16
Vector Calculus

16.1 Vector Fields

This chapter is concerned with applying calculus in the context of vector fields. A two-dimensional vector field is a function \( f \) that maps each point \((x, y)\) in \( \mathbb{R}^2 \) to a two-dimensional vector \((u, v)\), and similarly a three-dimensional vector field maps \((x, y, z)\) to \((u, v, w)\). Since a vector has no position, we typically indicate a vector field in graphical form by placing the vector \(f(x,y)\) with its tail at \((x, y)\). Figure 16.1.1 shows a representation of the vector field \( f(x, y) = (-x/\sqrt{x^2 + y^2 + 4}, y/\sqrt{x^2 + y^2 + 4}) \). For such a graph to be readable, the vectors must be fairly short, which is accomplished by using a different scale for the vectors than for the axes. Such graphs are thus useful for understanding the sizes of the vectors relative to each other but not their absolute size.

Vector fields have many important applications, as they can be used to represent many physical quantities: the vector at a point may represent the strength of some force (gravity, electricity, magnetism) or a velocity (wind speed or the velocity of some other fluid).

We have already seen a particularly important kind of vector field—the gradient. Given a function \( f(x, y) \), recall that the gradient is \( \langle f_x(x, y), f_y(x, y) \rangle \), a vector that depends on \((x, y)\) (is a function of) \( x \) and \( y \). We usually picture the gradient vector with its tail at \((x, y)\), pointing in the direction of maximum increase. Vector fields that are gradients have some particularly nice properties, as we will see. An important example is

\[
F = \left( \frac{-x}{(x^2 + y^2 + 2)^{3/2}}, \frac{-y}{(x^2 + y^2 + 2)^{3/2}}, \frac{-z}{(x^2 + y^2 + z^2)^{3/2}} \right),
\]

which points from the point \((x, y, z)\) toward the origin and has length

\[
\left(\frac{x^2 + y^2 + 2}{(x^2 + y^2 + z^2)^{3/2}}\right)^{1/2} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}
\]

which is the reciprocal of the square of the distance from \((x, y, z)\) to the origin—in other words, \( F \) is an “inverse square law”. The vector \( F \) is a gradient:

\[
F = \nabla \frac{1}{\sqrt{x^2 + y^2 + z^2}} \tag{16.1.1}
\]

which turns out to be extremely useful.

Examples 16.1.

Sketch the vector fields; check your work with Sage’s plot_vector_field function.

1. \((x, y)\)
2. \((x, -y)\)
3. \((x, y)\)
4. \((x, y)\)
5. \((y, y, x)\)
6. \((x + y, x + 3)\)

16.2 Line Integrals

We have so far integrated “over” intervals, areas, and volumes with single, double, and triple integrals. We now investigate integration over or “along” a curve—“line integrals” are really “curve integrals”. As with other integrals, a geometric example may be easiest to understand. Consider the function \( f = x + y \) and the parabola \( y = x^2 \) in the \( x, y \) plane, for \( 0 \leq x \leq 2 \). Imagine that we extend the parabola up to the surface \( f \), to form a curved wall or curtain, as in figure 16.2.1. What is the area of the surface thus formed? We already know one way to compute surface area, but here we take a different approach that is more useful for the problems to come.

Typically the curve is in vector form, or can easily be put in vector form; in this example we have \( \mathbf{v}(t) = (t, t^2) \). Then as we have seen in section 13.3 on arc length, the length of one of the straight line segments in the approximation is approximately

\[
ds = |\mathbf{v}'(t)|\,dt = \sqrt{1 + 4t^2} \,dt.
\]

As an integral is

\[
\int_a^b f(x, y)\,ds = \int_a^b f(t, t^2)\sqrt{1 + 4t^2} \,dt = \frac{107}{35} \sqrt{17} - \frac{1}{17} \ln(4 + \sqrt{17}).
\]

This integral of a function along a curve \( C \) is often written in abbreviated form as

\[
\int_C f(x, y)\,ds.
\]

Example 16.2.1 Compute \( \int_C ye^t\,ds \) where \( C \) is the line segment from \((1, 2)\) to \((4, 7)\).

We write the line segment as a vector function: \( \mathbf{v} = (1 + t(3, 5), 0 \leq t \leq 1, \) or in parametric form \( x = 1 + 3t, y = 2 + 5t \). Then

\[
\int_C ye^t\,ds = \int_0^1 (2 + 5t)e^{1 + 3t(1 + t)}\sqrt{1 + 4e^t} \,dt = \frac{16}{9}\sqrt{3e} - \frac{1}{9}\ln(4 + \sqrt{17}).
\]

All of these extend to three dimensions in the obvious way.

Example 16.2.2 Compute \( \int_C z^2\,ds \) where \( C \) is the line segment from \((0, 6, -1)\) to \((4, 5, 6)\).

We write the line segment as a vector function: \( \mathbf{v} = (0, 6, -1) + t(4, 5, 6), 0 \leq t \leq 1, \) or in parametric form \( x = 4t, y = 6 - 5t, z = -1 + 6t \). Then

\[
\int_C z^2\,ds = \int_0^1 (4t)^2(-1 + 6t)(\sqrt{16 + 25 + 36} - 16 \sqrt{17})\sqrt{1 + 4t^2 + 36t^4} \,dt = \frac{56}{3}\sqrt{17}.
\]

Now we turn to a perhaps more interesting example. Recall that in the simplest case, the work done by a force on an object is equal to the magnitude of the force times the distance the object moves; this assumes that the force is constant and in the direction of motion. We have already dealt with examples in which the force is not constant; now we are prepared to examine what happens when the force is not parallel to the direction of motion.
The projection of $\mathbf{F}$ onto $\mathbf{v}$: The length of this vector, that is, the magnitude of the force in the direction of $\mathbf{v}$ is

$$F \cdot \frac{\mathbf{v}}{|\mathbf{v}|}.$$ 

the scalar projection of $\mathbf{F}$ onto $\mathbf{v}$. If an object moves subject to this (constant) force, in the direction of $\mathbf{v}$, over a distance equal to the length of $\mathbf{v}$, the work done is

$$F \cdot \frac{\mathbf{v}}{|\mathbf{v}|} |\mathbf{v}| = F \cdot \mathbf{v}.$$ 

Thus, work in the vector setting is still “force times distance”, except that “times” means “dot product”.

If the force varies from point to point, it is represented by a vector field $\mathbf{F}$, the displacement vector $\mathbf{v}$ may also change, as an object may follow a curving path in two or three dimensions. Suppose that the path of an object is given by a vector function $\mathbf{r}(t)$; at any point along the path, the (small) tangent vector $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ gives an approximation to its motion over a short time $\Delta t$, so the work done during that time is approximately $F \cdot \mathbf{v} \Delta t$; the total work over some time period is then

$$\int_{t_0}^{t} F \cdot \mathbf{v} \, dt.$$ 

It is useful to rewrite this in various ways at different times. We start with

$$\int_{t_0}^{t} F \cdot \mathbf{v} \, dt = \int_{C} F \cdot d\mathbf{r},$$ 

abbreviating $\mathbf{v} \, dt$ by $d\mathbf{r}$. Or we can write

$$\int_{t_0}^{t} F \cdot \mathbf{v} \, dt = \int_{C} F \cdot \mathbf{T} \, ds,$$

using the unit tangent vector $\mathbf{T}$, abbreviating $\mathbf{v} \, dt$ as $d\mathbf{s}$, and indicating the path of the object by $C$. In other words, work is computed using a particular line integral of the form

$$\int_{C} F \cdot \mathbf{T} \, ds.$$ 

10. Compute $\int_{C} (1/z, 1/(x+y)) \cdot d\mathbf{r}$ along the path from $(1,1)$ to $(3,0)$ by using straight line segments $\Rightarrow$

11. Compute $\int_{C} (1/z, 1/(x+y)) \cdot d\mathbf{r}$ along the curve $r(t) = (t^2, t^2)$, $1 \leq t \leq 4. \Rightarrow$

12. Compute $\int_{C} (1/z, 1/(x+y)) \cdot d\mathbf{r}$ along the curve of $r(t) = (t^2, t^2)$, $1 \leq t \leq 4. \Rightarrow$

13. Compute $\int_{C} (1/z, 1/(x+y)) \cdot d\mathbf{r}$ along the curve of $r(t) = (t^2, t^2)$, $0 \leq t \leq 1. \Rightarrow$

14. Compute $\int_{a}^{b} \mathbf{F} \cdot d\mathbf{r}$ along the curve of $r(t) = (t^2, t^2)$, $0 \leq t \leq 1. \Rightarrow$

15. An object moves from $(1,1)$ to $(4,8)$ along the path $r(t) = (t^2, t^2)$, subject to the force $\mathbf{F} = (x^2, y^2)$. Find the work done. $\Rightarrow$

16. An object moves along the line segment from $(1,1)$ to $(2,5)$, subject to the force $\mathbf{F} = (x^2, y^2)$. Find the work done. $\Rightarrow$

17. An object moves along the parabola $r(t) = (t^2, t^2)$, $0 \leq t \leq 1$, subject to the force $\mathbf{F} = (1/(y+1), -(1/(y+1))$. Find the work done. $\Rightarrow$

18. An object moves along the line segment from $(0,0,0)$ to $(3,6,10)$, subject to the force $\mathbf{F} = (x^2, y^2, z^2)$. Find the work done. $\Rightarrow$

19. An object moves along the line segment from $(1,0,0)$ to $(1,-1,0)$, subject to the force $\mathbf{F} = (y, z, x)$. Find the work done. $\Rightarrow$

20. An object moves from $(1,1,2)$ to $(2,4,8)$ along the path $r(t) = (t, t^2, t^2)$, subject to the force $\mathbf{F} = (x^2, y^2, z^2)$. Find the work done. $\Rightarrow$

21. An object moves from $(1,0,0)$ to $(1,-1,0,0)$ along the path $r(t) = (\cos(t), \sin(t), t, t)$, subject to the force $\mathbf{F} = (x^2, y^2, z^2)$. Find the work done. $\Rightarrow$

22. Give an example of a non-trivial force field $\mathbf{F}$ and non-trivial path $r(t)$ for which the total work done moving along the path is zero.

16.3 The Fundamental Theorem of Line Integrals

One way to write the Fundamental Theorem of Calculus (7.2.1) is:

$$\int_{a}^{b} f'(x) \, dx = f(b) - f(a).$$ 

That is, to compute the integral of a derivative $f'$ we need only compute the values of $f$ at the endpoints. Something similar is true for line integrals of a certain form.

**Theorem 16.3.1 Fundamental Theorem of Line Integrals** Suppose a curve $C$ is given by the vector function $r(t)$, with $a = r(a)$ and $b = r(b)$. Then

$$\int_{C} F \cdot dr = f(b) - f(a),$$

provided that $r$ is sufficiently nice.

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Another immediate consequence of the Fundamental Theorem involves closed paths. A path \( C \) is closed if it forms a loop, so that traveling over the curve \( C \) brings you back to the starting point. If \( C \) is a closed path, we can integrate around it starting at any point, since the starting and ending points are the same.

\[ \oint_C \vec{F} \cdot d\vec{r} = \int_C (\vec{f} \cdot \vec{n}) \, ds = 0. \]

For example, in a gravitational field (an inverse square law field) the amount of work required to move an object around a closed path is zero. Of course, it's only the net amount of work that is zero. It may well take a great deal of work to get from point \( a \) to point \( b \), but then the return trip will "produce" work. For example, it takes work to pump water from a lower to a higher elevation, but if you then let gravity pull the water back down, you can recover work by running a water wheel or generator. (In the real world you won't recover all the work because of various losses along the way.)

To make use of the Fundamental Theorem of Line Integrals, we need to be able to spot conservative vector fields \( \vec{F} \) and to compute \( f \) so that \( \vec{F} = \nabla f \). Suppose that \( \vec{F} = (P, Q) = \nabla f \). Then \( P = f_y \) and \( Q = f_x \), and provided that \( f \) is sufficiently nice, we know from Clairaut's Theorem (14.6.2) that \( P_y = f_{yy} = f_{xy} = Q_x \). If we compute \( P_y \) and \( Q_x \) and find that they are not equal, then \( \vec{F} \) is not conservative. If \( P_y = Q_x \), then, again provided that \( \vec{F} \) is sufficiently nice, we can be assured that \( \vec{F} \) is conservative. Ultimately, what's important is that we be able to find \( f \), as this amounts to finding anti-derivatives, we may not always succeed.

**EXAMPLE 16.3.3** Find an \( f \) so that \( (3 + 2xy, x^2 - 3y^2) = \nabla f \).

First, note that

\[
\frac{\partial}{\partial y} (3 + 2xy) = 2x \quad \text{and} \quad \frac{\partial}{\partial x} (x^2 - 3y^2) = 2x,
\]

so the desired \( f \) does exist. This means that \( f_x = 3 + 2xy \) and \( f_y = g(y) \); the first two terms are needed to get \( 3 + 2xy \), and the \( g(y) \) could be any function of \( y \), as it would disappear upon taking a derivative with respect to \( x \). Likewise, since \( f_y = x^2 - 3y^2 \),

\[
f = xy^2 - y^3 + h(x).\]

This question now becomes, is it possible to find \( g(y) \) and \( h(x) \) so that

\[
3x + x^2y + g(y) = x^2 - y^3 + h(x),
\]

and of course the answer is yes: \( g(y) = -y^3, h(x) = 3x \). Thus, \( f = 3x + x^2y - y^3 \).

We can test a vector field \( \vec{F} = (P, Q, R) \) in a similar way. Suppose that \( (P, Q, R) = (f_x, f_y, f_z) \). If we temporarily hold \( z \) constant, then \( f(x, y, z) \) is a function of \( x \) and \( y \).

16.4 Green's Theorem 429

**THEOREM 16.4.1 Green's Theorem** If the vector field \( \vec{F} = (P, Q) \) and the region \( D \) is sufficiently nice, and if \( C \) is the boundary of \( D \) (a closed curve), then

\[
\int_C (Q_x - P_y) \, dx + (P_x - Q_y) \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA,
\]

provided the integration on the right is done counter-clockwise around \( C \).

To indicate that an integral \( \int_C \) is being done over a closed curve in the counter-clockwise direction, we usually write \( \oint_C \). We also use the notation \( \nabla \theta \) to mean the boundary of \( D \) oriented in the counter-clockwise direction. With this notation, \( \oint_C \) means \( \int_C \).

We already know one case, not particularly interesting, in which this theorem is true: If \( \vec{F} \) is conservative, we know that the integral \( \oint_C \vec{F} \cdot d\vec{r} = 0 \), because any integral of a conservative vector field around a closed curve is zero. We also know in this case that \( \partial P/\partial y = \partial Q/\partial x \), so both integrals in the theorem are simply the integral of the zero function, namely, 0. So in the case that \( \vec{F} \) is conservative, the theorem says simply that \( 0 = 0 \). In the correct direction. Now:

\[
\iint_D 1 \, dA
\]

computes the area of region \( D \). If we can find \( P \) and \( Q \) so that \( \partial Q/\partial x - \partial P/\partial y = 1 \), then the area is also

\[
\oint_C P \, dx + Q \, dy.
\]

It is quite easy to do this: \( P = 0, Q = x \) works, as do \( P = -y, Q = 0 \) and \( P = -y/2, Q = x/2 \).

**EXAMPLE 16.4.2** An ellipse centered at the origin, with its two principal axes aligned with the \( x \) and \( y \) axes, is given by

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
\]

We find the area of the interior of the ellipse via Green's theorem. To do this we need a vector equation for the boundary; one such equation is \( (a \cos t, b \sin t) \), as \( t \) ranges from 0 to \( 2\pi \). We can easily verify this by substitution:

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2 \cos^2 t}{a^2} + \frac{b^2 \sin^2 t}{b^2} = \cos^2 t + \sin^2 t = 1.
\]

Let's consider the three possibilities for \( P \) and \( Q \) above: Using \( 0 \) and \( x \) gives

\[
\oint_C 0 \, dx + x \, dy = \int_0^{2\pi} a \cos t \, (b \cos t) \, dt = \int_0^{2\pi} ab \cos^2 t \, dt.
\]
Using \( -q \) and 0 gives

\[
\oint_C -y \, dx + 0 \, dy = \int_0^{2\pi} -\sin(t)(-a \sin(t)) \, dt = \int_0^{2\pi} ab \sin^2(t) \, dt.
\]

Finally, using \(-y/2\) and 2/2 gives

\[
\oint_C \frac{y}{2} \, dx + \frac{x}{2} \, dy = \int_0^{2\pi} -\sin(t)(-a \sin(t)) + \frac{a \cos(t) \cos(t)}{2} \, dt = \int_0^{2\pi} ab \sin^2(t)/2 + \cos^2(t)/2 \, dt = \int_0^{2\pi} ab \, dt = \pi ab.
\]

The first two integrals are not particularly difficult, but the third is very easy, though the choice of \( P \) and \( Q \) seems more complicated.

![Figure 16.4.1: A “standard” ellipse.](http://mathinsight.org/divergence_idea)

**Proof of Green’s Theorem.** We cannot here prove Green’s Theorem in general, but we can do a special case. We seek to prove that

\[
\oint_C P \, dx + Q \, dy = \iint_D \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \, dA.
\]

It is sufficient to show that

\[
\oint_C P \, dx = \iint_D \frac{\partial P}{\partial y} \, dA \quad \text{and} \quad \oint_C Q \, dy = \iint_D \frac{\partial Q}{\partial x} \, dA,
\]

which can be done if we can compute the double integral in both possible ways, that is, using \( dA = dy \, dx \) and \( dA = dx \, dy \).

### 16.4 Green’s Theorem

\[
\oint_C \left( \sum_{k=1}^n a_k (y_1, y_2) \right) \, dx_k = \int_D \left( \sum_{k=1}^n \frac{\partial}{\partial x_k} a_k \right) \, dA.
\]

**Theorem 16.5.1**

\[
\oint_C (Q \, dx + P \, dy) = \iint_D \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \, dA.
\]

### 16.5 Divergence and Curl

Divergence and curl are two measurements of vector fields that are very useful in a variety of applications. Both are most easily understood by thinking of the vector field as representing a flow of a liquid or gas, that is, each vector in the vector field should be interpreted as a velocity vector. Roughly speaking, divergence measures the tendency of the fluid to collect or disperse at a point, and curl measures the tendency of the fluid to swirl around the point. Divergence is a scalar, that is, a single number, while curl is itself a vector. The magnitude of the curl measures how much the fluid is swirling, the direction indicates the axis around which it tends to swirl. These ideas are somewhat subtle in practice, and are beyond the scope of this course. You can find additional information on the web, for example at http://mathinsight.org/curl_idea and http://mathinsight.org/divergence_idea and in many books including Div, Grad, Curl, and All That: An Informal Text on Vector Calculus, by H. M. Schey.

Recall that if \( f \) is a function, the gradient of \( f \) is given by

\[
\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).
\]

A useful mnemonic for this (and for the divergence and curl, as it turns out) is to let

\[
\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right),
\]

that is, we pretend that \( \nabla \) is a vector with rather odd looking entries. Recalling that \( (x, u, v) = (u, u, v) \), we can then think of the gradient as

\[
\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) f = \left( \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \frac{\partial f}{\partial z} \right).
\]

That is, we simply multiply the \( f \) into the vector.

The divergence and curl can now be defined in terms of this same odd vector \( \nabla \) by using the cross product and dot product. The divergence of a vector field \( \mathbf{F} = (f, g, h) \) is

\[
\nabla \cdot \mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z},
\]

and the curl of \( \mathbf{F} \) is

\[
\nabla \times \mathbf{F} = \left( \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \mathbf{i} + \left( \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \mathbf{j} + \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathbf{k}.
\]

The divergence is given by

\[
\nabla \cdot \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}.
\]

Here are two simple but useful facts about divergence and curl.

**Theorem 16.5.1**

\[
\nabla \cdot (\nabla \times \mathbf{F}) = 0.
\]

In words, this says that the divergence of the curl is zero.

**Theorem 16.5.2**

\[
\nabla \times (\nabla f) = 0.
\]

That is, the curl of a gradient is the zero vector. Recalling that gradients are conservative vector fields, this says that the curl of a conservative vector field is the zero vector. Under suitable conditions, it is also true that if the curl of \( \mathbf{F} \) is 0 then \( \mathbf{F} \) is conservative.

(Note that this is exactly the same test that we discussed on page 427.)

**Example 16.5.3**

Let \( \mathbf{F} = (e^x, e^x) \). Then \( \nabla \times \mathbf{F} = (0, 0, -e^x) = 0 \). Thus, \( \mathbf{F} \) is conservative, and we can exhibit this directly by finding the corresponding \( f \).

Since \( f_x = e^x \), \( f = e^x + g(y, z) \). Since \( f_y = 1 \), it must be that \( g_y = 0 \), so \( g(y, z) = y + h(z) \). Thus \( f = e^x + y + h(z) \) and \( e^x = f_x = -f_y + 0 + h'(z) \), so \( h'(z) = 0 \), i.e., \( h(z) = C \), and \( f = e^x + y + C \).

We can rewrite Green’s Theorem using these new ideas; these rewritten versions in turn are closer to some later theorems we will see.

Suppose we write a two dimensional vector field in the form \( \mathbf{F} = (P, Q, 0) \), where \( P \) and \( Q \) are functions of \( x \) and \( y \). Then

\[
\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & P_y - P_z \\ 0 & Q_x - Q_z & 0 \\ 0 & 0 & 0 \end{vmatrix}.
\]

and so \( \nabla \times \mathbf{F} = \mathbf{k} \cdot (0, Q_x - P_z). \) So Green’s Theorem says

\[
\iint_D \mathbf{F} \cdot d\mathbf{r} = \iint_D P \, dx + Q \, dy = \int_D Q_x - P_z \, dA = \int_D \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA. \qquad (16.5.1)
\]
Roughly speaking, the rightmost integral adds up the curl (tendency to swirl) at each point in the region; the leftmost integral adds up the tangential components of the vector field around the entire boundary. Green’s Theorem says these are equal, or roughly, that the sum of the “microscopic” swirls over the region is the same as the “macroscopic” swirl around the boundary.

Next, suppose that the boundary $\partial D$ has a vector form $\mathbf{r}(t)$, so that $\mathbf{r}'(t)$ is tangent to the boundary, and $\mathbf{T} = \mathbf{r}'(t)/||\mathbf{r}'(t)||$ is the usual unit tangent vector. Writing $\mathbf{r} = (x(t), y(t))$ we get

$$T = \left(\frac{\dot{x}}{||\mathbf{r}'(t)||}, \frac{\dot{y}}{||\mathbf{r}'(t)||}\right),$$

and then

$$N = \left(-\frac{\dot{y}}{||\mathbf{r}'(t)||}, \frac{\dot{x}}{||\mathbf{r}'(t)||}\right)$$

is a unit vector perpendicular to $\mathbf{T}$, that is, a unit normal to the boundary. Now

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{N} \, ds = \int_{\partial D} (P \dot{y} - Q \dot{x}) \, dt = \int_{\partial D} P \dot{y} \, dt - Q \dot{x} \, dt$$

$$= \int_{\partial D} P \, dy - Q \, dx = \int_{\partial D} -Q \, dx + P \, dy.$$

So far, we’ve just rewritten the original integral using alternate notation. The last integral looks just like the right side of Green’s Theorem (16.4.1) except that

$$\int_{\partial D} \mathbf{T} \cdot \mathbf{N} \, ds = \int_{\partial D} \mathbf{F} \cdot \mathbf{N} \, ds.$$

We have dealt extensively with vector equations for curves, $\mathbf{r}(t) = (x(t), y(t), z(t))$. A similar technique can be used to represent surfaces in a way that is more general than the equations for surfaces we have used so far. Recall that when we use $\mathbf{r}(t)$ to represent a curve, we imagine the vector $\mathbf{r}(t)$ with its tail at the origin, and then we follow the head of the arrow as $t$ changes. The vector “draws” the curve through space as $t$ varies.

Suppose instead we have a vector function of two variables, $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$. As both $u$ and $v$ vary, we again imagine the vector $\mathbf{r}(u, v)$ with its tail at the origin, and its head sweeps out a surface in space. A useful analogy is the technology of CRT video screens, in which an electron gun fires electrons in the direction of the screen. The gun’s direction sweeps horizontally and vertically to “paint” the screen with the desired image. In practice, the gun moves horizontally through an entire line, then moves vertically to the next line and repeats the operation. In the same way, it can be useful to imagine fixing a
We have previously examined surfaces given in the form \( f(x, y) \). It is sometimes useful to represent such surfaces in the more general vector form, which is quite easy:

\[
\mathbf{r}(u, v) = (u, v, f(u, v)).
\]

The names of the variables are not important of course; instead of designating \( x \) and \( y \), we could simply write \( \mathbf{r}(x, y) = (x, y, f(x, y)) \).

We have also previously dealt with surfaces that are not functions of \( x \) and \( y \); many of those are easy to represent in vector form. One common type of surface that cannot be represented as \( z = f(x, y) \) is a surface given by an equation involving only \( x \) and \( y \). For example, \( x^2 + y^2 = 1 \) is a circle. Yet another sort of example is the sphere, say \( x^2 + y^2 + z^2 = 1 \) becomes

\[
\mathbf{r}(u, v) = (u \cos v, u \sin v, \sqrt{1 - u^2}).
\]

This emphasizes the tube structure, as it is naturally viewed as drawing a circle of radius \( 1 \) in the \( u \)-plane and rotating it around the \( z \)-axis.

EXAMPLE 16.6.4 We find the area of the surface \( r(u, v) = (u \cos v, u \sin v, u) \) for \( 0 \leq u \leq 1 \), \( 0 \leq v \leq \pi \). This is a portion of the helical surface in figure 16.6.2. We compute

\[
A = \iint_D \sqrt{1 + (du/dv)^2} \, du \, dv.
\]

1. Describe or sketch the surface with the given vector function.
   a. \( \mathbf{r}(u, v) = (u + 3, v - 2, 4u + 5v) \)
   b. \( \mathbf{r}(u, v) = (2u \cos v, 3u \sin v, v) \)
   c. \( \mathbf{r}(u) = (u, u^2 - x^2) \)
   d. \( \mathbf{r}(u) = (u \cos(2u), u \sin(2u)) \)
2. Find a vector function \( r(u, v) \) for the surface.
   a. The plane that passes through the point \((1, 2, -3)\) and is parallel to the vectors \((1, 1, -1)\) and \((1, 1, 1)\).
   b. The lower half of the ellipsoid \( 2x^2 + 4y^2 + z^2 = 1 \).
   c. The part of the sphere of radius 4 centered at the origin that lies between the planes \( z = -2 \) and \( z = 2 \).
3. Find the area of the portion of \( z = 2x + 3y + 4z \) that lies below \( z = 2 \).
4. Find the area of the portion of \( 2x + 4y + 4 = 0 \) inside \( x^2 + y^2 = 1 \).
5. Find the area of \( z = x^2 + y^2 \) that lies below \( z = 1 \).
6. Find the area of \( z = x^2 + y^2 \) that lies below \( z = 2 \).
7. Find the area of the portion of \( x^2 + y^2 + z^2 = 1 \) that lies in the first octant.
8. Find the area of the portion of \( x^2 + y^2 + z^2 = 1 \) that lies below \( x^2 + y^2 \leq 1 \), \( z \geq 0 \).
9. Find the area of \( z = x^2 + y^2 \) that lies inside \( x^2 + y^2 = 1 \).
10. Find the area of \( z = xy \) that lies inside \( x^2 + y^2 = 2 \).
11. Find the area of \( x^2 + y^2 + z^2 = 1 \) that lies above the interior of the circle given in polar coordinates by \( r = \sin \theta \).
12. Find the area of the cone \( z = \sqrt{x^2 + y^2} \) that lies above the interior of the circle given in polar coordinates by \( r = \sin \theta \).
13. Find the area of the plane \( z = x + y \) that lies over a region \( D \) with area \( A \).
14. Find the area of the cone \( z = \sqrt{x^2 + y^2} \) that lies over a region \( D \) with area \( A \).
15. Find the area of the cylinder \( x^2 + y^2 = 1 \) that lies inside the cylinder \( x^2 + z^2 = 1 \).
16. The surface \( f(x, y) \) can be represented with the vector function \( (x, y, f(x, y)) \). Set up the surface area integral using this vector function and compare to the integral of section 15.4.

\[ \int_D \iint_D \sqrt{1 + (du/dv)^2} \, du \, dv \]

The first vector gets to the point \((x_0, y_0, z)\) and then by varying \( u \) and \( v \), \( u \mathbf{a} + v \mathbf{b} \) gets to every point in the plane.

Returning to \( x + y + z = 1 \), the points \((1, 0, 0), (0, 1, 0), \) and \((0, 0, 1)\) are all on the plane. By subtracting coordinates we see that \((-1, 0, 1)\) and \((-1, 1, 0)\) are parallel to the plane, so a third vector form for this plane is

\[ \langle 1, 0, 0 \rangle + w \langle -1, 0, 1 \rangle + v \langle -1, 1, 0 \rangle = (1 - w - v, v, w) \]

This is clearly quite similar to the first form we found.

We have already seen (section 15.4) how to find the area of a surface when it is defined in the form \( f(x, y) \). Finding the area when the surface is given as a vector function is very similar. Looking at the plots of surfaces we have just seen, it is evident that the two sets of curves that fill out the surface divide it into a grid, and that the spaces in the grid are approximately parallelograms. As before this is the key: we can write down the area of a typical little parallelogram and add them all up with an integral.

Suppose we want to approximate the area of the surface \( r(u, v) \) near \( r(u_0, v_0) \). The functions \( r(u_0, v) \) and \( r(u, v_0) \) define two curves that intersect at \( r(u_0, v_0) \). The derivatives of \( r \) give us vectors tangent to these two curves, \( r_u(u_0, v_0) \) and \( r_v(u_0, v_0) \), and then \( r_u(u_0, v_0) \times r_v(u_0, v_0) \) are two small tangent vectors, whose lengths can be used as the lengths of the sides of an approximating parallelogram. Finally, the area of this parallelogram is \( |r_u \times r_v| \, du \, dv \) and so the total surface area is

\[ \int_D \int_D |r_u \times r_v| \, du \, dv. \]
We write the hemisphere as \( r(\phi, \theta) = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \phi), \) \( 0 \leq \phi \leq \pi/2 \) and \( 0 \leq \theta \leq 2\pi. \) So \( r_\phi = (-\sin \phi \sin \theta, \cos \phi \sin \theta, 0) \) and \( r_\theta = (-\cos \phi \sin \theta, \sin \phi \sin \theta, -\cos \phi) \). Then
\[
\langle r_\phi \times r_\theta, -\sin \phi \sin \theta, \cos \phi \sin \theta \rangle = \langle \sin \phi \sin \theta, -\cos \phi \sin \theta, 0 \rangle.
\]
and since we are interested only in \( 0 \leq \phi \leq \pi/2 \), we must choose the normal vectors to point in the "same way" through the surface. As usual, certain conditions must be met for this to work out; chief among them is the nature of the surface. As we integrate over the surface, we must choose the center of mass is at \((0,0,0/3)\).

Now suppose that \( \mathbf{F} \) is a vector field; imagine that it represents the velocity of some fluid at each point in space. We would like to measure how much fluid is passing through a surface \( D \) with flux across \( D \). As usual, we imagine computing the flux across a very small section of the surface, with area \( dS \), and then adding up all such small fluxes over \( D \) with an integral. Suppose that vector \( \mathbf{N} \) is a unit normal to the surface at a point; \( \mathbf{F} \cdot \mathbf{N} \) is the scalar projection of \( \mathbf{F} \) onto the direction of \( \mathbf{N} \), so it measures how fast the fluid is moving across the surface. In one unit of time the fluid moving across the surface will fill a volume of \( \mathbf{F} \cdot dS \), which is the rate at which the fluid is moving across a small patch of the surface. Thus, the total flux across \( D \) is
\[
\int_D \mathbf{F} \cdot dS = \int_D \mathbf{F} \cdot \mathbf{N} \, dS,
\]
defining \( dS = \mathbf{N} \, dS \). As usual, certain conditions must be met for this to work out; chief among them is the nature of the surface. As we integrate over the surface, we must choose the normal vectors \( \mathbf{N} \) in such a way that they point "the same way" through the surface. If the surface is roughly horizontal in orientation, we might want to measure \( \mathbf{F} \cdot \mathbf{N} \) in such a way that they point "the same way" through the surface. As usual, certain conditions must be met for this to work out; chief among them is the nature of the surface. As we integrate over the surface, we must choose the normal vectors \( \mathbf{N} \) in such a way that they point "the same way" through the surface.

**16.7 Surface Integrals**

\[
\langle \cos u \cos v, \sin u, -v \rangle.
\]

The third coordinate \(-v\) is negative, which is exactly what we desire, that is, the normal vector points down through the surface. Then
\[
\int_0^{2\pi} \int_0^1 \langle x, y, z \rangle : \langle \cos u \cos v, \sin u, -v \rangle \, du \, dv = \int_0^{2\pi} \int_0^1 \cos u \cos v \, du \, dv = \int_0^{2\pi} \cos u \, du = 2\pi.
\]

**Exercises 16.7.**

1. Find the center of mass of an object that occupies the upper hemisphere of \( x^2 + y^2 + z^2 = 1 \) and has density \( x^2 + y^2 \).

2. Find the center of mass of an object that occupies the surface \( z = xy, 0 \leq x \leq 1, 0 \leq y \leq 1 \) and has density \( z^2 \).

3. Find the centroid of the surface of a right circular cone of height \( h \) and base radius \( r \), not including the base.

4. Evaluate \( \int \int_{D_k} (x, y, z) : \langle x, y, z \rangle \, dS \), where \( D \) is given by \( x = z^2 + y^2, -1 \leq z \leq 1, -1 \leq y \leq 1 \), oriented up.

5. Evaluate \( \int \int_{D_k} (x, y, z) : \langle x, y, z \rangle \, dS \), where \( D \) is given by \( x = 3z - 5y, 1 \leq x \leq 2, 0 \leq y \leq 2 \), oriented up.

6. Evaluate \( \int \int_{D_k} (x, y, z) : \langle x, y, z \rangle \, dS \), where \( D \) is given by \( x = 1 - y^2 - z^2, -1 \leq y \leq 1, -1 \leq z \leq 1 \), oriented up.

7. Evaluate \( \int \int_{D_k} (x, y, z) : \langle x, y, z \rangle \, dS \), where \( D \) is given by \( x = z^2 + y^2, 0 \leq z \leq 1, -1 \leq y \leq 1 \), oriented up.

8. Evaluate \( \int \int_{D_k} (x, y, z) : \langle x, y, z \rangle \, dS \), where \( D \) is given by \( x = z^2 + y^2, 0 \leq z \leq 1, -1 \leq y \leq 1 \), oriented up.

9. Evaluate \( \int \int_{D_k} (x, y, z) : \langle x, y, z \rangle \, dS \), where \( D \) is given by \( x = z^2 + y^2, 0 \leq z \leq 1, -1 \leq y \leq 1 \), oriented up.

10. Evaluate \( \int \int_{D_k} (x, y, z) : \langle x, y, z \rangle \, dS \), where \( D \) is given by \( x = z^2 + y^2, 0 \leq z \leq 1, -1 \leq y \leq 1 \), oriented up.

11. A fluid has density \( 870 \text{ kg/m}^3 \) and flows with velocity \( v = (y^2, x^2, x^2) \), where distances are in meters and the components of \( v \) are in meters per second. Find the rate of flow outward through the portion of the cylinder \( x^2 + y^2 = 4, 0 \leq z \leq 1 \) for which \( y \neq 0 \).

**16.8 Stokes’s Theorem**

Recall that one version of Green’s Theorem (see equation 16.5.1) is
\[
\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_D (\nabla \times \mathbf{F}) \cdot d\mathbf{A}.
\]
Here \( D \) is a region in the \( x,y \)-plane and \( k \) is a unit normal to \( D \) at every point. If \( D \) is instead an orientable surface in space, there is an obvious way to alter this equation, and it turns out still to be true:

**THEOREM 16.8.1 Stokes’s Theorem** Provided that the quantities involved are sufficiently nice, and in particular if \( D \) is orientable,
\[
\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_D (\nabla \times \mathbf{F}) \cdot d\mathbf{S},
\]
if \( \partial D \) is oriented counter-clockwise relative to \( \mathbf{N} \).

Note how little has changed: \( \mathbf{k} \) becomes \( \mathbf{N} \), a unit normal to the surface, and \( d\mathbf{A} \) becomes \( d\mathbf{S} \), since this is now a general surface integral. The phrase “counter-clockwise relative to \( \mathbf{N} \)” means roughly that if we take the direction of \( \mathbf{N} \) to be “up”, then we go around the boundary counter-clockwise when viewed from above. In many cases, this description is inadequate. A slightly more complicated but general description is this: imagine standing on the side of the surface considered positive, walk to the boundary and turn left. You are now following the boundary in the correct direction.

**EXAMPLE 16.8.2** Let \( \mathbf{F} = (x^2 \cos x, z^2, x^2) \) and the surface \( D \) be \( z = \sqrt{1 - x^2 - y^2} \), oriented in the positive \( x \)-direction. It quickly becomes apparent that the surface integral in Stokes’s Theorem is intractable, so we try the line integral. The boundary of \( D \) is the unit circle in the \( y,z \)-plane, \( r = (0, \cos \phi, \sin \phi), 0 \leq \phi \leq 2\pi \). The integral is
\[
\int_C (x^2 \cos x, z^2, x^2) : (0, -\sin \phi, \cos \phi) \, du = \int_0^{2\pi} 0 \, du = 0.
\]
EXAMPLE 16.8.4 Consider the cylinder $r = (\cos u, \sin u, v)$, $0 \leq u \leq 2\pi$, $0 \leq v \leq 2r$, oriented outward, and $F = (y, z, xy)$. We compute

$$\int_D \nabla \cdot F \cdot \mathbf{N} dS = \int_{\partial D} F \cdot d\mathbf{r}$$

in two ways.

First, the divergence integral is

$$\int_0^{2\pi} \int_0^{2r} \phi \left( -\sin^2 u \right) du \, dr = 2\pi \int_0^{2r} -\sin^2 u \, du = -2\pi r^2.$$

The boundary consists of two parts, the bottom circle $(\cos t, \sin t, 0)$, with $t$ ranging from 0 to $2\pi$, and $(\cos t, \sin t, 2)$, with $t$ ranging from $2\pi$ to $0$. We compute the corresponding integrals and add the results:

$$\int_0^{2\pi} -\sin^2 t \, dt + \int_{2\pi}^{0} -\sin^2 t + 2 \cos^2 t = -\pi - \pi = -2\pi,$$

as before.

An interesting consequence of Stokes's Theorem is that if $D$ and $E$ are two orientable surfaces with the same boundary, then

$$\int_D (\nabla \times F) \cdot \mathbf{N} dS = \int_{\partial D} F \cdot d\mathbf{r} = \int_{\partial E} F \cdot d\mathbf{r} = \int_{\partial D} (\nabla \times F) \cdot \mathbf{N} dS.$$

Sometimes both of the integrals

$$\int_{\partial D} (\nabla \times F) \cdot \mathbf{N} dS$$

are difficult, but you may be able to find a second surface $E$ so that

$$\int_{\partial E} (\nabla \times F) \cdot \mathbf{N} dS$$

has the same value but is easier to compute.

EXAMPLE 16.8.5 In example 16.8.2 the line integral was easy to compute. But we might also notice that another surface $E$ with the same boundary is the flat disk $y^2 + z^2 \leq 1$, given by $r = (0, \cos u, \sin u)$. The normal is $\mathbf{N} = (0, -\sin z, -\cos z)$. We compute the curl:

$$\nabla \times F = \left( -x^2 - e^{2\pi} \sin y, 2xz - 2x^2 \cos z \right).$$

Since $z = 0$ everywhere on the surface,

$$(\nabla \times F) \cdot \mathbf{N} = (0, -\sin z - y) \cdot (0, 0, 0) = 0,$$

so the surface integral is

$$\int_E F \cdot d\mathbf{S} = 0,$$

as before. In this case, of course, it is still somewhat easier to compute the linear integral, avoiding $\nabla \times F$ entirely.

Proof of Stokes's Theorem. We can prove here a special case of Stokes’s Theorem, which perhaps not too surprisingly uses Green’s Theorem.

Suppose the surface $D$ of interest can be expressed in the form $z = g(x, y)$, and let $F = (P, Q, R)$. Using the vector function $r = (x, y, g(x, y))$ for the surface we get the surface integral

$$\int_D \nabla \times F \cdot \mathbf{N} dS = \int_D \left( 0 \right) \cdot \mathbf{N} dS = \int_D \left( \frac{\partial Q}{\partial y} - \frac{\partial R}{\partial z} \right) dA.$$

Here $E$ is the region in the $xy$ plane directly below the surface $D$.

For the line integral, we need a vector function for $\partial D$. If $(x(t), y(t))$ is a vector function for $\partial D$ then we may use $r(t) = (x(t), y(t), g(x(t), y(t)))$ to represent $\partial D$. Then

$$\int_{\partial D} F \cdot d\mathbf{r} = \int_a^b F_{x} \, dx + Q_{y} \, dy + R_{z} \, dz = \int_a^b \frac{\partial Q}{\partial y} \, dy + \frac{\partial R}{\partial z} \, dz.$$

The curl of $F$ is $\nabla \times F = (0, 0, \frac{\partial Q}{\partial y} - \frac{\partial R}{\partial z})$. Thus the line integral is

$$\int_{\partial D} F \cdot d\mathbf{r} = \int_D \left( \frac{\partial Q}{\partial y} - \frac{\partial R}{\partial z} \right) dA.$$

The two integrals are equal. Therefore, $$\int_{\partial D} F \cdot d\mathbf{r} = \int_D \nabla \times F \cdot \mathbf{N} dS.$$

Exercises 16.8.

1. Let $F = (x, y, y)$, and the plane $z = -2 + y^2 + x^2$ intersect a closed curve. Stokes’s Theorem implies that

$$\int_D (\nabla \times F) \cdot \mathbf{N} dS = \int_{\partial D} F \cdot d\mathbf{r} = \int_D \nabla \times F \cdot \mathbf{N} dS.$$

where the line integral is computed over the intersection $C$ of the plane and the cylinder, and the two surface integrals are computed over the portions of the two surfaces that have boundary $C$ (provided, of course, that the orientations all match). Compute all three integrals.

2. Let $D$ be the portion of $z = 1 - x^2 - y^2$ above the $xy$-plane, oriented up, and let $F = (x, y, x^2 + y^2)$. Compute $$\int_{\partial D} (\nabla \times F) \cdot \mathbf{N} dS.$$

3. Let $D$ be the portion of $z = 2x + 5y$ inside $x^2 + y^2 = 1$, oriented inside, and let $F = (y, x, x z)$. Compute $$\int_{\partial D} F \cdot d\mathbf{r}.$$

4. Compute $$\int_C (x^2 + 3x dy - y^2 dx)$$. where $C$ is the unit circle $x^2 + y^2 = 1$ oriented clockwise.

5. Let $D$ be the portion of $z = x + y + z$ over a region in the $x+y+z$ plane that has area $A$, oriented up, and let $F = (ax + by + cz, ax + by + cz, ax + by + cz)$. Compute $$\int_{\partial D} F \cdot d\mathbf{r}.$$

6. Let $D$ be any surface and let $F = (P(x, y, z), Q(x, y, z), R(x, y, z))$ (P depends only on $x$, Q only on $y$, and R only on $z$). Show that $\int_{\partial D} F \cdot d\mathbf{r} = 0$.

7. Show that $\int_D \nabla \phi \cdot d\mathbf{S} = \int_{\partial D} \phi \cdot d\mathbf{r}$, where $\phi$ describes a closed curve $C$ to which Stokes’s Theorem applies. (See theorem 12.4.1 and 16.5.2)

The Divergence Theorem

The third version of Green’s Theorem (equation 16.5.2) we saw was:

$$\int_{\partial D} F \cdot d\mathbf{r} = \int_D \nabla \cdot F \, dA.$$

With minor changes this turns into another equation, the Divergence Theorem:

THEOREM 16.9.1 Divergence Theorem Under suitable conditions, if $E$ is a region of three dimensional space and $D$ is its boundary surface, oriented outward, then

$$\int_D F \cdot d\mathbf{S} = \int_E \nabla \cdot F \, dV.$$
Proof. Again this theorem is too difficult to prove here, but a special case is easier. In the proof of a special case of Green’s Theorem, we need to know that we could describe the region of integration in both possible orders, so that we could set up a double integral using $dx\,dy$ and another using $dy\,dx$. Similarly, here, we need to be able to describe the three-dimensional region $E$ in different ways.

We start by rewriting the triple integral:

$$\iiint_E \mathbf{F} \, dV = \iiint_E (P_x \, dx + Q_y \, dy + R_z \, dz) \, dV = \iiint_E P_x \, dx + \iiint_E Q_y \, dy + \iiint_E R_z \, dz.$$

The double integral may be rewritten:

$$\iint_D \mathbf{F} \cdot \mathbf{N} \, dS = \iint_D (P \, dx + Q \, dy + R \, dz) \cdot \mathbf{N} \, dS = \iint_D P \, N_x \, dx + \iint_D Q \, N_y \, dy + \iint_D R \, N_z \, dz.$$

To prove that these give the same value it is sufficient to prove that

$$\iint_D P \, N_x \, dx = \iint_D P_x \, dx \quad \text{and} \quad \iint_D Q \, N_y \, dy = \iint_D Q_y \, dy \quad \text{(16.9.1)}$$

Not surprisingly, these are all pretty much the same; we'll do the first one.

We set the triple integral up with $dV$ oriented:

$$\iiint_E P_x \, dx = \iiint_E P_x \, dx \cdot dx \cdot dy \cdot dz = \iiint_E P \, dx \, dy \, dz.$$

where $D$ is the region in the $y$-$z$ plane over which we integrate. The boundary surface of $E$ consists of a “top” $z = g(y, z)$, a “bottom” $z = g_2(y, z)$, and a “wrap-around side” that is vertical to the $y$-$z$ plane. To integrate over the entire boundary surface, we can integrate over each of these (top, bottom, side) and add the results. Over the side surface, the vector $\mathbf{N}$ is perpendicular to the vector $\mathbf{i}$, so

$$\iint_D P \, N_x \, dx = 0 \quad \text{and} \quad \iint_D Q \, N_y \, dy = 0.$$

Thus, we are left with just the surface integral over the top plus the surface integral over the bottom. For the top, we use the vector function $\mathbf{r} = (g_2(y, z), y, z)$ which gives

$$\iint_D P \, N_x \, dx = \int_0^a \int_0^b \int_{g_1(y, z)}^{g_2(y, z)} P_x \, dx \, dy \, dz.$$

Computing $\iint_D Q \, N_y \, dy$, we get:

$$\iint_D Q \, N_y \, dy = \int_0^a \int_0^b \int_{g_1(y, z)}^{g_2(y, z)} Q_y \, dy \, dx \, dz.$$

Therefore, the triple integral is equal to the surface integrals.

**EXAMPLE 16.9.3** Let $\mathbf{F} = (x^3, y^3, z^3)$, and consider the cylindrical volume $x^2 + y^2 \leq 9$, $0 \leq z \leq 2$. The triple integral (using cylindrical coordinates) is

$$\int_0^{2\pi} \int_0^3 \int_0^2 \mathbf{F} \cdot \mathbf{r}_z \, dz \, dr \, \theta = 279\pi.$$

For the surface we need three integrals. The top of the cylinder can be represented by $r = (r \cos \theta, r \sin \theta, 0)$, $r_x \times r_z = (0, -r, 0)$, which points down into the cylinder, so we convert it to $(0, 0, -r)$. Then

$$\int_0^2 \int_0^{2\pi} \int_0^2 r \cos \theta \, dr \, d\theta \, dz = 36\pi.$$

The bottom is $\mathbf{r} = (r \cos \theta, r \sin \theta, 0)$, $r_x \times r_z = (0, -r, 0)$, and

$$\int_0^2 \int_0^{2\pi} \int_0^2 r \cos \theta \, dr \, d\theta \, dz = 0.$$

The side of the cylinder is $\mathbf{r} = (3 \cos \theta, 3 \sin \theta, 0)$, $r_x \times r_z = (3 \cos \theta, 3 \sin \theta, 0)$ which does point outward, so

$$\int_0^2 \int_0^{2\pi} \int_0^2 (27 \cos^2 \theta, 27 \sin^2 \theta, 0) \, dr \, d\theta \, d\phi = 243\pi.$$

The total surface integral is thus $36\pi + 0 + 243\pi = 279\pi$.

**Exercises 16.9.**

1. Using $\mathbf{F} = (xy, 2z)$ and the region bounded by $x^2 + y^2 = 9$, $z = 0$, and $z = 6$, compute both integrals from the Divergence Theorem.

2. Let $\mathbf{E}$ be the volume described by $0 \leq x \leq 2$, $0 \leq y \leq 2$, $0 \leq z \leq 1$, and $\mathbf{F} = (x, y, z)$. Compute $\iiint_D \mathbf{F} \cdot \mathbf{N} \, dS$.

3. Let $\mathbf{E}$ be the volume described by $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$, and $\mathbf{F} = (2y, 3y, 2z)$. Compute $\iiint_D \mathbf{F} \cdot \mathbf{N} \, dS$.

4. Let $\mathbf{E}$ be the volume described by $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq x$, and $\mathbf{F} = (x, 2y, 3z)$. Compute $\iiint_D \mathbf{F} \cdot \mathbf{N} \, dS$. 

5. Let $\mathbf{E}$ be the volume described by $x^2 + y^2 + z^2 \leq 4$, and $\mathbf{F} = (x^3, y^3, z^3)$. Compute $\iiint_D \mathbf{F} \cdot \mathbf{N} \, dS$.

6. Let $\mathbf{E}$ be the hemisphere described by $0 \leq z \leq \sqrt{4 - x^2 - y^2}$, and $\mathbf{F} = (x, y, z)$. Compute $\iiint_D \mathbf{F} \cdot \mathbf{N} \, dS$.

7. Let $\mathbf{E}$ be the volume described by $x^2 + y^2 \leq 1$, $0 \leq z \leq 4$, and $\mathbf{F} = (xy^2, yz^2, x^2z)$. Compute $\iiint_D \mathbf{F} \cdot \mathbf{N} \, dS$.

8. Let $\mathbf{E}$ be the solid cone above the $xy$-plane and inside $z = 1 - \sqrt{x^2 + y^2}$, and $\mathbf{F} = (x, y, z)$. Compute $\iiint_D \mathbf{F} \cdot \mathbf{N} \, dS$.

9. Prove the other two equations in the display 16.9.1.

10. Suppose $\mathbf{D}$ is a closed surface, and that $\mathbf{D}$ and $\mathbf{F}$ are sufficiently nice. Show that $\iiint_D \mathbf{F} \cdot \mathbf{N} \, dS = 0$.

11. Suppose $\mathbf{D}$ is a closed surface, and $\mathbf{F}$ is sufficiently nice, and $\mathbf{F} = (a, b, c)$ is a constant vector field. Show that $\iiint_D \mathbf{F} \cdot \mathbf{N} \, dS = 0$.

12. We know that the volume of a region $\mathbf{E}$ may often be computed as $\iiint_D dV$. Show that this volume may also be computed as $\frac{1}{3} \iiint_D (x, y, z) \cdot \mathbf{N} \, dS$ where $\mathbf{N}$ is the outward pointing unit normal to $\partial \mathbf{E}$. 

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