15

Multiple Integration

15.1 Volume and Average Height

Consider a surface \( f(x, y) \); you might temporarily think of this as representing physical topography—a hilly landscape, perhaps. What is the average height of the surface (or average altitude of the landscape) over some region?

As with most such problems, we start by thinking about how we might approximate the answer. Suppose the region is a rectangle, \([a, b] \times [c, d]\). We can divide the rectangle into a grid, \(m \times n\) subdivisions in one direction and \(n\) in the other, as indicated in figure 15.1.1. We pick \(x\) values \(x_0, x_1, \ldots, x_m\), in each subdivision in the \(x\) direction, and similarly in the \(y\) direction. At each of the points \((x_i, y_j)\) in one of the smaller rectangles in the grid, we compute the height of the surface: \(f(x_i, y_j)\). Now the average of these heights should be (depending on the fineness of the grid) close to the average height of the surface:

\[
\frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_i, y_j)
\]

As both \(m\) and \(n\) go to infinity, we expect this approximation to converge to a fixed value, the actual average height of the surface. For reasonably nice functions this does indeed happen.

The next question, of course, is: How do we compute these double integrals? You might temporarily think of this as representing physical values

\[
\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_i, y_j) \Delta y
\]

The quantity \(f(x_i, y_j)\Delta y\) is of course the area of one of the small rectangles; see figure 15.1.2. When we add all of these up, we get an approximation to the volume under the surface and above the rectangle \(R = [a, b] \times [c, d]\). When we take the limit as \(m\) and \(n\) go to infinity, the double sum becomes the actual volume under the surface, which we divide by \((b - a)(d - c)\) to get the average height.

Double sums like this come up in many applications, so in a way it is the most important part of this example; dividing by \((b - a)(d - c)\) is a simple extra step that allows the computation of an average. As we did in the single variable case, we introduce a special notation for the limit of such a double sum:

\[
\lim_{m,n \to \infty} \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} f(x_i, y_j) \Delta y = \iint_R f(x, y) \, dx \, dy = \int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dx,
\]

Figure 15.1.2 Approximating the volume under a surface.

the double integral of \(f\) over the region \(R\). The notation \(dA\) indicates a small bit of area, without specifying any particular order for the variables \(x\) and \(y\); it is shorter and more “generic” than writing \(dx \, dy\). The average height of the surface in this notation is

\[
\frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) \, dx \, dy.
\]

The notation \(\int_a^b \, dx\) is of course the area of the interval \([a, b]\). Similarly the notation \(\int_c^d \, dy\) is of course the area of the interval \([c, d]\). The quantity \(\int_0^t \int_0^s f(x, y) \, dy \, dx\) can be interpreted as the volume of a solid with face area \(G(y)\) and thickness \(\Delta y\). Think of the surface \(f(x, y)\) as the top of a loaf of sliced bread. Each slice has a cross-sectional area and thickness \(\Delta y\); the volume \(\int_0^t \int_0^s f(x, y) \, dy \, dx\) of the loaf is the sum of the volume of each slice. As we did in the single variable case, we introduce a special notation for the limit of such a double sum:

\[
\lim_{m,n \to \infty} \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} G(y_j) \Delta y = \int_a^b \left( \int_c^d G(y) \, dy \right) \, dx,
\]

the double integral of \(G\) over the region \(R\). The section 9.3, except that there we need the cross-sections to be plane regions under the surface, which we divide by \((b - a)(d - c)\) to get the average height.

EXAMPLE 15.1.1 Figure 15.1.2 shows the function \(\sin(xy) + 2/5\) on \([0, 5] \times [3, 5.2]\). The volume under this surface is

\[
\int_0^3 \int_0^5 \sin(xy) + \frac{2}{5} \, dx \, dy.
\]

The inner integral is

\[
\int_0^5 \sin(xy) + \frac{2}{5} \, dx = -\frac{\cos(xy)}{y} + \frac{1}{5}\left[ \sin(5.2y) - \sin(0.5y) \right] + \frac{18}{5} y.
\]

Unfortunately, this gives a function for which we can’t find a simple anti-derivative. To complete the problem we could use Sage or similar software to approximate the integral.
Doing this gives a volume of approximately 8.84, so the average height is approximately 8.84/6 ⩾ 1.47.

Because addition and multiplication are commutative and associative, we can rewrite the original double sum:

\[ \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_j, y_i) \Delta x \Delta y = \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} f(x_j, y_i) \Delta x \Delta y. \]

Now if we repeat the development above, the inner sum turns into an integral:

\[ \lim_{m \to \infty} \frac{1}{m} \sum_{i=0}^{m-1} f(x_j, y_i) \Delta y = \int_a^b f(x_j, y) \, dy. \]

and the outer sum turns into an integral:

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \int_a^b f(x, y) \, dy \, dx. \]

In other words, we can compute the integrals in either order, first with respect to \( y \) and then \( x \), or vice versa. Thinking of the loaf of bread, this corresponds to slicing the loaf in a direction perpendicular to the first.

We haven’t really proved that the value of a double integral is equal to the value of the corresponding two single integrals in either order of integration, but provided the function is reasonably nice, this is true; the result is called Fubini’s Theorem.

\[ \int_0^1 \int_0^1 (x - 1)^2 + 4y^2 \, dx \, dy = \int_0^1 \int_0^1 y + (x - 1)^2 y + \frac{4}{3} y^3 \, dx \]

\[ = \int_0^1 \left[ 2 + 2(x - 1)^2 + \frac{32}{3} y^3 \right] \, dx \]

\[ = 2x + \frac{2}{3} (x - 1)^3 + \frac{32}{3} y^3 \bigg|_0^1 \]

\[ = 6 + \frac{2}{3} \cdot 8 + \frac{32}{3} \cdot \frac{1}{3} - \left( 0 - \frac{1}{3} \cdot \frac{2}{3} \right) \]

\[ = 44. \]

In the other order:

\[ \int_0^1 \int_0^1 (x - 1)^2 + 4y^2 \, dx \, dy = \int_0^1 \int_0^1 \frac{4}{3} y + (x - 1)^2 y^3 \, dy \]

\[ = \int_0^1 \left[ 3 + \frac{8}{3} y + 4y^3 + \frac{1}{3} y^5 \right] \, dy \]

\[ = 6 + \frac{32}{3} \cdot \frac{1}{3} - \frac{2}{3} \]

\[ = 44. \]

In this example there is no particular reason to favor one direction over the other; in some cases, one direction might be much easier than the other, so it’s usually worth considering the two different possibilities.

Frequently we will be interested in a region that is not simply a rectangle. Let’s compute the volume under the surface \( z = x^2 + y^2 \) above the region described by \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1 \), shown in figure 15.1.4.

\[ \int_0^1 \int_0^1 x^2 + y^2 \, dx \, dy = \int_0^1 x^2 \, dx \int_0^1 y^2 \, dy = \frac{1}{3} \cdot \frac{1}{2} \cdot 1 \]

\[ = \frac{1}{6}. \]

In principle there is nothing more difficult about this problem. If we imagine the three-dimensional region under the surface and above the parabolic region as an oddly shaped loaf of bread, we can still slice it up, approximate the volume of each slice, and add these volumes up. For example, if we slice perpendicular to the \( x \) axis at \( x_i \), the thickness of a slice will be \( \Delta x \) and the area of the slice will be:

\[ \int_0^1 x_i + 2y^3 \, dy. \]

When we add these up and take the limit as \( \Delta x \) goes to 0, we get the double integral

\[ \int_0^1 \int_0^1 x + 2y^3 \, dy \, dx \]

\[ = \int_0^1 \left[ \frac{1}{2} x^2 + 2y^4 \right] \bigg|_0^1 \, dx \]

\[ = \int_0^1 \frac{1}{2} x^2 + \frac{2}{3} \, dx \]

\[ = \frac{1}{3} + \frac{1}{2} \]

\[ = \frac{5}{6}. \]

We could just as well slice the solid perpendicular to the \( y \) axis, in which case we get:

\[ \int_0^1 \int_0^1 x + 2y^3 \, dx \, dy \]

\[ = \int_0^1 \left[ \frac{1}{2} y^2 + 2y^4 \right] \bigg|_0^1 \, dy \]

\[ = \int_0^1 \frac{1}{2} y^2 + \frac{2}{3} \, dy \]

\[ = \frac{1}{3} + \frac{2}{3} \]

\[ = \frac{1}{3}. \]

What is the average height of the surface over this region? As before, it is the volume divided by the area of the base, but now we need to use integration to compute the area of the base, since it is not a simple rectangle. The area is

\[ \int_0^1 x^2 \, dx = \frac{1}{3}. \]

so the average height is 29/28.

**Example 15.1.3** Find the volume under the surface \( z = \sqrt{1 - x^2} \) and above the triangle formed by \( y = x \), \( x = 1 \), and the \( x \)-axis.

Let’s consider the two possible ways to set this up:

\[ \int_0^1 \int_0^1 \sqrt{1 - x^2} \, dy \, dx \quad \text{or} \quad \int_0^1 \int_0^1 \sqrt{1 - x^2} \, dx \, dy. \]

While this appears easier? In the first, the first (inner) integral is easy, because we need an anti-derivative with respect to \( y \), and the entire integrand \( \frac{1}{2} \sqrt{1 - x^2} \) is constant with respect to \( y \). Of course, the second integral may be more difficult. In the second, the first integral is mildly unpleasant—a trig substitution. So let’s try the first one, since the first step is easy, and see where that leaves us.

\[ \int_0^1 \int_0^1 \sqrt{1 - x^2} \, dy \, dx = \int_0^1 \left[ y \sqrt{1 - x^2} \right] \bigg|_0^1 \, dx = \int_0^1 \sqrt{1 - x^2} \, dx. \]

This is quite easy, since the substitution \( u = 1 - x^2 \) works:

\[ \int \sqrt{1 - x^2} \, dx = -\frac{1}{2} \sqrt{1 - x^2} \, dx = -\frac{1}{2} \left[ 1 - x^2 \right]^{1/2}. \]

Then

\[ \int_0^1 \sqrt{1 - x^2} \, dx = \frac{1}{2} \left[ 1 - x^2 \right]^{1/2} \bigg|_0^1 = \frac{1}{2}. \]

This is a good example of how the order of integration can affect the complexity of the problem. In this case it is possible to do the other order, but it is a bit messier. In some cases one order may lead to a very difficult or impossible integral; it’s usually worth considering both possibilities before going very far.
Exercises 15.1.
1. Compute \( \int_0^1 (y + x^2) \, dx \).
2. Compute \( \int_0^1 (y + x) \, dx \).
3. Compute \( \int_0^1 (x + y) \, dx \).
4. Compute \( \int_1^2 (x^2) \, dx \).
5. Compute \( \int_1^2 (x^2) \, dx \).
6. Compute \( \int_1^2 (y^2) \, dy \).
7. Compute \( \int_1^2 (y^2) \, dy \).
8. Compute \( \int_1^2 (y^2) \, dy \).
9. Compute \( \int_1^2 (y^2) \, dy \).
10. Compute \( \int_1^2 (y^2) \, dy \).
11. Compute \( \int_1^2 (y^2) \, dy \).
12. Compute \( \int_1^2 (y^2) \, dy \).
13. Compute \( \int_1^2 (y^2) \, dy \).
14. Compute \( \int_1^2 (y^2) \, dy \).
15. Compute \( \int_1^2 (y^2) \, dy \).
16. Evaluate \( \int_0^1 x \, dx \) over the region in the first quadrant bounded by the hyperbola \( xy = 16 \).

and the lines \( y = x \), \( y = 0 \), and \( x = 8 \).
17. Find the volume below \( z = 1 - y \) above the region \( -1 \leq x \leq 1 \), \( 0 \leq y \leq 1 - x^2 \).
18. Find the volume bounded by \( z = x^2 + y^2 \) and \( z = 4 \).
19. Find the volume in the first octant bounded by \( y^2 = 4 - x \) and \( y = 2x \).
20. Find the volume in the first octant bounded by \( y^2 = 4x \), \( x^2 + y = 4 \), \( z = y \), and \( y = 0 \).

Exercises 15.2
1. Determine the volume of a cylinder with radius \( r \) and height \( h \).
2. Find the volume of a sphere with radius \( R \).
3. Compute the volume of a cone with base radius \( r \) and height \( h \).
4. Find the volume of a right circular cylinder with radius \( r \) and height \( h \).
5. Compute the volume of a hemisphere with radius \( R \).
6. Find the volume of an ellipsoid with semi-axes \( a, b, \) and \( c \).
7. Find the volume of a paraboloid with upper bound \( z = x^2 + y^2 \) and lower bound \( z = 0 \).
8. Compute the volume of a tetrahedron with vertices at \( (0,0,0) \), \( (a,0,0) \), \( (0,b,0) \), and \( (0,0,c) \).
9. Find the volume of a pyramid with base \( B \) and height \( h \).
10. Determine the volume of a frustum of a pyramid with bases \( B_1 \) and \( B_2 \) and height \( h \).

Exercises 15.3
1. Compute the triple integrals for the function \( f(x,y,z) \) over the region bounded by the planes:
   a. \( x = 0 \), \( y = 0 \), \( z = 0 \), \( x + y + z = 1 \)
   b. \( x = 0 \), \( y = 0 \), \( z = 0 \), \( x + y + z = 2 \)
   c. \( x = 0 \), \( y = 0 \), \( z = 0 \), \( x + y + z = 3 \)
   d. \( x = 0 \), \( y = 0 \), \( z = 0 \), \( x + y + z = 4 \)
   e. \( x = 0 \), \( y = 0 \), \( z = 0 \), \( x + y + z = 5 \)

2. Find the volume between the cylinder \( x^2 + y^2 = a^2 \) and the plane \( z = k \).
3. Compute the volume of a sphere with radius \( R \).
4. Find the volume of a right circular cylinder with radius \( r \) and height \( h \).
5. Compute the volume of a hemisphere with radius \( R \).
6. Find the volume of an ellipsoid with semi-axes \( a, b, \) and \( c \).
7. Compute the volume of a paraboloid with upper bound \( z = x^2 + y^2 \) and lower bound \( z = 0 \).
8. Compute the volume of a tetrahedron with vertices at \( (0,0,0) \), \( (a,0,0) \), \( (0,b,0) \), and \( (0,0,c) \).
9. Find the volume of a pyramid with base \( B \) and height \( h \).
10. Determine the volume of a frustum of a pyramid with bases \( B_1 \) and \( B_2 \) and height \( h \).

Exercises 15.4
1. Find the volume of the solid bounded by the paraboloid \( z = x^2 + y^2 \) and the plane \( z = 1 \).
2. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 4 - x^2 - y^2 \).
3. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 9 - x^2 - y^2 \).
4. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 16 - x^2 - y^2 \).
5. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 25 - x^2 - y^2 \).
6. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 36 - x^2 - y^2 \).
7. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 49 - x^2 - y^2 \).
8. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 64 - x^2 - y^2 \).

Exercises 15.5
1. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 9 - x^2 - y^2 \).
2. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 16 - x^2 - y^2 \).
3. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 25 - x^2 - y^2 \).
4. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 36 - x^2 - y^2 \).
5. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 49 - x^2 - y^2 \).
6. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 64 - x^2 - y^2 \).

Exercises 15.6
1. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 1 \).
2. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 4 \).
3. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 9 \).
4. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 16 \).
5. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 25 \).
6. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 36 \).
7. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 49 \).
8. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 64 \).

Exercises 15.7
1. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 1 \).
2. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 4 \).
3. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 9 \).
4. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 16 \).
5. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 25 \).
6. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 36 \).
7. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 49 \).
8. Find the volume of the solid bounded by the paraboloids \( z = x^2 + y^2 \) and \( z = 64 \).
EXAMPLE 15.2.2 Find the volume under \( z = \sqrt{4-x^2-r^2} \) above the region enclosed by the curve \( r = 2 \cos \theta \), \(-\pi/2 \leq \theta \leq \pi/2\); see figure 15.2.2. The region is described in polar coordinates by the inequalities \(-\pi/2 \leq \theta \leq \pi/2\) and \(0 \leq r \leq 2 \cos \theta\); so the double integral is

\[
\int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} \sqrt{4-x^2-r^2} \, r \, dr \, d\theta.
\]

We can rewrite the integral as shown because of the symmetry of the volume; this avoids a complication during the evaluation. Proceeding:

\[
2 \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} (4-r^2)^{1/2} r \, dr \, d\theta
= 2 \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} (4-r^2)^{1/2} \frac{8}{3} \, dr \, d\theta
= 2 \left( \frac{8 \cos^3 \theta - \cos \theta}{3} \right)^{1/2} \bigg|_0^{2 \cos \theta}
= \frac{16 \sqrt{2}}{3}.
\]

Figure 15.2.2 Volume over a region with non-constant limits.

You might have learned a formula for computing areas in polar coordinates. It is possible to compute areas as volumes, so that you need only remember one technique. Consider the surface \( z = 1 \), a horizontal plane. The volume under this surface and above a region in the \( xy \) plane is simply 1 (area of the region), so computing the volume really just computes the area of the region.

15.3 Moment and Center of Mass

Using a single integral we were able to compute the center of mass for a one-dimensional object with variable density, and a two-dimensional object with constant density. With a double integral we can handle two dimensions and variable density.

\[ \bar{x} = \frac{1}{M} \int_{\text{region}} x \, dM, \quad \bar{y} = \frac{1}{M} \int_{\text{region}} y \, dM. \]

where \( M \) is the total mass, \( x_\text{c} \) is the moment around the \( y \)-axis, and \( y_\text{c} \) is the moment around the \( x \)-axis. (You may want to review the concepts in section 9.6.)

The key to the computation, just as before, is the approximation of mass. In the two-dimensional case, we treat density \( \sigma \) as mass per square area, so when density is constant, mass is (density)(area). If we have a two-dimensional region with varying density given by \( \sigma(x, y) \), and we divide the region into small subregions with area \( dA \), then the mass of one subregion is approximately \( \sigma(x, y)dA \), the total mass is approximately the sum of many of these, and as usual the sum turns into an integral in the limit:

\[ M = \int_{\text{region}} \sigma(x, y) \, dA, \]

and similarly for computations in cylindrical coordinates. Then as before:

\[ M_x = \int_{\text{region}} y \, \sigma(x, y) \, dA, \]
\[ M_y = \int_{\text{region}} x \, \sigma(x, y) \, dA. \]

EXAMPLE 15.3.1 Find the center of mass of a thin, uniform plate whose shape is the region between \( y = \cos x \) and the \( x \)-axis between \( x = -\pi/2 \) and \( x = \pi/2 \). Since the density is constant, we may take \( \sigma(x, y) = 1 \).

It is clear that \( \bar{x} = 0 \), but for practice let’s compute it anyway. First we compute the mass:

\[ M = \int_{-\pi/2}^{\pi/2} \int_{-1}^{1} 1 \, dy \, dx = 2 \int_{-\pi/2}^{\pi/2} \cos x \, dx = \sin \pi = 2. \]

Next,
\[ M_x = \int_{-\pi/2}^{\pi/2} \int_{-1}^{1} y \, \sigma(x, y) \, dy \, dx = \frac{\pi}{4} - \frac{\pi}{4} \cos x \, dx = \frac{\pi}{2}. \]

Finally,
\[ M_y = \int_{-\pi/2}^{\pi/2} \int_{-1}^{1} x \, \sigma(x, y) \, dy \, dx = \frac{\pi}{2} \cos x \, dx = 0. \]

So \( \bar{x} = 0 \) as expected, and \( \bar{y} = \pi/4 \). This is the same problem as in example 9.6.4, it may be helpful to compare the two solutions.
EXAMPLE 15.3.2 Find the center of mass of a two-dimensional plate that occupies the quarter circle $x^2 + y^2 \leq 1$ in the first quadrant and has density $k(x^2 + y^2)$. It seems clear that because of the symmetry of both the region and the density function (both are important), $\bar{x} = \bar{y}$. We'll do both to check our work.

Jumping right in:

$$M = \int_0^{\pi/2} \int_0^1 [k(x^2 + y^2)] \, r \, dr \, d\theta = k \int_0^{\pi/2} \int_0^1 (x^2 + y^2) \, r \, dr \, d\theta = k \int_0^{\pi/2} \frac{1}{3} \theta \, d\theta = \frac{k \pi}{6}.$$  

This integral is something we can do, but it's a bit unpleasant. Since everything in sight is related to a circle, let's back up and try polar coordinates. Then $x^2 + y^2 = r^2$ and

$$M = \int_0^{\pi/2} \int_0^1 [k(x^2 + y^2)] \, r \, dr \, d\theta = k \int_0^{\pi/2} \int_0^1 r^2 \frac{1}{2} \theta \, dr \, d\theta = k \int_0^{\pi/2} \frac{1}{2} \theta \, d\theta = \frac{k \pi}{4}.$$  

Much better. Next, since $y = r \sin \theta$,

$$M_x = k \int_0^{\pi/2} \int_0^1 r^3 \sin \theta \, dr \, d\theta = k \int_0^{\pi/2} \int_0^1 \frac{1}{2} \sin \theta \, d\theta = k \int_0^{\pi/2} \frac{1}{2} \sin \theta \, d\theta = \frac{k \pi}{4}.$$  

Similarly,

$$M_y = k \int_0^{\pi/2} \int_0^1 r^3 \cos \theta \, dr \, d\theta = k \int_0^{\pi/2} \int_0^1 \frac{1}{2} \cos \theta \, d\theta = k \int_0^{\pi/2} \frac{1}{2} \cos \theta \, d\theta = \frac{k \pi}{4}.$$  

Finally, $\bar{x} - \bar{y} = \frac{8 \pi}{3x}$.

Exercises 15.3.

1. Find the center of mass of a two-dimensional plate that occupies the square $[0,1] \times [0,1]$ and has density function $xy$.  
2. Find the center of mass of a two-dimensional plate that occupies the triangle $0 \leq x \leq 1$, $0 \leq y \leq x$, and has density function $xy$.  
3. Find the center of mass of a two-dimensional plate that occupies the upper unit semicircle centered at $(0,0)$ and has density function $x^2$.  
4. Find the center of mass of a two-dimensional plate that occupies the upper unit semicircle centered at $(0,0)$ and has density function $x^2$.  
5. Find the center of mass of a two-dimensional plate that occupies the triangle formed by $x = 2$, $y = 2x$, and $y = 2x$ and has density function $2x$.  
6. Find the center of mass of a two-dimensional plate that occupies the triangle formed by $x = 1$, $y = x$, and $2x + y = 6$ and has density function $x^2$.  

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Figure 15.4.1 Small parallelograms at points of tangency. (AP)

EXAMPLE 15.4.1 We find the area of the hemisphere $z = \sqrt{1 - x^2 - y^2}$. We compute the derivatives

$$f_x = \frac{-x}{\sqrt{1 - x^2 - y^2}} \quad f_y = \frac{-y}{\sqrt{1 - x^2 - y^2}},$$

and then the area is

$$\int_0^1 \int_0^{\sqrt{1 - x^2}} \sqrt{1 - x^2 - y^2} + \frac{y^2}{1 - x^2} - 1 \, dy \, dx.$$  

This is a bit on the messy side, but we can use polar coordinates:

$$\int_0^{2\pi} \int_0^1 \frac{1}{2} r \, dr \, d\theta.$$  

This integral is improper, since the function is undefined at the limit 1. We therefore compute

$$\lim_{a \to 1} \int_0^1 \frac{1}{2} r \, dr = \lim_{a \to 1} -\frac{1}{2} \sqrt{a^2} + 1 = 1,$$

using the substitution $u = 1 - r^2$. Then the area is

$$\int_0^{2\pi} \frac{1}{2} \, d\theta = \pi.$$  

You may recall that the area of a sphere of radius $r$ is $4\pi r^2$, so half the area of a unit sphere is $(1/2)4\pi r = 2\pi r$, in agreement with our answer.
The whole problem comes down to correctly describing the region by inequalities: 
\[ 0 \leq z \leq 2, \quad 3z/2 \leq y \leq 3, \quad 0 \leq z \leq 5y/2. \]
The lower limit on \( y \) comes from the equation of the line \( y = 3z/2 \) that forms one edge of the tetrahedron in the \( x-y \) plane; the upper \( z \) limit comes from the equation of the plane \( z = 5y/2 \) that forms the “upper” side of the tetrahedron; see figure 15.5.1. Now the volume is
\[
\int_{0}^{3} \int_{3z/2}^{3} \int_{0}^{2} dz
dy
dx = \int_{0}^{3} \int_{3z/2}^{3} \frac{5y}{2} \ dy
dx = \int_{0}^{3} \left[ \frac{5y^2}{2} \right]_{3z/2}^{3} \ dx = \int_{0}^{3} \frac{9}{2} \ dx - \int_{0}^{3} \frac{15z^2}{4} \ dx = \frac{15z^2}{8} - \frac{15z^3}{4} dx = \frac{15z^2}{8} - \frac{15z^3}{4} = 15 - 10 = 5.
\]
Pretty much just the way we did for two dimensions we can use triple integration to compute mass, center of mass, and various average quantities.

EXAMPLE 15.5.3 Suppose the temperature at a point is given by \( T = xyz \). Find the average temperature in the cube with opposite corners at \((0, 0, 0)\) and \((2, 2, 2)\). In two dimensions we add up the temperatures and divide by the area; here we add up the temperatures and divide by the volume, 8:
\[
\frac{1}{8} \int_{0}^{2} \int_{0}^{2} \int_{0}^{2} xyz \ dz
dy
dx = \frac{1}{8} \int_{0}^{2} \int_{0}^{2} \left[ \frac{xyz^2}{2} \right]_{0}^{2} \ dx = \frac{1}{8} \int_{0}^{2} \left( \frac{4y}{2} \right) \ dx = \frac{1}{8} \int_{0}^{2} 2y \ dx = \frac{1}{8} \left[ yx \right]_{0}^{2} = \frac{1}{8} (2x - 0) = \frac{1}{8} (2 \times 2) = \frac{1}{8} (4) = \frac{1}{2}.
\]

EXAMPLE 15.5.4 Suppose the density of an object is given by \( xz \), and the object occupies the tetrahedron with corners \((0, 0, 0), (1, 0, 0), (1, 1, 0), \) and \((0, 1, 1)\). Find the mass and center of mass of the object.

Exercises 15.5.

1. Evaluate \( \iiint z + 2x + y + 1 \ dV \) over the region.
2. Evaluate \( \iiint z^2 x \ dV \) over the region.
3. Evaluate \( \iiint z^2 x^2 e^{-z^2} \ dV \) over the region.
4. Evaluate \( \iiint x^2 y^2 \ dV \) over the region.
5. Evaluate \( \iiint r \cos \theta z \ dV \) over the region.
6. Evaluate \( \iiint z^2 x \ dV \) over the region.
7. Evaluate \( \iiint z^2 x^3 \ dV \) over the region.
8. Compute \( \iiint z^2 x^3 z + x \ dV \) over the region.
9. For each of the integrals in the previous exercise, give a description of the volume (both algebraic and geometric) that is in the domain of integration.
10. Compute \( \iiint z \ dV \) over the region \( x^2 + y^2 + z^2 \leq 1 \) in the first octant.
11. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one corner.
12. Find the mass of a cube with edge length 2 and density equal to the square of the distance from one edge.
13. An object occupies the volume of the upper hemisphere of \( x^2 + y^2 + z^2 = 4 \) and has density \( xz \) at \((x, y, z)\). Find the center of mass.
14. An object occupies the volume of the pyramid with corners at \((1, 1, 0), (1, -1, 0), (-1, -1, 0), \) and \((0, 0, 2)\) and has density \( y^2 \) at \((x, y, z)\). Find the center of mass.
15. Verify the moments \( M_{xy}, M_{yz}, \) and \( M_{xz} \) of example 15.5.6 by evaluating the integrals.
16. Find the region \( E \) for which \( \iiint_{E} (1 - x^2 - y^2 - z^2) \ dV \) is a maximum.

15.6 Cylindrical and Spherical Coordinates

We have seen that sometimes double integrals are simplified by doing them in polar coordinates; not surprisingly, triple integrals are sometimes simpler in cylindrical coordinates or spherical coordinates. To set up integrals in polar coordinates, we had to understand the shape and area of a typical small region into which the region of integration was divided. We needed to do the same thing here, for three-dimensional regions.
Find the average temperature in the unit sphere centered at the origin.

Suppose the temperature at \((x, y, z)\) in the unit sphere is \(T = 1/(1 + x^2 + y^2 + z^2)\). Find the average temperature in the unit sphere centered at the origin.

An object occupies the region inside the unit sphere at the origin, and has density equal to \(\rho \sin \phi \, d\phi \, d\theta \, d\rho\), or in the limit \(\rho^2 \sin \phi \, d\phi \, d\theta \, d\rho\). The function is converted to \(\sin \phi \, d\phi \, d\theta \, d\rho\), as indicated in the left graph.

The upshot is that the volume of the little box is approximately \(\Delta \phi \rho \sin \Delta \phi \Delta \theta \Delta \rho\), or in the limit \(\rho^2 \sin \phi \, d\phi \, d\theta \, d\rho\).

**Example 15.6.3** Suppose the temperature at \((x, y, z)\) is \(T = 1/(1 + x^2 + y^2 + z^2)\). Find the average temperature in the unit sphere centered at the origin.

**Exercises 15.6.**

1. Evaluate \(
\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2 + \rho^2} \, dx \, dy \, dz
\)

2. Evaluate \(
\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{x y z}{x^2 + y^2 + z^2} \, dx \, dy \, dz
\)

3. Evaluate \(
\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{x y z}{x^2 + y^2 + z^2} \, dx \, dy \, dz
\)

4. Evaluate \(
\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{x y}{x^2 + y^2 + z^2} \, dx \, dy \, dz
\)

5. Evaluate \(
\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{x z}{x^2 + y^2 + z^2} \, dx \, dy \, dz
\)

6. Evaluate \(
\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{y z}{x^2 + y^2 + z^2} \, dx \, dy \, dz
\)

7. Evaluate \(
\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{x y^2}{x^2 + y^2 + z^2} \, dx \, dy \, dz
\)

8. Evaluate \(
\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{x y^2}{x^2 + y^2 + z^2} \, dx \, dy \, dz
\)

9. Compute \(
\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{x y^2}{x^2 + y^2 + z^2} \, dx \, dy \, dz
\)

10. Find the mass of a right circular cone of height \(h\) and base radius \(a\) if the density is proportional to the distance from the base.

11. Find the mass of a right circular cone of height \(h\) and base radius \(a\) if the density is proportional to the distance from its axis of symmetry.

12. An object occupies the region inside the unit sphere at the origin, and has density equal to the distance from the \(x\)-axis. Find the mass.

### 15.7 Change of Variables

One of the most useful techniques for evaluating integrals is substitution, both "u-substitution" and trigonometric substitution, in which we change the variable to something more convenient. As we have seen, sometimes changing from rectangular coordinates to another coordinate system is helpful, and this too changes the variables. This is certainly a more complicated change, since instead of changing one variable for another we change an entire suite of variables, but as it turns out it is really very similar to the kinds of change of variables we already know as substitution.

Let's examine the single variable case again, from a slightly different perspective than we have previously used. Suppose we start with the problem \(\int_{0}^{1} \sqrt{1 - x^2} \, dx\),

this computes the area in the left graph of figure 15.7.1. We use the substitution \(x = \sin u\) to transform the function from \(\sqrt{1 - x^2}\) to \(\sin u \sqrt{1 - \sin^2 u}\), and we also convert \(dx\) to \(\cos u \, du\). Finally, we convert the limits 0 and 1 to 0 and \(\pi/2\). This transforms the integral to

\(\int_{0}^{\pi/2} \sin u \sqrt{1 - \sin^2 u} \, du\).

We want to notice that there are three different conversions: the main function, the differential \(dx\), and the interval of integration. The function is converted to \(\sin u \sqrt{1 - \sin^2 u}\),

\(\frac{1}{2}\), between 0.7 and 0.8. The function on the right is situated between the corresponding values around 0.8 and 0.9 so that \(\Delta u = \arcsin(0.8) = \arcsin(0.9)\). To make the widths match, and the areas therefore the same, we can multiply \(\Delta u\) by a correction factor; in this case the correction factor is approximately \(\cos\), which we compute when we convert \(dx\) to \(\cos u \, du\).

Now let's move to functions of two variables. Suppose we want to convert an integral

\(\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} f(x, y, z) \, dx \, dy \, dz\)

to use new variables \(u\) and \(v\). In the single variable case, there's typically just one reason to want to change the variable: to make the function "nicer" so that we can find an antiderivative. In the two variable case, there is a second potential reason: the two-dimensional region over which we need to integrate is somehow unpleasant, and we want the region in terms of \(u\) and \(v\) to be nicer—to be a rectangle, for example. Ideally, of course, the new function and the new region will be no worse than the originals, and at least one of them will be better; this doesn't always pan out.

As before, there are three parts to the conversion: the function itself must be rewritten in terms of \(u\) and \(v\), \(du \, dv\) must be converted to \(du \, dv\), and the old region must be converted to the new region. We will develop the necessary techniques by considering a particular example, and we will use an example we already know how to do by other means.

Consider

\(\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \sqrt{x^2 + y^2} \, dx \, dy \, dz\).

The limits correspond to integrating over the top half of a circular disk, and we recognize that the function will simplify in polar coordinates, so we would normally convert to polar coordinates.
In this case, the region in the x-y plane is approximately a rectangle with dimensions $\Delta x \times \Delta y$, but in general the corner angles will not be right angles, so the region will typically be (almost) a parallelogram. We need to compute the area of this parallelogram.

**EXAMPLE 15.7.1**

Integrate $xy$ over the region $x^2 + y^2 \leq 2$.

The equation $x^2 + y^2 = 2$ describes an ellipse as in figure 15.7.5; the region of integration is the interior of the ellipse. We will use the transformation $x = \sqrt{2u} - \sqrt{2v}$, $y = \sqrt{2u} + \sqrt{2v}$.

Substituting into the function itself we get $x^2 - xy + y^2 = 2u^2 + 2v^2$. The boundary of the ellipse is $x^2 - xy + y^2 = 2$, so the boundary of the corresponding region in the u-v plane is $2u^2 + 2v^2 = 2$ or $u^2 + v^2 = 1$, the unit circle, so this substitution makes the region of integration simpler.

Next, we compute the Jacobian, using $f = \sqrt{2u} - \sqrt{2v}$ and $g = \sqrt{2u} + \sqrt{2v}$:

$$\frac{\partial (g, f)}{\partial (u, v)} = \frac{\partial (\sqrt{2u} + \sqrt{2v}, \sqrt{2u} - \sqrt{2v})}{\partial (u, v)} = \frac{-2\sqrt{2}u + 2\sqrt{2}v}{\sqrt{2u} + \sqrt{2v}} = \frac{-2u + 2v}{\sqrt{2u} + \sqrt{2v}}$$

Hence the new integral is

$$\int\int_{R} \frac{(2u + 2v)^2}{4} \frac{1}{\sqrt{2}} \frac{1}{u^2 + v^2} du dv$$

where $R$ is the interior of the unit circle. This is still not an easy integral, but it is easily transformed to polar coordinates, and then easily integrated.

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**Chapter 15 Multiple Integration**

and called the Jacobian. Note that this is the absolute value of the two by two determinant

$$\begin{vmatrix} f_x & g_x \\ f_y & g_y \end{vmatrix}$$

which may be easier to remember. (Confusingly, the matrix, the determinant of the matrix, and the absolute value of the determinant are all called the Jacobian by various authors.)

Because there are two things to worry about, namely, the form of the function and the region of integration, transformations in two (or more) variables are quite tricky to discover.

**EXAMPLE 15.7.1**

Integrate $xy$ over the region $x^2 + y^2 \leq 2$.

The equation $x^2 + y^2 = 2$ describes an ellipse as in figure 15.7.5; the region of integration is the interior of the ellipse. We will use the transformation $x = \sqrt{2u} - \sqrt{2v}$, $y = \sqrt{2u} + \sqrt{2v}$.

Substituting into the function itself we get $x^2 - xy + y^2 = 2u^2 + 2v^2$. The boundary of the ellipse is $x^2 - xy + y^2 = 2$, so the boundary of the corresponding region in the u-v plane is $2u^2 + 2v^2 = 2$ or $u^2 + v^2 = 1$, the unit circle, so this substitution makes the region of integration simpler.

Next, we compute the Jacobian, using $f = \sqrt{2u} - \sqrt{2v}$ and $g = \sqrt{2u} + \sqrt{2v}$:

$$\frac{\partial (g, f)}{\partial (u, v)} = \frac{\partial (\sqrt{2u} + \sqrt{2v}, \sqrt{2u} - \sqrt{2v})}{\partial (u, v)} = \frac{-2\sqrt{2}u + 2\sqrt{2}v}{\sqrt{2u} + \sqrt{2v}} = \frac{-2u + 2v}{\sqrt{2u} + \sqrt{2v}}$$

Hence the new integral is

$$\int\int_{R} \frac{(2u + 2v)^2}{4} \frac{1}{\sqrt{2}} \frac{1}{u^2 + v^2} du dv$$

where $R$ is the interior of the unit circle. This is still not an easy integral, but it is easily transformed to polar coordinates, and then easily integrated.
There is a similar change of variables formula for triple integrals, though it is a bit more difficult to derive. Suppose we use three substitution functions, 

\[ x = f(u, v, w), \quad y = g(u, v, w), \quad z = h(u, v, w). \]

The Jacobian determinant is now

\[
\begin{vmatrix}
\frac{\partial (x, y, z)}{\partial (u, v, w)} \\
\end{vmatrix} = \begin{vmatrix}
f_u & g_u & h_u \\
f_v & g_v & h_v \\
f_w & g_w & h_w \\
\end{vmatrix}.
\]

Then the integral is transformed in a similar fashion:

\[
\int \int \int_R F(x, y, z) \, dV = \int \int \int_S F(f(u, v, w), g(u, v, w), h(u, v, w)) \left| \frac{\partial (x, y, z)}{\partial (u, v, w)} \right| \, du \, dv \, dw,
\]

where of course the region \( S \) in \( uvw \) space corresponds to the region \( R \) in \( xyz \) space.

**Exercises 15.7.**

1. Complete example 15.7.1 by converting to polar coordinates and evaluating the integral. ⇒

2. Evaluate \( \int \int xy \, dx \, dy \) over the square with corners \((0, 0), (1, 1), (1, -1)\) in two ways: directly, and using \( x = (u + v)/2, y = (u - v)/2. \) ⇒

3. Evaluate \( \int \int x^2 + y^2 \, dx \, dy \) over the square with corners \((-1, 0), (0, 1), (0, -1)\) in two ways: directly, and using \( x = (u + v)/2, y = (u - v)/2. \) ⇒

4. Evaluate \( \int \int (x + y)e^{x+y} \, dx \, dy \) over the triangle with corners \((0, 0), (1, 1), (1, 0)\) in two ways: directly, and using \( x = (u + v)/2, y = (u - v)/2. \) ⇒

5. Evaluate \( \int \int y(x - y) \, dx \, dy \) over the parallelogram with corners \((0, 0), (3, 3), (7, 3), (4, 0)\) in two ways: directly, and using \( x = u + v, y = v. \) ⇒

6. Evaluate \( \int \int \sqrt{x^2 + y^2} \, dx \, dy \) over the triangle with corners \((0, 0), (4, 4), (4, 0)\) using \( x = u, \ y = uv. \) ⇒

7. Evaluate \( \int \int y \sin(xy) \, dx \, dy \) over the region bounded by \( xy = 1, \ xy = 4, \ y = 1, \ \) and \( y = 4 \) using \( x = u/v, \ y = v. \) ⇒

8. Evaluate \( \int \int \sin(9x^2 + 4y^2) \, dA \) over the region in the first quadrant bounded by the ellipse \( 9x^2 + 4y^2 = 1. \) ⇒

9. Compute the Jacobian for the substitutions \( x = \rho \sin \phi \cos \theta, \ y = \rho \sin \phi \sin \theta, \ z = \rho \cos \phi. \)