14 Partial Differentiation

14.1 Functions of Several Variables

In single-variable calculus we were concerned with functions that map the real numbers \( \mathbb{R} \) to \( \mathbb{R} \), sometimes called “real functions of one variable”, meaning the “input” is a single real number and the “output” is likewise a single real number. In the last chapter we considered functions taking a real number to a vector, which may also be viewed as functions \( f: \mathbb{R} \rightarrow \mathbb{R}^3 \), that is, for each input value we get a position in space. Now we turn to functions of several variables, meaning several input variables, functions \( f: \mathbb{R}^n \rightarrow \mathbb{R} \). We will deal primarily with \( n = 2 \) and to a lesser extent \( n = 3 \), in fact many of the techniques we discuss can be applied to larger values of \( n \) as well.

A function \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \) maps a pair of values \((x, y)\) to a single real number. The three-dimensional coordinate system we have already used is a convenient way to visualize such functions: above each point \((x, y)\) in the \(xy\)-plane we graph the point \((x, y, z)\), where of course \(z = f(x, y)\).

EXAMPLE 14.1.1 Consider \( f(x, y) = 3x + 4y - 5 \). Writing this as \( z = 3x + 4y - 5 \) and then \( 3x + 4y - z = 5 \) we recognize the equation of a plane. In the form \( f(x, y) = 3x + 4y + 5 \) the emphasis has shifted: we now think of \( x \) and \( y \) as independent variables and \( z \) as a variable dependent on them, but the geometry is unchanged.

EXAMPLE 14.1.2 We have seen that \( x^2 + y^2 + z^2 = 4 \) represents a sphere of radius 2. We cannot write this in the form \( f(x, y) \), since for each \( x \) and \( y \) in the disk \( x^2 + y^2 < 4 \) there are two corresponding points on the sphere. As with the equation of a circle, we can resolve this equation into two functions: \( f(x, y) = \sqrt{4 - x^2 - y^2} \) and \( f(x, y) = -\sqrt{4 - x^2 - y^2} \), representing the upper and lower hemispheres. Each of these is an example of a function with a restricted domain: only certain values of \( x \) and \( y \) make sense (namely, those for which \( x^2 + y^2 \leq 4 \)) and the graphs of these functions are limited to a small region of the plane.

EXAMPLE 14.1.3 Consider \( f = \sqrt{x} + \sqrt{y} \). This function is defined only when both \( x \) and \( y \) are non-negative. When \( y = 0 \) we get \( f(x, y) = \sqrt{x} \) the familiar square root function in the \( xz\)-plane, and when \( x = 0 \) we get the same curve in the \( yz\)-plane. Generally speaking, we see that starting from \( f(0, 0) = 0 \) this function gets larger in every direction in roughly the same way that the square root function gets larger. For example, if we restrict attention to the line \( x = y \), we get \( f(x, y) = 2\sqrt{x} \) and along the line \( y = 2x \) we have \( f(x, y) = \sqrt{x} + \sqrt{2x} \) which is also a parabola, but it does not really “represent” the cross-section along \( y = kx \), because the cross-section has the line \( y = 4x \) where the horizontal axis should be.

A computer program that plots such surfaces can be very useful, as it is often difficult to get a good idea of what they look like. Still, it is valuable to be able to visualize relatively simple surfaces without such aids. As is the previous example, it is often a good idea to examine the function on restricted subsets of the plane, especially lines. It can also be useful to identify those points \((x, y)\) that share a common \( z\)-value.

EXAMPLE 14.1.4 Consider \( f(y, x) = x^2 + y^2 \). When \( x = 0 \) this becomes \( f = y^2 \), a parabola in the \( yz\)-plane: when \( y = 0 \) we get the “same” parabola \( f = x^2 \) in the \( xz\)-plane. Now consider the line \( y = kx \). If we simply replace \( y \) by \( kx \) we get \( f(x, y) = (1 + k^2)x^2 \), which is a parabola, but it does not really “represent” the cross-section along \( y = kx \), because the cross-section has the line \( y = 4x \) where the horizontal axis should be.

14.1 Functions of Several Variables

As in this example, the points \((x, y)\) such that \( f(x, y) = k \) usually form a curve, called a level curve of the function. A graph of some level curves can give a good idea of the shape of the surface; it looks much like a topographic map of the surface. In figure 14.1.2 both the surface and its associated level curves are shown. Note that, as with a topographic map, the heights corresponding to the level curves are evenly spaced, so that curves closer together the surface is steeper.

Functions \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \) behave much like functions of two variables; we will on occasion discuss functions of three variables. The principal difficulty with such functions is visualizing them, as they do not “live” in the three dimensions we are familiar with. For three variables there are various ways to interpret functions that make them easier to understand. For example, \( f(x, y, z) \) could represent the temperature at the point \((x, y, z)\), or the pressure, or the strength of a magnetic field. It remains useful to consider those points at which \( f(x, y, z) = k \), where \( k \) is some constant value. If \( f(x, y, z) \) is temperature, the set of points \((x, y, z)\) such that \( f(x, y, z) = k \) is the collection of points in space with temperature \( k \); in general this is called a level set; for three variables, a level set is typically a surface, called a level surface.

EXAMPLE 14.1.5 Suppose the temperature at \((x, y, z)\) is \( T(x, y, z) = e^{-x^2-y^2-\frac{1}{z^2}} \). This function has a maximum value of 1 at the origin, and tends to 0 in all directions. If \( k \) is positive and at most 1, the set of points for which \( T(x, y, z) = k \) is those points satisfying \( x^2 + y^2 + z^2 = -\ln k \), a sphere centered at the origin. The level surfaces are the concentric spheres centered at the origin.

Exercises 14.1.

1. Let \( f(x, y) = (x - y)^2 \). Determine the equations and shapes of the cross-sections when \( x = 0 \), \( y = 0 \), and describe the level curves. Use a three-dimensional graphing tool to graph the surface.

2. Let \( f(x, y) = |x| + |y| \). Determine the equations and shapes of the cross-sections when \( x = 0 \), \( y = 0 \), and describe the level curves. Use a three-dimensional graphing tool to graph the surface.

3. Let \( f(x, y) = e^{-x^2 - y^2} \sin(x^2 + y^2) \). Determine the equations and shapes of the cross-sections when \( x = 0 \), \( y = 0 \), \( x = y \), and describe the level curves. Use a three-dimensional graphing tool to graph the surface.

4. Let \( f(x, y) = \sin(x - y) \). Determine the equations and shapes of the cross-sections when \( x = 0 \), \( y = 0 \), \( x = y \), and describe the level curves. Use a three-dimensional graphing tool to graph the surface.

5. Let \( f(x, y) = (x^2 - y^2)^2 \). Determine the equations and shapes of the cross-sections when \( x = 0 \), \( y = 0 \), \( x = y \), and describe the level curves. Use a three-dimensional graphing tool to graph the surface.

6. Find the domain of each of the following functions of two variables:
   a. \( \sqrt{x^2 - x^2 - y^2} \)
   b. \( \arctan(x^2 + y^2 - 2) \)
   c. \( \sqrt{x^2 - x^2 - y^2} \)

7. Below are two sets of level curves. One is for a cone, one is for a paraboloid. Which is which? Explain.
14.2 Limits and Continuity

To develop calculus for functions of one variable, we needed to make sense of the concept of a limit, which we needed to understand continuous functions and to define the derivative. Limits involving functions of two variables can be considerably more difficult to deal with; fortunately, most of the functions we encounter are fairly easy to understand.

The potential difficulty is largely due to the fact that there are many ways to “approach” a point in the x-y plane. If we want to say that \[ \lim_{(x,y)\to(a,b)} f(x,y) = L, \] we need to capture the idea that as \((x, y)\) gets close to \((a, b)\) then \(f(x, y)\) gets close to \(L\). For functions of one variable, \(f(x)\), there are only two ways that \(x\) can approach \(a\): from the left or right. But there are an infinite number of ways to approach \((a, b)\), along any one of an infinite number of lines, or an infinite number of parabolas, or an infinite number of sine curves, and so on. We might hope that it’s really not so bad—suppose, for example, that along every possible line through \((a, b)\) the value of \(f(x, y)\) gets close to \(L\); surely this means that “\(f(x, y)\) approaches \(L\) as \((x, y)\) approaches \((a, b)\)”? Sadly, no.

**EXAMPLE 14.2.1** Consider \(f(x, y) = \sqrt{x^2 + y^2}\). When \(x = 0\) or \(y = 0\), \(f(x, y)\) is 0, so the limit of \(f(x, y)\) approaching the origin along either the x or y axis is 0. Moreover, along the line \(y = mx\) \(f(x, y) = m^2x^2/(x^2 + m^2y^2)\). As \(x\) approaches 0 this expression approaches 0 as well. So along every line through the origin \(f(x, y)\) approaches 0. Now suppose we approach the origin along \(x = y^2\). Then

\[
f(x, y) = y^2/2 + y^2 = 1/2,
\]

so the limit is 1/2. Looking at figure 14.2.1, it is apparent that there is a ridge above \(x = y^2\). Approaching the origin along a straight line, we go over the ridge and then drop down toward 0, but approaching along the ridge the height is a constant 1/2. Thus, there is no limit at \((0, 0)\).

Fortunately, we can define the concept of limit without needing to specify how a particular point is approached—indeed, in definition 2.3.2, we didn’t need the concept of “approach.” Roughly, that definition says that when \(x\) is close to \(a\) then \(f(x)\) is close to \(L\); there is no mention of “how” we get close to \(a\). We can adapt that definition to two variables quite easily:

**DEFINITION 14.2.2 Limit** Suppose \(f(x, y)\) is a function. We say that

\[
\lim_{(x,y)\to(a,b)} f(x,y) = L
\]

if for every \(\varepsilon > 0\) there is a \(\delta > 0\) so that whenever \(0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta\), \(|f(x, y) - L| < \varepsilon\).

We want to force this to be less than \(\varepsilon\) by picking \(\delta\) “small enough.” If we choose \(\delta = \varepsilon/3\) then

\[
|4\sqrt{2}y|/|x^2 + y^2| < 1/3. \Rightarrow \varepsilon = \varepsilon.
\]

Recall that a function \(f(x)\) is continuous at \(x = a\) if \(\lim_{x\to a} f(x) = f(a)\); roughly this says that there is no “hole” or “jump” at \(x = a\). We can say exactly the same thing about a function of two variables.

**DEFINITION 14.2.4** \(f(x, y)\) is continuous at \((a, b)\) if \(\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)\).

**EXAMPLE 14.2.5** The function \(f(x, y) = 3x^2y/(x^2 + y^2)\) is not continuous at \((0, 0)\), because \(f(0,0)\) is not defined. However, we know that \(\lim_{(x,y)\to(0,0)} f(x,y) = 0\), so we can easily “fix” the problem, by extending the definition of \(f\) so that \(f(0,0) = 0\). This surface is shown in figure 14.2.2.

![Figure 14.2.2](image)

**EXAMPLE 14.2.3** We show that \(\lim_{(x,y)\to(0,0)} x^2/(x^2 + y^2) = 0\). Suppose \(\varepsilon > 0\). Then

\[
|x^2/(x^2 + y^2)| = \frac{x^2}{x^2 + y^2} < \frac{\varepsilon}{1 + \varepsilon}
\]

Note that \(x^2/(x^2 + y^2) \leq 1\) and \(|y| = \sqrt{x^2 + y^2} < \delta\). So

\[
\frac{x^2}{x^2 + y^2} \frac{\varepsilon}{1 + \varepsilon}\]

This says that we can make \(|f(x, y) - 0| < \varepsilon\), no matter how small \(\varepsilon\) is, by making the distance from \((x, y)\) to \((0, 0)\) “small enough”.

**EXERCISES 14.2**

Determine whether each limit exists. If it does, find the limit and prove that it is the limit; if it does not, explain how you know.

1. \(\lim_{(x,y)\to(0,0)} x^2 + y^2\)
2. \(\lim_{(x,y)\to(0,0)} xy\)
3. \(\lim_{(x,y)\to(0,0)} x^2 + y^2\)
4. \(\lim_{(x,y)\to(0,0)} xy\)
5. \(\lim_{(x,y)\to(0,0)} \sin(x^2 + y^2)/x^2 + y^2\)
6. \(\lim_{(x,y)\to(0,0)} \sqrt{x^2 + y^2}\)
7. \(\lim_{(x,y)\to(0,0)} x^2 + y^2\)
8. \(\lim_{(x,y)\to(0,0)} x^2 + y^2\)
9. \(\lim_{(x,y)\to(0,0)} x^2 + \sin^2 y\)
10. \(\lim_{(x,y)\to(0,0)} (x - 1)^2\ln x\)
11. \(\lim_{(x,y)\to(0,0)} (x - 1)^2 + y^2\)
12. \(\lim_{(x,y)\to(0,0)} x^2 + y^2\)
13. Does the function \( f(x,y) = \frac{x - y}{1 + x^2 + y^2} \) have any discontinuities? What about \( f(x,y) = \frac{z - y}{1 + x^2 + y^2} \)? Explain.

### 14.3 Partial Differentiation

When first considered what the derivative of a vector function might mean, there was really not much difficulty in understanding either how such a thing might be computed or what it might measure. In the case of functions of two variables, things are a bit harder to understand. If we think of a function of two variables in terms of its graph, a surface, there is a more-or-less obvious derivative-like question we might ask, namely, how "steep" is the surface. But it's not clear that this has a simple answer, nor how we might proceed. We will start with what seem to be very small steps toward the goal, surprisingly, it turns out that these simple ideas hold the keys to a more general understanding.

#### Example 14.3.1

The partial derivative with respect to \( x \) of the function \( f(x,y) = x^2 + y^2 \) is

\[
\frac{\partial f}{\partial x} = 2x. 
\]

The partial derivative with respect to \( y \) is

\[
\frac{\partial f}{\partial y} = 2y. 
\]

The gradient vector of \( f(x,y) \) at \( (a,b) \) is

\[
\nabla f(a,b) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2a, 2b). 
\]

This gives the rate of change of \( f(x,y) \) along the tangent line to the level curve \( f(x,y) = k \) at \( (a,b) \). The equation of this tangent line is

\[
y - b = 2y(y - b). 
\]

Let's start by looking at some particularly easy lines: those parallel to the \( x \) or \( y \) axis. Suppose we are interested in the cross-section of \( f(x,y) \) above the line \( y = b \). If we substitute \( b \) for \( y \) in \( f(x,y) \), we get a function in one variable, describing the height of the cross-section as a function of \( x \). Because \( y = b \) is parallel to the \( x \)-axis, if we view it from a vantage point on the negative \( y \)-axis, we will see what appears to be simply an ordinary curve in the \( x \)-\( z \) plane.

Consider again the parabolic surface \( f(x,y) = x^2 + y^2 \). The cross-section above the line \( y = 2 \) consists of all points \((x, 2, x^2 + 4)\). Looking at this cross-section from somewhere on the negative \( y \)-axis, we see what appears to be just the curve \( f(x) = x^2 + 4 \). At any point on the cross-section, \((x, 2, x^2 + 4)\), the steepness of the surface in the direction of the line \( y = 2 \) is simply the slope of the curve \( f(x) = x^2 + 4 \), namely 2x.

Example 14.3.2

The partial derivative with respect to \( y \) of \( f(x,y) = \sin(xy) + 3xy \) is

\[
\frac{\partial f}{\partial y} = \cos(xy) + 3x. 
\]

So far, using no new techniques, we have succeeded in measuring the slope of a surface in two quite special directions. For functions of one variable, the derivative is closely linked to the notion of tangent line. For surfaces, the analogous idea is the tangent plane—a plane that just touches a surface at a point, and has the same "steepness" as the surface in all directions. Even though we haven't yet figured out how to compute the slope in all directions, we have enough information to find tangent planes. Suppose we want the plane tangent to a surface at a particular point \( (a,b,c) \). If we compute the two partial derivatives of the function for that point, we get enough information to determine two lines tangent to the surface, both through \( (a,b,c) \) and both tangent to the surface in their respective directions. These two lines determine a plane, that is, there is exactly one plane containing the two lines: the tangent plane. Figure 14.3.3 shows (part of) two tangent lines at a point, and the tangent plane containing them.

How can we discover an equation for this tangent plane? We know a point on the plane, \( (a,b,c) \), we need a vector normal to the plane. If we can find two vectors, one-parallel to each of the tangent lines we know how to find, then the cross product of those vectors will give the desired normal vector.

Example 14.3.4

A tangent vector.

How can we find vectors parallel to the tangent lines? Consider first the line tangent to the surface above the line \( y = b \). A vector \( \langle u, v, w \rangle \) parallel to this tangent line must have \( y \)-component \( v = 0 \), and we may as well take the \( x \)-component to be \( u = 1 \). The ratio
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right side does the same, because as \((x,y)\) approaches \((x_0,y_0)\), \(c_1\) approaches 0. Essentially the same calculation works for \(f_y\).

Almost all of the functions we will encounter are differentiable at points we will be interested in, and often at all points. This is usually because the functions satisfy the hypotheses of this theorem.

**Example 14.3.3** Find the plane tangent to \(x^2 + y^2 + z^2 = 4\) at \((1,1,\sqrt{2})\). This point is on the upper hemisphere, so we use \(f(x,y) = \sqrt{4-x^2-y^2}\). Then \(f_x(x,y) = -x(4-x^2-y^2)^{-1/2}\) and \(f_y(x,y) = -y(4-x^2-y^2)^{-1/2}\); so \(f_x(1,1) = f_y(1,1) = -1/\sqrt{2}\) and the equation is

\[
z = \frac{1}{\sqrt{2}}(x-1) - \frac{1}{\sqrt{2}}(y-1) + \sqrt{2}
\]

The hemisphere and this tangent plane are pictured in figure 14.3.3.

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### 14.4 The Chain Rule

Consider the surface \(z = x^2y + y^2z\), and suppose that \(x = 2+ t^4 \) and \(y = 1 - t^2\). We can think of the latter two equations as describing how \(x\) and \(y\) change relative to, say, time. Then

\[
z = x^2y + y^2z = (2 + t^4)^2(1 - t^2) + (2 + t^4)(1 - t^2)^2
\]

we’ll tell us explicitly how the \(z\) coordinate of the corresponding point on the surface depends on \(t\). If we want to know \(dz/dt\) we can compute it more or less directly—it’s actually a bit simpler to use the chain rule:

\[
dz/dt = x^2y' + 2xy'z + y^2z' = (2 + t^4)(4t^3) + 2t^4(1 - t^2)^2(4t)
\]

whereas the result would be shorter if we treated the variables not being differentiated as constants.

**Proof.** If \(f\) is differentiable, then

\[
\Delta z = f(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + c_1\Delta x + c_2\Delta y,
\]

where \(c_1\) and \(c_2\) approach 0 as \((x,y)\) approaches \((x_0,y_0)\). This definition takes a bit of absorbing. Let’s rewrite the central equation a bit: \(z = f(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0) + c_1\Delta x + c_2\Delta y\).

The first three terms on the right are the equation of the tangent plane, that is,

\[
f(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + f(x_0,y_0).
\]

The second is the value of the point on the plane above \((x,y)\). Equation 14.3.1 says that the \(z\)-value of a point on the surface is equal to the \(z\)-value of a point on the plane plus a “little bit,” namely \(c_1\Delta x + c_2\Delta y\). As \((x,y)\) approaches \((x_0,y_0)\), both \(\Delta x\) and \(\Delta y\) approach 0, so this little bit \(c_1\Delta x + c_2\Delta y\) also approaches 0, and the \(z\)-values on the surface and the plane get close to each other. But that by itself is not very interesting: since the surface and the plane both contain the point \((x_0,y_0,z_0)\), the \(z\) value will approach \(z_0\) and hence get close to each other whether the tangent plane is “tangent” to the surface or not. The extra condition in the definition says that as \((x,y)\) approaches \((x_0,y_0)\), the \(\epsilon\) value approaches 0—this means that \(c_1\Delta x + c_2\Delta y\) approaches 0 much, much faster, because \(c_1\Delta x\) is much smaller than either \(c_1\) or \(\Delta x\). It is this extra condition that makes the plane a tangent plane.

We can see that the extra condition on \(c_1\) and \(c_2\) is just what is needed if we look at partial derivatives. Suppose we temporarily fix \(y = y_0\), so \(\Delta y = 0\). Then the equation from the definition becomes

\[
\Delta z = f(x_0,y_0)\Delta x + c_1\Delta x
\]

or

\[
\Delta z = f(x_0,y_0) + c_1\Delta x
\]

Noting the limit of the two sides as \(\Delta x\) approaches 0, the left side turns into the partial derivative of \(z\) with respect to \(x\) at \((x_0,y_0)\), or in other words \(f_x(x_0,y_0)\), and the
We can write the chain rule in a way that is somewhat closer to the single variable chain rule:

\[ \frac{df}{dt} = (f_x, f_y) \cdot (x', y'). \]

or (roughly) the derivatives of the outside function "times" the derivatives of the inside functions.

Not surprisingly, essentially the same chain rule works for functions of more than two variables, for example, given a function of three variables \( f(x, y, z) \), where each of \( x, y \) and \( z \) is a function of \( t \):

\[ \frac{df}{dt} = (f_x, f_y, f_z) \cdot (x', y', z'). \]

We can even extend the idea further. Suppose that \( f(x, y) \) is a function and \( x = g(s, t) \) and \( y = h(s, t) \) are functions of two variables \( s \) and \( t \). Then \( f \) is "really" a function of \( s \) and \( t \) as well, and

\[ \frac{df}{ds} = f_x g_s + f_y h_s, \quad \frac{df}{dt} = f_x g_t + f_y h_t. \]

The natural extension of this to \( f(x, y, z) \) works as well.

Recall that we used the ordinary chain rule to do implicit differentiation. We can do the same with the new chain rule.

**EXAMPLE 14.4.2**

\( x^2 + y^2 + z^2 = 4 \) defines a sphere, which is not a function of \( x \) and \( y \), though it can be thought of as two functions, the top and bottom hemispheres. We can think of \( z \) as one of these two functions, so really \( z = z(x, y) \), and we can think of \( x \) and \( y \) as particularly simple functions of \( x \) and \( y \), and let \( f(x, y, z) = x^2 + y^2 + z^2 \). Since

\( f(x, y, z) = 4, \frac{df}{dx} = 0 \), but using the chain rule:

\[ 0 = \frac{df}{dx} = \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y + \frac{\partial f}{\partial z} z \]

\[ = (2x)(1) + (2y)(0) + (2z)(\frac{\partial z}{\partial x}) \]

noting that \( z \) is temporarily held constant its derivative \( \frac{\partial z}{\partial x} = 0 \). Now we can solve for \( \frac{df}{dx} \):

\[ \frac{\partial z}{\partial x} = \frac{2x}{2z} = \frac{x}{z} \]

In a similar manner we can compute \( \frac{df}{dy} \).

### 14-5 Directional Derivatives

So we need to somehow "mark off" units on the line, and we need a convenient way to refer to the line in calculations. It turns out that we can accomplish both by using the vector form of a line. Suppose that \( u \) is a unit vector \( \langle x_u, y_u \rangle \) in the direction of interest. A vector equation for the line through \((x_0, y_0)\) in this direction is \( g(v) = \langle u_1 x + u_2 y + y_0 \rangle \). The height of the surface above the point \((u_1 x + u_2 y + y_0)\) is \( g'(v) = f(u_1 x + u_2 y + y_0) \).

Because \( u \) is a unit vector, the value of \( v \) is precisely the distance along the line from \((x_0, y_0)\) to \((u_1 x + u_2 y + y_0)\); this means that the line is effectively a \( t \) axis, with origin at the point \((x_0, y_0)\), so the slope we seek is \( g'(0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \langle 1, 0 \rangle = f_x(x_0, y_0) / u_x \) or \( -\nabla f \cdot u \).

Here we have used the chain rule and the derivative \( \frac{d}{dt} (u_1 x + u_2 y + y_0) = u_1 \) and \( \frac{d}{dt} (u_1 x + u_2 y + y_0) = u_2 \).

The vector \( \langle f_x, f_y \rangle \) is very useful, so it has its own symbol, \( \nabla f \), pronounced "del \( f \);" it is also called the gradient of \( f \).

**EXAMPLE 14.5.1** Find the slope of \( z = x^2 + y^2 \) at \((1, 2)\) in the direction of the vector \((3, 4)\).

We first compute the gradient at \((1, 2)\): \( \nabla f = \langle 2x, 2y \rangle \), which is \((2, 4)\) at \((1, 2)\). A unit vector in the desired direction is \((3/5, 4/5)\), and the desired slope is then \((2, 4) \cdot (3/5, 4/5) = 6/5 + 16/5 = 22/5\).

**EXAMPLE 14.5.2** Find a tangent vector to \( z = x^2 + y^2 \) at \((1, 2)\) in the direction of the vector \((3, 4)\) and show that it is parallel to the tangent plane at that point.

Since \((3/5, 4/5)\) is a unit vector in the desired direction, we can easily expand it to a tangent vector simply by adding the third coordinate computed in the previous example: \((3/5, 4/5, 22/5)\). To see that this vector is parallel to the tangent plane, we compute its dot product with a normal to the plane. We know that a normal to the tangent plane is \( \langle f_x, f_y, -1 \rangle = (2, 4, -1) \), and the dot product is \((2, 4, -1) \cdot (3/5, 4/5, 22/5) = 6/5 + 16/5 - 22/5 = 0 \), so the two vectors are perpendicular. (Note that the vector normal to the surface, namely \( \langle f_x, f_y, -1 \rangle \), is simply the gradient with a \(-1\) tacked on as the third component.)

The slope of a surface given by \( z = f(x, y) \) in the direction of a (two-dimensional) vector \( u \) is called the directional derivative of \( f \), written \( D_u f \). The directional derivative immediately provides us with some additional information. We know that

\[ D_u f = \nabla f \cdot u = \nabla f \cdot \langle u_x, u_y \rangle \cos \theta = \langle \nabla f \rangle \cdot \langle 
abla f \rangle \cos \theta \]

if \( u \) is a unit vector; \( \theta \) is the angle between \( \nabla f \) and \( u \) and this tells us immediately that the largest value of \( D_u f \) occurs when \( \cos \theta = 1 \), namely, when \( \theta = 0 \), so \( \nabla f \) is parallel to \( u \). In other words, the gradient \( \nabla f \) points at the direction of steepest ascent of the surface, and \( -\nabla f \) in the direction of descent. Likewise, the smallest value of \( D_u f \) occurs when \( \cos \theta = -1 \), namely, when \( \theta = \pi \) so \( \nabla f \) is anti-parallel to \( u \). In other words, \( -\nabla f \) points in the direction of steepest descent of the surface, and \( -\nabla f \) is the slope in that direction.

**EXAMPLE 14.5.3** Investigate the direction of steepest ascent and descent for \( z = x^2 + y^2 \).

The gradient is \((2x, 2y) = (2, 2)(x, y)\); this is a vector parallel to the vector \((x, y)\), so the steepest direction is directly away from the origin, starting at the point \((x, y)\). The direction of steepest descent is thus directly toward the origin from \((x, y)\). Note that at \((0, 0)\) the gradient vector is \((0, 0)\), which has no direction, and it is clear from the plot of this surface that there is a minimum point at the origin, and tangent vectors in all directions are parallel to the \(x\)-axis.

If \( f \) is perpendicular to \( u \), \( D_u f = \langle \nabla f \rangle \cdot \langle \nabla f \rangle / 2 = 0 \), since \( \cos(\pi/2) = 0 \). This means that in either of the two directions perpendicular to \( \nabla f \), the slope of the surface is 0; this implies that a vector in either of those directions is tangent to the level curve at that point. Starting with \( f(x, y) = f_x(x_0, y_0) \), it is easy to find a vector perpendicular to it: either \( \langle f_y(x_0, y_0), -f_x(x_0, y_0) \rangle \). If \( f(x, y, z) \) is a function of three variables, all the calculations proceed in essentially the same way. The rate at which \( f \) changes in a particular direction is \( \nabla f \cdot u \), where now \( \nabla f = \langle f_x, f_y, f_z \rangle \) and \( u = \langle u_x, u_y, u_z \rangle \) is a unit vector. Again \( \nabla f \) points in the direction of maximum rate of increase, \( -\nabla f \) points in the direction of maximum rate of decrease, and any vector perpendicular to \( \nabla f \) is tangent to the level surface \( f(x, y, z) = k \) at the point in question. Of course there are no longer just two such vectors; the vectors perpendicular to \( \nabla f \) describe the tangent plane to the level surface, or in other words \( \nabla f \) is normal to the tangent plane.

**EXAMPLE 14.5.4** Suppose the temperature at a point in space is given by \( f(x, y, z) = T_0 / (1 + x^2 + y^2 + z^2) \); at the origin the temperature in Kelvin is \( T_0 > 0 \), and it decreases in every direction from there. It might be, for example, that there is a source of heat at the...
origin, and as we get further from the source, the temperature decreases. The gradient is
\[ \nabla T = \frac{-2T_x x}{(1 + x^2 + y^2 + z^2)^2} + \frac{-2T_y y}{(1 + x^2 + y^2 + z^2)^2} + \frac{-2T_z z}{(1 + x^2 + y^2 + z^2)^2} \]
and its components are
\[ x, y, z. \]
The gradient points directly at the origin from the point \((x, y, z)\) by moving directly toward the heat source, we increase the temperature as quickly as possible.

**EXAMPLE 14.5.5** Find the points on the surface defined by \(x^2 + 2y^2 + 3z^2 = 1\) where the tangent plane is parallel to the plane defined by \(3x - y - 3z = 1\).

Two planes are parallel if their normals are parallel or anti-parallel, so we want to find the points on the surface with normal parallel or anti-parallel to \((3, -1, -3)\). Let \(f = x^2 + 2y^2 + 3z^2\), the gradient of \(f\) is normal to the level surface at every point, so we are looking for a gradient parallel or anti-parallel to \((3, -1, -3)\). The gradient is \((2x, 4y, 6z)\); if it is parallel or anti-parallel to \((3, -1, -3)\), then
\[ (2x, 4y, 6z) = \pm (3, -1, -3) \]
for some \(k\). This means we need a solution to the equations
\[ 2x = 3k \quad 4y = -k \quad 6z = 3k \]
but this is three equations in four unknowns—we need another equation. What we haven't used so far is that the points we seek are on the surface \(x^2 + 2y^2 + 3z^2 = 1\); this is the fourth equation. If we solve the first three equations for \(x, y, \) and \(z\) and substitute into the fourth equation we get
\[ 1 = \left( \frac{3k}{7} \right)^2 + \left( \frac{-k}{7} \right)^2 + \left( \frac{3k}{7} \right)^2 \]
\[ = \frac{9}{7} \quad \frac{2}{7} \quad \frac{3}{7} \]
\[ = \frac{36k^2}{49} \]
so \(k = \pm \frac{7}{6}\). The desired points are \(\left( \frac{7k}{6}, \frac{-k}{6}, \frac{3k}{6} \right)\) and \(\left( \frac{-7k}{6}, \frac{k}{6}, \frac{-3k}{6} \right)\). The ellipsoid and the three planes are shown in figure 14.5.1.

### 14.6 Higher order derivatives

**Exercise 14.5.**

1. Find \(D^3 f\) for \(f = x^2 + x^3 y + y^2\) in the direction of \(v = (2, 1)\) at the point \((1, 1)\).
2. Find \(D^4 f\) for \(f = \sin(xy)\) in the direction of \(v = (-1, -1)\) at the point \((3, 1)\).
3. Find \(D^3 f\) for \(f = e^x \cos(y)\) in the direction 30 degrees from the positive x axis at the point \((1, \pi/4)\).
4. The temperature of a thin plate in the \(xy\) plane is \(T = x^2 + y^2\). How fast does temperature change at the point \((1, 1, 1)\) moving in a direction 30 degrees from the positive x axis? Answer: \(\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} \times T\).
5. Suppose the density of a thin plate at \((x, y)\) is \(\sqrt{x^2 + y^2}\). Find the rate of change of the density at \((2, 1)\) in a direction \(\pi/3\) radians from the positive x axis.
6. Suppose the electric potential at \((x, y)\) is \(\ln(x^2 + y^2)\). Find the rate of change of the potential at the point \((1, 1)\) moving in a direction 45 degrees from the positive x axis.
7. A plane perpendicular to the \(xy\) plane contains the point \((2, 1, 8)\) on the paraboloid \(z = x^2 + 4y^2\). The cross-section of the paraboloid created by this plane has slope 0 at this point. Find an equation of the plane.
8. A plane perpendicular to the \(xy\) plane contains the point \((3, 2, 2)\) on the paraboloid \(z = x^2 + 4y^2\). The cross-section of the paraboloid created by this plane has slope 0 at this point. Find an equation of the plane.
9. Suppose the temperature at \((x, y, z)\) is given by \(T = x^2 + \sin(yz)\). In what direction should you go from the point \((1, 1, 1)\) to decrease the temperature as quickly as possible? What is the rate of change of temperature in this direction?

**THEOREM 14.6.2 Clairaut's Theorem**  If the mixed partial derivatives are continuous, they are equal.

### 14.6.1 Compute all four second derivatives of \(f(x, y) = x^2 y^2\).

Using an obvious notation, we get:
\[ f_{xx} = 2y^2 \quad f_{yy} = 4x^2 \quad f_{xy} = 4xy \quad f_{yx} = 2x^2. \]

You will have noticed that two of these are the same, the “mixed partials” computed by taking partial derivatives with respect to both variables in the two possible orders. This is not an accident—as long as the function is reasonably nice, this will always be true.

### 14.6.2 Clairaut’s Theorem

If the mixed partial derivatives are continuous, they are equal.

**Exercise 14.4.6**

1. Find first and second partial derivatives of \(f = x^2 y^3 + x^2 y^3\).
2. Find first and second partial derivatives of \(z^3 y^2 + x^2 z^2\).
3. Find all first and second partial derivatives of \(4x^4 + 3y^5 + 2z\).
4. Find all first and second partial derivatives of \(x^2 y^2 + z^3\).
5. Find all first and second partial derivatives of \(\sin(2x) \cos(2y)\).
6. Find all first and second partial derivatives of \(e^{x^2 + y^2}\).
7. Find all first and second partial derivatives of \(\sqrt{x^2 + y^2}\).
8. Find all first and second partial derivatives of \(z^3 y^2 + x^2 z^2\).
9. Find all first and second partial derivatives of \(x + y + z\) with respect to \(x, y\), and \(z\) if \(x^2 + 4y^2 + 16z^2 - 64 = 0\).
10. Let \(u\) and \(v\) be constants. Prove that the function \(v(x, y) = x^2 + y^2 + \sin(x^2)\) is a solution to the heat equation \(\nabla^2 u + \nabla^2 v = 0\).
11. Let \(u\) be a constant. Prove that \(v(x, y) = \sin(x^2) + \ln(x^2) + e^{-x^2} + \sin(x^2)\) is a solution to the wave equation \(\nabla^2 u + \nabla^2 v = 0\).

12. How many third-order derivatives does a function of 2 variables have? How many of these are distinct?
MAXIMA AND MINIMA

Suppose a surface given by \( f(x, y) \) has a local maximum at \((x_0, y_0, z_0)\); geometrically, this point on the surface looks like the top of a hill. If we look at the cross-section in the plane \( y = y_0 \), we will see a local maximum on the curve at \((x_0, z_0)\), and we know from single-variable calculus that \( \frac{dz}{dx} = 0 \) at this point. Likewise, in the plane \( x = x_0 \), both partial derivatives at the point must be zero, and likewise for a local minimum. Thus, to find local maximum and minimum points, we need only consider those points at which both partial derivatives are 0. As in the single-variable case, it is possible for the derivatives to be 0 at a point that is neither a maximum or a minimum, so we need to test these points further.

You will recall that in the single variable case, we examined three methods to identify maximum and minimum points: the most useful is the second derivative test, though it does not always work. For functions of two variables there is also a second derivative test; again it is by far the most useful test, though it doesn’t always work:

**THEOREM 14.7.1** Suppose that the second partial derivatives of \( f(x, y) \) are continuous near \((x_0, y_0)\), and \( f_x(x_0, y_0) = f_y(x_0, y_0) = 0 \). We denote by \( D \) the discriminant:

\[
D(x_0, y_0) = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2.
\]

If \( D > 0 \):
- If \( f_{xx}(x_0, y_0) > 0 \) there is a local maximum at \((x_0, y_0)\);
- If \( f_{xx}(x_0, y_0) < 0 \) there is a local minimum at \((x_0, y_0)\).

If \( D < 0 \): there is neither a maximum nor a minimum at \((x_0, y_0)\).

If \( D = 0 \): the test fails.

**EXAMPLE 14.7.2** Verify that \( f(x, y) = x^2 + y^2 \) has a minimum at \((0, 0)\).

First, we compute all the needed derivatives:

\[
f_x = 2x, \quad f_y = 2y, \quad f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 0.
\]

The discriminant:

\[
D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 0 > 0.
\]

so we get no information. However, in this case it is easy to see that there is a minimum at \((0, 0)\), because \( f(0, 0) = 0 \) and at all other points \( f_x(x, y) = 2x \) and \( f_y = 2y \), so the discriminant is \( D = 0 \).

**EXAMPLE 14.7.5** Find all local maxima and minima for \( f(x, y) = x^2 + y^3 \).

The derivatives:

\[
f_x = 2x, \quad f_y = 3y^2, \quad f_{xx} = 2, \quad f_{yy} = 6y, \quad f_{xy} = 0.
\]

Again there is a single critical point, at \((0, 0)\), and

\[
D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 0 \cdot 0 = 0,
\]

so we get no information. In this case, a little thought shows there is neither a maximum nor a minimum at \((0, 0)\): when \( x \) and \( y \) are both negative, \( f(x, y) > 0 \), and when \( x = y = 0 \), \( f(x, y) = 0 \), and there are points of both kinds arbitrarily close to \((0, 0)\). Alternatively, if we look at the cross-section when \( y = 0 \), we get \( f(x, 0) = x^2 \), which does not have either a maximum or minimum at \( x = 0 \).

**EXAMPLE 14.7.6** Suppose a box with no top is to hold a certain volume \( V \). Find the dimensions for the box that result in the minimum surface area.

The area of the box is \( A = 2hw + 2hl + 2w \), and the volume is \( V = lwh \), so we can write the area as a function of two variables:

\[
A(l, w) = 2V \frac{l}{w} + 2V \frac{w}{l} + l^2.
\]

Then

\[
A_l = -2V \frac{1}{w^2} + l \quad \text{and} \quad A_w = 2V \frac{1}{l^2} + l.
\]

If we set these equal to zero and solve, we find \( w = (2V)^{1/3} \) and \( l = (2V)^{1/3} \), and the corresponding height is \( h = V/(2V)^{2/3} \).

The second derivatives are

\[
A_{ll} = 4V, \quad A_{lw} = 4V, \quad A_{ww} = 4V, \quad A_{wl} = 1,
\]

so the discriminant is

\[
D = 4V - 4V - 1 = 1 > 0.
\]

Since \( A_{ll} > 0 \), there is a local minimum at the critical point. Is this a global minimum? It is, but it is difficult to see this analytically; physically and graphically it is clear that there is a minimum, in which case it must be at the single critical point. This applets shows an example of such a graph. Note that we must choose a value for \( V \) in order to graph it.

---

**Chapter 14 Partial Differentiation**

**EXAMPLE 14.7.3** Find all local maxima and minima for \( f(x, y) = x^2 - y^2 \).

The derivatives:

\[
f_x = 2x, \quad f_y = -2y, \quad f_{xx} = 2, \quad f_{yy} = -2, \quad f_{xy} = 0.
\]

Again there is a single critical point, at \((0, 0)\), and

\[
D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 2 \cdot 2 - 0 = 4 < 0,
\]

so there is neither a maximum nor minimum there, and so there are no local maxima or minima. The surface is shown in figure 14.7.1.

---

**EXAMPLE 14.7.4** Find all local maxima and minima for \( f(x, y) = x^4 + y^4 \).

The derivatives:

\[
f_x = 4x^3, \quad f_y = 4y^3, \quad f_{xx} = 12x^2, \quad f_{yy} = 12y^2, \quad f_{xy} = 0.
\]

Again there is a single critical point, at \((0, 0)\), and

\[
D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 0 > 0 - 0 = 0,
\]

so we get no information. However, in this case it is easy to see that there is a minimum at \((0, 0)\), because \( f(0, 0) = 0 \) and at all other points \( f(x, y) > 0 \).

---

**Exercises 14.7**

1. Find all local maximum and minimum points of \( f = x^2 + 4y^2 - 2x + 8y - 1 \).
2. Find all local maximum and minimum points of \( f = x^2 - y^2 + 6x - 10y + 2 \).
3. Find all local maximum and minimum points of \( f = x^2 + y^2 \).
4. Find all local maximum and minimum points of \( f = 9 + 4x - 2x^2 - 3y^2 \).
unknowns, which typically results in many solutions (as we expected). A third equation will be
interested in those points where two level curves are tangent—but there are many such
gradients. The curve $100 = 2\lambda x + y^2$ is the sum of the squares of the distances from point $(x, y)$ to the
three points. Find $x$ and $y$ so that this quantity is minimized. 

Suppose that $f(x, y) = x^2 + y^2 + \varepsilon y$. Find and classify the critical points, and discuss how
they change when $\varepsilon$ takes on different values.

Find the shortest distance from the point $(0, 0, 0)$ to the paraboloid $z = x^2 + y^2$. 

Consider the function $f(x, y) = x^2 - y^2 + \varepsilon y$. 

a. Show that $(0, 0)$ is the only critical point of $f$.

b. Show that the discriminant test is inconclusive for $f$.

c. Determine the cross-sections of $f$ obtained by setting $y = \pm z$ for various values of $\varepsilon$.

d. What kind of critical point is $(0, 0, 0)$?

Find the volume of the largest rectangular box with edges parallel to the axes that can be
inscribed in the ellipsoid $2x^2 + 7y^2 + 18z^2 = 288$. 

### 14.8 Lagrange Multipliers

Many applied max/min problems take the form of the last two examples: we want to
find an extreme value of a function, like $V = x^2z$, subject to a constraint, like $S =
\sqrt{x^2 + y^2 + z^2}$. Often this can be done, as we have, by explicitly combining the equations
and then finding critical points. There is another approach that is often convenient, the
method of Lagrange multipliers.

It is somewhat easier to understand two variable problems, so we begin with one as an
example. Suppose the perimeter of a rectangle is to be $100$ units. Find the rectangle
with largest area. This is a fairly straightforward problem from single variable calculus.

We write down the two equations: $A = zP = 100 = x + 2+y$, solve the second of these for $y$ (or $x$), substitute into the first, and end up with a one-variable maximization problem.

Let’s now think of it differently: the equation $A = zP$ defines a surface, and the
equation $100 = 2x + 2y$ defines a curve (a line, in this case) in the $x-y$ plane. If we graph
both of these in the three-dimensional coordinate system, we can phrase the problem like
this: what is the highest point on the surface above the line? The solution we already
understand effectively produces the equation of the cross-section of the surface above the
line and then treats it as a single variable problem. Instead, imagine that we draw the
level curves (the contour lines) for the surface in the $x-y$ plane, along with the line.

Imagine that the line represents a hiking trail and the contour lines are, as on a
topographic map, the lines of constant altitude. How could you estimate, based on the
graph, the high (or low) points on the path? As the path crosses contour lines, you know
the path must be increasing or decreasing in elevation. At some point you will see the path
just touch a contour line (tangant to it), and then begin to cross contours in the opposite
order—that point of tangency must be a maximum or minimum point. If we can identify
all such points, we can then check them to see which gives the maximum and which gives
the minimum value. As usual, we also need to check boundary points; in this problem, we
know that $x$ and $y$ are positive, so we are interested in just the portion of the line in the
first quadrant, as shown. The endpoints of the path, the two points on the axes, are not
points of tangency, but they are the two places that the function $xy$ is a minimum in the
first quadrant.

How can we actually make use of this? At the points of tangency that we seek, the
constraint curve (in this case the line) and the level curve have the same slope—their
tangent lines are parallel. This also means that the constraint curve is perpendicular to
the gradient vector of the function; going a bit further, if we can express the constraint
curve itself as a level curve, then we seek the points at which the two level curves have
parallel gradients. The curve $100 = 2x + 2y$ can be thought of as a level curve of the
function $2x + 2y$; figure 14.8.2 shows both sets of level curves on a single graph. We are
interested in those points where two level curves are tangent—but there are many such
points, in fact an infinite number, as we’ve only shown a few of the level curves. All along
the line $y = x$ are points at which two level curves are tangent. While this might seem to
be a show-stopper, it is not.

The gradient of $2x + 2y$ is $(2, 2)$, and the gradient of $xy$ is $(y, x)$. They are parallel
when $(2, 2) = \lambda(y, x)$, that is, when $2 = \lambda y$ and $2 = \lambda x$. We have two equations in three
unknowns, which typically results in many solutions (as we expected). A third equation will
reduce the number of solutions; the third equation is the original constraint, $100 = 2x + 2y$. 

#### 14.8.1 Constraint line with contour plot of the surface $xy$.

So we have the following system to solve:

\[ 2 = \lambda y \quad 2 = \lambda x \quad 100 = 2x + 2y. \]

In the first two equations, $\lambda$ can’t be $0$, so we may divide by it to get $x = y = 2/\lambda$. Substituting into the third equation we get

\[ \frac{2}{\lambda^2} + \frac{2}{\lambda^2} = 100 \]

so $x = y = 25$. Note that we are not really interested in the value of $\lambda$—it is a clever
tool, the Lagrange multiplier, introduced to solve the problem. In many cases, as here, it is
easier to find $\lambda$ than to find everything else without using $\lambda$.

The same method works for functions of three variables, except of course everything is
one dimension higher: the function to be optimized is a function of three variables and
the constraint represents a surface—for example, the function may represent temperature,
and we may be interested in the maximum temperature on some surface, like a sphere.

The points we seek are those at which the constraint surface is tangent to a level surface of
the function. Once again, we consider the constraint surface to be a level surface of some
function, and we look for points at which the two gradients are parallel, giving us three
equations in four unknowns. The constraint provides a fourth equation.

**EXAMPLE 14.8.1** Recall example 14.7.8: the diagonal of a box is $1$, we seek to
maximize the volume. The constraint is $1 = \sqrt{x^2 + y^2 + z^2}$, which is the same as $x^2 + y^2 + z^2 = 1$. 

#### 14.8.2 Contour plots for $2x + 2y$ and $xy$. 

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1. Find the points on the surface $x^2 + 2x + 3$ that are closest to the origin.

10. Find the maximum and minimum values of $ \nabla f(x, y, z) = 4x^2 + 2y^2 + 3z^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 50$.

11. Find the maximum and minimum values of $ \nabla f(x, y, z) = x^2 + 2y^2 + 5z^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 8$.

12. Find the maximum and minimum values of $ \nabla f(x, y, z) = 2x^2 + 3y^2 + z^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 16$.

13. Find the maximum and minimum values of $ \nabla f(x, y, z) = 2x^2 + 3y^2 + z^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 20$.

14. Find the maximum and minimum values of $ \nabla f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 25$.

15. Find the maximum and minimum values of $ \nabla f(x, y, z) = 2x^2 + 3y^2 + z^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 36$.

16. Find the shape for a given volume $V$ that will minimize surface area.

17. Find the shape for a given volume $V$ that will minimize cost.

18. Find the shape for a given volume $V$ that will minimize cost.

19. Find the maximum and minimum volumes of $f(x, y, z) = 3x^2 + y^2 + 2z^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 16$.

20. Find the maximum and minimum volumes of $f(x, y, z) = x^2 + 2y^2 + 3z^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 8$.

21. Find the maximum and minimum volumes of $f(x, y, z) = 2x^2 + 3y^2 + 5z^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 20$.