see equation 16.5.1) is

\[
\int_{\partial D} F \cdot dr = \int_{D (\nabla \times F)} \cdot \mathbf{k} \, dA.
\]

Here \( D \) is a region in the \( x-y \) plane and \( \mathbf{k} \) is a unit normal to \( D \) at every point. If \( D \) is instead an orientable surface in space, there is an obvious way to alter this equation, and it turns out still to be true:

\[
\text{THEOREM 16.8.1 Stokes’s Theorem} \quad \text{Provided that the quantities involved are sufficiently nice, and in particular if \( D \) is orientable,}
\]

\[
\int_{\partial D} F \cdot dr = \int_{D (\nabla \times F)} \cdot N \, dS.
\]

if \( \partial D \) is oriented counter-clockwise relative to \( N \). \( \triangleright \)

Note how little has changed: \( k \) becomes \( N \), a unit normal to the surface, and \( dA \) becomes \( dS \), since this is now a general surface integral. The phrase “counter-clockwise relative to \( N \)” means roughly that if we take the direction of \( N \) to be “up”, then we go around the boundary counter-clockwise when viewed from above”. In many cases, this description is inadequate. A slightly more complicated but general description is that: imagine standing on the side of the surface considered positive; walk to the boundary and turn left. You are now following the boundary in the correct direction.

\[
\text{EXAMPLE 16.8.2 Let } F = (\cos x, x^2, z^2) \text{ and the surface } D \text{ be } x = \sqrt{1 - y^2 - z^2}, \text{ oriented in the positive } x \text{ direction. It quickly becomes apparent that the surface integral in Stokes’s Theorem is intractable, so we try the line integral. The boundary of } D \text{ is the unit circle in the } y-z \text{ plane, } r = (0, \cos u, \sin u), 0 \leq u \leq 2\pi \text{. The integral is}
\]

\[
\int_{\partial D} F \cdot dr = \int_0^{2\pi} (\cos u, u, 0) \cdot (0, -\sin u, \cos u) \, du = \int_0^{2\pi} 0 \, du = 0.
\]
EXAMPLE 16.8.4
Consider the cylinder \( r = (\cos u, \sin u, v), 0 \leq u \leq 2\pi, 0 \leq v \leq 2 \), oriented outward, and \( \mathbf{F} = (y, x, z) \). We compute

\[
\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{S_1} \mathbf{F} \cdot d\mathbf{r}
\]

in two ways.

First, the double integral is

\[
\iint_{S_1} [(\partial_y - \partial_z) Q - (\partial_z - \partial_x) P + (\partial_x - \partial_y) R] \, dA.
\]

The boundary consists of two parts, the bottom circle \((\cos u, \sin u, 0)\), with \( t \) ranging from 0 to \( 2\pi \), and \((\cos u, \sin u, 2), \) with \( t \) ranging from \( 2\pi \) to 0. We compute the corresponding integrals and add the results:

\[
\int_{0}^{2\pi} (-\sin^2 t \, dt + \partial_t (-\sin^2 t + 2\cos^2 t - \pi) = -2\pi.
\]

as before. An interesting consequence of Stokes's Theorem is that if \( D \) and \( E \) are two orientable surfaces with the same boundary, then

\[
\int_D \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{S_1} \mathbf{F} \cdot d\mathbf{r} = \int_{S_2} \mathbf{F} \cdot d\mathbf{r} = \int_E (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.
\]

Sometimes both of the integrals

\[
\int_D \nabla \times \mathbf{F} \cdot d\mathbf{S} \quad \text{and} \quad \int_{S_1} \mathbf{F} \cdot d\mathbf{r}
\]

difficult, but you may be able to find a second surface \( E \) so that

\[
\int_E (\nabla \times \mathbf{F}) \cdot d\mathbf{S}
\]

has the same value but is easier to compute.

EXAMPLE 16.8.5
Let \( \mathbf{F} = (-y^2, x, z^2) \), and let the curve \( C \) be the intersection of the cylinder \( x^2 + y^2 = 1 \) with the plane \( y + z = 2 \), oriented counter-clockwise when viewed from above. We compute \( \int_C \mathbf{F} \cdot d\mathbf{r} \) in two ways.

First we do it directly: a vector function for \( C \) is \( \mathbf{r} = (\cos u, \sin u, 2 - \sin u) \), so \( \mathbf{r}_t = (-\sin u, \cos u, -\cos u) \), and the integral is then

\[
\int_0^{2\pi} y^2 \sin u \cos u + x^2 \cos u - z^2 \cos u \, du = \int_0^{2\pi} \sin^3 u + \cos^3 u - (2 - \sin^2 u) \, du = \pi.
\]

To use Stokes's Theorem, we pick a surface with \( C \) as the boundary, the simplest such surface is that portion of the plane \( y + z = 2 \) inside the cylinder. This has vector equation \( \mathbf{r} = (\cos u, \sin u, 2 - \sin u) \), \( \mathbf{r}_t = (-\sin u, \cos u, -\cos u) \), and \( \mathbf{r}_u \times \mathbf{r}_t = (0, -\cos u, -\sin u) \). To match the orientation of \( C \) we need to use the normal \((0, 0, 1)\). The curl of \( \mathbf{F} \) is \((0, 1 + 2y) = (0, 0, 1 + 2\sin u) \), and the surface integral from Stokes's Theorem is

\[
\int_0^{2\pi} (1 + 2\sin u) \, du = \pi.
\]

In this case the surface integral was more work to set up, but the resulting integral is somewhat easier.

Proof of Stokes's Theorem. We can prove here a special case of Stokes's Theorem, which perhaps not too surprisingly uses Green’s Theorem.

Suppose the surface \( D \) of interest can be expressed in the form \( z = g(x, y) \), and let \( \mathbf{F} = (P, Q, R) \). Using the vector function \( \mathbf{r} = (x, y, g(x, y)) \) for the surface we get the surface integral

\[
\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_S (\partial_y Q - \partial_z R + \partial_x P) \, dA.
\]

Here \( E \) is the region in the \( x-y \) plane directly below the surface \( D \).

For the line integral, we need a vector function for \( \partial D \). If \((x(t), y(t))\) is a vector function for \( \partial D \) then we may use \( \mathbf{r}(t) = (x(t), y(t), g(x(t), y(t))) \) to represent \( \partial D \). Then

\[
\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_a^b P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \, dt = \int_a^b P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \, dt.
\]

using the chain rule for \( dz/dt \). Now we continue to manipulate this

\[
\int_{\partial D} P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \, dt = \int_{\partial D} \left[ P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right] \, dt,
\]

which now looks just like the line integral of Green’s Theorem, except that the functions \( P \) and \( Q \) of Green’s Theorem have been replaced by the more complicated \( P + R(\partial g/\partial y) \) and \( Q + R(\partial g/\partial x) \). We can apply Green’s Theorem to get

\[
\int_{\partial D} P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \, dt - \int_{\partial E} \left[ \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \right] \, dA.
\]

Now we can use the chain rule again to evaluate the derivatives inside this integral, and it becomes

\[
\int_E (Q_1 + Q_2) + (P_2 + P_3) - (P_1 + P_3) \, dA
\]

which is the same expression we obtained for the surface integral.

\[\square\]
Proof. Again this theorem is too difficult to prove here, but a special case is easier. In the proof of a special case of Green’s Theorem, we needed to know that we could describe the region of integration in both possible orders, so that we could set up one double integral using \( dx \, dy \) and another using \( dy \, dx \). Similarly here, we need to be able to describe the three-dimensional region \( E \) in different ways.

We start by rewriting the triple integral:
\[
\iint_D \nabla \cdot \mathbf{F} \, dV = \iiint_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) \, dx \, dy \, dz = \iint_D P_x \, dV + \iint_D Q_y \, dV + \iint_D R_z \, dV.
\]

The double integral may be rewritten:
\[
\int_{D_y} \int_{D_z} \mathbf{N} \cdot d\mathbf{S} = \iint_D P \, d\mathbf{S} + \iint_{D_y} Q_y \, d\mathbf{S} + \iint_{D_z} R_z \, d\mathbf{S}.
\]

To prove that these give the same value it is sufficient to prove that
\[
\int_{D_y} \int_{D_z} \mathbf{N} \cdot d\mathbf{S} = \iint_D P \, d\mathbf{S},
\]

and (16.9.1)
\[
\int_{D_y} \int_{D_z} \mathbf{N} \cdot d\mathbf{S} = \iint_D Q_y \, d\mathbf{S}.
\]

Not surprisingly, these are all pretty much the same; we’ll do the first one.

We set the triple integral up with \( dz \) innermost:
\[
\int_{D_y} \int_{D_z} P_x \, dV = \int_{D_y} \int_{D_z} P \, d\mathbf{S} + \int_{D_y} \int_{D_z} Q_y \, d\mathbf{S} + \int_{D_y} \int_{D_z} R_z \, d\mathbf{S},
\]

where \( D \) is the region in the \( yz \)-plane over which we integrate. The boundary surface of \( E \) consists of a “top” \( x = g_1(y, z) \), a “bottom” \( x = g_2(y, z) \), and a “wrap-around side” that is vertical to the \( yz \)-plane. To integrate over the entire boundary surface, we can integrate over each of these (top, bottom, side) and add the results. Over the side surface, the vector \( \mathbf{N} \) is perpendicular to the vector \( \mathbf{l} \), so
\[
\iint_{D_y} \mathbf{N} \cdot d\mathbf{S} = \int_{D_y} \mathbf{N} \cdot \mathbf{l} = 0.
\]

Thus, we are left with just the surface integral over the top plus the surface integral over the bottom. For the top, we use the vector function \( \mathbf{r} = (g_1(y, z), y, z) \), which gives
\[
\int_{D_y} \int_{D_z} P \, d\mathbf{S} = \int_{D_y} \int_{D_z} P \, d\mathbf{S} = \int_{D_y} \int_{D_z} Q_y \, d\mathbf{S}.
\]

16.9 The Divergence Theorem

**Example 16.9.3** Let \( \mathbf{F} = (x^3, y^3, z^3) \), and consider the cylindrical volume \( x^2 + y^2 \leq 9 \), \( 0 \leq z \leq 2 \). The triple integral (using cylindrical coordinates) is
\[
\int_0^{2\pi} \int_0^3 \int_0^2 (2r^2 + 2r) \, dr \, dz \, d\theta = 279\pi.
\]

For the surface we need three integrals. The top of the cylinder can be represented by \( r = (r \cos u, r \sin u, 2) \), \( r_u \times r_v = (0, 0, -r) \), which points down into the cylinder, so we convert it to \( (0, 0, 1) \). Then
\[
\int_0^{2\pi} \int_0^3 \int_0^2 \frac{1}{4} r^3 \, dr \, dz \, d\theta = 36\pi.
\]

The bottom is \( r = (r \cos u, r \sin u, 0) \), \( r_u \times r_v = (0, 0, -r) \) and
\[
\int_0^{2\pi} \int_0^3 \int_0^2 \frac{1}{4} r^3 \, dr \, dz \, d\theta = 36\pi.
\]

The side of the cylinder is \( r = (3 \cos u, 3 \sin u, v) \), \( r_u \times r_v = (3 \cos u, 3 \sin u, 0) \) which does point outward, so
\[
\int_0^{2\pi} \int_0^3 \int_0^2 \frac{1}{8} (27 \cos^2 u + 27 \sin^2 u, 1) \cdot (3 \cos u, 3 \sin u, 0) \, dr \, dz \, d\theta = 243\pi.
\]

The total surface integral is thus 36\pi + 0 + 243\pi = 279\pi.

**Exercises 16.9.**

1. Using \( \mathbf{F} = (x^3, y^3, -z^2) \) and the region bounded by \( x^2 + y^2 = 9 \), \( z = 0 \), and \( z = 5 \), compute both integrals from the Divergence Theorem.

2. Let \( \mathbf{F} \) be the vector described by \( 0 \leq x \leq 0 \), \( 0 \leq y \leq 0 \), \( 0 \leq z \leq 0 \), and \( x = y, z = 0 \). Compute \( \iint_D \mathbf{F} \cdot d\mathbf{S} \).

3. Let \( \mathbf{F} \) be the vector described by \( 0 \leq x \leq 1 \), \( 0 \leq y \leq 0 \), \( 0 \leq z \leq 1 \), and \( y = 0 \). Compute \( \iint_D \mathbf{F} \cdot d\mathbf{S} \).

4. Let \( \mathbf{F} \) be the vector described by \( 0 \leq x \leq 1 \), \( 0 \leq y \leq 0 \), \( 0 \leq z \leq x + y \), and \( \mathbf{F} = (x, 2y, 3z) \). Compute \( \iint_D \mathbf{F} \cdot d\mathbf{S} \).

5. Let \( \mathbf{F} \) be the vector described by \( x^2 + y^2 + z^2 \leq 4 \), and \( \mathbf{F} = (x^3, y^3, z^2) \). Compute \( \iint_D \mathbf{F} \cdot d\mathbf{S} \).

6. Let \( \mathbf{F} \) be the hemisphere described by \( 0 \leq z \leq \sqrt{1-x^2-y^2} \), and \( \mathbf{F} = (\sqrt{x^2+y^2}+z, \sqrt{x^2+y^2}+z, -1) \). Compute \( \iint_D \mathbf{F} \cdot d\mathbf{S} \).

7. Let \( \mathbf{F} \) be the vector described by \( x^2 + y^2 \leq 1 \), \( 0 \leq z \leq 4 \), and \( \mathbf{F} = (xy^2, yz^2, x^2z) \). Compute \( \iint_D \mathbf{F} \cdot d\mathbf{S} \).

8. Let \( \mathbf{F} \) be the solid cone above the \( xy \)-plane and inside \( z = 1 - \sqrt{x^2+y^2} \), and \( \mathbf{F} = (x \cos z, y \sin z, \sqrt{x^2+y^2}) \). Compute \( \iint_D \mathbf{F} \cdot d\mathbf{S} \).

9. Prove the other two equations in the display 16.9.1.

10. Suppose \( D \) is a closed surface, and that \( D \) and \( F \) are sufficiently nice. Show that
\[
\iint_D (\nabla \cdot \mathbf{F}) \cdot d\mathbf{S} = 0.
\]

where \( \mathbf{N} \) is the outward pointing unit normal.

11. Suppose \( D \) is a closed surface, \( F \) is sufficiently nice, and \( \mathbf{F} = (a, b, c) \) is a constant vector field. Show that
\[
\iint_D \mathbf{F} \cdot d\mathbf{S} = 0.
\]

where \( \mathbf{N} \) is the outward pointing unit normal.

12. We know that the volume of a region \( E \) may often be computed as \( \iiint_D dx \, dy \, dz \). Show that this volume may also be computed as \( \frac{1}{4} \iint_D (x^2+y^2) \cdot d\mathbf{S} \) where \( \mathbf{N} \) is the outward pointing unit normal to \( \partial E \).