16 Vector Calculus

16.1 Vector Fields

This chapter is concerned with applying calculus in the context of vector fields. A two-dimensional vector field is a function \( f \) that maps each point \((x, y)\) in \( \mathbb{R}^2 \) to a two-dimensional vector \((u, v)\), and similarly a three-dimensional vector field maps \((x, y, z)\) to \((u, v, w)\). Since a vector has no position, we typically indicate a vector field in graphical form by placing the vector \( f(x, y) \) with its tail at \((x, y)\). Figure 16.1.1 shows a representation of the vector field \( f(x, y) = (-x/\sqrt{x^2 + y^2} + 1, y/\sqrt{x^2 + y^2} + 1) \). For such a graph to be readable, the vectors must be fairly short, which is accomplished by using a different scale for the vectors than for the axes. Such graphs are thus useful for understanding the sizes of the vectors relative to each other but not their absolute size.

Vector fields have many important applications, as they can be used to represent many physical quantities: the vector at a point may represent the strength of some force (gravity, electricity, magnetism) or a velocity (wind speed or the velocity of some other fluid).

We have so far integrated “over” intervals, areas, and volumes with single, double, and triple integrals. We now investigate integration over or “along” a curve—“line integrals” are really “curve integrals”.

As with other integrals, a geometric example may be easiest to understand. Consider the function \( f = x + y \) and the parabola \( y = x^2 \) in the \( xy \)-plane, for \( 0 \leq x \leq 2 \). Imagine that we extend the parabola up to the surface \( f \), to form a curved wall or curtain, as in figure 16.2.1. What is the area of the surface thus formed? We already know one way to compute surface area, but here we take a different approach that is more useful for the problems to come.

Typically the curve is in vector form, or can easily be put in vector form; in this example we have \( \mathbf{v}(t) = (t, t^2) \). Then as we have seen in section 13.3 on arc length, the length of one of the straight line segments in the approximation is approximately

\[ ds = |\mathbf{v}'(t)| \, dt = \sqrt{1 + t^4} \, dt, \]

so the integral is

\[ \int_0^2 f(t, t^2) \sqrt{1 + t^4} \, dt = \int_0^2 \left( t + t^4 \right) \sqrt{1 + t^4} \, dt = \frac{167}{35} \, \sqrt{77} - \frac{1}{64} \, \ln(4 + \sqrt{77}). \]

This integral of a function along a curve \( C \) is often written in abbreviated form as

\[ \int_C f(x, y) \, ds. \]

EXAMPLE 16.2.1 Compute \( \int_C ye^t \, ds \) where \( C \) is the line segment from \((1, 2)\) to \((4, 7)\).

We write the line segment as a vector function: \( \mathbf{v} = (1, 2) + t(3, 5), \) \( 0 \leq t \leq 1, \) or in parametric form \( x = 1 + 3t, \) \( y = 2 + 5t. \)

Then

\[ \int_C ye^t \, ds = \int_0^1 \left( 2 + 5t \right) e^{1 + 3t} \sqrt{3^2 + 5^2} \, dt = 16 \, e^{13/4} - \frac{1}{9} \, \sqrt{77} e. \]

All of these extend to three dimensions in the obvious way.

EXAMPLE 16.2.2 Compute \( \int_C z^2 \, ds \) where \( C \) is the line segment from \((0, 6, -1)\) to \((4, 1, 5)\).

We write the line segment as a vector function: \( \mathbf{v} = (0, 6, -1) + t(4, -5, 6), \) \( 0 \leq t \leq 1, \) or in parametric form \( x = 4t, \) \( y = 6 - 5t, \) \( z = -1 + 6t. \)

Then

\[ \int_C z^2 \, ds = \int_0^1 (4t)^2(-1 + 6t) \sqrt{16 + 25 + 36} \, dt = 16 \, e^{17} \int_0^1 t^2 + 6t^3 \, dt = \frac{56}{3} \, \sqrt{77}. \]

Now we turn to a perhaps more interesting example. Recall that in the simplest case, the work done by a force on an object is equal to the magnitude of the force times the distance the object moves; this assumes that the force is constant and in the direction of motion. We have already dealt with examples in which the force is not constant; now we are prepared to examine what happens when the force is not parallel to the direction of motion.
We have already examined the idea of components of force, in example 12.3.4: the component of a force \( \mathbf{F} \) in the direction of a vector \( \mathbf{v} \) is
\[
\mathbf{F} \cdot \frac{\mathbf{v}}{|\mathbf{v}|},
\]
the projection of \( \mathbf{F} \) onto \( \mathbf{v} \). The length of this vector, that is, the magnitude of the force in the direction of \( \mathbf{v} \), is
\[
\mathbf{F} \cdot \frac{\mathbf{v}}{|\mathbf{v}|},
\]
the scalar projection of \( \mathbf{F} \) onto \( \mathbf{v} \). If an object moves subject to this (constant) force, in the direction of \( \mathbf{v} \), over a distance equal to the length of \( \mathbf{v} \), the work done is
\[
\mathbf{F} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = \mathbf{F} \cdot \mathbf{v}.
\]
Thus, work in the vector setting is still "force times distance", except that "times" means "dot product".

If the force varies from point to point, it is represented by a vector field \( \mathbf{F} \), the displacement vector \( \mathbf{v} \) may also change, as an object may follow a curving path in two or three dimensions. Suppose that the path of an object is given by a vector function \( \mathbf{r}(t) \); at any point along the path, the (small) tangent vector \( \mathbf{r}'(t) \) gives an approximation to its motion over a short time \( \Delta t \); so the work done during that time is approximately \( \mathbf{F} \cdot \mathbf{r}'(t) \delta t \); the total work over some time period is then
\[
\int_a^b \mathbf{F} \cdot \mathbf{r}'(t) \, dt.
\]
It is useful to rephrase this in various ways at different times. We start with
\[
\int_a^b \mathbf{F} \cdot \mathbf{r}'(t) \, dt = \int_C \mathbf{F} \cdot d\mathbf{r},
\]
abbreviating \( r'(t) \, dt \) by \( d\mathbf{r} \). Or we can write
\[
\int_a^b \mathbf{F} \cdot \mathbf{r}'(t) \, dt = \int_a^b \mathbf{F} \cdot \frac{\mathbf{r}'(t) \, dt}{|\mathbf{r}'(t)|} = \int_a^b \mathbf{F} \cdot \mathbf{T} \, |\mathbf{r}'(t)| \, dt = \int_C \mathbf{F} \cdot \mathbf{T} \, ds,
\]
using the unit tangent vector \( \mathbf{T} \), abbreviating \( |\mathbf{r}'(t)| dt \) as \( ds \), and indicating the path of the object by \( C \). In other words, work is computed using a particular line integral of the form

\subsection*{16.3 The Fundamental Theorem of Line Integrals}

10. Compute \( \int_C \langle 1 + xy, 1 + xy \rangle \, ds \) along the path from \((1,1)\) to \((3,6)\) using straight line segments.

11. Compute \( \int_C \langle 1 + xy, 1 + xy \rangle \, ds \) along the curve \((2t^5, t)\), \(1 \leq t \leq 4\).

12. Compute \( \int_C \langle 1 + xy, 1 + xy \rangle \, ds \) along the curve \((t, t^3)\), \(1 \leq t \leq 4\).

13. Compute \( \int_C x \, dx + xz \, dy + yz \, dx \) along the curve \((x, x^2, x^3)\), \(0 \leq x \leq 1\).

14. Compute \( \int_C yz \, dx + xz \, dy + yz \, dx \) along the curve \((cos(t), sin(t), t)\), \(0 \leq t \leq \pi\).

15. An object moves from \((1,1)\) to \((4,8)\) along the path \( r(t) = \langle 1 + 3t, 1 + 4t \rangle \), subject to the force \( \mathbf{F} = \langle x^2, y^2 \rangle \). Find the work done.

16. An object moves along the line segment from \((1,1)\) to \((2,5)\), subject to the force \( \mathbf{F} = \langle y^2, y^2 \rangle \). Find the work done.

17. An object moves along the parabola \( r(t) = \langle t, t^3, t \rangle \), \(0 \leq t \leq 1\), subject to the force \( \mathbf{F} = \langle x^2, y^2, z^2 \rangle \). Find the work done.

18. An object moves along the line segment from \((0,0,0)\) to \((3,6,10)\), subject to the force \( \mathbf{F} = \langle x^2, y^2, z^2 \rangle \). Find the work done.

19. An object moves along the curve \( r(t) = \langle \sqrt{t}, \sqrt{t}, \sqrt{t} \rangle \), \(1 \leq t \leq 4\), subject to the force \( \mathbf{F} = \langle y, z, x \rangle \). Find the work done.

20. An object moves from \((1,1,1)\) to \((2,4,8)\) along the path \( r(t) = \langle t, t^3, t \rangle \), subject to the force \( \mathbf{F} = \langle y^2, y^2, z^2 \rangle \). Find the work done.

21. An object moves from \((1,0,0)\) to \((-1,0,\pi)\) along the path \( r(t) = \langle cos(t), sin(t), t \rangle\), subject to the force \( \mathbf{F} = \langle x^2, y^2, z^2 \rangle \). Find the work done.

22. Give an example of a non-trivial force field \( \mathbf{F} \) and non-trivial path \( r(t) \) for which the total work done moving along the path is zero.

\section*{16.3 The Fundamental Theorem of Line Integrals}

One way to write the Fundamental Theorem of Calculus (7.2.1) is:
\[
\int_a^b f'(x) \, dx = f(b) - f(a)
\]
That is, to compute the integral of a derivative \( f' \) we need only compute the values of \( f \) at the endpoints. Something similar is true for line integrals of a certain form.

\textbf{Theorem 16.3.1 Fundamental Theorem of Line Integrals}

Suppose a curve \( C \) is given by the vector function \( r(t) \), with \( a = r(a) \) and \( b = r(b) \). Then
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \mathbf{F}(b) - \mathbf{F}(a)
\]
provided that \( r \) is sufficiently nice.
Another immediate consequence of the Fundamental Theorem involves closed paths. A path $C$ is closed if it forms a loop, so that traveling over the curve $C$ brings you back to the starting point. If $C$ is a closed path, we can integrate around it starting at any point $a$, since the starting and ending points are the same,

$$\int_C P \, dx + Q \, dy = f(a) - f(a) = 0.$$

For example, in a gravitational field (an inverse square law field) the amount of work required to move an object around a closed path is zero. Of course, it’s only the net amount of work that is zero. It may well take a great deal of work to get from point $a$ to point $b$, but then the return trip will “produce” work. For example, it takes work to pump water from a lower to a higher elevation, but if you then let gravity pull the water back down, you can recover work by running a water wheel or generator. (In the real world you won’t recover all the work because of various losses along the way.)

To make use of the Fundamental Theorem of Line Integrals, we need to be able to spot conservative vector fields $F$ and to compute $f$ so that $F = -\nabla f$. Suppose $F = (P, Q)$, and provided that $f$ is sufficiently nice, we know from Clairaut’s Theorem (14.6.2) that $P_x = f_y = Q_x = f_y$. If we compute $P_x$ and $Q_y$ and find that they are not equal, then $F$ is not conservative. If $P_y = Q_x$, then, again provided that $F$ is sufficiently nice, we can be assured that $F$ is conservative. Ultimately, what’s important is that we be able to find $f$; as this amounts to finding anti-derivatives, we may not always succeed.

**EXAMPLE 16.3.3** Find an $f$ so that $(3 + 2xy, x^2 - 3y^3) = -\nabla f$.

First, note that

$$\frac{\partial}{\partial y} (3 + 2xy) = 2x \quad \text{and} \quad \frac{\partial}{\partial x} (x^2 - 3y^3) = 2x,$$

so the desired $f$ does exist. This means that $f_y = 3 + 2xy$, so that $f = 3x + x^2y + g(y)$; the first two terms were needed to get $3 + 2xy$, and the $g(y)$ could be any function of $y$, as it would disappear upon taking a derivative with respect to $x$. Likewise, since $f_x = x^2 - 3y^3$, $f = x^2y - y^3 + h(x)$. The question now becomes, is it possible to find $g(y)$ and $h(x)$ so that $3x + x^2y + g(y) = x^2y - y^3 + h(x)$, and of course the answer is yes: $g(y) = -y^3$, $h(x) = 3x$. Thus, $f = 3x + x^2y - y^3$.

We can test a vector field $F = (P, Q, R)$ in a similar way. Suppose that $(P, Q, R) = (f_x, f_y, f_z)$. If we temporarily hold $x$ constant, then $f(x, y, z)$ is a function of $x$ and $y$.

### 16.4 Green’s Theorem

**THEOREM 16.4.1 Green’s Theorem** If the vector field $F = (P, Q, R)$ and the region $D$ are sufficiently nice, and if $C$ is the boundary of $D$ (the path $C$ is a closed curve), then

$$\int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad \text{is the line integral} \quad \oint_C P \, dx + Q \, dy,$$

provided the integration on the right is done counter-clockwise around $C$.

To indicate that an integral $\int_C$ is being done over a closed curve in the counter-clockwise direction, we usually write $\int_C$. We also use the notation $\partial D$ to mean the boundary of $D$ oriented in the counter-clockwise direction. With this notation, $\oint_C = \int_{\partial D}$.

We already know one case, not particularly interesting, in which this theorem is true: if $F$ is conservative, we know that the integral $\int_C F \cdot dr = 0$, because any integral of a conservative vector field around a closed curve is zero. We also know in this case that $\partial P/\partial y - \partial Q/\partial x = 0$, so the double integral in the theorem is simply the integral of the zero function, namely, 0. In so the case that $F$ is conservative, the theorem says simply that $0 = 0$.

**EXAMPLE 16.4.2** We illustrate the theorem by computing both sides of

$$\int_{x^2 + y^2 = 1} x^2 \, dx + xy \, dy = \int_0^1 0 \, dA,$$

where $D$ is the triangular region with corners $(0, 0)$, $(1, 0)$, $(0, 1)$.

Starting with the double integral:

$$\int_D y - 0 \, dA = \int_0^1 \int_0^1 y \, dx \, dy = \int_0^1 \left[ \int_0^1 x^2 \frac{dx}{2} - \frac{(1 - x)^3}{6} \right] dy = \frac{1}{3},$$

There is no single formula to describe the boundary of $D$, so to compute the left side directly we need to compute three separate integrals corresponding to the three sides of the triangle, and each of these integrals we break into two integrals, the “$dx$” part and the “$dy$” part. The three sides are described by $y = 0$, $y = 1 - x$, and $x = 0$. The integrals are then

$$\int_{x^2} x^2 \, dx + xy \, dy = \int_0^1 x^2 \, dx + \int_0^1 0 \, dy = \int_0^1 x^2 \, dx + \int_0^{1 - x} (1 - y) \, dy + \int_0^1 0 \, dx + \int_0^1 0 \, dy = \frac{1}{3} + \frac{1}{2} + 0 + 0 = \frac{1}{6}.$$"
Using $-y$ and $0$ gives
\[ \int_C -y \, dx + 0 \, dy = \int_0^1 -\sin(t)(-\sin(t)) \, dt = \int_0^1 \sin^2(t) \, dt. \]

Finally, using $-y$ and $2$ gives
\[ \int_C \frac{y}{2} \, dx + \frac{x}{2} \, dy = \int_0^1 -\sin(t)(-\sin(t)) \, dt + \frac{1}{2} \cos(t)(\cos(t)) \, dt = \int_0^1 \frac{1}{2} \, dt = \frac{1}{2}. \]

The first two integrals are not particularly difficult, but the third is very easy, though the choice of $P$ and $Q$ seems more complicated.

![Figure 16.4.1: A "standard" ellipse.](image)

**Proof of Green's Theorem.** We cannot here prove Green's Theorem in general, but we can do a special case. We seek to prove that
\[ \int_C P \, dx + Q \, dy = \iint_D \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \, dA. \]

It is sufficient to show that
\[ \int_C P \, dx = \iint_D \frac{\partial P}{\partial y} \, dA \quad \text{and} \quad \int_C Q \, dy = \iint_D \frac{\partial Q}{\partial x} \, dA, \]

which can be done if we can compute the double integral in both possible ways, that is, using $dA = dy \, dx$ and $dA = dx \, dy$.

### 16.5 Divergence and Curl

Divergence and curl are two measurements of vector fields that are very useful in a variety of applications. Both are most easily understood by thinking of the vector field as representing a flow of a liquid or gas; that is, each vector in the vector field should be interpreted as a velocity vector. Roughly speaking, divergence measures the tendency of the fluid to collect or disperse at a point, and curl measures the tendency of the fluid to swirl around the point. Divergence is a scalar, that is, a single number, while curl is itself a vector. The magnitude of the curl measures how much the fluid is swirling, the direction indicates the axis around which it tends to swirl. These ideas are somewhat subtle in practice, and are beyond the scope of this course. You can find additional information on the web, for example at
http://mathinsight.org/curl_idea and http://mathinsight.org/divergence_idea and in many books including Div, Grad, Curl, and All That: An Informal Text on Vector Calculus, by H. M. Schey.

Recall that if $f$ is a function, the gradient of $f$ is given by
\[ \nabla f = \left( \frac{\partial f}{\partial x} \right) \hat{i} + \left( \frac{\partial f}{\partial y} \right) \hat{j}. \]

A useful mnemonic for this (and for the divergence and curl, as it turns out) is to let
\[ \nabla f = \left( \frac{\partial f}{\partial x} \right) \hat{i} + \left( \frac{\partial f}{\partial y} \right) \hat{j} = \left( \frac{\partial f}{\partial x} \right) \hat{i} + \left( \frac{\partial f}{\partial y} \right) \hat{j}, \]

that is, we pretend that $\nabla$ is a vector with rather odd looking entries. Recalling that $\Delta (u, v, w) = (uw, vu, wu)$, we can then think of the gradient as
\[ \nabla f = \left( \frac{\partial f}{\partial x} \right) \hat{i} + \left( \frac{\partial f}{\partial y} \right) \hat{j} = \left( \frac{\partial f}{\partial x} \right) \hat{i} + \left( \frac{\partial f}{\partial y} \right) \hat{j}, \]

that is, we simply multiply the $f$ into the vector.

### 16.4 Green's Theorem

For the first equation, we start with
\[ \iint_D \frac{\partial P}{\partial y} \, dA - \iint_D \frac{\partial Q}{\partial x} \, dA = \int_C P(x, y) \, dx - P(x_0, y) \, dx. \]

Here we have simply used the ordinary Fundamental Theorem of Calculus, since for the inner integral we are integrating a derivative with respect to $y$; it is simply $P(x, y)$, and then we substitute $y_0$ and $y$, and $y$ and $y_0$.

Now we need to manipulate $\int_C \frac{\partial P}{\partial y} \, dy$. The boundary of region $D$ consists of 4 parts, given by the equations $y = y_0(x), x = x_0(y)$, and $x = x_0(y), y = y_0(x)$.

For the other two portions, we use the parametric forms $x = t, y = y_0(t), a \leq t \leq b$, and $x = x_0(t), y = y_0(t), a \leq t \leq b$. Letting $t$ range from $b$ to $a$, since we are integrating counter-clockwise around the boundary. The resulting integrals give us
\[ \int_C \frac{\partial P}{\partial y} \, dy = \int_{a}^{b} P(t, y_0(t)) \, dt + \int_{b}^{a} P(t, y_0(t)) \, dt - \int_{a}^{b} P(t, y_0(t)) \, dt = \int_{a}^{b} P(t, y_0(t)) \, dt. \]

which is the result of the double integral times $-1$, as desired.

The equation involving $Q$ is essentially the same, and left as an exercise.

### Exercises

1. Compute $\iint_D x \, dA$, where $D$ is described by $0 \leq x \leq 1, 0 \leq y \leq 1.$
2. Compute $\int_C x \, dy + y \, dx$, where $C$ is described by $0 \leq x \leq 1, 0 \leq y \leq 1.$
3. Compute $\int_C x^2 \, dy + y \, dx$, where $D$ is described by $0 \leq x \leq 2, -1 \leq y \leq 1.$
4. Compute $\int_C y(x, y) \, dx + y \, dx$, where $D$ is described by $0 \leq x \leq y, 0 \leq y \leq 2.$
5. Compute $\int_C x^2 \, dy + y \, dx$, where $D$ is described by $0 \leq x \leq 1, 0 \leq y \leq y.$
6. Compute $\int_C y(x, y) \, dx + y \, dx$, where $D$ is described by $0 \leq x \leq 2, 0 \leq y \leq 1.$
7. Compute $\int_C y(x, y) \, dx + (2x-3y) \, dy$, where $D$ is described by $1 \leq x \leq 2, 1 \leq y \leq 2.$
8. Compute $\int_C x^2 + y \, dx + y \, dy$, where $D$ is described by $0 \leq x \leq \pi/2, 0 \leq y \leq 2.$
9. Compute $\int_C x \, dy + y \, dx$, where $D$ is described by $1 \leq x \leq 2, 1 \leq y \leq e^x.$

### 16.3 Chapter 16 Vector Calculus

The divergence and curl can now be defined in terms of this same odd vector $\mathbf{F}$ by using the cross product and dot product. The divergence of a vector field $\mathbf{F} = (f, g, h)$ is
\[ \nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}. \]

The curl of $\mathbf{F}$ is
\[ \nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}. \]

These are two simple but useful facts about divergence and curl.

### Theorem 16.5.1

**Theorem 16.5.1** \( \nabla \times (\nabla \times \mathbf{F}) = 0 \)

In words, this says that the divergence of the curl is zero.

**THEOREM 16.5.2** \( \nabla \cdot (\nabla \times \mathbf{F}) = 0 \)

That is, the curl of a gradient is the zero vector. Recalling that gradients are conservative vector fields, this says that the curl of a conservative vector field is the zero vector. Under suitable conditions, it is also true that if the curl of $\mathbf{F}$ is $\mathbf{0}$ then $\mathbf{F}$ is conservative. (Note that this is exactly the same test that we discussed on page 437.)

**EXAMPLE 16.5.3** Let $\mathbf{F} = (e^x, 1, x^2)$. Then $\nabla \times \mathbf{F} = (0, e^x - e^x, 0) = 0$. Thus, $\mathbf{F}$ is conservative, and we can exhibit this directly by finding the corresponding $f$.

Since $f_x = e^x$, $f = e^x + y(g(x))$. Since $f_y = 1$, it must be that $g_y = 1$, so $g(y, z) = y + h(z)$. Thus $f = e^x + y + h(z)$ and $e^x = f_x = f_x + 0 + h''(z)$. so $h''(z) = 0$, i.e., $h(z) = C$, and $f = e^x + y + C$.

We can rewrite Green's Theorem using these new ideas; these rewritten versions in turn are closer to some later theorems we will see.

Suppose we write a two dimensional vector field in the form $\mathbf{F} = (P, Q, 0)$, where $P$ and $Q$ are functions of $x$ and $y$. Then
\[ \nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = (0, 0, \nabla \cdot \mathbf{F}), \]

and $\nabla \times \mathbf{F} = (0, 0, \nabla \cdot \mathbf{F}) = 0$. So Green's Theorem says
\[ \int_D \nabla \times \mathbf{F} \cdot d\mathbf{r} = \int_D P \, dx + Q \, dy = \iint_D \nabla \cdot \mathbf{F} \, dA = \iint_D (\nabla \times \mathbf{F}) \cdot d\mathbf{A}. \]
Roughly speaking, the right-most integral adds up the curl (tendency to swirl) at each point in the region; the left-most integral adds up the tangential components of the vector field around the entire boundary. Green’s Theorem says these are equal, or roughly, that the sum of the “microscopic” swirls over the region is the same as the “macroscopic” swirl around the boundary.

Next, suppose that the boundary $\partial D$ has a vector form $\mathbf{r}(t)$, so that $\mathbf{r}'(t)$ is tangent to the boundary, and $\mathbf{T} = \mathbf{r}'(t)/|\mathbf{r}'(t)|$ is the usual unit tangent vector. Writing $\mathbf{r} = \langle x(t), y(t) \rangle$, we get

$$\mathbf{T} = \frac{\langle x', y' \rangle}{|\mathbf{r}'(t)|},$$

and then

$$\mathbf{N} = \frac{\langle x'' - y', -x', 0 \rangle}{|\mathbf{r}'(t)|}$$

is a unit vector perpendicular to $\mathbf{T}$, that is, a unit normal to the boundary. Now

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{N} \, ds = \int_{\partial D} (P, Q) \cdot \left( \frac{\langle x'' - y', -x', 0 \rangle}{|\mathbf{r}'(t)|} \right) \, ds = \int_{\partial D} P y' \, dt - Q x' \, dt$$

$$= \int_{\partial D} P \, dy - Q \, dx = \int_{D} -Q \, dx + P \, dy.$$

So far, we’ve just rewritten the original integral using alternate notation. The last integral looks just like the right side of Green’s Theorem (16.4.1) except that $\mathbf{F}$ has acquired a negative sign. Then applying Green’s Theorem we get

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{N} \, ds = \int_{D} \int_{\partial D} \mathbf{F} \cdot \mathbf{N} \, dA.$$

Summarizing the long string of equalities,

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{N} \, ds = \int_{D} \int_{\partial D} \mathbf{F} \cdot \mathbf{N} \, dA. \quad (16.5.2)$$

Roughly speaking, the first integral adds up the flow across the boundary of the region, from inside to out, and the second sums the divergence (tendency to spread) at each point in the interior. The theorem roughly says that the sum of the “microscopic” spreads is the same as the total spread across the boundary and out of the region.

### 16.5.1 Vector Functions for Surfaces

Consider the function $\mathbf{r}(u, v) = \langle u \cos v, v \sin u, v \rangle$. For a fixed value of $v$, as $u$ varies from $0$ to $2\pi$, this traces a circle of radius $v$ at height $v$ above the $x$-$y$ plane. Put lots and lots of these together and they form a cone, as in figure 16.6.1.

**Example 16.6.1**

The value of $v$ and letting $\mathbf{r}(u, v)$ sweep out a curve as changes. Then $v$ can change a bit, and $\mathbf{r}(u, v)$ sweeps out a new curve very close to the first. Put enough of these curves together and they form a surface.

**Example 16.6.2**

Let $\mathbf{r} = \langle \cos u \cos v, \sin u \sin v, u \rangle$. If $v$ is constant, the resulting curve is a helix (as in figure 13.1.1). If $u$ is constant, the resulting curve is a straight line at height $u$ in the direction $v$ radians from the positive $z$ axis. Note in figure 16.6.2 how the helixes and the lines both paint the same surface in a different way.

This technique allows us to represent many more surfaces than previously.

**Example 16.6.3**

The curve given by

$$\mathbf{r} = \langle (2 + \cos(3v/2)) \sin u, (2 + \cos(3u/2)) \sin u, \sin(3u/2) \rangle$$

called is a trefoil knot. Recall that from the vector equation of the curve we can compute the unit tangent $\mathbf{T}$, the unit normal $\mathbf{N}$, and the binormal vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$; you may want to review section 13.3. The binormal is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$; one way to interpret this is that $\mathbf{N}$ and $\mathbf{B}$ define a plane perpendicular to $\mathbf{T}$, that is, perpendicular to the curve; since $\mathbf{N}$ and $\mathbf{B}$ are perpendicular to each other, they can function just as in

### 16.6 Vector Functions for Surfaces

We have dealt extensively with vector equations for curves, $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. A similar technique can be used to represent surfaces in a way that is more general than the equations for surfaces we have used so far. Recall that when we use $\mathbf{r}'(t)$ to represent a curve, we imagine the vector $\mathbf{r}(t)$ with its tail at the origin, and then we follow the head of the arrow as $t$ changes. The vector “draws” the curve through space as $t$ varies.

Suppose we instead have a vector function of two variables,

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle.$$
We have previously examined surfaces given in the form \( f(x, y) \). It is sometimes useful to represent such surfaces in the more general vector form, which is quite easy: \( r(u, v) = \langle u, v, f(u, v) \rangle \). The names of the variables are not important of course; instead of designating \( x \) and \( y \), we could simply write \( r(x, y) = \langle x, y, f(x, y) \rangle \).

We have also previously dealt with surfaces that are not functions of \( x \) and \( y \); many of these are easy to represent in vector form. One common type of surface that cannot be represented as \( z = f(x, y) \) is a surface given by an equation involving only \( x \) and \( y \). For example, \( x + y = 1 \) and \( -y^2 + 1 = x \) are “vertical” surfaces. For every point \((x, y)\) in the plane that satisfies the equation, the point \((x, y, z)\) lies on the surface, for every value of \( z \). Thus, a corresponding vector form for the surface is something like \( \langle f(x, y), v, u \rangle \); for example, \( x + y = 1 \) becomes \( \langle 0, 1, u \rangle \) and \( y^2 = x^2 + 1 \) becomes \( \langle u, 0, v \rangle \).

Yet another sort of example is the sphere, say \( x^2 + y^2 + z^2 = 1 \). This cannot be written in the form \( z = f(x, y) \), but it is easy to write in vector form; indeed this particular surface is much like the cone, since it has circular cross-sections, or we can think of it as a tube around a portion of the \( z \)-axis, with a radius that varies depending on where along the axis we are. One vector expression for the sphere is \( \langle \sin u \cos v, \sin u \sin v, \cos u \rangle \); this emphasizes the tube structure, as it is naturally viewed as drawing a circle of radius \( \sqrt{1 - \cos^2 u} \) around the \( z \)-axis at height \( v \). We could also take a cue from spherical coordinates, and write \( \langle \sin u \cos v \cos \theta, \sin u \sin v \cos \theta, \sin u \sin \theta \rangle \), where in effect \( u \) and \( v \) are \( \phi \) and \( \theta \) in disguise.

It is quite simple in Sage to plot any surface for which you have a vector representation. Using different vector functions sometimes gives different looking plots, because Sage in effect draws the surface by holding one variable constant and then the other. For example, you might have noticed in figure 16.6.2 that the curves in the two right-hand graphs are superimposed on the left-hand graph; the graph of the surface is just the combination of the two sets of curves, with the spaces filled in with color.

Here’s a simple but striking example: the plane \( x + y + z = 1 \), which is the plane \( \langle x, y, f(u, v) \rangle \) where in effect \( u \) and \( v \) are \( \phi \) and \( \theta \) in disguise.

11. Find the area of the portion of \( x^2 + y^2 = 1 \) for \( 0 \leq x \leq 1 \).
12. Find the area of the portion of \( y = x^2 \) for \( 0 \leq x \leq 1 \).
13. Find the area of the portion of \( y = x^2 \) for \( 0 \leq x \leq 1 \).
14. Find the area of the portion of \( y = x^2 \) for \( 0 \leq x \leq 1 \).

16.6 Vector Functions for Surfaces

EXAMPLE 16.6.4 We find the area of the surface \( \langle \cos u \cos v, \sin u \cos v, \sin v \rangle \) for \( 0 \leq u \leq \pi \) and \( 0 \leq v \leq \pi \) \( v \leq \pi \). This is a portion of the helical surface in figure 16.6.2. We compute \( r_u = (-\sin u \cos v, \cos u \cos v, 0) \) and \( r_v = (\sin u \sin v, \cos u \sin v, 0) \). The cross product of these two vectors is \langle \sin u \sin v, \cos u \sin v, 0 \rangle \rangle with length \( \sqrt{1 + \sin^2 v} \), and the surface area is

\[
\iint_D \sqrt{1 + \sin^2 v} \, dv \, du = \frac{\pi}{2} \left( 1 + \frac{\pi}{2} \right) \frac{\pi}{2}.
\]

16.7 Surface Integrals

In the integral for surface area,

\[
\iint_D \oint_{\partial S} \langle r_u \times r_v \rangle \, du \, dv,
\]

the integrand \( \langle r_u \times r_v \rangle \, du \, dv \) is the area of a tiny parallelogram, that is, a very small surface area, so it is reasonable to abbreviate it \( dS \); then a shortened version of the integral is

\[
\iint_D \iint_{\partial S} 1 \, dS.
\]

We have already seen (section 15.4) how to find the area of a surface when it is defined in the form \( f(x, y) \). Finding the area when the surface is given as a vector function is very similar. Looking at the plots of surfaces we have just seen, it is evident that the two sets of curves that fill out the surface divide it into a grid, and that the spaces in the grid are approximately parallelograms. As before this is the key: we can write down the area of a typical little parallelogram and add them all up with an integral.

Suppose we want to approximate the area of the surface \( r(u, v) \) near \( r(u_0, v_0) \). The functions \( r(u, v) \) and \( r(u, v) \) define two curves that intersect at \( r(u_0, v_0) \). The derivatives of \( r \) give us vectors tangent to these two curves, \( r_u(u_0, v_0) \) and \( r_v(u_0, v_0) \), and then \( \langle r_u(u_0, v_0) \times r_v(u_0, v_0) \rangle \) is the area of a small parallelogram, whose lengths can be used as the lengths of the sides of an approximating parallelogram. Finally, the area of this parallelogram is \( \langle r_u \times r_v \rangle \, du \, dv \) and so the total surface area is

\[
\iint_D \langle r_u \times r_v \rangle \, du \, dv.
\]

That is, we express everything in terms of \( u \) and \( v \), and then we can do an ordinary double integral.

EXAMPLE 16.7.1 Suppose a thin object occupies the upper hemisphere of \( x^2 + y^2 + z^2 = 1 \) and has density \( \sigma(x, y, z) = z \). Find the mass and center of mass of the object. (Note that the object is just a thin shell; it does not occupy the interior of the hemisphere.)
We write the hemisphere as \( x(\phi, \theta) = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \phi), \) \( 0 \leq \phi \leq \pi/2 \) and \( 0 \leq \theta \leq 2\pi. \) So \( r_0 = (\sin \phi \sin \theta, \cos \phi \sin \phi, \sin \phi \phi, -\sin \phi) \) and \( r_\theta = (\cos \phi \sin \phi, \sin \phi \sin \phi, \cos \phi \sin \phi, -\cos \phi \sin \phi) \)
and
\[ [r_\phi \times r_\theta] = (\sin \phi \phi, -\sin \phi \phi, \sin \phi \phi, -\cos \phi \sin \phi) \]

since we are interested only in \( 0 \leq \phi \leq \pi/2. \) Finally, the density is \( \chi = \cos \phi \) and the integral for mass is
\[
\int_0^{2\pi} \int_0^{\pi/2} \int_0^2 \rho \, r_\phi \cdot r_\theta \, d\phi \, d\theta = \pi. 
\]

By symmetry, the center of mass is clearly on the \( z \)-axis, so we only need to find the \( z \)-coordinate of the center of mass. The moment around the \( x-y \) plane is
\[
\int_0^{2\pi} \int_0^{\pi/2} \int_0^2 z \rho \cos \phi \sin \phi \, r_\phi \cdot r_\theta \, d\phi \, d\theta = \frac{2\pi}{3},
\]

so the center of mass is at \( (0, 0, 2/3). \)

Now suppose that \( \mathbf{F} \) is a vector field; imagine that it represents the velocity of some fluid at each point in space. We would like to know how much fluid is passing through a surface \( D, \) the flux across \( D. \) As usual, we imagine computing the flux across a very small section of the surface, with area \( dS, \) and then adding up all such small fluxes over \( D \) with an integral. Suppose that vector \( \mathbf{N} \) is a unit normal to the surface at a point; \( \mathbf{N} \) is the scalar projection of \( \mathbf{F} \) onto the direction of \( \mathbf{N}, \) so it measures how fast the fluid is moving across the surface. In one unit of time the fluid moving across the surface will fill a volume of \( \mathbf{N} \cdot dS, \) which is the rate at which the fluid is moving across a small patch of the surface. Thus, the total flux across \( D \) is
\[
\int_D \mathbf{F} \cdot dS = \int_D \mathbf{F} \cdot \mathbf{N} dS
\]
defining \( dS = \mathbf{N} dS. \) As usual, certain conditions must be met for this to work out; chief among them is the nature of the surface. To integrate over the surface, we must choose the normal vectors \( \mathbf{N} \) in such a way that they point “the same way” through the surface. For example, if the surface is roughly horizontal in orientation, we might want to measure the flux in the “upwards” direction, or if the surface is closed, like a sphere, we might want to measure the flux “outwards” across the surface. In the first case we need to have positive \( z \)-component, in the second we would make sure that \( \mathbf{N} \) points away from the origin.

Assuming that the quantities involved are well behaved, however, the flux of the vector field across the surface \( \mathbf{r}(x, y) \) is
\[
\int_D \mathbf{F} \cdot dS = \int_D \mathbf{F} \cdot \mathbf{N} dS = \int_D \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) \cdot dA = \int_D \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) \cdot dA.
\]

In practice, we may have to use \( \mathbf{r}_x \times \mathbf{r}_y \) or even something a bit more complicated to make sure that the normal vector points in the desired direction.

**EXAMPLE 16.7.2** Compute the flux of \( \mathbf{F} = (x, y, z) \) across the cone \( z = \sqrt{x^2 + y^2}, \) \( 0 \leq z \leq 1, \) in the downward direction.

We write the cone as a vector function: \( \mathbf{r} = (\cos u \sin v, \sin u, v), \) \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq 1. \) Then \( \mathbf{r}_u = (-\sin u \sin v, \cos u, 0) \) and \( \mathbf{r}_v = (\cos u \cos v, \sin u, 1) \) and \( \mathbf{r}_u \times \mathbf{r}_v =

Note how little has changed: \( \mathbf{k} \) becomes \( \mathbf{N}, \) a unit normal to the surface, and \( dA \) becomes \( dS, \) since this is now a general surface integral. The phrase “clockwise-counter-clockwise relative to \( \mathbf{N} \)” means roughly that if we take the direction of \( \mathbf{N} \) to be “up,” then we go around the boundary counter-clockwise when viewed from “above.” In many cases, this description is inadequate. A slightly more complicated but general description is this: imagine standing on the side of the surface considered positive; walk to the boundary and turn left. You are now following the boundary in the correct direction.

**EXAMPLE 16.8.2** Let \( \mathbf{F} = (\cos x, y^2, xz) \) and the surface \( D \) be \( x = \sqrt{1 - y^2 + z^2}, \) oriented in the positive \( x \) direction. It quickly becomes apparent that the surface integral in Stokes’s Theorem is intractable, so we try the line integral. The boundary of \( D \) is the unit circle in the \( y-z \) plane, \( \mathbf{r} = (0, \cos u, \sin u), \) \( 0 \leq u \leq 2\pi. \) The integral is
\[
\int_0^{2\pi} (\cos u \cos v, y^2, xz) \cdot \mathbf{r}_u \, du = 0 du = 0.
\]
EXAMPLE 16.8.4 Consider the cylinder \( r = (\cos u, \sin u, v), 0 \leq u \leq 2\pi, 0 \leq v \leq 2 \), oriented outward, and \( \mathbf{F} = (y, x, z) \). We compute 
\[
\int_0^{2\pi} \int_0^2 \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r}
\]
in two ways.

First, the double integral is
\[
\int_0^{2\pi} \int_0^2 (\cos u, \sin u, 0) \cdot (-\sin u, \cos u, 0) \, dv \, du = \int_0^{2\pi} \int_0^2 -\sin^2 u \, dv \, du = -2\pi.
\]

The boundary consists of two parts, the bottom circle \((\cos t, \sin t, 0)\), with \( t \) ranging from 0 to 2\( \pi \), and \((\cos t, \sin t, 2)\), with \( t \) ranging from 2\( \pi \) to 0. We compute the corresponding integrals and add the results:
\[
\int_0^{2\pi} -\sin^2 t \, dt + \int_{2\pi}^0 -\sin^2 t + 2 \cos^2 t = -\pi - \pi = -2\pi,
\]
as before.

An interesting consequence of Stokes’s Theorem is that if \( D \) and \( E \) are two orientable surfaces with the same boundary, then
\[
\int_D \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_D \mathbf{F} \cdot d\mathbf{r} = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial E} \mathbf{F} \cdot d\mathbf{r} = \int_E \nabla \times \mathbf{F} \cdot d\mathbf{S}.
\]

Sometimes both of the integrals
\[
\int_D \nabla \times \mathbf{F} \cdot d\mathbf{S} \quad \text{and} \quad \int_{\partial D} \mathbf{F} \cdot d\mathbf{r}
\]
are difficult, but you may be able to find a second surface \( E \) so that
\[
\int_E \nabla \times \mathbf{F} \cdot d\mathbf{S} \quad \text{has the same value but is easier to compute.}
\]

EXAMPLE 16.8.4 In example 16.8.2 the line integral was easy to compute. But we might also notice that another surface \( E \) with the same boundary is the flat disk \( y^2 + z^2 \leq 1 \),...
Proof. Again this theorem is too difficult to prove here, but a special case is easier. In the proof of a special case of Green’s Theorem, we needed to know that we could describe the region of integration in both possible orders, so that we could set up one double integral using $dz\,dy$ and another using $dy\,dz$. Similarly here, we need to be able to describe the three-dimensional region $E$ in different ways.

We start by rewriting the triple integral:

$$
\iiint_E \nabla \cdot \mathbf{F} \, dV = \iiint_E \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) \, dx\,dy\,dz = \iiint_E P_x \, dx\,dy\,z + \iiint_E Q_y \, dy\,dz\,x + \iiint_E R_z \, dz\,dx\,y.
$$

The double integral may be rewritten:

$$
\iiint_D P \, N_x \, dS = \iiint_D \left( P(x, y, z) + P(y, z, x) + P(z, x, y) \right) \, dx\,dy\,dz = \iiint_D \left( P(x, y, z) + P(y, z, x) + P(z, x, y) \right) \, dx\,dy\,dz
$$

To prove that these give the same value it is sufficient to prove that

$$
\iiint_D P \, N_x \, dS = \iiint_D P \, N_x \, dS.
$$

To prove this, we set the triple integral up with three-dimensional region $E$.

$$
\iiint_D P \, N_x \, dS = \iiint_D P \, N_x \, dS
$$

Thus, we are left with just the surface integral over the top plus the surface integral over the bottom. For the top, we use the vector function $\mathbf{r} = (y^2, y, z)$ which gives

$$
\mathbf{r}_x \times \mathbf{r}_y = (1, -g_y, -g_z).
$$

and another using $\mathbf{r}_y \times \mathbf{r}_z = (1, -g_z, -g_x)$. The dot product of this with $\mathbf{i} = (1, 0, 0)$ is 1. Then

$$
\iiint_D P \, N_x \, dS = \iiint_D P(y^2, y, z) \, dA.
$$

In almost identical fashion we get

$$
\iiint_D P \, N_x \, dS = \iiint_D P(y^2, y, z) \, dA
$$

where the negative sign is needed to make $\mathbf{N}$ point in the negative $x$ direction. Now

$$
\iiint_D P \, N_x \, dS = \iiint_D P(y^2, y, z) \, dA
$$

which is the same as the value of the triple integral above.

\begin{example}
Let $\mathbf{F} = (2x, 3y, 2z)$, and consider the three-dimensional volume inside the cube with faces parallel to the principal planes and opposite corners at $(0, 0, 0)$ and $(1, 1, 1)$. We compute the two integrals of the divergence theorem.

The triple integral is the easier of the two:

$$
\iiint_D \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^1 \int_0^1 2 + 2z \, dx\,dy\,dz = 279.
$$

The surface integral must be separated into six parts, one for each face of the cube. One face is $z = 0$ or $r = (u, v, 0), 0 \leq u, v \leq 1$. Then $\mathbf{r}_x = (1, 0, 0), \mathbf{r}_y = (0, 1, 0)$, and $\mathbf{r}_x \times \mathbf{r}_y = (0, 1, 0)$. We need to this be oriented downward (out of the cube), so we use $(0, 0, -1)$ and the corresponding integral is

$$
\iiint_D \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^1 \int_0^1 3 \, dV = 36.
$$

Another face is $y = 1$ or $r = (u, 1, v)$. Then $\mathbf{r}_x = (1, 0, 0), \mathbf{r}_v = (0, 0, 1)$, and the corresponding integral is

$$
\iiint_D \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^1 \int_0^1 3 \, dV = 36.
$$

The remaining four integrals have values 0, 0, 2, and 1, and the sum of these is 6, in agreement with the triple integral.

\end{example}