16
Vector Calculus

16.1 Vector Fields

This chapter is concerned with applying calculus in the context of vector fields. A two-dimensional vector field is a function $f$ that maps each point $(x, y)$ in $\mathbb{R}^2$ to a two-dimensional vector $\langle u, v \rangle$, and similarly a three-dimensional vector field maps $(x, y, z)$ to $\langle u, v, w \rangle$. Since a vector has no position, we typically indicate a vector field in graphical form by placing the vector $f(x, y)$ with its tail at $(x, y)$. Figure 16.1.1 shows a representation of the vector field $f(x, y) = \langle -\frac{x}{\sqrt{x^2 + y^2 + 4}}, \frac{y}{\sqrt{x^2 + y^2 + 4}} \rangle$. For such a graph to be readable, the vectors must be fairly short, which is accomplished by using a different scale for the vectors than for the axes. Such graphs are thus useful for understanding the sizes of the vectors relative to each other but not their absolute size.

Vector fields have many important applications, as they can be used to represent many physical quantities: the vector at a point may represent the strength of some force (gravity, electricity, magnetism) or a velocity (wind speed or the velocity of some other fluid).

We have already seen a particularly important kind of vector field—the gradient. Given a function $f(x, y)$, recall that the gradient is $(f_x(x, y), f_y(x, y))$, a vector that depends on (is a function of) $x$ and $y$. We usually picture the gradient vector with its tail at $(x, y)$, pointing in the direction of maximum increase. Vector fields that are gradients have some particularly nice properties, as we will see. An important example is

$$
F = \begin{bmatrix}
-x \\
-x/\sqrt{x^2 + y^2 + z^2} \\
-y/\sqrt{x^2 + y^2 + z^2} \\
-z/\sqrt{x^2 + y^2 + z^2}
\end{bmatrix},
$$

which points from the point $(x, y, z)$ toward the origin and has length

$$
\frac{\sqrt{x^2 + y^2 + z^2}}{(x^2 + y^2 + z^2)^{3/2}} = \frac{1}{(\sqrt{x^2 + y^2 + z^2})^2},
$$

which is the reciprocal of the square of the distance from $(x, y, z)$ to the origin—in other words, $F$ is an “inverse square law”. The vector $F$ is a gradient:

$$
F = \nabla \frac{1}{\sqrt{x^2 + y^2 + z^2}},
$$

(16.1.1)

which turns out to be extremely useful.

Exercises 16.1.

Sketch the vector fields; check your work with Sage’s `plot_vector_field` function.

1. $(x, y)$
2. $(-x, -y)$
3. $(x, -y)$
4. $(\sin x, \cos y)$
5. $(y, 1/x)$
6. $(x + 1, x + 3)$
16.2 Line Integrals

We have so far integrated “over” intervals, areas, and volumes with single, double, and triple integrals. We now investigate integration over or “along” a curve—“line integrals” are really “curve integrals”.

As with other integrals, a geometric example may be easiest to understand. Consider the function \( f = x + y \) and the parabola \( y = x^2 \) in the \( x\)-\( y \) plane, for \( 0 \leq x \leq 2 \). Imagine that we extend the parabola up to the surface \( f \), to form a curved wall or curtain, as in figure 16.2.1. What is the area of the surface thus formed? We already know one way to compute surface area, but here we take a different approach that is more useful for the problems to come.

![Figure 16.2.1 Approximating the area under a curve. (AP)](image)

As usual, we start by thinking about how to approximate the area. We pick some points along the part of the parabola we’re interested in, and connect adjacent points by straight lines; when the points are close together, the length of each line segment will be close to the length along the parabola. Using each line segment as the base of a rectangle, we choose the height to be the height of the surface \( f \) above the line segment. If we add up the areas of these rectangles, we get an approximation to the desired area, and in the limit this sum turns into an integral.

Typically the curve is in vector form, or can easily be put in vector form; in this example we have \( \mathbf{v}(t) = (t, t^2) \). Then as we have seen in section 13.3 on arc length, the length of one of the straight line segments in the approximation is approximately

\[
\int \sqrt{1 + (t^2)^2} \, dt = \int \sqrt{1 + 4t^2} \, dt, \quad \text{so the integral is}
\]

\[
\int_0^2 \sqrt{1 + 4t^2} \, dt = \int_0^2 (t^2 + t^2) \sqrt{1 + 4t^2} \, dt = \frac{167}{45} \sqrt{17} - \frac{1}{12} - \frac{1}{64} \ln(4 + \sqrt{17}).
\]

This integral of a function along a curve \( C \) is often written in abbreviated form as

\[
\int_C \mathbf{f}(x, y) \, ds.
\]

EXAMPLE 16.2.1 Compute \( \int_C y e^x \, ds \) where \( C \) is the line segment from \((1, 2)\) to \((4, 7)\).

We write the line segment as a vector function: \( \mathbf{v} = (1, 2) + t(3, 5), \ 0 \leq t \leq 1 \), or in parametric form \( x = 1 + 3t, \ y = 2 + 5t \). Then

\[
\int_C y e^x \, ds = \int_0^1 (2 + 5t)e^{1 + 3t} \sqrt{3^2 + 5^2} \, dt = \frac{16}{9} \sqrt{34} e^4 - \frac{1}{9} \sqrt{34} e.
\]

All of these ideas extend to three dimensions in the obvious way.

EXAMPLE 16.2.2 Compute \( \int_C x^2 z \, ds \) where \( C \) is the line segment from \((0, 6, -1)\) to \((4, 1, 5)\).

We write the line segment as a vector function: \( \mathbf{v} = (0, 6, -1) + t(4, -5, 6), \ 0 \leq t \leq 1 \), or in parametric form \( x = 4t, \ y = 6 - 5t, \ z = -1 + 6t \). Then

\[
\int_C x^2 z \, ds = \int_0^1 (4t)^2(-1 + 6t) \sqrt{16 + 25 + 36} \, dt = 16\sqrt{77} \int_0^1 -t^2 + 6t^3 \, dt = \frac{56}{3} \sqrt{77}.
\]

Now we turn to a perhaps more interesting example. Recall that in the simplest case, the work done by a force on an object is equal to the magnitude of the force times the distance the object moves; this assumes that the force is constant and in the direction of motion. We have already dealt with examples in which the force is not constant; now we are prepared to examine what happens when the force is not parallel to the direction of motion.
We have already examined the idea of components of force, in example 12.3.4: the component of a force \( F \) in the direction of a vector \( v \) is

\[
\frac{F \cdot v}{|v|^2} v.
\]

the projection of \( F \) onto \( v \). The length of this vector, that is, the magnitude of the force in the direction of \( v \), is

\[
\frac{F \cdot v}{|v|},
\]

the scalar projection of \( F \) onto \( v \). If an object moves subject to this (constant) force, in the direction of \( v \), over a distance equal to the length of \( v \), the work done is

\[
\frac{F \cdot v}{|v|} |v| = F \cdot v.
\]

Thus, work in the vector setting is still “force times distance”, except that “times” means “dot product”.

If the force varies from point to point, it is represented by a vector field \( F \); the displacement vector \( v \) may also change, as an object may follow a curving path in two or three dimensions. Suppose that the path of an object is given by a vector function \( \mathbf{r}(t) \); at any point along the path, the (small) tangent vector \( \mathbf{r}' \Delta t \) gives an approximation to its motion over a short time \( \Delta t \), so the work done during that time is approximately \( F \cdot \mathbf{r}' \Delta t \); the total work over some time period is then

\[
\int_{t_0}^{t_1} F \cdot \mathbf{r}' \, dt.
\]

It is useful to rewrite this in various ways at different times. We start with

\[
\int_{t_0}^{t_1} F \cdot \mathbf{r}' \, dt = \int_{t_0}^{t_1} F \cdot d\mathbf{r},
\]

abbreviating \( \mathbf{r}' \, dt \) by \( d\mathbf{r} \). Or we can write

\[
\int_{t_0}^{t_1} F \cdot \mathbf{r}' \, dt = \int_{t_0}^{t_1} \frac{\mathbf{r}'}{|\mathbf{r}'|} |\mathbf{r}'| \, dt = \int_{t_0}^{t_1} F \cdot T |\mathbf{r}'| \, dt = \int_{C} F \cdot T \, ds,
\]

using the unit tangent vector \( T \), abbreviating \( |\mathbf{r}'| \, dt \) as \( ds \), and indicating the path of the object by \( C \). In other words, work is computed using a particular line integral of the form

we have considered. Alternately, we sometimes write

\[
\int_{C} F \cdot \mathbf{r}' \, dt = \int_{C} (f, g, h) \cdot (x', y', z') \, dt = \int_{C} \left( f \frac{dx}{dt} + g \frac{dy}{dt} + h \frac{dz}{dt} \right) \, dt
\]

\[
= \int_{C} f \, dx + g \, dy + h \, dz = \int_{C} f \, dx + \int_{C} g \, dy + \int_{C} h \, dz,
\]

and similarly for two dimensions, leaving out references to \( z \).

EXAMPLE 16.2.3 Suppose an object moves from \((-1,1)\) to \((2,4)\) along the path \( \mathbf{r}(t) = (t, t^2) \), subject to the force \( \mathbf{F} = (x \sin y) \). Find the work done.

We can write the force in terms of \( t \) as \( \langle t \sin(t^2), t^2 \rangle \), and compute \( \mathbf{r}'(t) = (1, 2t) \), and then the work is

\[
\int_{-1}^{2} (t \sin(t^2), t^2) \cdot (1, 2t) \, dt = \int_{-1}^{2} t \sin(t^2) + 2t^3 \, dt = \frac{15}{2} + \cos(1) - \cos(4).
\]

Alternatively, we might write

\[
\int_{C} x \sin y \, dx + \int_{C} y \, dy = \int_{-1}^{2} x \sin(x^2) \, dx + \int_{-1}^{4} y \, dy = \frac{-\cos(1)}{2} + \frac{\cos(1)}{2} + \frac{16}{2} - \frac{1}{2},
\]

getting the same answer.

**Exercises 16.2.**

1. Compute \( \int_{C} xy^2 \, ds \) along the line segment from \((1, 2, 0)\) to \((2, 1, 3)\).

2. Compute \( \int_{C} \sin x \, ds \) along the line segment from \((-2, 1, 2)\) to \((2, 1, 5)\).

3. Compute \( \int_{C} x \cos(xy) \, ds \) along the line segment from \((1, 0, 1)\) to \((2, 2, 3)\).

4. Compute \( \int_{C} \sin x \, dx + \cos y \, dy \) along the top half of the unit circle, from \((1, 0)\) to \((-1, 0)\).

5. Compute \( \int_{C} x e^x + x^2 \, dy \) along the line segment \( y = 3, 0 \leq x \leq 2 \).

6. Compute \( \int_{C} x e^x + x^2 \, dy \) along the line segment \( x = 4, 0 \leq y \leq 4 \).

7. Compute \( \int_{C} x e^x + x^2 \, dy \) along the line segment \( x = 3t, y = t^2, 0 \leq t \leq 1 \).

8. Compute \( \int_{C} x e^x + x^2 \, dy \) along the curve \( x = 3t, y = t^2, 0 \leq t \leq 1 \).

9. Compute \( \int_{C} (x \cos y) \, ds \) along the curve \( (t, t), 0 \leq t \leq 1 \).
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10. Compute \( \int_C (1/zy, 1/(x + y)) \cdot dr \) along the path from (1, 1) to (3, 1) to (3, 3) using straight line segments.

11. Compute \( \int_C (1/xy, 1/(y + z)) \cdot dr \) along the curve \((2t, 5t), 1 \leq t \leq 4\).

12. Compute \( \int_C (1/xy, 1/(x + y)) \cdot dr \) along the curve \((t, t^2), 1 \leq t \leq 4\).

13. Compute \( \int_C xyz \cdot dx + xz \cdot dy + yz \cdot dz \) along the curve \((t, t^2, t^3), 0 \leq t \leq 1\).

14. Compute \( \int_C yz \cdot dx + xz \cdot dy + yz \cdot dz \) along the curve \((t, t^2, t^3), 0 \leq t \leq 1\).

15. An object moves from \((1, 1)\) to \((4, 8)\) along the path \(r(t) = (t^2, t^3)\), subject to the force \(F = (x^2, y)\). Find the work done. \(\Rightarrow\)

16. An object moves along the line segment from \((1, 1)\) to \((2, 5)\), subject to the force \(F = (x/(x^2 + y^2), y/(x^2 + y^2))\). Find the work done. \(\Rightarrow\)

17. An object moves along the parabola \(r(t) = (t, t^2), 0 \leq t \leq 1\), subject to the force \(F = (1/(y + 1), -1/(x + 1))\). Find the work done. \(\Rightarrow\)

18. An object moves along the line segment from \((0, 0, 0)\) to \((3, 6, 10)\), subject to the force \(F = (x^2, y^2, z^2)\). Find the work done. \(\Rightarrow\)

19. An object moves along the curve \(r(t) = (\sqrt{t}, 1, \sqrt{t}), 1 \leq t \leq 4\), subject to the force \(F = (\sqrt{t}, y, z)\). Find the work done. \(\Rightarrow\)

20. An object moves from \((1, 1, 1)\) to \((2, 4, 8)\) along the path \(r(t) = (t, t^2, t^3)\), subject to the force \(F = (\sin x, \sin y, \sin z)\). Find the work done. \(\Rightarrow\)

21. An object moves from \((1, 0, 0)\) to \((-1, 0, x)\) along the path \(r(t) = (\cos t, \sin t, t)\), subject to the force \(F = (y, y, z)\). Find the work done. \(\Rightarrow\)

22. Give an example of a non-trivial force field \(F\) and non-trivial path \(r(t)\) for which the total work done moving along the path is zero.

16.3 The Fundamental Theorem of Line Integrals

One way to write the Fundamental Theorem of Calculus (7.2.1) is:

\[
\int_a^b f'(x) \, dx = f(b) - f(a).
\]

That is, to compute the integral of a derivative \(f'\) we need only compute the values of \(f\) at the endpoints. Something similar is true for line integrals of a certain form.

**Theorem 16.3.1 Fundamental Theorem of Line Integrals** Suppose a curve \(C\) is given by the vector function \(r(t), with \(a = r(a) and \(b = r(b)\). Then

\[
\int_C \nabla f \cdot dr = f(b) - f(a),
\]

provided that \(r\) is sufficiently nice.

**Proof.** We write \(r = (x(t), y(t), z(t))\), so that \(r' = (x'(t), y'(t), z'(t))\). Also, we know that \(\nabla f = (f_x, f_y, f_z)\). Then

\[
\int_C \nabla f \cdot dr = \int_a^b (f_x, f_y, f_z) \cdot (x'(t), y'(t), z'(t)) \, dt = \int_a^b f_x \, x' + f_y \, y' + f_z \, z' \, dt.
\]

By the chain rule (see section 14.4), \(f_x \cdot x' + f_y \cdot y' + f_z \cdot z' = df/dt\), where \(f\) in this context means \(f(x(t), y(t), z(t))\), a function of \(t\). In other words, all we have is

\[
\int_a^b f(t) \, dt = f(b) - f(a).
\]

In this context, \(f(a) = f(x(a), y(a), z(a))\). Since \(a = r(a) = (x(a), y(a), z(a))\), we can write \(f(a) = f(a)\)—this is a bit of a cheat, since we are simultaneously using \(f\) to mean \(f(x(t), y(t), z(t))\) and \(f(x(t), y(t), z(t))\), but the concepts are clear and the different uses are compatible. Doing the same for \(b\), we get

\[
\int_C \nabla f \cdot dr = \int_a^b f(t) \, dt = f(b) - f(a) = f(b) - f(a).
\]

This theorem, like the Fundamental Theorem of Calculus, says roughly that if we integrate a "derivative-like function" \((f'\) or \(\nabla f\)) the result depends only on the values of the original function \((f)\) at the endpoints.

If a vector field \(F\) is the gradient of a function, \(F = \nabla f\), we say that \(F\) is a conservaive vector field. If \(F\) is a conservative force field, then the integral for work, \(\int_C F \cdot dr\), is in the form required by the Fundamental Theorem of Line Integrals. This means that in a conservative force field, the amount of work required to move an object from point \(a\) to point \(b\) depends only on those points, not on the path taken between them.

**Example 16.3.2** An object moves in the force field

\[
F = \left\langle \begin{array}{c}
\frac{-x}{(x^2 + y^2 + z^2)^{3/2}} \\
\frac{-y}{(x^2 + y^2 + z^2)^{3/2}} \\
\frac{-z}{(x^2 + y^2 + z^2)^{3/2}}
\end{array} \right\rangle,
\]

along the curve \(r = (1 + t, t^2, t \cos t)\) as \(t\) ranges from 0 to 1. Find the work done by the force on the object.

The straightforward way to do this involves substituting the components of \(r\) into \(F\), forming the dot product \(F \cdot r'\), and then trying to compute the integral, but this integral is extraordinarily messy, perhaps impossible to compute. But since \(F = \nabla (1/\sqrt{x^2 + y^2 + z^2})\) we need only substitute:

\[
\int_C F \cdot dr = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left| \begin{array}{c}
(2, 1, -1) \\
(1, 0, 0)
\end{array} \right| = \frac{1}{\sqrt{6}} - 1.
\]
Another immediate consequence of the Fundamental Theorem involves closed paths. A path $C$ is closed if it forms a loop, so that traveling over the $C$ curve brings you back to the starting point. If $C$ is a closed path, we can integrate around it starting at any point $a$; since the starting and ending points are the same,

$$\int_C \nabla f \cdot dr = f(b) - f(a) = 0.$$ 

For example, in a gravitational field (an inverse square law field) the amount of work required to move an object around a closed path is zero. Of course, it’s only the net amount of work that is zero. It may well take a great deal of work to get from point $a$ to point $b$, but then the return trip will “produce” work. For example, it takes work to pump water from a lower to a higher elevation, but if you then let gravity pull the water back down, you can recover work by running a water wheel or generator. (In the real world you won’t recover all the work because of various losses along the way.)

To make use of the Fundamental Theorem of Line Integrals, we need to be able to spot conservative vector fields $\mathbf{F}$ and to compute $f$ so that $\mathbf{F} = \nabla f$. Suppose that $\mathbf{F} = (P, Q, R)$ and provided that $f$ is sufficiently nice, we know from Clairaut’s Theorem (14.6.2) that $P_y = f_{xy} = f_{yx} = Q_x$. If we compute $P_x$ and $Q_x$ and find that they are not equal, then $\mathbf{F}$ is not conservative. If $P_y = Q_x$, then, again provided that $\mathbf{F}$ is sufficiently nice, we can be assured that $\mathbf{F}$ is conservative. Ultimately, what’s important is that we be able to find $f$; as this amounts to finding anti-derivatives, we may not always succeed.

**Example 16.3.3** Find an $f$ so that $(3 + 2xy, x^2 - 3y^2) = \nabla f$.

First, note that

$$\frac{\partial}{\partial y}(3 + 2xy) = 2x \quad \text{and} \quad \frac{\partial}{\partial x}(x^2 - 3y^2) = 2x,$$

so the desired $f$ does exist. This means that $f_x = 3 + 2xy$, so that $f = 3x + x^2y + g(y)$; the first two terms are needed to get $3 + 2xy$, and the $g(y)$ could be any function of $y$, as it would disappear upon taking a derivative with respect to $x$. Likewise, since $f_y = x^2 - 3y^2$, $f = x^2y - y^3 + h(x)$. The question now becomes, is it possible to find $g(y)$ and $h(x)$ so that

$$3x + x^2y + g(y) = x^2y - y^3 + h(x),$$

and of course the answer is yes: $g(y) = -y^3$, $h(x) = 3x$. Thus, $f = 3x + x^2y - y^3$.

We can test a vector field $\mathbf{F} = (P, Q, R)$ in a similar way. Suppose that $\mathbf{F} = (f_x, f_y, f_z)$. If we temporarily hold $z$ constant, then $f(x, y, z)$ is a function of $x$ and $y$,

and by Clairaut’s Theorem $P_y = f_{xy} = f_{yx} = Q_x$. Likewise, holding $y$ constant implies $P_z = f_{yz} = f_{zy} = R_x$, and with $x$ constant we get $Q_z = f_{zy} = f_{yz} = R_y$. Conversely, if we find that $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ then $\mathbf{F}$ is conservative.

### Exercises 16.3

1. Find an $f$ so that $\nabla f = (2x + y^2, 2y + x^4)$, or explain why there is no such $f$. \(=\)

2. Find an $f$ so that $\nabla f = (x^3 - y^3)$, or explain why there is no such $f$. \(=\)

3. Find an $f$ so that $\nabla f = (xe^y, ye^x)$, or explain why there is no such $f$. \(=\)

4. Find an $f$ so that $\nabla f = (y\cos x, y\sin x)$, or explain why there is no such $f. \(=\)

5. Find an $f$ so that $\nabla f = (y\cos x, \sin x)$, or explain why there is no such $f. \(=\)

6. Find an $f$ so that $\nabla f = (x^2y^3, xy^4)$, or explain why there is no such $f. \(=\)

7. Find an $f$ so that $\nabla f = (y, x, xz, yx)$, or explain why there is no such $f. \(=\)

8. Evaluate $\int_C (10x^3 - 2y^3) \, dx - 3x^2y^2 \, dy$ where $C$ is the part of the curve $x^3 - 5x^2y^2 - 7x^2 = 0$ from $(3, -2)$ to $(3, 2)$. \(=\)

9. Let $\mathbf{F} = (yz, xz, xy)$. Find the work done by this force field on an object that moves from $(1, 0, 2)$ to $(1, 2, 3)$. \(=\)

10. Let $\mathbf{F} = (e^x, xe^y + \sin z, y \cos z)$. Find the work done by this force field on an object that moves from $(0, 0, 0)$ to $(1, -1, 3)$. \(=\)

11. Let

$$\mathbf{F} = \left( \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-z}{(x^2 + y^2 + z^2)^{3/2}} \right).$$

Find the work done by this force field on an object that moves from $(1, 1, 1)$ to $(4, 5, 6)$. \(=\)
THEOREM 16.4.1 Green’s Theorem If the vector field $\mathbf{F} = \langle P, Q \rangle$ and the region $D$ are sufficiently nice, and if $C$ is the boundary of $D$ ($C$ is a closed curve), then
\[
\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \oint_C P \, dx + Q \, dy,
\]
promised the integration on the right is done counter-clockwise around $C.$

To indicate that an integral $\int_C$ is being done over a closed curve in the counter-clockwise direction, we usually write $\oint_C.$ We also use the notation $\partial D$ to mean the boundary of $D$ oriented in the counter-clockwise direction. With this notation, $\int_C = \int_{\partial D}.$

We already know one case, not particularly interesting, in which this theorem is true: If $\mathbf{F}$ is conservative, we know that the integral $\int_C \mathbf{F} \cdot d\mathbf{r} = 0,$ because any integral of a conservative vector field around a closed curve is zero. We also know in this case that $\partial P/\partial y = \partial Q/\partial x,$ so the double integral in the theorem is simply the integral of the zero function, namely, 0. So in the case that $\mathbf{F}$ is conservative, the theorem says simply that $0 = 0.$

EXAMPLE 16.4.2 We illustrate the theorem by computing both sides of
\[
\int_{\partial D} x^4 \, dx + xy \, dy = \iint_D y \, -0 \, dA,
\]
where $D$ is the triangular region with corners $(0,0), (1,0), (0,1).$

Starting with the double integral:
\[
\iint_D y \, -0 \, dA = \int_0^1 \int_0^{1-x} y \, dy \, dx = \int_0^1 \frac{(1-x)^2}{2} \, dx = \left[ -\frac{(1-x)^3}{3} \right]_0^1 = \frac{1}{6}.
\]

There is no single formula to describe the boundary of $D,$ so to compute the left directly we need to compute three separate integrals corresponding to the three sides of the triangle, and each of these integrals we break into two integrals, the “$dx$” part and the “$dy$” part. The three sides are described by $y = 0, y = 1 - x,$ and $x = 0.$ The integrals are then
\[
\int_{\partial D} x^4 \, dx + xy \, dy = \int_0^1 x^4 \, dx + \int_0^1 0 \, dy + \int_0^1 x^4 \, dx + \int_0^1 (1-y) \, y \, dy + \int_0^1 0 \, dx + \int_0^1 0 \, dy = \frac{1}{5} + 0 - \frac{1}{5} + \frac{1}{6} + 0 + 0 = \frac{1}{6}.
\]

Alternately, we could describe the three sides in vector form as $(t,0),$ $(1-t,t),$ and $(0,1-t).$ Note that in each case, as $t$ ranges from 0 to 1, we follow the corresponding side

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in the correct direction. Now
\[
\int_{\partial D} x^4 \, dx + xy \, dy = \int_0^1 t^4 + t \cdot 0 \, dt + \int_0^1 (1-t)^4 + (1-t) \cdot t \, dt + \int_0^1 0 + 0 \, dt
= \int_0^1 t^4 \, dt + \int_0^1 (1-t)^4 + (1-t) \cdot t \, dt = \frac{1}{6}.
\]

In this case, none of the integrations are difficult, but the second approach is somewhat tedious because of the necessity to set up three different integrals. In different circumstances, either of the integrals, the single or the double, might be easier to compute. Sometimes it is worthwhile to turn a single integral into the corresponding double integral, sometimes exactly the opposite approach is best.

Here is a clever use of Green’s Theorem: We know that areas can be computed using double integrals, namely,
\[
\iint_D 1 \, dA
\]
computes the area of region $D.$ If we can find $P$ and $Q$ so that $\partial Q/\partial x - \partial P/\partial y = 1,$ then the area is also
\[
\int_{\partial D} P \, dx + Q \, dy.
\]
It is quite easy to do this: $P = 0, Q = x$ works, as do $P = -y, Q = 0$ and $P = -y/2, Q = x/2.$

EXAMPLE 16.4.3 An ellipse centered at the origin, with its two principal axes aligned with the $x$ and $y$ axes, is given by
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
\]
We find the area of the interior of the ellipse via Green’s theorem. To do this we need a vector equation for the boundary; one such equation is $(a \cos t, b \sin t),$ as $t$ ranges from 0 to $2\pi.$ We can easily verify this by substitution:
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2 \cos^2 t}{a^2} + \frac{b^2 \sin^2 t}{b^2} = \cos^2 t + \sin^2 t = 1.
\]

Let’s consider the three possibilities for $P$ and $Q$ above: Using 0 and $x$ gives
\[
\oint_C 0 \, dx + x \, dy = \int_0^{2\pi} a \cos(t) b \sin(t) dt = \int_0^{2\pi} ab \cos^2(t) dt.
\]
Using \(-y\) and 0 gives

\[
\int_C -y\,dx + 0\,dy = \int_0^{2\pi} b\sin(t)(-a\sin(t))\,dt = \int_0^{2\pi} ab\sin^2(t)\,dt.
\]

Finally, using \(-y/2\) and \(x/2\) gives

\[
\int_C -\frac{y}{2}\,dx + \frac{x}{2}\,dy = \int_0^{2\pi} \frac{b\sin(t)}{2}(-a\sin(t))\,dt + \frac{a\cos(t)}{2}(b\cos(t))\,dt = \int_0^{2\pi} \frac{ab}{2}\,dt = \pi ab.
\]

The first two integrals are not particularly difficult, but the third is very easy, though the choice of \(P\) and \(Q\) seems more complicated.

\[\square\]

**Figure 16.4.1** A “standard” ellipse, \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\).

**Proof of Green’s Theorem.** We cannot here prove Green’s Theorem in general, but we can do a special case. We seek to prove that

\[
\int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA.
\]

It is sufficient to show that

\[
\int_C P\,dx = \iint_D -\frac{\partial P}{\partial y}\,dA \quad \text{and} \quad \int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA,
\]

which we can do if we can compute the double integral in both possible ways, that is, using \(dA = dy\,dx\) and \(dA = dx\,dy\).

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For the first equation, we start with

\[
\iint_D \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy\,dx = \int_a^b P(x, g_2(x)) - P(x, g_1(x))\,dx.
\]

Here we have simply used the ordinary Fundamental Theorem of Calculus, since for the inner integral we are integrating a derivative with respect to \(y\); an antiderivative of \(\frac{\partial P}{\partial y}\) with respect to \(y\) is simply \(P(x, y)\), and then we substitute \(g_1\) and \(g_2\) for \(y\) and subtract.

Now we need to manipulate \(\int_C P\,dx\). The boundary of region \(D\) consists of 4 parts, given by the equations \(y = g_1(x), x = b, y = g_2(x),\) and \(x = a\). On the portions \(x = b\) and \(x = a, dx = 0\,dt\), so the corresponding integrals are zero. For the other two portions, we use the parametric forms \(x = t, y = g_1(t), a \leq t \leq b,\) and \(x = t, y = g_2(t),\) letting \(t\) range from \(b\) to \(a\), since we are integrating counter-clockwise around the boundary. The resulting integrals give us

\[
\int_C P\,dx = \int_a^b P(t, g_1(t))\,dt + \int_b^a P(t, g_2(t))\,dt = \int_a^b P(t, g_1(t))\,dt - \int_a^b P(t, g_2(t))\,dt
\]

which is the result of the double integral times \(-1\), as desired.

The equation involving \(Q\) is essentially the same, and left as an exercise. \[\square\]

**Exercises 16.4.**

1. Compute \(\int_D 2y\,dx + 3x\,dy\), where \(D\) is described by \(0 \leq x \leq 1, 0 \leq y \leq 1\).
2. Compute \(\int_D xy\,dx + xy\,dy\), where \(D\) is described by \(0 \leq x \leq 1, 0 \leq y \leq 1\).
3. Compute \(\int_D x^2 + 3y\,dx + e^{xy}\,dy\), where \(D\) is described by \(-2 \leq x \leq 2, -1 \leq y \leq 1\).
4. Compute \(\int_D y\cos x\,dx + y\sin x\,dy\), where \(D\) is described by \(0 \leq x \leq \pi/2, 0 \leq y \leq 2\).
5. Compute \(\int_D x^2 y\,dx + xy^3\,dy\), where \(D\) is described by \(0 \leq x \leq 1, 0 \leq y \leq x\).
6. Compute \(\int_D x\sqrt{\gamma}\,dx + \sqrt{x^2 + y^2}\,dy\), where \(D\) is described by \(1 \leq x \leq 2, 2x \leq y \leq 4\).
7. Compute \(\int_D (x/y)\,dx + (2 + 3x)\,dy\), where \(D\) is described by \(1 \leq x \leq 2, 1 \leq y \leq x^2\).
8. Compute \(\int_D \sin y\,dx + \sin x\,dy\), where \(D\) is described by \(0 \leq x \leq \pi/2, 0 \leq y \leq \pi/2\).
9. Compute \(\int_D x\ln y\,dx\), where \(D\) is described by \(1 \leq x \leq 2, e^x \leq y \leq e^{e^x}\).


16.5 Divergence and Curl

The divergence and curl can now be defined in terms of this same odd vector $\nabla$ by

$$\nabla \cdot \mathbf{F} = \left( \frac{\partial f}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial h}{\partial z} \right) \cdot (f, g, h) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

The curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \left| \begin{array}{ccc} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial h}{\partial z} \end{array} \right| = \left( \frac{\partial g}{\partial z} - \frac{\partial h}{\partial y}, \frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}, \frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} \right).$$

Here are two simple but useful facts about divergence and curl.

**Theorem 16.5.1** $\nabla \cdot (\nabla \times \mathbf{F}) = 0.$

In words, this says that the divergence of the curl is zero.

**Theorem 16.5.2** $\nabla \times (\nabla f) = 0.$

That is, the curl of a gradient is the zero vector. Recalling that gradients are conservative vector fields, this says that the curl of a conservative vector field is the zero vector.

Under suitable conditions, it is also true that if the curl of $\mathbf{F}$ is 0 then $\mathbf{F}$ is conservative. (Note that this is exactly the same test that we discussed on page 427.)

**Example 16.5.3** Let $\mathbf{F} = (e^x, 1, x^z)$. Then $\nabla \times \mathbf{F} = (0, e^x - e^x, 0) = 0$. Thus, $\mathbf{F}$ is conservative, and we can exhibit this directly by finding the corresponding $f$. Since $f_x = e^x$, $f = e^x + g(y, z)$. Since $f_y = 1$, it must be that $g_y = 1$, so $g(y, z) = y + h(z)$. Thus $f = e^x + y + h(z)$ and

$$x e^x = f_x = x e^x + h'(z),$$

so $h'(z) = 0$, i.e., $h(z) = C$, and $f = x e^x + y + C$.

We can rewrite Green’s Theorem using these new ideas; these rewritten versions in turn are closer to some later theorems we will see.

Suppose we write a two dimensional vector field in the form $\mathbf{F} = (P, Q, 0)$, where $P$ and $Q$ are functions of $x$ and $y$. Then

$$\nabla \times \mathbf{F} = \left| \begin{array}{ccc} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial P}{\partial x} & \frac{\partial Q}{\partial y} & 0 \end{array} \right| = (0, 0, Q_x - P_y),$$

and so $(\nabla \times \mathbf{F}) \cdot \mathbf{k} = (0, 0, Q_x - P_y) \cdot (0, 0, 1) = Q_x - P_y$. So Green’s Theorem says

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial D} P \, dx + Q \, dy = \iint_D Q_x - P_y \, dA = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA. \quad (16.5.1)$$
Roughly speaking, the right-most integral adds up the curl (tendency to swirl) at each point in the region; the left-most integral adds up the tangential components of the vector field around the entire boundary. Green’s Theorem says these are equal, or roughly, that the sum of the “microscopic” swirls over the region is the same as the “macroscopic” swirl around the boundary.

Next, suppose that the boundary ∂D has a vector form r(t), so that r'(t) is tangent to the boundary, and T = r'(t)/|r'(t)| is the usual unit tangent vector. Writing r = (x(t), y(t)) we get

\[ T = \frac{\langle x', y' \rangle}{|r'(t)|} \]

and then

\[ N = \frac{\langle y', -x' \rangle}{|r'(t)|} \]

is a unit vector perpendicular to T, that is, a unit normal to the boundary. Now

\[ \int_{\partial D} F \cdot N \, ds = \int_{\partial D} \langle P, Q \rangle \cdot \frac{\langle y', -x' \rangle}{|r'(t)|} |r'(t)| \, dt = \int_{\partial D} P y' \, dt - Q x' \, dt \]

\[ = \int_{\partial D} P \, dy - Q \, dx = \int_{\partial D} -Q \, dx + \int_{\partial D} P \, dy. \]

Summarizing the long string of equalities,

\[ \int_{\partial D} -Q \, dx + \int_{\partial D} P \, dy = \iint_D P_x + Q_y \, dA = \iint_D \nabla \cdot F \, dA. \]

Roughly speaking, the first integral adds up the flow across the boundary of the region, from inside to out, and the second sums the divergence (tendency to spread) at each point in the interior. The theorem roughly says that the sum of the “microscopic” spreads is the same as the total spread across the boundary and out of the region.
value of \( v \) and letting \( r(u, v) \) sweep out a curve as \( u \) changes. Then \( v \) can change a bit, and \( r(u, v) \) sweeps out a new curve very close to the first. Put enough of these curves together and they form a surface.

**EXAMPLE 16.6.1**  Consider the function \( r(u, v) = (v \cos u, v \sin u, v) \). For a fixed value of \( v \), as \( u \) varies from 0 to \( 2\pi \), this traces a circle of radius \( v \) at height \( v \) above the \( x\)-\( y \) plane. Put lots and lots of these together, and they form a cone, as in figure 16.6.1.

![Figure 16.6.1  Tracing a surface.](image1)

**EXAMPLE 16.6.2**  Let \( r = (v \cos u, v \sin u, u) \). If \( v \) is constant, the resulting curve is a helix (as in figure 13.1.1). If \( u \) is constant, the resulting curve is a straight line at height \( u \) radians from the positive \( x \) axis. Note in figure 16.6.2 how the helixes and the lines both paint the same surface in a different way.

![Figure 16.6.2  Tracing a surface.](image2)

This technique allows us to represent many more surfaces than previously.

**EXAMPLE 16.6.3**  The curve given by

\[
r = ((2 + \cos(3u/2)) \cos u, (2 + \cos(3u/2)) \sin u, \sin(3u/2))
\]

is called a trefoil knot. Recall that from the vector equation of the curve we can compute the unit tangent \( T \), the unit normal \( N \), and the binormal vector \( B = T \times N \); you may want to review section 13.3. The binormal is perpendicular to both \( T \) and \( N \); one way to interpret this is that \( N \) and \( B \) define a plane perpendicular to \( T \), that is, perpendicular to the curve; since \( N \) and \( B \) are perpendicular to each other, they can function just as \( i \) and \( j \) do for the \( x\)-\( y \) plane. Of course, \( N \) and \( B \) are functions of \( u \), changing as we move along the curve \( r(u) \). So, for example, \( c(u, v) = N \cos v + B \sin v \) is a vector equation for a unit circle in a plane perpendicular to the curve described by \( r \), except that the usual interpretation of \( c \) would put its center at the origin. We can fix that simply by adding \( c \) to the original \( r \); let \( f = r(u) + c(u, v) \). For a fixed \( u \) this draws a circle around the point \( r(u) \); as \( u \) varies we get a sequence of such circles around the curve \( r \), that is, a tube of radius 1 with \( r \) at its center. We can easily change the radius; for example \( r(u) + ac(u, v) \) gives the tube radius \( a \); we can make the radius vary as we move along the curve with \( r(u) + g(u)c(u, v) \), where \( g(u) \) is a function of \( u \). As shown in figure 16.6.3, it is hard to see that the plain knot is knotted; the tube makes the structure apparent. Of course, there is nothing special about the trefoil knot in this example; we can put a tube around (almost) any curve in the same way.

![Figure 16.6.3  Tubes around a trefoil knot, with radius 1/2 and 3cos(u)/4.](image3)
We have previously examined surfaces given in the form \( f(x, y) \). It is sometimes useful to represent such surfaces in the more general vector form, which is quite easy: \( \mathbf{r}(u, v) = (u, v, f(u, v)) \). The names of the variables are not important of course; instead of disguising \( x \) and \( y \), we could simply write \( \mathbf{r}(x, y) = (x, y, f(x, y)) \).

We have also previously dealt with surfaces that are not functions of \( x \) and \( y \); many of these are easy to represent in vector form. One common type of surface that cannot be represented as \( z = f(x, y) \) is a surface given by an equation involving only \( x \) and \( y \). For example, \( x + y = 1 \) and \( y = x^2 \) are “vertical” surfaces. For every point \((x, y)\) in the plane that satisfies the equation, the point \((x, y, z)\) is on the surface, for every value of \( z \). Thus, a corresponding vector form for the surface is something like \( \langle f(u), g(u), v \rangle \); for example, \( x + y = 1 \) becomes \( \langle u, 1 - u, v \rangle \) and \( y = x^2 \) becomes \( \langle u, u^2, v \rangle \).

Yet another sort of example is the sphere, say \( x^2 + y^2 + z^2 = 1 \), which cannot be written quite naturally as \( \langle f(u), g(u), v \rangle \). Computer renderings of the plane using these two functions are shown in figure 16.6.4.

Suppose we want to approximate the area of the surface \( \mathbf{r}(u, v) \) and then by varying \( u \) and \( v \), \( uu + vv \) gets to every point in the plane.

Returning to \( x + y + z = 1 \), the points \((1, 0, 0), (0, 1, 0), \) and \((0, 0, 1)\) are all on the plane. By subtracting coordinates we see that \((-1, 0, 1)\) and \((-1, 1, 0)\) are parallel to the plane, so a third vector form for this plane is \( (1, 0, 0) + u(-1, 0, 1) + v(-1, 1, 0) = \langle 1 - u - v, v, u \rangle \).

This is clearly quite similar to the first form we found.

We have already seen (section 15.4) how to find the area of a surface when it is defined in the form \( f(x, y) \). Finding the area when the surface is given as a vector function is very similar. Looking at the plots of surfaces we have just seen, it is evident that the two sets of curves that fill out the surface divide it into a grid, and that the spaces in the grid are approximately parallelograms. As before this is the key: we can write down the area of a typical little parallelogram and add them all up with an integral.

Suppose we want to approximate the area of the surface \( \mathbf{r}(u, v) \) near \( \mathbf{r}(u_0, v_0) \). The functions \( \mathbf{r}(u, v_0) \) and \( \mathbf{r}(u_0, v) \) define two curves that intersect at \( \mathbf{r}(u_0, v_0) \). The derivatives of \( \mathbf{r} \) give us vectors tangent to these two curves: \( \mathbf{r}_u(u_0, v_0) \) and \( \mathbf{r}_v(u_0, v_0) \), and then \( \mathbf{r}_u(u_0, v_0) \) \( du \) and \( \mathbf{r}_v(u_0, v_0) \) \( dv \) are two small tangent vectors, whose lengths can be used as the lengths of the sides of an approximating parallelogram. Finally, the area of this parallelogram is \( |\mathbf{r}_u \times \mathbf{r}_v| \) \( du \) \( dv \) and so the total surface area is

\[
\int_a^b \int_c^d |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv.
\]
EXAMPLE 16.6.4 We find the area of the surface \( \langle v \cos u, v \sin u, u \rangle \) for \( 0 \leq u \leq \pi \) and \( 0 \leq v \leq 1 \); this is a portion of the helical surface in Figure 16.6.2. We compute \( r_u = (-v \sin u, v \cos u, 1) \) and \( r_v = (\cos u, \sin u, 0) \). The cross product of these two vectors is \( (\sin u, -v, \cos u) \) with length \( \sqrt{1 + v^2} \), and the surface area is
\[
\int_0^\pi \int_0^1 \sqrt{1 + v^2} \, dv \, du = \frac{\pi \sqrt{2}}{2} + \frac{\pi \ln(\sqrt{2} + 1)}{2}.
\]

Exercises 16.6.
1. Describe or sketch the surface with the given vector function.
   a. \( r(u, v) = (u + v, 3 - v, 1 + 4u + 5v) \)
   b. \( r(u, v) = (2\sin u, 3\cos u, v) \)
   c. \( r(s, t) = (s, t, t^2 - x^2) \)
   d. \( r(s, t) = (s \sin 2t, s^2, s \cos 2t) \)
2. Find a vector function \( r(u, v) \) for the surface.
   a. The plane that passes through the point \((1, 2, -3)\) and is parallel to the vectors \((1, 1, -1)\) and \((1, -1, 1)\).
   b. The lower half of the ellipsoid \( 2x^2 + 4y^2 + z^2 = 1 \).
   c. The part of the sphere of radius 4 centered at the origin that lies between the planes \( z = -2 \) and \( z = 2 \).
3. Find the area of the portion of \( x + 2y + 4z = 10 \) in the first octant.
4. Find the area of the portion of \( 2x + 4y + z = 0 \) inside \( x^2 + y^2 = 1 \).
5. Find the area of \( z = x^2 + y^2 \) that lies below \( z = 1 \).
6. Find the area of \( z = \sqrt{x^2 + y^2} \) that lies below \( z = 2 \).
7. Find the area of the portion of \( x^2 + y^2 + z^2 = a^2 \) that lies in the first octant.
8. Find the area of the portion of \( x^2 + y^2 + z^2 = a^2 \) that lies above \( x^2 + y^2 \leq b^2 \), \( b \leq a \).
9. Find the area of \( z = x^2 - y^2 \) that lies inside \( x^2 + y^2 = a^2 \).
10. Find the area of \( z = xy \) that lies inside \( x^2 + y^2 = a^2 \).
11. Find the area of \( x^2 + y^2 + z^2 = a^2 \) that lies above the interior of the circle given in polar coordinates by \( r = a \cos \theta \).
12. Find the area of the cone \( z = k \sqrt{x^2 + y^2} \) that lies above the interior of the circle given in polar coordinates by \( r = a \cos \theta \).
13. Find the area of the plane \( z = ax + by + c \) that lies over a region \( D \) with area \( A \).
14. Find the area of the cone \( z = k \sqrt{x^2 + y^2} \) that lies over a region \( D \) with area \( A \).
15. Find the area of the cylinder \( x^2 + z^2 = a^2 \) that lies inside the cylinder \( x^2 + y^2 = a^2 \).
16. The surface \( f(x, y) \) can be represented with the vector function \( \langle x, y, f(x, y) \rangle \). Set up the surface area integral using this vector function and compare to the integral of section 15.4.

16.7 Surface Integrals
In the integral for surface area,
\[
\int_a^b \int_c^d |r_u \times r_v| \, du \, dv,
\]
the integrand \( |r_u \times r_v| \, du \, dv \) is the area of a tiny parallelogram, that is, a very small surface area, so it is reasonable to abbreviate it \( dS \); then a shortened version of the integral is
\[
\iint_D 1 \, dS.
\]
We have already seen that if \( D \) is a region in the plane, the area of \( D \) may be computed with
\[
\iint_D 1 \, dA,
\]
so this is really quite familiar, but the \( dS \) hides a little more detail than does \( dA \).

Just as we can integrate functions \( f(x, y) \) over regions in the plane, using
\[
\iint_D f(x, y) \, dA,
\]
so we can compute integrals over surfaces in space, using
\[
\iiint_D f(x, y, z) \, dS.
\]
In practice this means that we have a vector function \( r(u, v) = (x(u, v), y(u, v), z(u, v)) \) for the surface, and the integral we compute is
\[
\int_a^b \int_c^d f(x(u, v), y(u, v), z(u, v)) |r_u \times r_v| \, du \, dv.
\]
That is, we express everything in terms of \( u \) and \( v \), and then we can do an ordinary double integral.

EXAMPLE 16.7.1 Suppose a thin object occupies the upper hemisphere of \( x^2 + y^2 + z^2 = 1 \) and has density \( \sigma(x, y, z) = z \). Find the mass and center of mass of the object. (Note that the object is just a thin shell; it does not occupy the interior of the hemisphere.)
We write the hemisphere as \( \mathbf{r}(\phi, \theta) = \langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle \), \( 0 \leq \phi \leq \pi/2 \) and \( 0 \leq \theta \leq 2\pi \). So \( \mathbf{r}_\theta = \langle -\sin \theta \sin \phi, \cos \theta \sin \phi, 0 \rangle \) and \( \mathbf{r}_\phi = \langle \cos \theta \cos \phi, \sin \theta \cos \phi, -\sin \phi \rangle \). Then
\[
\mathbf{r}_\theta \times \mathbf{r}_\phi = \langle -\cos \theta \sin^2 \phi, -\sin \theta \sin^2 \phi, \cos \phi \rangle
\]
and
\[
|\mathbf{r}_\theta \times \mathbf{r}_\phi| = |\sin \phi| = \sin \phi,
\]
since we are interested only in \( 0 \leq \phi \leq \pi/2 \). Finally, the density is \( z = \cos \phi \) and the integral for mass is
\[
\int_0^{2\pi} \int_0^{\pi/2} \cos \phi \sin \phi \, d\phi \, d\theta = \pi.
\]

By symmetry, the center of mass is clearly on the \( z \)-axis, so we only need to find the \( z \)-coordinate of the center of mass. The moment around the \( x-y \) plane is
\[
\int_0^{2\pi} \int_0^{\pi/2} z \cos \phi \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/2} \cos^2 \phi \sin \phi \, d\phi \, d\theta = \frac{2\pi}{3},
\]
so the center of mass is at \((0, 0, 2/3)\).

Now suppose that \( \mathbf{F} \) is a vector field; imagine that it represents the velocity of some fluid at each point in space. We would like to measure how much fluid is passing through a surface \( \mathcal{D} \), the flux across \( \mathcal{D} \). As usual, we imagine computing the flux across a very small section of the surface, with area \( \, d\mathcal{S} \), and then adding up all such small fluxes over \( \mathcal{D} \) with an integral. Suppose that vector \( \mathbf{N} \) is a unit normal to the surface at a point; \( \mathbf{F} \cdot \mathbf{N} \) is the scalar projection of \( \mathbf{F} \) onto the direction of \( \mathbf{N} \), so it measures how fast the fluid is moving across the surface. In one unit of time the fluid moving across the surface will fill a volume of \( \mathbf{F} \cdot \mathbf{N} \, d\mathcal{S} \), which is therefore the rate at which the fluid is moving across a small patch of the surface. Thus, the total flux across \( \mathcal{D} \) is
\[
\iint_{\mathcal{D}} \mathbf{F} \cdot \mathbf{N} \, d\mathcal{S} = \iint_{\mathcal{D}} \mathbf{F} \cdot \, d\mathcal{S},
\]
defining \( d\mathcal{S} = \mathbf{N} \, d\mathcal{S} \). As usual, certain conditions must be met for this to work out; chief among them is the nature of the surface. As we integrate over the surface, we must choose the normal vectors \( \mathbf{N} \) in such a way that they point “the same way” through the surface. For example, if the surface is roughly horizontal in orientation, we might want to measure the flux in the “upwards” direction, or if the surface is closed, like a sphere, we might want to measure the flux “outwards” across the surface. In the first case we would choose \( \mathbf{N} \) to have positive \( z \) component, in the second we would make sure that \( \mathbf{N} \) points away from the origin. Unfortunately, there are surfaces that are not orientable: they have only one side, so that it is not possible to choose the normal vectors to point in the “same way” through the surface. The most famous such surface is the Möbius strip shown in figure 16.7.1. It is quite easy to make such a strip with a piece of paper and some tape. If you have never done this, it is quite instructive; in particular, you should draw a line down the center of the strip until you return to your starting point. No matter how unit normal vectors are assigned to the points of the Möbius strip, there will be normal vectors very close to each other pointing in opposite directions.

Assuming that the quantities involved are well behaved, however, the flux of the vector field across the surface \( \mathbf{r}(u, v) \) is
\[
\iint_{\mathcal{D}} \mathbf{F} \cdot \mathbf{N} \, d\mathcal{S} = \int \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA = \iint_{\mathcal{D}} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA.
\]
In practice, we may have to use \( \mathbf{r}_u \times \mathbf{r}_v \) or even something a bit more complicated to make sure that the normal vector points in the desired direction.

**EXAMPLE 16.7.2** Compute the flux of \( \mathbf{F} = (x, y, z^4) \) across the cone \( z = \sqrt{x^2 + y^2} \), \( 0 \leq z \leq 1 \), in the downward direction.

We write the cone as a vector function: \( \mathbf{r} = (v \cos u, v \sin u, v) \), \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq 1 \). Then \( \mathbf{r}_u = (-v \sin u, v \cos u, 0) \) and \( \mathbf{r}_v = (\cos u, \sin u, 1) \) and \( \mathbf{r}_u \times \mathbf{r}_v = (v \cos u, v \sin u, 1) \). The enclosed area is \( \iint_{\mathcal{D}} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA = \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = (x, y, z^4) \cdot (v \cos u, v \sin u, 1) = (v \cos u, v \sin u, v^4) \). Integrating gives
\[
\int_0^{2\pi} \int_0^1 (v \cos u, v \sin u, v^4) \, dv \, du = (v^2/2, v^2/2, v^5/5) \bigg|_{0}^{1} = (0, 0, 1/5).
\]
\[ \langle v \cos u, v \sin u, -v \rangle \cdot \langle v \cos u, v \sin u, -v \rangle \] 

The third coordinate \(-v\) is negative, which is exactly what we desire, that is, the normal vector points down through the surface. Then

\[
\int_0^{2\pi} \int_0^1 \langle x, y, z \rangle \cdot \langle v \cos u, v \sin u, -v \rangle \, dv \, du = \int_0^{2\pi} \int_0^1 x v \cos u + y v \sin u - z^4 v \, dv \, du
\]

\[
= \int_0^{2\pi} \int_0^1 v^2 \cos^2 u + v^2 \sin^2 u - v^5 \, dv \, du
\]

\[
= \int_0^{2\pi} \int_0^1 v^2 - v^5 \, dv \, du = \frac{\pi}{3}.
\]

\section*{Exercises 16.7.}

1. Find the center of mass of an object that occupies the upper hemisphere of \(x^2 + y^2 + z^2 = 1\) and has density \(x^2 + y^2\). \(\Rightarrow\)

2. Find the center of mass of an object that occupies the surface \(z = xy, 0 \leq x \leq 1, 0 \leq y \leq 1\) and has density \(\sqrt{1 + x^2 + y^2}\). \(\Rightarrow\)

3. Find the center of mass of an object that occupies the surface \(z = \sqrt{x^2 + y^2}, 1 \leq z \leq 4\) and has density \(x^2 z\). \(\Rightarrow\)

4. Find the centroid of the surface of a right circular cone of height \(h\) and base radius \(r\), not including the base. \(\Rightarrow\)

5. Evaluate \(\int\int_D \langle 2, -3, 4 \rangle \cdot \mathbf{N} \, dS\), where \(D\) is given by \(z = x^2 + y^2, -1 \leq x \leq 1, -1 \leq y \leq 1\), oriented up. \(\Rightarrow\)

6. Evaluate \(\int\int_D \langle x, y, 3 \rangle \cdot \mathbf{N} \, dS\), where \(D\) is given by \(z = 3x - 5y, 1 \leq x \leq 2, 0 \leq y \leq 2\), oriented up. \(\Rightarrow\)

7. Evaluate \(\int\int_D \langle x, y, -2 \rangle \cdot \mathbf{N} \, dS\), where \(D\) is given by \(z = 1 - x^2 - y^2, x^2 + y^2 \leq 1\), oriented up. \(\Rightarrow\)

8. Evaluate \(\int\int_D \langle xy, yz, xz \rangle \cdot \mathbf{N} \, dS\), where \(D\) is given by \(z = x + y^2 + 2, 0 \leq x \leq 1, x \leq y \leq 1\), oriented up. \(\Rightarrow\)

9. Evaluate \(\int\int_D \langle e^x, e^y, z \rangle \cdot \mathbf{N} \, dS\), where \(D\) is given by \(z = xy, 0 \leq x \leq 1, -x \leq y \leq x\), oriented up. \(\Rightarrow\)

10. Evaluate \(\int\int_D \langle xz, yz, z \rangle \cdot \mathbf{N} \, dS\), where \(D\) is given by \(z = a^2 - x^2 - y^2, x^2 + y^2 \leq b^2\), oriented up. \(\Rightarrow\)

11. A fluid has density 870 kg/m\(^3\) and flows with velocity \(v = \langle z, y^2, z^2 \rangle\), where distances are in meters and the components of \(v\) are in meters per second. Find the rate of flow outward through the portion of the cylinder \(x^2 + y^2 = 4, 0 \leq z \leq 1\) for which \(y \geq 0\). \(\Rightarrow\)

\section*{Chapter 16 Vector Calculus}

12. Gauss’s Law says that the net charge, \(Q\), enclosed by a closed surface, \(S\), is

\[ Q = \varepsilon_0 \int \mathbf{E} \cdot \mathbf{N} \, dS \]

where \(\mathbf{E}\) is an electric field and \(\varepsilon_0\) (the permittivity of free space) is a known constant; \(\mathbf{N}\) is oriented outward. Use Gauss’s Law to find the charge contained in the cube with vertices \((\pm 1, \pm 1, \pm 1)\) if the electric field is \(\mathbf{E} = \langle x, y, z \rangle\). \(\Rightarrow\)

\section*{16.8 Stokes’s Theorem}

Recall that one version of Green’s Theorem (see equation 16.5.1) is

\[ \int_D \mathbf{F} \cdot \mathbf{dr} = \int \nabla \times \mathbf{F} \cdot \mathbf{k} \, dA. \]

Here \(D\) is a region in the \(x\)-\(y\) plane and \(\mathbf{k}\) is a unit normal to \(D\) at every point. If \(D\) is instead an orientable surface in space, there is an obvious way to alter this equation, and it turns out still to be true:

\textbf{THEOREM 16.8.1 Stokes’s Theorem} Provided that the quantities involved are sufficiently nice, and in particular if \(D\) is orientable,

\[ \int_D \mathbf{F} \cdot \mathbf{dr} = \int \nabla \times \mathbf{F} \cdot \mathbf{N} \, dS, \]

if \(\partial D\) is oriented counter-clockwise relative to \(\mathbf{N}\). \(\Rightarrow\)

Note how little has changed: \(\mathbf{k}\) becomes \(\mathbf{N}\), a unit normal to the surface, and \(\mathbf{dA}\) becomes \(dS\), since this is now a general surface integral. The phrase “counter-clockwise relative to \(\mathbf{N}\)” means roughly that if we take the direction of \(\mathbf{N}\) to be “up”, then we go around the boundary counter-clockwise when viewed from “above”. In many cases, this description is inadequate. A slightly more complicated but general description is this: imagine standing on the side of the surface considered positive; walk to the boundary and turn left. You are now following the boundary in the correct direction.

\textbf{EXAMPLE 16.8.2} Let \(\mathbf{F} = \langle e^x \cos z, x^2 z, xy \rangle\) and the surface \(D\) be \(x = \sqrt{1 - y^2 - z^2}\), oriented in the positive \(x\) direction. It quickly becomes apparent that the surface integral in Stokes’s Theorem is intractable, so we try the line integral. The boundary of \(D\) is the unit circle in the \(y\)-\(z\) plane, \(\mathbf{r} = \langle 0, \cos u, \sin u \rangle, 0 \leq u \leq 2\pi\). The integral is

\[ \int_0^{2\pi} \langle e^x \cos z, x^2 z, xy \rangle \cdot \langle 0, -\sin u, \cos u \rangle \, du = \int_0^{2\pi} 0 \, du = 0, \]

because \(x = 0\). \(\Rightarrow\)
EXAMPLE 16.8.3  Consider the cylinder \( r = (\cos u, \sin u, v), \) \( 0 \leq u \leq 2\pi, \) \( 0 \leq v \leq 2, \) oriented outward, and \( F = (y, zx, xy) . \) We compute

\[
\int_D \nabla \times F \cdot dS = \int_{\partial D} F \cdot dr
\]

in two ways.

First, the double integral is

\[
\int_0^{2\pi} \int_0^2 (0, -\sin u, v - 1) \cdot (\cos u, \sin u, 0) \, dv \, du = \int_0^{2\pi} \int_0^2 -\sin^2 u \, dv \, du = -2\pi.
\]

The boundary consists of two parts, the bottom circle \( \langle \cos t, \sin t, 0 \rangle, \) with \( t \) ranging from 0 to \( 2\pi, \) and \( \langle \cos t, \sin t, 2 \rangle, \) with \( t \) ranging from \( 2\pi \) to 0. We compute the corresponding integrals and add the results:

\[
\int_0^{2\pi} -\sin^2 t \, dt + \int_0^{2\pi} -\sin^2 t + 2\cos^2 t = -\pi - \pi = -2\pi,
\]

as before.

An interesting consequence of Stokes’s Theorem is that if \( D \) and \( E \) are two orientable surfaces with the same boundary, then

\[
\int_D (\nabla \times F) \cdot N \, dS = \int_{\partial D} F \cdot dr = \int_{\partial D} \nabla \times F \cdot N \, dS.
\]

Sometimes both of the integrals

\[
\int_D (\nabla \times F) \cdot N \, dS \quad \text{and} \quad \int_{\partial D} F \cdot dr
\]

are difficult, but you may be able to find a second surface \( E \) so that

\[
\int_E (\nabla \times F) \cdot N \, dS
\]

has the same value but is easier to compute.

EXAMPLE 16.8.4  In example 16.8.2 the line integral was easy to compute. But we might also notice that another surface \( E \) with the same boundary is the flat disk \( y^2 + z^2 \leq 1, \)

\[
given by \( r = (0, v \cos u, v \sin u) . \) The normal is \( r_u \times r_v = (v, 0, 0) . \) We compute the curl:

\[
\nabla \times F = (x - x^2, -e^y \sin z - y, 2xz - xe^y \cos z).
\]

Since \( x = 0 \) everywhere on the surface,

\[
(\nabla \times F) \cdot N = (0, -\sin z - y, 0) \cdot (v, 0, 0) = 0,
\]

so the surface integral is

\[
\int_E 0 \, dS = 0,
\]

as before. In this case, of course, it is still somewhat easier to compute the line integral, avoiding \( \nabla \times F \) entirely.

\[
\]

EXAMPLE 16.8.5  Let \( F = (-y^2, x, z^2) , \) and let the curve \( C \) be the intersection of the cylinder \( x^2 + y^2 = 1 \) with the plane \( y + z = 2, \) oriented counter-clockwise when viewed from above. We compute \( \int_C F \cdot dr \) in two ways.

First we do it directly: a vector function for \( C \) is \( r = (\cos u, \sin u, 2 - \sin u), \) so \( r' = (-\sin u, \cos u, -\cos u), \) and the integral is then

\[
\int_0^{2\pi} y^2 \sin u + x \cos u - z^2 \cos u \, du = \int_0^{2\pi} \sin^3 u + \cos^2 u - (2 - \sin u)^2 \cos u \, du = \pi.
\]

To use Stokes’s Theorem, we pick a surface with \( C \) as the boundary; the simplest such surface is that portion of the plane \( y + z = 2 \) inside the cylinder. This has vector equation \( r = (v \cos u, v \sin u, 2 - v \sin u) . \) We compute \( r_u = (-v \sin u, v \cos u, -v \cos u), \)

\[
r_v = (\cos u, \sin u, -\sin u), \quad \text{and} \quad r_u \times r_v = (0, -v, v).
\]

To match the orientation of \( C \) we need to use the normal \( (0, v, v) . \) The curl of \( F \) is \( (0, 0, 1 + 2y) = (0, 0, 1 + 2v \sin u), \) and the surface integral from Stokes’s Theorem is

\[
\int_0^{2\pi} \int_0^1 (1 + 2v \sin u) v \, dv \, du = \pi.
\]

In this case the surface integral was more work to set up, but the resulting integral is somewhat easier.
Proof of Stokes’s Theorem. We can prove here a special case of Stokes’s Theorem, which perhaps not too surprisingly uses Green’s Theorem. Suppose the surface \( D \) of interest can be expressed in the form \( z = g(x, y) \), and let \( \mathbf{F} = \langle P, Q, R \rangle \). Using the vector function \( \mathbf{r} = \langle x, y, g(x, y) \rangle \) for the surface we get the surface integral
\[
\iint_D \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_E \nabla \times \mathbf{F} \cdot \mathbf{N} \, dA
\]
where \( N \) is a vector normal to \( \partial D \) pointing outward. This is the same as the expression we obtained for the integral above.

Exercises 16.8.

1. Let \( \mathbf{F} = \langle x, y \rangle \). The plane \( z = 2x + 2y - 1 \) and the paraboloid \( z = x^2 + y^2 \) intersect in a closed curve. Stokes’s Theorem implies that
\[
\iint_D \nabla \times \mathbf{F} \cdot \mathbf{N} \, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C (\nabla \times \mathbf{F}) \cdot \mathbf{N} \, dS,
\]
where the line integral is computed over the intersection \( C \) of the plane and the paraboloid, and the two surface integrals are computed over the portions of the two surfaces that have boundary \( C \) (provided, of course, that the orientations all match). Compute all three integrals. \( \Rightarrow \)

2. Let \( D \) be the portion of \( z = 1 - x^2 - y^2 \) above the \( x \)-\( y \) plane, oriented up, and let \( \mathbf{F} = \langle xy^2, -x^2y, z \rangle \). Compute \( \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{N} \, dS \). \( \Rightarrow \)

3. Let \( D \) be the portion of \( z = 2x + 5y \) inside \( x^2 + y^2 = 1 \), oriented up, and let \( \mathbf{F} = \langle y, z, -x \rangle \). Compute \( \iint_D \mathbf{F} \cdot d\mathbf{r} \). \( \Rightarrow \)

4. Compute \( \iint_C x^2 \, dx + 3x \, dy - y^3 \, dz \), where \( C \) is the unit circle \( x^2 + y^2 = 1 \) oriented counterclockwise. \( \Rightarrow \)

5. Let \( D \) be the portion of \( z = px + qy + r \) over a region in the \( x \)-\( y \) plane that has area \( A \), oriented up, and let \( \mathbf{F} = \langle ax + by + cz, az + by + cz, ax + by + cz \rangle \). Compute \( \iint_D \mathbf{F} \cdot d\mathbf{r} \). \( \Rightarrow \)

6. Let \( D \) be any surface and let \( \mathbf{F} = \langle P(x), Q(y), R(z) \rangle \) (\( P \) depends only on \( x \), \( Q \) only on \( y \), and \( R \) only on \( z \)). Show that \( \iint_D \mathbf{F} \cdot d\mathbf{r} = 0 \). \( \Rightarrow \)

7. Show that \( \iint_C f \nabla g + g \nabla f \cdot d\mathbf{r} = 0 \), where \( \mathbf{r} \) describes a closed curve \( C \) to which Stokes’s Theorem applies. (See theorems 12.4.1 and 16.5.2.)

16.9 The Divergence Theorem

The third version of Green’s Theorem (equation 16.5.2) we saw was:
\[
\iiint_D \mathbf{F} \cdot \mathbf{N} \, dS = \iiint_D \nabla \cdot \mathbf{F} \, dV.
\]
With minor changes this turns into another equation, the Divergence Theorem:

THEOREM 16.9.1 Divergence Theorem Under suitable conditions, if \( E \) is a region of three dimensional space and \( D \) is its boundary surface, oriented outward, then
\[
\iiint_D \mathbf{F} \cdot \mathbf{N} \, dS = \iiint_E \nabla \cdot \mathbf{F} \, dV.
\]
16.9 The Divergence Theorem

**Proof.** Again this theorem is too difficult to prove here, but a special case is easier. In the proof of a special case of Green’s Theorem, we needed to know that we could describe the region of integration in both possible orders, so that we could set up one double integral using \(dx\ dy\) and another using \(dy\ dz\). Similarly here, we need to be able to describe the three-dimensional region \(E\) in different ways.

We start by rewriting the triple integral:

\[
\iiint_E \nabla \cdot \mathbf{F} \, dV = \iiint_E (P_x + Q_y + R_z) \, dV = \iiint_E P_x \, dV + \iiint_E Q_y \, dV + \iiint_E R_z \, dV.
\]

The double integral may be rewritten:

\[
\iint_D \mathbf{F} \cdot \mathbf{N} \, dS = \iint_D (P_i + Q_j + R_k) \cdot \mathbf{N} \, dS = \iint_D P_i \cdot \mathbf{N} \, dS + \iint_D Q_j \cdot \mathbf{N} \, dS + \iint_D R_k \cdot \mathbf{N} \, dS.
\]

To prove that these give the same value it is sufficient to prove that

\[
\iint_D P_i \cdot \mathbf{N} \, dS = \iiint_E P_x \, dV,
\]

\[
\iint_D Q_j \cdot \mathbf{N} \, dS = \iiint_E Q_y \, dV,
\]

\[
\iint_D R_k \cdot \mathbf{N} \, dS = \iiint_E R_z \, dV.
\]

(16.9.1)

Not surprisingly, these are all pretty much the same; we’ll do the first one.

We set the triple integral up with \(dz\) innermost:

\[
\iiint_E P_x \, dV = \iint_{g_2(y,z)}^{g_1(y,z)} P_x \, dx \, dA = \iint_B P(g_2(y,z), y, z) - P(g_1(y,z), y, z) \, dA,
\]

where \(B\) is the region in the \(y\)-\(z\) plane over which we integrate. The boundary surface of \(E\) consists of a “top” \(x = g_2(y,z)\), a “bottom” \(x = g_1(y,z)\), and a “wrap-around side” that is vertical to the \(y\)-\(z\) plane. To integrate over the entire boundary surface, we can integrate over each of these (top, bottom, side) and add the results. Over the side surface, the vector \(\mathbf{N}\) is perpendicular to the vector \(i\), so

\[
\iint_{\text{side}} P_i \cdot \mathbf{N} \, dS = \iint_{\text{side}} 0 \, dS = 0.
\]

Thus, we are left with just the surface integral over the top plus the surface integral over the bottom. For the top, we use the vector function \(\mathbf{r} = (g_2(y,z), y, z)\) which gives

\[
\mathbf{r}_y \times \mathbf{r}_z = (1, -g_2, -g_2);
\]

the dot product of this with \(i = (1, 0, 0)\) is 1. Then

\[
\iint_{\text{top}} P_i \cdot \mathbf{N} \, dS = \iint_B P(g_2(y,z), y, z) \, dA.
\]

In almost identical fashion we get

\[
\iint_{\text{bottom}} P_i \cdot \mathbf{N} \, dS = -\iint_B P(g_1(y,z), y, z) \, dA,
\]

where the negative sign is needed to make \(\mathbf{N}\) point in the negative \(x\) direction. Now

\[
\iint_D P_i \cdot \mathbf{N} \, dS = \iint_B P(g_2(y,z), y, z) \, dA - \iint_B P(g_1(y,z), y, z) \, dA,
\]

which is the same as the value of the triple integral above.

**EXAMPLE 16.9.2** Let \(\mathbf{F} = (2x, 3y, z^2)\), and consider the three-dimensional volume inside the cube with faces parallel to the principal planes and opposite corners at \((0, 0, 0)\) and \((1, 1, 1)\). We compute the two integrals of the divergence theorem.

The triple integral is the easier of the two:

\[
\int_0^1 \int_0^1 \int_0^1 2 + 3 + 2z \, dx \, dy \, dz = 6.
\]

The surface integral must be separated into six parts, one for each face of the cube. One face is \(z = 0\) or \(\mathbf{r} = (u, v, 0)\), \(0 \leq u, v \leq 1\). Then \(\mathbf{r}_u = (1, 0, 0)\), \(\mathbf{r}_v = (0, 1, 0)\), and \(\mathbf{r}_u \times \mathbf{r}_v = (0, 0, 1)\). We need this to be oriented downward (out of the cube), so we use \((0, 0, -1)\) and the corresponding integral is

\[
\int_0^1 \int_0^1 \int_0^1 -z^2 \, du \, dv = \int_0^1 \int_0^1 0 \, du \, dv = 0.
\]

Another face is \(y = 1\) or \(\mathbf{r} = (u, 1, v)\). Then \(\mathbf{r}_u = (1, 0, 0)\), \(\mathbf{r}_v = (0, 0, 1)\), and \(\mathbf{r}_u \times \mathbf{r}_v = (0, -1, 0)\). We need a normal in the positive \(y\) direction, so we convert this to \((0, 1, 0)\), and the corresponding integral is

\[
\int_0^1 \int_0^1 3y \, du \, dv = \int_0^1 \int_0^1 3 \, du \, dv = 3.
\]

The remaining four integrals have values 0, 0, 2, and 1, and the sum of these is 6, in agreement with the triple integral.
EXERCISES 16.9.3 Let \( F = (x^3, y^3, z^3) \), and consider the cylindrical volume \( x^2 + y^2 \leq 9, 0 \leq z \leq 2 \). The triple integral (using cylindrical coordinates) is

\[
\int_0^{2\pi} \int_0^3 \int_0^2 (3r^2 + 2z)r \, dz \, dr \, d\theta = 279\pi.
\]

For the surface we need three integrals. The top of the cylinder can be represented by \( r = \langle r \cos u, r \sin u, 2 \rangle \); \( r_u \times r_v = \langle 0, 0, -v \rangle \), which points down into the cylinder, so we convert it to \( \langle 0, 0, v \rangle \). Then

\[
\int_0^{2\pi} \int_0^3 \int_0^2 (v^3 \cos^3 u, v^3 \sin^3 u, 4) \cdot \langle 0, 0, v \rangle \, dv \, du = \int_0^{2\pi} \int_0^3 4v \, dv \, du = 36\pi.
\]

The bottom is \( r = \langle v \cos u, v \sin u, 0 \rangle \); \( r_u \times r_v = \langle 0, 0, -v \rangle \) and

\[
\int_0^{2\pi} \int_0^3 \int_0^2 (v^3 \cos^3 u, v^3 \sin^3 u, 0) \cdot \langle 0, 0, -v \rangle \, dv \, du = \int_0^{2\pi} \int_0^3 0 \, dv \, du = 0.
\]

The side of the cylinder is \( r = \langle 3 \cos u, 3 \sin u, v \rangle \); \( r_u \times r_v = \langle 3 \cos u, 3 \sin u, 0 \rangle \) which does point outward, so

\[
\int_0^{2\pi} \int_0^2 \left( 27 \cos^2 u, 27 \sin^2 u, v^2 \right) \cdot \langle 3 \cos u, 3 \sin u, 0 \rangle \, dv \, du = \int_0^{2\pi} \int_0^2 81 \cos^4 u + 81 \sin^4 u \, dv \, du = 243\pi.
\]

The total surface integral is thus \( 36\pi + 0 + 243\pi = 279\pi \).

\[
\text{Exercises 16.9.}
\]

1. Using \( F = (3x, y^3, -2z^2) \) and the region bounded by \( x^2 + y^2 = 9 \), \( z = 0 \), and \( z = 5 \), compute both integrals from the Divergence Theorem. 

2. Let \( E \) be the volume described by \( 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c \), and \( F = (x^3, y^3, z^3) \). Compute \( \int_{\partial E} F \cdot N \, dS \).

3. Let \( E \) be the volume described by \( 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1 \), and \( F = (2xy, 3xz, xe^y) \). Compute \( \int_{\partial E} F \cdot N \, dS \).

4. Let \( E \) be the volume described by \( 0 \leq x \leq 1, 0 \leq y \leq z, 0 \leq z \leq x + y \), and \( F = (x, 2y, 3z) \). Compute \( \int_{\partial E} F \cdot N \, dS \).