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Multiple Integration

15.1 Volume and Average Height

Consider a surface $f(x, y)$; you might temporally think of this as representing physical
topography—a hilly landscape, perhaps. What is the average height of the surface (or
average altitude of the landscape) over some region?

As with most such problems, we start by thinking about how we might approximate
the answer. Suppose the region is a rectangle, $[a, b] \times [c, d]$. We can divide the rectangle
into a grid, $m$ subdivisions in one direction and $n$ in the other, as indicated in figure 15.1.1.

Let $x_i$ values $x_0, x_1, \ldots, x_m$ in each subdivision in the $x$ direction, and similarly in
the $y$ direction. At each of the points $(x_i, y_j)$ in one of the smaller rectangles in the grid,
we compute the height of the surface: $f(x_i, y_j)$. Now the average of these heights should
be (depending on the fineness of the grid) close to the average height of the surface:

$$
\frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_i, y_j)
$$

As both $m$ and $n$ go to infinity, we expect this approximation to converge to a fixed value,
the actual average height of the surface. For reasonably nice functions this does indeed
happen.

![Figure 15.1.2 Approximating the volume under a surface.](image)

The next question, of course, is: How do we compute these double integrals? You
might think that we will need some two-dimensional version of the Fundamental Theorem
of Calculus, but as it turns out we can get away with just the single variable version,
applied twice.

Going back to the double sum, we can rewrite it to emphasize a particular order in
which we want to add the terms:

$$
\sum_{i=0}^{m-1} \left( \sum_{j=0}^{n-1} f(x_i, y_j) \Delta x \right) \Delta y
$$

In the sum in parentheses, only the value of $x_i$ is changing; $y_j$ is temporarily constant. As
$m$ goes to infinity, this sum has the right form to turn into an integral:

$$
\lim_{m \to \infty} \sum_{i=0}^{m-1} \left( \sum_{j=0}^{n-1} f(x_i, y_j) \Delta x \right) \Delta y = \int_a^b f(x, y) \, dx
$$

So after we take the limit as $m$ goes to infinity, the sum is

$$
\int_a^b f(x, y) \, dx
$$

The two parts of this product have useful meaning: $(b - a)(d - c)$ is of course the area
of the rectangle, and the double sum adds up the terms of the form $f(x_i, y_j) \Delta x \Delta y$, which is
the height of the surface at a point times the area of one of the small rectangles into which
we have divided the large rectangle. In short, each term $f(x_i, y_j) \Delta x \Delta y$ is the volume of
a tall, thin, rectangular box, and is approximately the volume under the surface and above
one of the small rectangles; see figure 15.1.2. When we add all of these up, we get an
approximation to the volume under the surface and above the rectangle $R = [a, b] \times [c, d]$.

Let's be clear about what this means: we first will compute the inner integral, temporarily
setting the $y$ variable to a constant. We will do this by finding an anti-derivative with respect to
$x$, then substituting $x = a$ and $x = b$ and subtracting, as usual. The result will be an
expression with no $x$ variable but some occurrences of $y$. The outer integral will be an
ordinary one-variable problem, with $y$ as the variable.

**EXAMPLE 15.1.1** Figure 15.2.2 shows the function $\sin(xy) + 6/5$ on $[0, 5] \times [0, 5, 2.5]$.
The volume under this surface is

$$
\int_0^5 \int_0^{2.5} \sin(xy) + \frac{6}{5} \, dy \, dx
$$

The inner integral is

$$
\int_0^{2.5} \sin(xy) + \frac{6}{5} \, dy = -\cos(xy) \Big|_0^{2.5} + \frac{6}{5} \int_0^{2.5} \cos(3.5y) \, dy = -\cos(2.5) + \frac{6}{5} \cos(10) + \frac{18}{5}.
$$

Unfortunately, this gives a function for which we can’t find a simple anti-derivative. To
complete the problem we could use Sage or similar software to approximate the integral.
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15.1.3 Approximating the volume under a surface with slices. (AP)

Doing this gives a volume of approximately 8.84, so the average height is approximately 8.84/6 ≈ 1.47.

Because addition and multiplication are commutative and associative, we can rewrite the original double sum:

$$\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_j, y_i) \Delta A = \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} f(x_j, y_i) \Delta A.$$

Now if we repeat the development above, the inner sum turns into an integral:

$$\int_{y=0}^{n} f(x_j, y) \Delta y = f(x_j, y) \left[ y \right]_{y=0}^{n}.$$

In other words, we can compute the integrals in either order, first with respect to $x$ and then the outer sum turns into an integral:

$$\lim_{m, n \to \infty} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_j, y_i) \Delta A = \int_{x=0}^{1} \int_{y=0}^{n} f(x, y) \, dy \, dx.$$

We haven’t really proved that the value of a double integral is equal to the value of the corresponding two single integrals in either order of integration, but provided the function is reasonably nice, this is true; the result is called Fubini’s Theorem.

EXAMPLE 15.1.2 We compute $\int_{R} 1 + (x - 1)^2 + 4y^2 \, dA$, where $R = [0, 3] \times [0, 2]$, in two ways.

First,

$$\int_{x=0}^{3} \int_{y=0}^{2} 1 + (x - 1)^2 + 4y^2 \, dy \, dx = \int_{x=0}^{3} \left[ y + (x - 1)^2 y + \frac{4}{3} y^3 \right]_{y=0}^{2} \, dx.$$

$$= \int_{x=0}^{3} 2 + 2(x - 1)^2 + \frac{32}{3} \, dx = 2x + \frac{2}{3} (x - 1)^3 + \frac{32}{3} \left[ x \right]_{0}^{3}$$

$$= 6 + \frac{2}{3} \left( 2 - \frac{1}{3} \right) + \frac{32}{3} \left( 3 - (0 - 1) \right) = 44.$$

In the other order,

$$\int_{y=0}^{2} \int_{x=0}^{3} 1 + (x - 1)^2 + 4y^2 \, dx \, dy = \int_{y=0}^{2} x + \left[ \frac{4}{3} y^3 \right]_{y=0}^{2} \, dy$$

$$= \int_{y=0}^{2} 3 + \frac{8}{3} + \frac{1}{3} \, dy = 3y + \frac{8}{3} + \frac{4}{3} \left[ y \right]_{0}^{2} = 3 + \frac{8}{3} + \frac{4}{3} \left( 2 - 0 \right) = 44.$$

In this example there is no particular reason to favor one direction over the other; in some cases, one direction might be much easier than the other, so it’s usually worth considering the two different possibilities.

Frequently we will be interested in a region that is not simply a rectangle. Let’s compute the volume under the surface $z = x + 2y^3$ above the region described by $0 \leq x \leq 1$ and $0 \leq y \leq x^2$, shown in figure 15.1.4.

In principle there is nothing more difficult about this problem. If we imagine the three-dimensional region under the surface and above the parabolic region as an oddly shaped loaf of bread, we can still slice it up, approximate the volume of each slice, and add these slices up. For example, if we slice perpendicular to the $x$ axis at $x_i$, the thickness of the slice will be $\Delta x$ and the area of the slice will be

$$\int_{x=0}^{x} x_i + 2y^3 \, dy.$$}

When we add these up and take the limit as $\Delta x$ goes to 0, we get the double integral

$$\int_{x=0}^{1} \int_{y=0}^{x^2} x + 2y^3 \, dy \, dx = \int_{x=0}^{1} xy + \frac{2}{4} y^4 \bigg|_{y=0}^{x^2} \, dx$$

$$= \int_{x=0}^{1} x^2 + \frac{1}{2} x^6 \, dx = \frac{1}{2} x^4 + \frac{1}{2} x^6 \bigg|_{x=0}^{1}$$

$$= \frac{1}{2} + \frac{1}{2} = 1.$$

We could just as well slice the solid perpendicular to the $y$ axis, in which case we get

$$\int_{y=0}^{2} \int_{x=0}^{y} x + 2y^3 \, dx \, dy = \int_{y=0}^{2} \frac{x^2}{2} + 2y^3 \bigg|_{x=0}^{y} \, dy$$

$$= \int_{y=0}^{2} \frac{1}{2} y^2 - \frac{1}{2} y^2 + 2y^3 \sqrt{y} \, dy$$

$$= \int_{y=0}^{2} \frac{1}{2} y^2 - \frac{1}{2} y^2 + 2y^3 \sqrt{y} \, dy$$

$$= \frac{1}{2} \left( \frac{1}{3} y^3 + \frac{1}{4} y^4 \right) \bigg|_{y=0}^{2} = \frac{1}{2} \left( \frac{1}{3} (2)^3 + \frac{1}{4} (2)^4 \right)$$

$$= \frac{1}{2} \left( \frac{8}{3} + \frac{16}{4} \right) = \frac{20}{6} = \frac{10}{3}.$$

What is the average height of the surface over this region? As before, it is the volume divided by the area of the base, but now we need to use integration to compute the area of the base, since it is not a simple rectangle. The area is

$$\int_{x=0}^{1} x^2 \, dx = \frac{1}{3},$$

so the average height is 29/82.

EXAMPLE 15.1.3 Find the volume under the surface $z = \sqrt{1 - x^2}$ and above the triangle formed by $y = x$, $x = 1$, and the $x$-axis.

Let’s consider the two possible ways to set this up:

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1 - x^2}} \sqrt{1 - x^2} \, dy \, dx \quad \text{or} \quad \int_{y=0}^{1} \int_{x=0}^{\sqrt{1 - y^2}} \sqrt{1 - y^2} \, dx \, dy.$$

Which appears easier? In the first, the first (inner) integral is easy, because we need an anti-derivative with respect to $y$, and the entire integrand $\sqrt{1 - y^2}$ is constant with respect to $y$. Of course, the second integral may be more difficult. In the second, the first integral is mildly unpleasant—a trig substitution. So let’s try the first one, since the first step is easy, and see where that leaves us.

$$\int_{x=0}^{1} \int_{y=0}^{\sqrt{1 - x^2}} \sqrt{1 - x^2} \, dy \, dx = \int_{x=0}^{1} \sqrt{1 - x^2} \, dy \bigg|_{y=0}^{\sqrt{1 - x^2}} = \int_{x=0}^{1} x \sqrt{1 - x^2} \, dx.$$

This is quite easy, since the substitution $u = 1 - x^2$ works:

$$\int x \sqrt{1 - x^2} \, dx = -\frac{1}{2} \sqrt{u} \bigg|_{u=0}^{u=1} - \frac{1}{2} \sqrt{1 - x^2} \bigg|_{x=0}^{x=1} = -\frac{1}{2} \sqrt{1 - x^2} \bigg|_{x=0}^{x=1}.$$

Then

$$\int x \sqrt{1 - x^2} \, dx = \int \frac{1}{2} \left( 1 - x^2 \right)^{3/2} \, dx = \frac{1}{2} \left( 1 - x^2 \right)^{3/2}.$$

This is a good example of how the order of integration can affect the complexity of the problem. In this case it is possible to do the other order, but it is a bit messier. In some cases one order may lead to a very difficult or impossible integral; it’s usually worth considering both possibilities before going very far.
1. Compute \( \int_0^1 \int_0^1 \frac{1}{x} \, dy \, dx \).
2. Compute \( \int_0^1 \int_0^1 \frac{1}{y} \, dx \, dy \).
3. Compute \( \int_0^1 \int_0^1 xy \, dx \, dy \).
4. Compute \( \int_0^1 \int_0^1 \frac{1}{x^2} \, dx \, dy \).
5. Compute \( \int_0^1 \int_0^1 \frac{1}{y^2} \, dy \, dx \).
6. Compute \( \int_0^1 \int_0^1 \frac{1}{xy} \, dx \, dy \).
7. Compute \( \int_0^1 \int_0^{\pi} \frac{1}{r} \, dr \, d\theta \).
8. Compute \( \int_0^1 \int_0^{\pi} \frac{1}{\sin \theta} \, d\theta \, d\phi \).
9. Compute \( \int_0^1 \int_0^{\pi} \frac{1}{\cos \theta} \, d\theta \, d\phi \).
10. Compute \( \int_0^1 \int_0^{\pi} \frac{1}{\sin \phi} \, d\phi \, d\theta \).
11. Compute \( \int_0^1 \int_0^{\pi} \frac{1}{\cos \phi} \, d\phi \, d\theta \).
12. Compute \( \int_0^1 \int_0^{\pi} \frac{1}{\sin \phi \cos \phi} \, d\phi \, d\theta \).
13. Compute \( \int_0^1 \int_0^{\pi} \frac{1}{\cos \phi \sin \phi} \, d\phi \, d\theta \).
14. Compute \( \int_0^1 \int_0^{\pi} \frac{1}{\sin \phi \cos \phi} \, d\phi \, d\theta \).
15. Compute \( \int_0^1 \int_0^{\pi} \frac{1}{\sin \phi \cos \phi} \, d\phi \, d\theta \).
16. Evaluate \( \int_0^1 \int_0^1 \frac{1}{xy} \, dx \, dy \).

17. Find the volume below \( z = 1 - y \) above the region \(-1 \leq x \leq 1, 0 \leq y \leq 1 - x^2 \).
18. Find the volume bounded by \( z = x^2 + y^2 \) and \( z = 4 \).
19. Find the volume in the first octant bounded by \( y^2 = 4 - x \) and \( y = 2x \).
20. Find the volume in the first octant bounded by \( y^2 = 4x, 2x + y = 4, z = y, \) and \( y = 0 \).

**Exercises 15.1.**

1. Find the volume in the first octant bounded by \( x + y + z = 9, 2x + 3y = 18, \) and \( x + 3y = 9 \).
2. Find the volume in the first octant bounded by \( x^2 + y^2 = a^2 \) and \( z = x + y \).
3. Find the volume bounded by \( 4x^2 + y^2 = 4c \) and \( z = 2 \).
4. Find the volume bounded by \( z = x^2 + y^2 \) and \( z = y \).
5. Find the volume under the surface \( z = xy \) above the triangle with vertices \((1, 1, 0), (4, 1, 0), (1, 2, 0)\).
6. Find the volume enclosed by \( y = x^2, y = 4, z = x^3, z = 0 \).
7. A swimming pool is circular with a 40 meter diameter. The depth is constant along east-west lines and increases linearly from 2 meters at the south end to 7 meters at the north end. Find the volume of the pool.
8. Find the average value of \( f(x, y) = e^{x+y^2} \) on the rectangle with vertices \((0, 0), (4, 0), (4, 1), \) and \((0, 1)\).

**Exercises 15.2 Double Integrals in Cylindrical Coordinates**

Suppose we have a surface given in cylindrical coordinates as \( z = f(r, \theta) \) and we wish to find the integral over some region. We could attempt to translate into rectangular coordinates and do the integration there, but it is often easier to stay in cylindrical coordinates.

How might we approximate the volume under such a surface in a way that uses cylindrical coordinates directly? The basic idea is the same as before: we divide the region into many small regions, multiply the area of each small region by the height of the surface somewhere in that little region, and add them up. \( \Delta r \) is the shape of the small regions; in order to have a nice representation in terms of \( r \) and \( \theta \), we use small pieces of ring-shaped areas, as shown in figure 15.2.1. Each small region is roughly rectangular, except that two sides are segments of a circle and the other two sides are not quite parallel. Near a point \((r, \theta)\), the length of either circular arc is about \( r \Delta \theta \) and the length of each straight side is simply \( \Delta r \). When \( r \Delta \theta \) and \( \Delta r \) are very small, the region is nearly a rectangle with area \( r \Delta r \Delta \theta \), and the volume under the surface is approximately

\[
\sum \sum f(r, \theta) r \Delta r \Delta \theta.
\]

**Figure 15.1.6** Intersection of three cylinders. (AP)

**Figure 15.2.1** A cylindrical coordinates “grid”.

**Example 15.2.1** Find the volume under \( z = \sqrt{4 - r^2} \) above the quarter circle bounded by the two axes and the circle \( x^2 + y^2 = 4 \) in the first quadrant.

In terms of \( r \) and \( \theta \), this region is described by the restrictions \( 0 \leq r \leq 2 \) and \( 0 \leq \theta \leq \pi/2 \); so we have

\[
\int_0^{\pi/2} \int_0^2 \frac{1}{2} (4 - r^2)^{1/2} r \, dr \, d\theta - \int_0^2 \frac{1}{2} (4 - r^2)^{1/2} r \, dr = \left[ \frac{\theta}{2} \right]_0^{\pi/2} - \left[ \frac{1}{2} \frac{\theta}{2} \right]_0^2 = \frac{\pi}{4}.
\]

The surface is a portion of the sphere of radius 2 centered at the origin, in fact exactly one-eighth of the sphere. We know the formula for volume of a sphere is \( \frac{4}{3} \pi r^3 \), so the volume we have computed is \( \frac{1}{8} \frac{4}{3} \pi r^3 = \frac{4\pi}{3} r \), in agreement with our answer.

This example is much like a simple one in rectangular coordinates: the region of interest may be described exactly by a constant range for each of the variables. As with rectangular coordinates, we can adapt the method to deal with more complicated regions.
EXAMPLE 15.2.2
Find the volume under \( z = \sqrt{r^2 + z^2} \) above the region enclosed by the curve \( r = 2 \cos \theta, \ -\pi/2 \leq \theta \leq \pi/2 \); see figure 15.2.2. The region is described in polar coordinates by the inequalities \(-\pi/2 \leq \theta \leq \pi/2\) and \(0 \leq r \leq 2 \cos \theta\), so the double integral is
\[
\int_{-\pi/2}^{\pi/2} \int_{0}^{2 \cos \theta} \sqrt{r^2 + z^2} \, r \, dr \, d\theta.
\]
We can rewrite the integral as shown because of the symmetry of the volume; this avoids a complication during the evaluation. Proceeding:
\[
2 \int_{0}^{\pi/2} \int_{0}^{2 \cos \theta} \sqrt{r^2 + z^2} \, r \, dr \, d\theta = 2 \int_{0}^{\pi/2} \int_{0}^{2 \cos \theta} \left(1 - \frac{1}{2} \left(\frac{x^2}{2} \right)^2 \right) \, d\theta
\]
\[
= 2 \int_{0}^{\pi/2} \left(8 \cos^2 \theta - 8 \cos \theta \frac{1}{3} \right)^{1/2} \, d\theta = \frac{8}{3} \sqrt{7}.
\]

Figure 15.2.2 Volume over a region with non-constant limits.

You might have learned a formula for computing areas in polar coordinates. It is possible to compute areas as volumes, so that you need only remember one technique. Consider the surface \( z = 1 \), a horizontal plane. The volume under this surface and above a region in the \( x \)-\( y \) plane is simply \( 1 \) (area of the region), so computing the volume really just computes the area of the region.

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14. Find the volume under \( z = x^2 + y^4 + 2 \) above the region \( x^2 + y^4 \leq 4 \) \( \Rightarrow \)

15. Find the volume between \( z = x^2 y^2 \) and \( z = 1 \) above the region \( x^2 + y^4 \leq 1 \) \( \Rightarrow \)

16. Find the volume inside \( x^2 + y^2 = 1 \) and \( x^2 + z^2 = 1 \) \( \Rightarrow \)

17. Find the volume under \( z = r \) above \( r^3 + \cos \theta \) \( \Rightarrow \)

18. Figure 15.2.4 shows the plot of \( r = 1 + 4 \sin \theta \).

Figure 15.2.4 \( r = 1 + 4 \sin \theta \)

a. Describe the behavior of the graph in terms of the given equation. Specifically, explain maximum and minimum values, number of leaves, and the leaves within leaves.

b. Give an integral or integrals to determine the area outside a smaller leaf but inside a larger leaf.

c. How would changing the value of \( a \) in the equation \( r = 1 + a \cos \phi \) change the relative sizes of the inner and outer leaves? Focus on values \( a \geq 1 \). (Hint: How would we change the maximum and minimum values?)

d. Consider the integral \( \int \frac{1}{\sqrt{1 - y^2}} \, dy \), where \( D \) is the unit disk centered at the origin. (See the graph here.)

a. Why might this integral be considered improper?

b. Calculate the value of the integral of the same function \( 1/\sqrt{1 - y^2} \) over the annulus with outer radius \( 1 \) and inner radius \( \lambda \).

c. Obtain a value for the integral on the whole disk by letting \( \lambda \) approach 0. \( \Rightarrow \)
d. For which values \( \lambda \) can we replace the denominator with \( (x^2 + y^2)^{1/2} \) in the original integral and still get a finite value for the improper integral?

15.3 Moment and Center of Mass

Using a single integral we were able to compute the center of mass for a one-dimensional object with variable density, and a two-dimensional object with constant density. With a double integral we can handle two dimensions and variable density.

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Just as before, the coordinates of the center of mass are
\[
\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M},
\]
where \( M \) is the total mass, \( M_x \) is the moment around the y-axis, and \( M_y \) is the moment around the x-axis. (You may want to review the concepts in section 9.6.)

The key to the computation, just as before, is the approximation of mass. In the two-dimensional case, we treat density \( \sigma \) as mass per square area, so when density is constant, mass is \( \text{(density)} \cdot \text{(area)} \). If we have a two-dimensional region with varying density given by \( \sigma(x, y) \), and we divide the region into small subregions with area \( \Delta A \), then the mass of one subregion is approximately \( \sigma(x, y) \Delta A \), the total mass is approximately the sum of these, and as usual the sum turns into an integral in the limit:
\[
M = \int_{A} \int_{A} \sigma(x, y) \, dy \, dx,
\]
and similarly for computations in cylindrical coordinates. Then as before,
\[
M_x = \int_{A} \int_{A} y \sigma(x, y) \, dy \, dx, \quad M_y = \int_{A} \int_{A} x \sigma(x, y) \, dy \, dx.
\]

EXAMPLE 15.3.1
Find the center of mass of a thin, uniform plate whose shape is the region between \( y = \cos x \) and the x-axis between \( x = -\pi/2 \) and \( x = \pi/2 \). Since the density is constant, we must take \( \sigma(x, y) = 1 \).

It is clear that \( \bar{x} = 0 \), but for practice let’s compute it anyway. First we compute the mass:
\[
M = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \cos x \, dx \, dy = \int_{-\pi/2}^{\pi/2} \cos x \, dx \big|_{-\pi/2}^{\pi/2} = 2.
\]
Next,
\[
M_y = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} y \cos x \, dx \, dy = \int_{-\pi/2}^{\pi/2} y \cos x \, dx \big|_{-\pi/2}^{\pi/2} = \frac{\pi}{4},
\]
Finally,
\[
M_x = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} x \cos x \, dx \, dy = \int_{-\pi/2}^{\pi/2} x \cos x \, dx = 0.
\]
So \( \bar{x} = 0 \) as expected, and \( y = \pi/4 \). This is the same problem as in example 9.6.4, it may be helpful to compare the two solutions.
EXAMPLE 15.3.2 Find the center of mass of a two-dimensional plate that occupies the quarter circle \( x^2 + y^2 \leq 1 \) in the first quadrant and has density \( \kappa(x^2 + y^2) \). It seems clear that because of the symmetry of both the region and the density function (both are important!), \( \bar{x} = \bar{y} \). We’ll do both to check our work.

Jumping right in:

\[
M = \int_{0}^{\pi/2} \int_{0}^{1} \kappa(x^2 + y^2) \, r \, dr \, d\theta = k \int_{0}^{\pi/2} \int_{0}^{1} r^2 \sqrt{1 - r^2} \, \left( 1 - \frac{r^2}{3} \right) \, dr = k\pi.
\]

This integral is something we can do, but it’s a bit unpleasant. Since everything in sight is related to a circle, let’s back up and try polar coordinates. Then \( x^2 + y^2 = r^2 \) and

\[
M = \int_{0}^{\pi/2} \int_{0}^{1} \kappa r^2 \, d\theta = k \int_{0}^{\pi/2} \frac{1}{2} \kappa \sin^2 \theta \, d\theta = k\pi.
\]

Much better. Note that \( y = r \sin \theta \),

\[
M_y = k \int_{0}^{\pi/2} \int_{0}^{1} r^3 \sin \theta \, d\theta = k \int_{0}^{\pi/2} \frac{1}{2} \kappa \cos^2 \theta \, d\theta = k\pi/2.
\]

Similarly,

\[
M_x = k \int_{0}^{\pi/2} \int_{0}^{1} r^3 \cos \theta \, d\theta = k \int_{0}^{\pi/2} \frac{1}{2} \kappa \sin^2 \theta \, d\theta = k\pi/2.
\]

Finally, \( \bar{x} = \bar{y} = \frac{8}{3\pi} \).

Exercises 15.3.

1. Find the center of mass of a two-dimensional plate that occupies the square \([0, 1] \times [0, 1]\) and has density function \( \kappa \).
2. Find the center of mass of a two-dimensional plate that occupies the triangle \(0 \leq x \leq 1, 0 \leq y \leq x\), and has density function \( \kappa(x, y) \).
3. Find the center of mass of a two-dimensional plate that occupies the upper unit semicircle centered at \((0, 0)\) and has density function \( \kappa(x, y) \).
4. Find the center of mass of a two-dimensional plate that occupies the upper unit semicircle centered at \((0, 0)\) and has density function \( \kappa(x, y) \).
5. Find the center of mass of a two-dimensional plate that occupies the triangle formed by \( x = 2, y = x, \) and \( y = 2x \) and has density function \( \kappa(x, y) \).
6. Find the center of mass of a two-dimensional plate that occupies the triangle formed by \( x = 0, y = x, \) and \( x + y = 6 \) and has density function \( \kappa(x, y) \).

15.4 Surface Area

Figure 15.4.1 Small parallelograms at points of tangency. (AP)

EXAMPLE 15.4.1 We find the area of the hemisphere \( z = \sqrt{1 - x^2 - y^2} \). We compute the derivatives

\[
f_x = \frac{-x}{\sqrt{1 - x^2 - y^2}}, \quad f_y = \frac{-y}{\sqrt{1 - x^2 - y^2}}\]

and then the area is

\[
\int_{x}^{1} \int_{y}^{1} \sqrt{1 - x^2 - y^2} + \frac{y^2}{1 - x^2 - y^2} + 1 \, dy \, dx.
\]

This is a bit on the messy side, but we can use polar coordinates:

\[
\int_{0}^{\pi/2} \int_{0}^{1} r \, dr \, d\theta.
\]

This integral is improper, since the function is undefined at the limit 1. We therefore compute

\[
\lim_{a \to 1^{-}} \int_{0}^{a} \frac{1}{\sqrt{1 - r^2}} \, dr = \lim_{a \to 1^{-}} -\sqrt{1 - r^2} + 1 = 1,
\]

using the substitution \( u = 1 - r^2 \). Then the area is

\[
\int_{0}^{\pi/2} 1 \, d\theta = 2\pi.
\]

You may recall that the area of a sphere of radius \( r \) is \( 4\pi r^2 \), so half the area of a unit sphere is \((1/2)4\pi r^2 = 2\pi r^2\), in agreement with our answer.

15.5 Triple Integrals

It will come as no surprise that we can also do triple integrals—integrals over a three- dimensional region. The simplest application allows us to compute volumes in an alternate way.

To approximate a volume in three dimensions, we can divide the three-dimensional region into small rectangular boxes, each \( \Delta x \times \Delta y \times \Delta z \) with volume \( \Delta x \Delta y \Delta z \). We then add them all up and take the limit, to get an integral:

\[
\int_{x}^{1} \int_{y}^{1} \int_{z}^{1} f(x, y, z) \, dx \, dy \, dz
\]

If the limits are constant, we are simply computing the volume of a rectangular box.

EXAMPLE 15.5.1 We use an integral to compute the volume of the box with opposite vertices at \((0, 0, 0)\) and \((1, 2, 3)\):

\[
\int_{0}^{1} \int_{0}^{2} \int_{0}^{3} f(x, y, z) \, dx \, dy \, dz = \int_{0}^{1} \int_{0}^{2} \int_{0}^{3} \, dx \, dy \, dz - \int_{0}^{1} \int_{0}^{2} \int_{0}^{3} 0 \, dx \, dy \, dz = \int_{0}^{1} \int_{0}^{2} \int_{0}^{3} \, dx \, dy \, dz = \int_{0}^{1} \int_{0}^{2} \int_{0}^{3} \, dx \, dy \, dz = \int_{0}^{1} 6 \, dx = 6.
\]

Of course, this is more interesting and useful when the limits are not constant.

EXAMPLE 15.5.2 Find the volume of the tetrahedron with corners at \((0, 0, 0)\), \((0, 3, 0)\), \((2, 3, 0)\), and \((2, 3, 5)\).
The whole problem comes down to correctly describing the region by inequalities: $0 \leq z \leq 2, \frac{3}{2} \leq y \leq 3, 0 \leq x \leq 5/2$. The lower $y$ limit comes from the equation of the line $y = 3x/2$ that forms one edge of the tetrahedron in the $x$-$y$ plane; the upper $z$ limit comes from the equation of the plane $z = 5x/2$ that forms the “upper” side of the tetrahedron; see figure 15.5.1. Now the volume is

\[
\iiint_{E} dz \, dy \, dx = \int_{0}^{3/2} \int_{2}^{3} \int_{0}^{2 \cos \alpha} r \, dr \, d\theta \, dz = \int_{0}^{3/2} \frac{3}{4} \sin^2 \alpha \, dz = \frac{3}{4} \int_{0}^{3/2} \sin^2 \alpha \, dz.
\]

Pretty much just the way we did for two dimensions we can use triple integration to compute mass, center of mass, and various average quantities.

**EXAMPLE 15.5.3** Suppose the temperature at a point is given by $T = xyz$. Find the average temperature in the cube with opposite corners at $(0,0,0)$ and $(2,2,2)$.

In two dimensions we add up the temperature at “each” point and divide by the area; here we add up the temperatures and divide by the volume, 8:

\[
\frac{1}{8} \int_{0}^{2} \int_{0}^{2} \int_{0}^{2} xyz \, dx \, dy \, dz = \frac{1}{8} \int_{0}^{2} \int_{0}^{2} \left( \frac{z}{4} \right) \, dy \, dz = \frac{1}{8} \int_{0}^{2} \left( \frac{z}{4} \right) \, dz = 1.
\]

**EXAMPLE 15.5.4** Suppose the density of an object is given by

\[\rho = \begin{cases} \frac{1}{2}, & \text{if } z \leq \frac{x}{2} + \frac{y}{2}, \\ 0, & \text{otherwise}. \end{cases}\]

Find the mass of a cube with edge length 2 and density equal to the square of the distance from the horizontal axis.

\[M = \iiint_{E} \rho \, dx \, dy \, dz = \iiint_{E} \left( \frac{1}{2} \right) \left( \sqrt{x^2 + y^2 + z^2} \right)^2 \, dx \, dy \, dz = \iiint_{E} \frac{1}{2} \left( x^2 + y^2 + z^2 \right) \, dx \, dy \, dz.
\]

\[= \frac{1}{2} \iiint_{E} \left( x^2 + y^2 + z^2 \right) \, dx \, dy \, dz = \frac{1}{2} \int_{0}^{2} \int_{0}^{2} \int_{0}^{2} \left( x^2 + y^2 + z^2 \right) \, dx \, dy \, dz = \frac{1}{2} \int_{0}^{2} \int_{0}^{2} \left( \frac{8}{3} + \frac{4}{3} + \frac{8}{3} \right) \, dx \, dy = \frac{16}{3}.
\]

Finally, the coordinates of the center of mass are $\bar{x} = M_{y\ell}/M = 1/3$, $\bar{y} = M_{x\ell}/M = 5/8$, and $\bar{z} = M_{x\ell}/M = 1/3$.

15.6 Cylindrical and Spherical Coordinates

We have seen that sometimes double integrals are simplified by doing them in polar coordinates; not surprisingly, triple integrals are sometimes simpler in cylindrical coordinates or spherical coordinates. To set up integrals in polar coordinates, we had to understand the shape and area of a typical small region into which the region of integration was divided. We needed to do the same thing here, for three dimensional regions.
EXAMPLE 15.6.3 Suppose the temperature at \((x, y, z)\) is \(T = 1/(1 + x^2 + y^2 + z^2)\). Find the average temperature in the unit sphere centered at the origin.

13. An object occupies the region inside the unit sphere at the origin, and has density equal to the square of the distance from the origin. Find the mass.

14. An object occupies the region between the unit sphere at the origin and a sphere of radius 2 with center at the origin, and has density equal to the distance from the origin. Find the mass.

15. An object occupies the region in the first octant bounded by the cones \(\phi = \pi/4\) and \(\phi = \pi/2\), and the sphere \(\rho = 4\sqrt{2}\), and has density proportional to the distance from the origin. Find the mass.

15.7 Change of Variables

One of the most useful techniques for evaluating integrals is substitution, both "u-substitution" and trigonometric substitution, in which we change the variable to something more convenient. As we have seen, sometimes changing from rectangular coordinates to another coordinate system is helpful, and this too changes the variables. This is certainly a more complicated change, since instead of changing one variable for another we change an entire suite of variables, but as it turns out it is really very similar to the kinds of change of variables we already know as substitution.

Let's examine the single variable case again, from a slightly different perspective than we have previously used. Suppose we start with the problem

\[
\int x^2 \sqrt{1-x^2} \, dx
\]

which computes the area in the left graph of figure 15.7.1. We use the substitution \(x = \sin u\) to transform the function from \(x^2 \sqrt{1-x^2}\) to \(\sin^3 u (1 - \sin^2 u)\), and we also convert \(dx\) to \(\cos u\, du\). Finally, we convert the limits 0 and 1 to \(0\) and \(\pi/2\). This transforms the integral to

\[
\int_0^1 x^2 \sqrt{1-x^2} \, dx = \int_{0}^{\pi/2} \sin^2 u \sqrt{1-\sin^2 u} \cos u \, du
\]

We want to notice that there are three different conversions: the main function, the differential \(dx\), and the interval of integration. The function is converted to \(\sin^2 u \sqrt{1-\sin^2 u}\),

\[
\frac{3}{4} \int_0^1 \int_0^{\pi/2} \int_{1/u}^{\sqrt{x^2+y^2}} \frac{1}{1+x^2+y^2} \, dy \, dx
\]

This looks quite messy, since everything in the problem is closely related to a sphere, we'll convert to spherical coordinates.

\[
\int_0^{2\pi} \int_0^{\pi/2} \int_0^{5} \sin \phi \, d\phi \, d\theta \, dr = \frac{3}{4} \left(\frac{1}{4}x^2+y^2\right) = \frac{3}{4} (x^2 - x^2 - 3) = \frac{3}{4}
\]

Exercises 15.6.

1. Evaluate \(\int_0^1 \int_{\sqrt{y}}^{1} \sqrt{x^2+y^2} \, dx \, dy\).

2. Evaluate \(\int_0^1 \int_0^{\pi/2} \int_0^{a} x^2 \sqrt{x^2+y^2} \, dy \, dx \, dz\).

3. Evaluate \(\int_0^1 \int_y^1 x^2 \sqrt{x^2+y^2} \, dx \, dy\).

4. Evaluate \(\int_0^1 \int_{\sqrt{y}}^{1} \sqrt{x^2+y^2} \, dx \, dy\).

5. Evaluate \(\int_0^1 \int_0^{\pi/2} \int_0^{1} \sin \phi \, d\phi \, d\theta \, dr\).

6. Evaluate \(\int_0^1 \int_0^{\pi/2} \int_0^{1} \sin \phi \, d\phi \, d\theta \, dr\).

7. Evaluate \(\int_0^1 \int_0^{\pi/2} \int_0^{a} x^2 + y^2 \, dx \, dy \, dz\).

8. Evaluate \(\int_0^1 \int_0^{\pi/2} \int_0^{a} x^2 + y^2 + z^2 \, dx \, dy \, dz\).

9. Evaluate \(\int_0^1 \int_0^{\pi/2} \int_0^{1} \sin \phi \, d\phi \, d\theta \, dr\).

10. Find the mass of a right circular cone of height \(a\) and base radius \(b\) if the density is proportional to the distance from the point \((0, 0, 0)\).

11. Find the mass of a right circular cone of height \(a\) and base radius \(b\) if the density is proportional to the distance from its axis of symmetry.

12. An object occupies the region inside the unit sphere at the origin, and has density equal to the distance from the \(z\)-axis. Find the mass.

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In two dimensions we add up the temperature at “each” point and divide by the area; here we add up the temperatures and divide by the volume; \((4/3)\pi\).

The limits correspond to integrating over the top half of a circular disk, and we recognize that the function will simplify in polar coordinates, so we would normally convert to polar coordinates.
coordinates: \[ \int_0^r \int_0^{\pi} r \, dr \, d\theta = \frac{\pi r^2}{2} \]

But let's instead approach this as a substitution problem, starting with \( x = r \cos \theta, y = r \sin \theta \). This pair of equations describes a function from \( r\theta \) space to \( xy \) space, and because it involves familiar concepts, it is not too hard to understand what it does. In figure 15.7.2 we have indicated geometrically a bit about how this function behaves. The four dots labeled \( a \) in the \( r\theta \) plane correspond to the three dots in the \( xy \) plane; dots \( a \) and \( b \) both go to the origin \( r = 0 \). The horizontal arrow in the \( r\theta \) plane has \( r = 1 \) everywhere and \( \theta \) ranges from 0 to \( \pi \), so the corresponding points \( x = r \cos \theta, y = r \sin \theta \) start at \((1,0)\) and follow the unit circle counter-clockwise. Finally, the vertical arrow has \( \theta = \pi/4 \) and \( r \) ranges from 0 to 1, so it maps to the straight arrow in the \( xy \) plane. Extrapolating from these few examples, it’s not hard to see that every vertical line in the \( r\theta \) plane is transformed to a line through the origin in the \( xy \) plane, and every horizontal line in the \( r\theta \) plane is transformed to a circle with center at the origin in the \( xy \) plane.

Since we are interested in integrating over the half-disc in the \( xy \) plane, we will integrate over the rectangle \([0, \pi] \times [0, r] \) in the \( r\theta \) plane, because we now see that the points in this rectangle are sent precisely to the upper half disc by \( x = r \cos \theta \) and \( y = r \sin \theta \).

Figure 15.7.2 Double change of variable.

At this point we are two-thirds done with the task: we know the \( r\theta \) limits of integration, and we can easily convert the function to the new variables:

\[ \sqrt{x^2 + y^2} = \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta} = r \sqrt{\cos^2 \theta + \sin^2 \theta} = r. \]

The final, and most difficult, task is to figure out what replaces \( dx \, dy \). (Of course, we actually know the answer, because we are in effect converting to polar coordinates. What we really want is a series of steps that gets to that right answer but that will also work for other substitutions that are not so familiar.)

Let’s take a step back and remember how integration arises from approximation. When we approximate the integral in the \( xy \) plane, we are computing the volumes of tall thin boxes, in this case boxes that are \( dx \times dy \times \sqrt{x^2 + y^2} \). We are aiming to come up with an integral in the \( r\theta \) plane that looks like this:

\[ \int_0^r \int_0^{\pi} r \, dr \, d\theta. \]

What we’re missing is exactly the right quantity to replace the \( \pi \) so that we get the correct answer. Of course, this integral is also the result of an approximation, in which we add up volumes of boxes that are \( \Delta r \times \Delta \theta \times \text{height} \); the problem is that the height that will give us the correct answer is not simply \( r \). Or put another way, we can think of the correct height as \( r \), but the area of the base \( \Delta r \Delta \theta \) as being wrong. The height \( r \) comes from equation 15.7.1, which is to say, it is precisely the same as the corresponding height in the \( x-y \) version of the integral. The problem is that the area of the base \( \Delta r \Delta \theta \) is not the same as the area of the base \( \Delta r \times \Delta \theta \). We can think of the \( \pi \) in the integral as a correction factor that is needed so that \( rd\theta \) to \( dx \, dy \).

So let’s think about what that little base \( \Delta \theta \times \Delta r \) corresponds to. We know that each bit of horizontal line in the \( r\theta \) plane corresponds to a bit of circular arc in the \( xy \) plane, and each bit of vertical line in the \( r\theta \) plane corresponds to a bit of “radial line” in the \( x-y \) plane. In figure 15.7.3 we show a typical rectangle in the \( r\theta \) plane and its corresponding area in the \( x-y \) plane.

Figure 15.7.3 Corresponding areas.

In this case, the region in the \( x-y \) plane is approximately a rectangle with dimensions \( \Delta r \times \Delta \theta \), but in general the corner angles will not be right angles, so the region will typically be (almost) a parallelogram. We need to compute the area of this parallelogram. We know a neat way to do this: compute the length of a certain cross product (page 315). If we can determine an appropriate two vectors we’ll be nearly done.

Fortunately, we’ve really done this before. The sides of the region in the \( x-y \) plane are formed by temporarily fixing either \( r \) or \( \theta \) and letting the other variable range over a small interval. In figure 15.7.4, for example, the upper right edge of the region is formed by fixing \( \theta = 2\pi/3 \) and letting \( r \) run from 0.5 to 0.75. In other words, we have a vector function \( \mathbf{v}(\theta) = (r \cos \theta, r \sin \theta) \), and we are interested in a restricted set of values for \( r \).

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A vector tangent to this path is given by the derivative \( \mathbf{v}'(\theta) = (\cos \theta, \sin \theta) \), and a small tangent vector, with length approximately equal to the side of the region, is \((\cos \theta, \sin \theta, 0) \, dr \). Likewise, if we fix \( r = r_0 = 0.5 \), we get the vector function \( \mathbf{w}(\theta) = (r_0 \cos \theta, r_0 \sin \theta, 0) \) with derivative \( \mathbf{w}'(\theta) = (-r_0 \sin \theta, r_0 \cos \theta, 0) \) and a small tangent vector \((-r_0 \sin \theta, r_0 \cos \theta, 0) \, d\theta \). The area of the parallelogram is not a particularly good approximation to the true area.

Figure 15.7.4 The approximating parallelogram.

The area of this parallelogram is the length of the cross product:

\[ \begin{vmatrix} -r_0 \sin \theta & -r_0 \cos \theta & 0 \\ r_0 \cos \theta & r_0 \sin \theta & 0 \\ 0 & 0 & 0 \end{vmatrix} \, d\theta \, dr \]

\[ = \begin{vmatrix} 0 \cos \theta & 0 \cos \theta & 0 \\ -r_0 \sin \theta & r_0 \cos \theta & 0 \\ 0 & 0 & 0 \end{vmatrix} \, dr \, d\theta \]

\[ = (0, 0, -r_0 \sin^2 \theta + r_0 \cos^2 \theta) \, dr \, d\theta \]

The length of this vector is \( r_0 \, dr \). So in general, for any values of \( r \) and \( \theta \), the area in the \( xy \) plane corresponding to a small rectangle anchored at \((0, r)\) in the \( r\theta \) plane is approximately \( r \, dr \). In other words, \( r \) replaces the \( \pi \) in equation 15.7.2.

In general, a substitution will start with equations \( x = f(u, v) \) and \( y = g(u, v) \). Again, it will be straightforward to convert the function being integrated. Converting the limits will require, as above, an understanding of just how the functions \( f \) and \( g \) transform the \( u-v \) plane into the \( xy \) plane. Finally, the small vectors we need to approximate an area will be \( \mathbf{f}_u \times \mathbf{f}_v \) (in \( du \, dv \)). The cross product of these is \((0, 0, f_u \cdot g_v - f_v \cdot g_u) \, du \, dv \), with length \( \sqrt{(f_u \cdot g_v - f_v \cdot g_u)^2} \). The quantity \( f_u \cdot g_v - f_v \cdot g_u \) is usually denoted \( \left. \frac{\partial(f, g)}{\partial(u, v)} \right|_{(x, y)} \).

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and called the Jacobian. Note that this is the absolute value of the two by two determinant

\[ \begin{vmatrix} f_u & g_u \\ f_v & g_v \end{vmatrix} \]

which may be easier to remember. (Confusingly, the matrix, the determinant of the matrix, and the absolute value of the determinant are all called the Jacobian by various authors.)

Because there are two things to worry about, namely, the form of the function and the region of integration, transformations in two (or more) variables are quite tricky to discover.

EXAMPLE 15.7.1 Integrate \( x^2 + y^2 \) over the region \( x^2 + y^2 \leq 2 \).

The equation \( x^2 + y^2 \leq 2 \) describes an ellipse as in figure 15.7.5; the region of integration is the interior of the ellipse. We will use the transformation \( x = \sqrt{2u - \sqrt{2}v} \) and \( y = \sqrt{2u + \sqrt{2}v} \). Substituting into the function itself we get \( f(x, y) = x^2 + y^2 \). The boundary of the ellipse is \( x^2 + y^2 = 2 \), so the boundary of the corresponding region in the \( u-v \) plane is \( 2u^2 + 2v^2 = 2 \); or \( u^2 + v^2 = 1 \), the unit circle, so this substitution makes the region of integration simpler.

Next, we compute the Jacobian, using \( f = \sqrt{2u - \sqrt{2}v} \) and \( g = \sqrt{2u + \sqrt{2}v} \):

\[ \begin{vmatrix} f_u & g_u \\ f_v & g_v \end{vmatrix} = \frac{\sqrt{2}uv}{\sqrt{2u^2 + \sqrt{2}v^2}} = \frac{4}{\sqrt{2u^2 + \sqrt{2}v^2}} \]

Hence the new integral is

\[ \int \int_R \left( \frac{2u^2 + 2v^2}{\sqrt{2u^2 + \sqrt{2}v^2}} \right) \frac{1}{4} \, du \, dv \]

where \( R \) is the interior of the unit circle. This is still not an easy integral, but it is easily transformed to polar coordinates, and then easily integrated.

Figure 15.7.5 \( x^2 + y^2 = 2 \)
There is a similar change of variables formula for triple integrals, though it is a bit more difficult to derive. Suppose we use three substitution functions, \( x = f(u, v, w), \)
\( y = g(u, v, w), \) and \( z = h(u, v, w). \) The Jacobian determinant is now

\[
\begin{vmatrix}
\frac{\partial (x, y, z)}{\partial (u, v, w)}
\end{vmatrix}
\]

Then the integral is transformed in a similar fashion:

\[
\int \int \int _{R} F(x, y, z) \, dV = \int \int \int _{S} F(f(u, v, w), g(u, v, w), h(u, v, w)) \left| \frac{\partial (x, y, z)}{\partial (u, v, w)} \right| \, du \, dv \, dw,
\]

where of course the region \( S \) in \( uvw \) space corresponds to the region \( R \) in \( xyz \) space.

**Exercises 15.7.**

1. Complete example 15.7.1 by converting to polar coordinates and evaluating the integral.
2. Evaluate \( \int \int xy \, dx \, dy \) over the square with corners \((0, 0), (1, 1), (1, -1)\) in two ways: directly, and using \( x = (u + v)/2, y = (u - v)/2. \)
3. Evaluate \( \int \int x^2 + y^2 \, dx \, dy \) over the square with corners \((-1, 0), (0, 1), (0, -1)\) in two ways: directly, and using \( x = (u + v)/2, y = (u - v)/2. \)
4. Evaluate \( \int \int (x + y)e^{-x+y} \, dx \, dy \) over the triangle with corners \((0, 0), (1, 1), (1, 0)\) in two ways: directly, and using \( x = (u + v)/2, y = (u - v)/2. \)
5. Evaluate \( \int \int y(x - y) \, dx \, dy \) over the parallelogram with corners \((0, 0), (3, 3), (7, 3), (4, 0)\) in two ways: directly, and using \( x = u + v, y = v. \)
6. Evaluate \( \int \int \sqrt{x^2 + y^2} \, dx \, dy \) over the triangle with corners \((0, 0), (4, 4), (4, 0)\) using \( x = u, y = uv. \)
7. Evaluate \( \int \int y \sin(xy) \, dx \, dy \) over the region bounded by \( xy = 1, \) \( xy = 4, \) and \( y = 4 \) using \( x = u/v, y = v. \)
8. Evaluate \( \int \int \sin(3x^2 + 4y^2) \, dx \, dy \) over the region in the first quadrant bounded by the ellipse \( 3x^2 + 4y^2 = 1. \)
9. Compute the Jacobian for the substitutions \( x = \rho \sin \phi \cos \theta, \) \( y = \rho \sin \phi \sin \theta, \) \( z = \rho \cos \phi. \)