14

Partial Differentiation

14.1 Functions of Several Variables

In single-variable calculus we were concerned with functions of the form $f : \mathbb{R} \rightarrow \mathbb{R}$, that is, for which $n = 1$. Now we turn to functions of several variables, meaning several input variables, functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$. We will deal primarily with $n = 2$ and to a lesser extent $n = 3$. In fact many of the techniques we discuss can be applied to larger values of $n$ as well.]

A function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ maps a pair of values $(x, y)$ to a single real number. The three-dimensional coordinate system we have already used is a convenient way to visualize such functions: above each point $(x, y)$ in the $x$-$y$ plane we graph the point $(x, y, z)$, where of course $z = f(x, y)$.

**EXAMPLE 14.1.1** Consider $f(x, y) = 3x + 4y - 5$. Writing this as $z = 3x + 4y - 5$ and then $3x + 4y - z = 5$ we recognize the equation of a plane. In the form $f(x, y) = 3x + 4y - 5$ the emphasis has shifted: we now think of $x$ and $y$ as independent variables and $z$ as a variable dependent on them, but the geometry is unchanged.

**EXAMPLE 14.1.2** We have seen that $x^2 + y^2 + z^2 = 4$ represents a sphere of radius 2.

As we cannot write this in the form $f(x, y, z)$, since for each $x$ and $y$ there are an infinite number of corresponding $z$-values, we refer to $x^2 + y^2 + z^2 = 4$ as a level surface.

If we restrict attention to the line $x = y$, we get $f(x, y) = 3x + 4y - 5$ along the line $y = 2x$; we have $f(x, y) = 2x^2 + 4y - 5$ and the graph of this function looks larger in every direction in roughly the same way that the square root function gets larger. For example, if we replace $x$ by $2x$, the graph gets stretched in every direction. Although the graph of $f(x, y, z) = 2x^2 + 4y - 5$ is a parabola, but it does not really "represent" the cross-section along $y = 2x$, because the cross-section has the line $y = 2x$ where the horizontal axis should be.

**EXAMPLE 14.1.3** Consider $f = \sqrt{\sigma + \sqrt{\tau}}$. This function is defined only when both $x$ and $y$ are non-negative. When $y = 0$ we get $f(x, y) = \sqrt{\sigma}$, the familiar square root function in the $x$-$y$ plane; and when $x = 0$ we get the same curve in the $y$-$z$ plane. Generally speaking, we see that starting from $f(0, 0) = 0$ this function gets larger in every direction in roughly the same way that the square root function gets larger.

A computer program that plots such surfaces can be very useful, as it is often difficult to get a good idea of what they look like. Still, it is valuable to be able to visualize relatively simple surfaces without such aids. As in the previous example, it is often a good idea to examine the function on restricted subsets of the plane, especially lines. It can also be useful to identify those points $(x, y)$ that share a common $z$-value.

**EXAMPLE 14.1.4** Consider $f(x, y) = x^2 + y^2$. When $z = 0$ this becomes $f = z^2$, a parabola in the $y$-$z$ plane; when $y = 0$ we get the "same" parabola $f = x^2$ in the $x$-$z$ plane. Now consider the line $y = kx$. If we simply replace $y$ by $kx$ we get $f(x, y) = (1 + k^2)x^2$ which is a parabola, but it does not really "represent" the cross-section along $y = kx$, because the cross-section has the line $y = kx$ where the horizontal axis should be.

As in this example, the points $(x, y)$ such that $f(x, y) = k$ usually form a curve, called a level curve of the function. A graph of some level curves can give a good idea of the shape of the surface; it looks much like a topographic map of the surface. In figure 14.1.2 both the surface and its associated level curves are shown. Note that, as with a topographic map, the heights corresponding to the level curves are evenly spaced, so that where curves are closer together the surface is steeper.

Functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ behave much like functions of two variables; we will on occasion discuss functions of several variables. The principal difficulty with such functions is visualizing them, as they do not "fit" in the three dimensions we are familiar with. For three variables there are various ways to interpret functions that make them easier to understand. For example, $f(x, y, z)$ could represent the temperature at the point $(x, y, z)$, or the pressure, or the strength of a magnetic field. It remains useful to consider those points at which $f(x, y, z) = k$, where $k$ is some constant value. If $f(x, y, z)$ is temperature, the set of points $(x, y, z)$ such that $f(x, y, z) = k$ is the collection of points in space with temperature

This equation into two functions, $f(x, y) = \sqrt{1 - x^2 - y^2}$ and $f(x, y) = -\sqrt{1 - x^2 - y^2}$, representing the upper and lower hemispheres. Each of these is an example of a function with a restricted domain: only certain values of $x$ and $y$ make sense (namely, those for which $x^2 + y^2 \leq 4$) and the graphs of these functions are limited to a small region of the plane.

**EXAMPLE 14.1.5** Suppose the temperature at $(x, y, z)$ is $T(x, y, z) = e^{-(x^2 + y^2 + z^2)}$. This function has a maximum value of 1 at the origin, and tends to 0 in all directions. If $k$ is a positive number and $r$ a point for which $(x, y, z) = k$ is those points satisfying $x^2 + y^2 + z^2 = k^2$, a sphere centered at the origin. The level surfaces are the concentric spheres centered at the origin.

**Exercises 14.1.**

1. Let $f(x, y) = (x - y)^2$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, $x = y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface.

2. Let $f(x, y) = |x| + |y|$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, $x = y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface.

3. Let $f(x, y) = e^{-(t^2 + s^2)}$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, $x = y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface.

4. Let $f(x, y) = \sin(x - y)$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, $x = y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface.

5. Let $f(x, y) = (x^2 - y^2)^2$. Determine the equations and shapes of the cross-sections when $x = 0$, $y = 0$, $x = y$, and describe the level curves. Use a three-dimensional graphing tool to graph the surface.

6. Find the domain of each of the following functions of two variables:
   - $\sqrt{x^2 + y^2} + \sqrt{x^2 + y^2}$
   - $\sqrt{x^2 + y^2 - 4}$
   - $\sqrt{x^2 - y^2}$

7. Below are two sets of level curves. One is for a cone, one is for a paraboloid. Which is which? Explain.
14.2 Limits and Continuity

To develop calculus for functions of one variable, we needed to make sense of the concept of a limit, which we needed to understand continuous functions and to define the derivative. Limits involving functions of two variables can be considerably more difficult to deal with; fortunately, most of the functions we encounter are fairly easy to understand.

The potential difficulty is largely due to the fact that there are many ways to "approach" a point in the $x$-$y$ plane. If we want to say that $\lim_{(x,y)\to(a,b)} f(x,y) = L$, we need to capture the idea that as $(x,y)$ gets close to $(a,b)$ then $f(x,y)$ gets close to $L$. For functions of one variable, $f(x)$, there are only two ways that $x$ can approach $a$: from the left or right. But there are an infinite number of ways to approach $(a,b)$: along any one of an infinite number of lines, or an infinite number of parabolas, or an infinite number of sine curves, and so on. We might hope that it’s really not so bad—suppose, for example, that along every possible line through $(a,b)$ the value of $f(x,y)$ gets close to $L$: surely this means that “$f(x,y)$ approaches $L$ as $(x,y)$ approaches $(a,b)$.” Sadly, no.

**Example 14.2.1** Consider $f(x,y) = x^2(y^2 + y^3)$. When $x = 0$ or $y = 0$, $f(x,y)$ is 0, so the limit of $f(x,y)$ approaching the origin along either the $x$ or $y$ axis is 0. Moreover, along the line $y = mx$, $f(x,y) = m^2x^4((x^2 + m^2x^2))$. As $x$ approaches 0 this expression approaches 0 as well. So along every line through the origin $f(x,y)$ approaches 0. Now suppose we approach the origin along $x = y^2$. Then

$$f(x,y) = \frac{y^4}{y^2 + y^3} = \frac{y^2}{y + y^2} = \frac{1}{2},$$

so the limit is 1/2. Looking at figure 14.2.1, it is apparent that there is a ridge above $x = y^2$. Approaching the origin along a straight line, we go over the ridge and then drop down toward 0, but approaching along the ridge the height is a constant 1/2. Thus, there is no limit at $(0,0)$.

Fortunately, we can define the concept of limit without needing to specify how a particular point is approached—indeed, in definition 2.3.2, we didn’t need the concept of “approach.” Roughly, that definition says that when $x$ is close to a then $f(x)$ is close to $L$, there is no mention of “how” we get close to $a$. We can adapt that definition to two variables quite easily:

**Definition 14.2.2 Limit** Suppose $f(x,y)$ is a function. We say that 

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every $\epsilon > 0$ there is a $\delta > 0$ so that whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, $|f(x,y) - L| < \epsilon$.

We want to force this to be less than $\epsilon$ by picking $\delta$ “small enough.” If we choose $\delta = \epsilon/3$ then

$$\frac{3\epsilon}{3} < 1 \cdot 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

Recall that a function $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$; roughly this says that there is no “hole” or “jump” at $x = a$. We can say exactly the same thing about a function of two variables.

**Definition 14.2.4** $f(x,y)$ is continuous at $(a,b)$ if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

**Example 14.2.5** The function $f(x,y) = 3x^2/y(x^2 + y^2)$ is not continuous at $(0,0)$, because $f(0,0)$ is not defined. However, we know that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, so we can easily “fix” the problem, by extending the definition of $f$ so that $f(0,0) = 0$. This surface is shown in figure 14.2.2.

This says that there is no “hole” or “jump” at $(a,b)$. Fortunately, most of the functions we encounter are fairly easy to understand.

**Table 14.2.6** Polynomials are continuous everywhere. Rational functions are continuous everywhere they are defined.

### Exercises 14.2

Determine whether each limit exists. If it does, find the limit and prove that it is the limit; if it does not, explain how you know:

1. $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}$
2. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$
3. $\lim_{(x,y)\to(0,0)} \frac{x}{x^2 + y^2}$
4. $\lim_{(x,y)\to(0,0)} \frac{y}{x^2 + y^2}$
5. $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$
6. $\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2 + y^2}}{x^2 + y^2}$
7. $\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 + y^2}$
8. $\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2 + y^2}$
9. $\lim_{(x,y)\to(0,0)} \frac{x^2 + \sin y}{x^2 + y^2}$
10. $\lim_{(x,y)\to(0,0)} \frac{(x-1)^2 y}{x^2 + y^2}$
11. $\lim_{(x,y)\to(0,0)} \frac{x^2 - y}{x^2 + y}$
12. $\lim_{(x,y)\to(0,0)} \frac{2x^3}{x^2 + y^2}$
13. Does the function \( f(x, y) = \frac{x - y}{1 + x^2 + y^2} \) have any discontinuities? What about \( f(x, y) = \frac{x}{1 + x^2 + y^2} \)? Explain.

14.3 Partial Differentiation

When we first considered what the derivative of a vector function might mean, there was really not much difficulty in understanding either how such a thing might be computed or what it might measure. In the case of functions of two variables, things are a bit harder to understand. If we think of a function of two variables in terms of its graph, a surface, there is more-or-less obvious derivative-like question we might ask, namely, how “steep” is the surface goes down; of course moving in the opposite direction, toward decreasing lines tangent to the surface, both through \((a, b, c)\) and \((0, 0, 0)\), then extend the line vertically into a plane. Look at the intersection of the surface with the plane. If we pay attention to just the plane, we see the chosen straight line where the x-axis would normally be, and the intersection with the surface shows up as a curve in the plane. Figure 14.3.1 shows the parabolic surface from figure 14.1.2, exposing its cross-section above the line \(x + y = 1\).

In principle, this is a problem we know how to solve: find the slope of a curve in a plane. Let’s start by looking at some particularly easy lines: those parallel to the x or y axis. Suppose we are interested in the cross-section of \(f(x, y)\) above the line \(y = b\). If we substitute \(b\) for \(y\) in \(f(x, y)\), we get a function in one variable, describing the height of the cross-section as a function of \(x\). Because \(y = b\) is parallel to the x-axis, if we view it from a vantage point on the negative y-axis, we will see what appears to be simply an ordinary curve in the x-z plane.

Consider again the parabolic surface \(f(x, y) = x^2 + y^2\). The cross-section above the line \(y = 1\) consists of all points \((x, 2, z^2 + 4)\). Looking at this cross-section from somewhere on the negative y axis, we see what appears to be just the curve \(f(x) = x^2 + 4\). At any point on the cross-section, \((x, z^2 + 4)\), the steepness of the surface in the direction of the line \(y = 1\) is simply the slope of the curve \(f(x) = x^2 + 4\), namely \(2x\). Figure 14.3.2 shows the same parabolic surface as before, but now cut by the plane \(y = 2\). The left graph shows the cut-off surface, the right shows just the cross-section, looking up from the negative y-axis toward the origin.

Imagine a particular point on a surface; what might we be able to say about how steep it is? We can limit the question to make it more familiar: how steep is the surface in a particular direction? What does this even mean? Here’s one way to think of it: Suppose we’re interested in the point \((a, b, c)\). Pick a straight line in the \(x-y\) plane through the point \((a, b, 0)\), then extend the line vertically into a plane. Look at the intersection of the plane with the surface. If we pay attention to just the plane, we see the chosen straight line where the x-axis would normally be, and the intersection with the surface shows up as a curve in the plane. Figure 14.3.1 shows the parabolic surface from figure 14.1.2, exposing its cross-section above the line \(x + y = 1\).

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If, say, we’re interested in the point \((-1, 2, 5)\) on the surface, then the slope in the direction of the line \(y = 2x - 2\) means that starting at \((-1, 2, 5)\) and moving on the surface, above the line \(y = 2\), in the direction of increasing \(x\) values, the surface goes down; of course moving in the opposite direction, toward decreasing \(x\) values, the surface will rise.

If we’re interested in some other line \(y = k\), there is really no change in the computation. The equation of the cross-section above \(y = k\) is \(x^2 + k^2\) with derivative \(2x\). We can save ourselves the effort, small as it is, of substituting \(k\) for \(y\). All we are in effect doing is temporarily assuming that \(y\) is some constant. With this assumption, the derivative \(\frac{d}{dx}(x^2 + k^2) = 2x\). To emphasize that we are only temporarily assuming \(y\) is constant, we use a slightly different notation: \(\frac{dy}{dx}(x^2 + y^2) = 2x\); the \(\partial y\) reminds us that there are more variables than \(x\), but that only \(x\) is being treated as a variable. We read the equation as “the partial derivative of \(x^2 + y^2\) with respect to \(x\) is \(2x\)”.

A convenient alternate notation for the partial derivative of \(f(x, y)\) with respect to \(x\) is \(f_x\).

**EXAMPLE 14.3.1** The partial derivative with respect to \(x\) of \(x^2 + 3xy\) is \(2x + 3y\). Note that the partial derivative includes the variable \(y\), unlike the example \(x^2 + y^2\). It is somewhat unusual for the partial derivative to depend on a single variable; this example is more typical.

Of course, we can do the same sort of calculation for lines parallel to the y-axis. We temporarily hold \(x\) constant, which gives us the equation of the cross-section above a line \(x = k\). We can then compute the derivative with respect to \(y\); this will measure the steepness of the curve in the \(y\) direction.

**EXAMPLE 14.3.2** The partial derivative with respect to \(y\) of \(f(x, y) = \sin(xy) + 3xy\) is \(f_y = \cos(xy) + 3x\).

So far, using no new techniques, we have succeeded in measuring the slope of a surface in two quite special directions. For functions of one variable, the derivative is closely linked to the notion of tangent line. For surfaces, the analogous idea is the tangent plane—a plane that just touches a surface at a point, and has the same “steepness” as the surface in all directions. Even though we haven’t yet figured out how to compute the slope in all directions, we have enough information to find tangent planes. Suppose we want the plane tangent to a surface at a particular point \((a, b, c)\). If we compute the two partial derivatives of the function for that point, we get enough information to determine two lines tangent to the surface, both through \((a, b, c)\) and both tangent to the surface in their respective directions. These two lines determine a plane, that is, there is exactly one plane containing the two lines: the tangent plane. Figure 14.3.3 shows (part of) two tangent lines at a point, and the tangent plane containing them.

How can we discover an equation for this tangent plane? We know a point on the plane, \((a, b, c)\); we need a vector normal to the plane. If we can find two vectors, one parallel to each of the tangent lines we know how to find, then the cross product of these vectors will give the desired normal vector.

How can we find vectors parallel to the tangent lines? Consider first the line tangent to the surface above the line \(y = 0\). A vector \((u, v, w)\) parallel to this tangent line must have \(y\) component \(v = 0\), and we may as well take the \(x\) component to be \(u = 1\). The ratio
of the z component to the x component is the slope of the tangent line, precisely what we know how to compute. The slope of the tangent line is \( f_z(a, b) \), so

\[
f_z(a, b) = \frac{w}{z - w}.
\]

In other words, a vector parallel to this tangent line is \( (1, 0, f_z(a, b)) \), as shown in figure 14.3.4. If we repeat the reasoning for the tangent line above \( x = a \), we get the vector \( (0, 1, f_y(a, b)) \).

Now to find the desired normal vector we compute the cross product, \( \langle 1, 0, f_z(a, b) \rangle \times \langle 0, 1, f_y(a, b) \rangle \). From our earlier discussion of planes, we can write down the equation we seek: \( f_z(a, b) x + f_y(a, b) y + \epsilon = k \), and \( k \) as usual can be computed by substituting a known point: \( f_x(a, b) x + f_y(a, b) y - c = k \). There are various more-or-less

right side does the same, because as \((x, y)\) approaches \((x_0, y_0)\), \( f\) approaches 0. Essentially the same calculation works for \( f\).

Almost all of the functions we will encounter are differentiable at points we will be interested in, and often at all points. This is usually because the functions satisfy the hypotheses of this theorem.

**THEOREM 14.3.5** If \( f(x, y) \) and its partial derivatives are continuous at a point \((x_0, y_0)\), then \( f\) is differentiable there.

**Exercises 14.3.**

1. Find \( f_x \) and \( f_y \) where \( f(x, y) = \cos(x^2 + y^2) \).
2. Find \( f_x \) and \( f_y \) where \( f(x, y) = \frac{x}{x^2 + y^2} \).
3. Find \( f_x \) and \( f_y \) where \( f(x, y) = e^{x^2 + y^2} \).
4. Find \( f_x \) and \( f_y \) where \( f(x, y) = y \tan(x^2) \).
5. Find \( f_x \) and \( f_y \) where \( f(x, y) = \sqrt{x^2 - y^2} \).
6. Find \( f_x \) and \( f_y \) where \( f(x, y) = \tan(x^2) \).
7. Find \( f_x \) and \( f_y \) where \( f(x, y) = \frac{1}{x^2 + y^2} \).
8. Find an equation for the plane tangent to \( x^2 + y^2 = 4 \) at \((1, 1, 0)\).
9. Find an equation for the plane tangent to \( f(x, y) = x^2 + y^2 \) at \((x, y) = (1, 2, 1)\).
10. Find an equation for the plane tangent to \( f(x, y) = y^2 + z^2 = 3 \) at \((1, 1, 0)\).
11. Find an equation for the plane tangent to \( f(x, y) = x \tan(x^2) \) at \((2, 1, 2, 0)\).
12. Find an equation for the line normal to \( x^2 + y^4 = 2z \) at \((2, 1, 4, 2)\).
13. Explain in your own words why, when taking a partial derivative of a function of multiple variables, we can treat the variables not being differentiated as constants.
14. Consider a differentiable function, \( f(x, y) \). Give physical interpretations of the meanings of \( f_x(a, b) \) and \( f_y(a, b) \) as they relate to the graph of \( f \).
15. In much the same way that we use the tangent line to approximate the value of a function from single variable calculus, we can use the tangent plane to approximate a function from multivariable calculus. Consider the tangent plane found in Exercise 11. Use this plane to approximate \( f(1.98, 0.4) \).
16. Suppose that one of your colleagues has calculated the partial derivatives of a given function, and reported to you that \( f_x(x, y) = 2x + 2y \) and that \( f_y(x, y) = x + 6y \). Do you believe them? Why or why not? If not, what answer might you have accepted for \( f_x \)?
17. Suppose \( f(1) \) and \( g(x) \) are single variable differentiable functions. Find \( dh/dt \) and \( dz/dy \) for each of the following two variable functions.

a. \( z = f(g(x)) \)

b. \( z = f(x^2) \)

c. \( z = f(x^2) \)

When does a surface have a tangent plane at a particular point? What we really want from a tangent plane, as from a tangent line, is that the plane be a “good” approximation of the surface near the point. Here is how we can make this precise:

**DEFINITION 14.3.4** Let \( \Delta z = z - z_0 \). \( \Delta y = y - y_0 \), and \( \Delta x = x - x_0 \) where \( z_0 = f(x_0, y_0) \). The function \( z = f(x, y) \) is differentiable at \((x_0, y_0)\) if

\[
\Delta z = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + c_1 \Delta x + c_2 \Delta y,
\]

and both \( c_1 \) and \( c_2 \) approach 0 as \((x, y)\) approaches \((x_0, y_0)\).

This definition takes a bit of absorbing. Let’s rewrite the central equation a bit:

\[
z = f(x_0, y_0)(x - x_0) + f_x(x_0, y_0)(y - y_0) + f(x_0, y_0) + \Delta x + \Delta y.
\]

The first three terms on the right are the equation of the tangent plane, that is,

\[
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0) + \Delta x + \Delta y.
\]

Now taking the limit of the two sides as \( \Delta x \) approaches 0, the left side turns into the partial derivative of \( z \) with respect to \( x \) at \((x_0, y_0)\), or in other words \( f_x(x_0, y_0) \), and the
EXAMPLE 14.4.2 \( x^2 + y^2 + z^2 = 4 \) defines a sphere, which is not a function of \( x \) and \( y \), though it can be thought of as two functions, the top and bottom hemispheres. We can think of \( z \) as one of these two functions, so really \( z = z(x, y) \), and we can think of \( x \) and \( y \) as particularly simple functions of \( x \) and \( y \), and let \( f(x, y, z) = x^2 + y^2 + z^2 \). Since \( f(x, y, z) = 4 \), \( \partial f/\partial z = 0 \), but using the chain rule:

\[
0 = \frac{\partial f}{\partial z} = f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z \frac{\partial z}{\partial z} = (2x)(1) + (2y)(0) + (2z) \frac{\partial z}{\partial z}.
\]

noting that since \( z \) is temporarily held constant its derivative \( \partial z/\partial z = 0 \). Now we can solve for \( \partial z/\partial z \):

\[
\frac{\partial z}{\partial z} = -\frac{2x}{2} = -2.
\]

In a similar manner we can compute \( \partial z/\partial y \).

14.5 Directional Derivatives

So we need to somehow “mark off” units on the line, and we need a convenient way to refer to the line in calculations. It turns out that we can accomplish both by using the vector form of a line. Suppose that \( \mathbf{u} \) is a unit vector \((u_x, u_y, u_z)\) in the direction of interest. A vector equation for the line through \((x_0, y_0, z_0)\) in this direction is \( x = x_0 + u_xt, y = y_0 + u_yt, z = z_0 + u_zt \).

The height of the surface above the point \((x_0, y_0, z_0)\) is \( g(x, y, z) = g(x_0, y_0, z_0) \). Because \( \mathbf{u} \) is a unit vector, the value of \( t \) is precisely the distance along the line from \((x_0, y_0)\) to \((x_t, y_t, z_t)\); this means that the line is effectively a \( t \) axis, with origin at the point \((x_0, y_0)\), so the slope we seek is:

\[
\theta' = \frac{\partial g}{\partial t} = \langle f_x, f_y \rangle \cdot \mathbf{u} = |\mathbf{u}| |\nabla f| \cos \theta = |\mathbf{u}| |\nabla f| \cos \theta.
\]

Here we have used the chain rule and the derivatives \( f_x(x_0 + u_xt, y_0 + u_yt, z_0 + u_zt) = u_x + f_x \) and \( f_y(x_0 + u_xt, y_0 + u_yt, z_0 + u_zt) = u_y + f_y \). The vector \( \langle f_x, f_y \rangle \) is very useful, so it has its own symbol, \( \nabla f \), pronounced “del \( f \);” it is also called the gradient of \( f \).

EXAMPLE 14.5.1 Find the slope of the line \( x = x^2 + y^2 \) at \((1, 2)\) in the direction of the vector \((3, 4)\).

We first compute the gradient \( \nabla f = \langle f_x, f_y \rangle \), which is \( \langle 2x, 2y \rangle \) at \((1, 2)\). A unit vector in the desired direction is \( \langle 3/\sqrt{5}, 4/\sqrt{5} \rangle \), and the desired slope is then \( \langle 2x, 4y \rangle = 6/5 \times \nabla f = 22/5 \).

EXAMPLE 14.5.2 Find a tangent vector to \( z = x^2 + y^2 \) at \((1, 2)\) in the direction of the vector \((3, 4)\); and show that it is parallel to the tangent plane at that point.

Since \( \langle 3/\sqrt{5}, 4/\sqrt{5} \rangle \) is a unit vector in the desired direction, we can easily expand it to a tangent vector simply by adding the third coordinate computed in the previous example: \( \langle 3/\sqrt{5}, 4/\sqrt{5}, 22/5 \rangle / \sqrt{5} \). To see that this vector is parallel to the tangent plane, we can compute its dot product with a normal to the plane. We know that a normal to the tangent plane is \( \langle 2x, 2y, 1 \rangle \) at \((1, 2, 5)\), and the dot product is \( \langle 2x, 2y, 1 \rangle \cdot \langle 2x, 2y, 1 \rangle = 22/5 = 6/5 \times \sqrt{5} = 22/5 \), which is zero, so the vectors are perpendicular. (Note that the vector normal to the surface, namely \( \langle 2x, 2y, 1 \rangle \), is simply the gradient with \(-1\) tacked on as the third component.)

The slope of a surface given by \( f = f(x, y) \) in the direction of a \((x, y)\)-vector \( \mathbf{u} \) is called the directional derivative of \( f \), written \( D_\mathbf{u} f \). The directional derivative immediately provides us with some additional information. We know that \( D_{\mathbf{u}} f = \langle \nabla f \rangle \cdot \mathbf{u} = |\mathbf{u}| |\nabla f| \cos \theta = |\mathbf{u}| |\nabla f| \cos \theta \) if \( \mathbf{u} \) is a unit vector; \( \theta \) is the angle between \( \nabla f \) and \( \mathbf{u} \). This tells us immediately that the largest value of \( D_{\mathbf{u}} f \) occurs when \( \cos \theta = 1 \), namely, when \( \theta = 0 \), so \( \nabla f \) is parallel to \( \mathbf{u} \).

In other words, the gradient \( \nabla f \) points in the direction of steepest ascent of the surface, and \( -\nabla f \) is the slope in that direction. Likewise, the smallest value of \( D_{\mathbf{u}} f \) occurs when \( \cos \theta = -1 \), namely, when \( \theta = \pi \), so \( \nabla f \) is anti-parallel to \( \mathbf{u} \). In other words, \( -\nabla f \) points in the direction of steepest descent of the surface, and \(-\nabla f \) is the slope in that direction.

EXAMPLE 14.5.3 Investigate the direction of steepest ascent and descent for \( z = x^2 + y^2 \).

The gradient is \( \langle 2x, 2y \rangle = 2z(x, y) \); this is a vector parallel to the vector \((x, y)\), so the direction of steepest ascent is directly away from the origin, starting at the point \((x, y)\). The direction of steepest descent is thus directly toward the origin from \((x, y)\). Note that at \((0, 0)\) the gradient vector is \((0, 0)\), which has no direction, and it is clear from the plot of this surface that there is a minimum point at the origin, and tangent vectors in all directions are parallel to the \(x\)-axis.

If \( \nabla f \) is perpendicular to \( \mathbf{u} \), \( D_{\mathbf{u}} f = |\nabla f| \cos \theta = 0 \), since \( \cos \theta = 0 \). This means that in either of the two directions perpendicular to \( \nabla f \), the slope of the surface is zero; this implies that a vector in either of these directions is tangent to the level curve at that point. Starting with \( \mathbf{u} = \langle f_x, f_y \rangle \), it is easy to find a vector perpendicular to it; either \( \langle -f_x, f_y \rangle \) or \( \langle -f_y, f_x \rangle \) will work.

If \( f(x, y, z) \) is a function of three variables, all the calculations proceed in essentially the same way. The rate at which \( f \) changes in a particular direction is \( \nabla f \), where now \( \nabla f = \langle f_x, f_y, f_z \rangle \) and \( \mathbf{u} = \langle u_x, u_y, u_z \rangle \) is a unit vector. Again \( \nabla f \) points in the direction of maximum rate of increase, \(-\nabla f \) points in the direction of maximum rate of decrease, and any vector perpendicular to \( \nabla f \) is tangent to the level surface \( f(x, y, z) = k \) at the point in question. Of course there are no longer just two such vectors; the vectors perpendicular to \( \nabla f \) describe the tangent plane to the level surface, or in other words \( \nabla f \) is a normal to the tangent plane.

EXAMPLE 14.5.4 Suppose the temperature at a point in space is given by \( T(x, y, z) = T_0/(1 + x^2 + y^2 + z^2) \); at the origin the temperature in Kelvin is \( T_0 > 0 \), and it decreases in every direction from there. It might be, for example, that there is a source of heat at the
14.5 Directional Derivatives

The gradient points directly at the origin from the point \((x, y, z)\) by moving directly toward the heat source, we increase the temperature as quickly as possible.

**EXAMPLE 14.5.5**

Find all first and second partial derivatives of \(f(x, y) = x^2y^2\).

Using an obvious notation, we get:

\[
f_{xx} = 2y^2 \\
\frac{\partial f}{\partial x} = 4xy \\
\frac{\partial f}{\partial y} = 4y \\
f_{yy} = 2x^2.
\]

You will have noticed that two of these are the same, the “mixed partials” computed by taking partial derivatives with respect to both variables in the two possible orders. This is not an accident—as long as the function is reasonably nice, this will always be true.

**THEOREM 14.6.2 Clairaut’s Theorem**

If the mixed partial derivatives are continuous, they are equal.

**EXAMPLE 14.6.3**

Compute the mixed partials of \(f(x, y) = xy(x^2 + y^2)\).

\[
f_{xx} = 2y^2 \\
\frac{\partial f}{\partial x} = 4xy \\
\frac{\partial f}{\partial y} = 4y \\
f_{yy} = 2x^2.
\]

We leave \(f_{xy}\) as an exercise.

**Exercises 14.6.**

1. Find \(f_{xx}\) and \(f_{yy}\) of \(f(x, y) = xy(x^2 + y^2)\).

2. Find \(f_{xx}\) and \(f_{yy}\) of \(f(x, y) = x^2y^2 + 10\).

3. Find all first and second partial derivatives of \(e^{x^2 + y^2}\).

4. Find all first and second partial derivatives of \(\sin(xy)\).

5. Find all first and second partial derivatives of \(\sin(xe^{y^2})\).

6. Find all first and second partial derivatives of \(e^{x^2+y^2}\).

7. Find all first and second partial derivatives of \(\ln(x^2+y^2)\).

8. Find all first and second partial derivatives of \(z\) with respect to \(x\) and \(y\) if \(x^2 + y^2 + 16z^2 - 4y = 0\).

9. Find all first and second partial derivatives of \(z\) with respect to \(x\) and \(y\) if \(x^2 + y^2 + z^2 = 1\).

10. Let \(u = x^2 + y^2 + z^2\) and \(\alpha = e^{x^2+y^2}\). Prove that the function \(u(x, y, z) = e^{-\alpha}\sin(\alpha)\) is a solution to the heat equation \(u_t = \alpha u_{xx} + \alpha u_{yy} + \alpha u_{zz}\).

11. Let \(u = \sin(x + ay) + \cos(x + ay)\) be a solution to the wave equation \(u_{tt} = \alpha u_{xx}\).

12. How many third-order derivatives does a function of 2 variables have? How many of these are distinct?
14.7 Maxima and minima

Suppose a surface given by \( f(x, y) \) has a local maximum at \( (x_0, y_0, z_0) \); geometrically, this point on the surface looks like the top of a hill. If we look at the cross-section in the plane \( y = y_0 \), we will see a local maximum on the curve at \( (x_0, z_0) \), and we know from single-variable calculus that \( \frac{dz}{dx} \leq 0 \) at this point. Likewise, in the plane \( x = x_0 \), so if there is a local maximum at \( (x_0, y_0, z_0) \), both partial derivatives at the point must be zero, and likewise for a local minimum. Thus, to find local maximum and minimum points, we need only consider those points at which both partial derivatives are \( 0 \). As in the single-variable case, it is possible for the derivative to be \( 0 \) at a point that is neither a maximum or a minimum, so we need to test these points further.

You will recall that in the single variable case, we examined three methods to identify maximum and minimum points; the most useful is the second derivative test, though it does not always work. For functions of two variables there is also a second derivative test; again it is by far the most useful test, though it doesn’t always work.

**THEOREM 14.7.1** Suppose that the second partial derivatives of \( f(x, y) \) are continuous near \( (x_0, y_0) \), and \( f_x(x_0, y_0) = f_y(x_0, y_0) = 0 \). We denote by \( D \) the discriminant:

\[
D(x_0, y_0) = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2.
\]

If \( D > 0 \):
- if \( f_{xx}(x_0, y_0) < 0 \) there is a local maximum at \( (x_0, y_0) \);
- if \( f_{xx}(x_0, y_0) > 0 \) there is a local minimum at \( (x_0, y_0) \);
If \( D < 0 \) there is neither a maximum nor a minimum at \( (x_0, y_0) \);
If \( D = 0 \) the test fails.

**EXAMPLE 14.7.2** Verify that \( f(x, y) = x^2 + y^2 \) has a minimum at \((0, 0)\).

First, we compute all the needed derivatives:

\[
f_x = 2x \quad f_y = 2y \quad f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = 0.
\]

The derivatives \( f_x \) and \( f_y \) are zero only at \((0, 0)\). Applying the second derivative test, we have:

\[
D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 2 - 0 - 4 = -2 > 0,
\]

so there is a local minimum at \((0, 0)\), and there are no other possibilities.

**EXAMPLE 14.7.5** Find all local maxima and minima for \( f(x, y) = x^2 + y^2 \).

The derivatives:

\[
f_x = 2x \quad f_y = 2y \quad f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = 0.
\]

Again there is a single critical point, at \((0, 0)\), and

\[
D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 2 - 0 - 4 = -2 < 0,
\]

so there is neither a maximum nor minimum there, and so there are no local maxima or minima. The surface is shown in figure 14.7.1.

![Figure 14.7.1 A saddle point, neither a maximum nor a minimum. (AP)](image)

**EXAMPLE 14.7.6** Suppose a box with no top is to hold a certain volume \( V \). Find the dimensions of the box that result in the minimum surface area.

The area of the box is \( A = 2hl + 2lh + 2lw \), and the volume is \( V = lwh \), so we can write the area as a function of two variables:

\[
A(l, w) = 2lw + 2lh + 2w = \frac{2V}{l} + \frac{2V}{w} + lw.
\]

Then \( A_l = \frac{2V}{l^2} + w \) and \( A_w = \frac{2V}{w^2} + l \).

If we set these equal to zero and solve, we find \( w = (2V)^{1/3} \) and \( l = (2V)^{1/3} \), and the corresponding height is \( h = \sqrt[3]{V/(2V)}^{2/3} \).

The second derivatives are

\[
A_{ll} = \frac{4V}{l^3}, \quad A_{ww} = \frac{4V}{w^3}, \quad A_{lw} = 1,
\]

so the discriminant is

\[
D = \frac{4V}{l^3} \frac{4V}{w^3} - 1 = 4 - 1 > 3 > 0.
\]

Since \( A_{ll} > 0 \) there is a local minimum at the critical point. Is this a global minimum? It is, but it is difficult to see this analytically; physically and graphically it is clear that there is a minimum, in which case it must be at the single critical point. This applet shows an example of such a graph. Note that we must choose a value for \( V \) in order to graph it.

**EXERCISES 14.7.**
1. Find all local maximum and minimum points of \( f = x^2 + 4y^2 - 2x + 8y - 1 \).
2. Find all local maximum and minimum points of \( f = x^2 - y^2 + 6x - 10y + 2 \).
3. Find all local maximum and minimum points of \( f = x^3 \).
4. Find all local maximum and minimum points of \( f = 9 + 4x - y - 2x^2 - 3y^2 \).
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Figure 14.7.2 The volume of a box with fixed length diagonal.

5. Find all local maximum and minimum points of \( f = x^3 + 4xy + y^2 - 6y + 1 \).

6. Find all local maximum and minimum points of \( f = x^3 - 9y + 2y^2 - 5x + 6y - 9 \).

7. Find the absolute maximum and minimum points of \( f = x^3 + 3y - 3xy \) over the region bounded by \( y = x \), \( y = 0 \), and \( x = 2 \).

8. A six-sided rectangular box is to hold 1/2 cubic meter; what shape should the box be to minimize surface area?

9. The post office will accept packages whose combined length and girth is at most 130 inches.

10. The bottom of a rectangular box costs twice as much per unit area as the sides and top.

11. Using the methods of this section, find the shortest distance from the origin to the plane \( x + y + z = 10 \).

12. Using the methods of this section, find the shortest distance from the point \((x_0, y_0, z_0)\) to the plane \(ax + by + cz = d\). You may assume that \(x \neq 0\); use of Sage or similar software is recommended.

13. A trough is to be formed by bending up two sides of a long metal rectangle so that the cross-section of the trough is an isosceles trapezoid, as in figure 6.2.6. If the width of the metal sheet is 2 meters, how should it be bent to maximize the volume of the trough?

Figure 14.8.2 Contour plots for 2x + 2y and xy.

So we have the following system to solve:

\[
\begin{align*}
2 &= 3y \\
2 &= 2z \\
100 &= 2x + 2y.
\end{align*}
\]

In the first two equations, \( \lambda \) can't be 0, so we may divide by it to get \( x = y = 2/\lambda \). Substituting into the third equation we get

\[
\frac{2^2}{\lambda} + \frac{2}{\lambda} = 100
\]

so \( x = y = 25 \). Note that we are not really interested in the value of \( \lambda \)—it is a clever tool, the Lagrange multiplier, introduced to solve the problem. In many cases, however, it is easier to find \( \lambda \) than to find everything else without using \( \lambda \).

The same method works for functions of three variables, except of course everything is one dimension higher: the function to be optimized is a function of three variables and the constraint represents a surface—for example, the function may represent temperature, and we may be interested in the maximum temperature on some surface, like a sphere. The points we seek are those at which the constraint surface is tangent to a level surface of the function. Once again, we consider the constraint surface to be a level surface of some function, and we look for points at which the two gradients are parallel, giving us three equations in four unknowns. The constraint provides a fourth equation.

EXAMPLE 14.8.1 Recall example 14.7.8: the diagonal of a box is 1, we seek to maximize the volume. The constraint is \( 1 = \sqrt{x^2 + y^2 + z^2} \), which is the same as \( 1 = \)
$x^2 + y^2 + z^2$. The function to maximize is $xyz$. The two gradient vectors are $(2x, 2y, 2z)$ and $(y, x, x)$, so the equations to be solved are:

$$
\begin{align*}
yz &= 2x\lambda \\
xz &= 2y\lambda \\
x^2 &= 2z\lambda \\
z &= 1 - x^2 + y^2 + z^2
\end{align*}
$$

If $\lambda = 0$ then at least two of $x, y, z$ must be 0, giving a volume of 0, which will not be the maximum. If we multiply the first two equations by $x$ and $y$ respectively, we get:

$$
\begin{align*}
x^2y &= 2x^2\lambda \\
x^2z &= 2y^2\lambda
\end{align*}
$$

so $2x\lambda = 2y\lambda$ or $x^2 = y^2$, in the same way we can show $x^2 = z^2$. Hence the fourth equation becomes $1 = x^2 + y^2 + z^2$ or $x = \sqrt{1/3}$, and so $x = y = z = 1/\sqrt{3}$ gives the maximum volume. This is of course the same answer we obtained previously.

Another possibility is that we have a function of three variables, and we want to find a maximum or minimum value not on a surface but on a curve; often the curve is the intersection of two surfaces, so that we really have two constraint equations, say $g(x, y, z) = c_1$ and $h(x, y, z) = c_2$. It turns out that at points on the intersection of the surfaces where $f$ has a maximum or minimum value, $
abla f = \lambda \nabla g + \mu \nabla h.$

As before, this gives us three equations, one for each component of the vectors, but now in five unknowns, $x, y, z, \lambda, \mu$. Since there are two constraint functions, we have a total of five equations in five unknowns, and so can usually find the solutions we need.

**EXAMPLE 14.8.2** The plane $x + y + z = 1$ intersects the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the points on the ellipse closest to and farthest from the origin.

We want the extreme values of $f = \sqrt{x^2 + y^2 + z^2}$ subject to the constraints $g = x^2 + y^2 = 1$ and $h = x + y + z = 1$. To simplify the algebra, we may use instead $f = x^2 + y^2 + z^2$, since this has a maximum or minimum value at exactly the points at which $\sqrt{x^2+y^2+z^2}$ does. The gradients are:

$$
\nabla f = (2x, 2y, 2z) \quad \nabla g = (2x, 2y, 0) \quad \nabla h = (1, 1, 1),
$$

so the equations we need to solve are:

$$
\begin{align*}
x^2 &= \lambda 2x + \mu \\
y^2 &= \lambda 2y + \mu \\
z &= 0 - \mu \\
1 &= x^2 + y^2 \\
1 &= x + y + z.
\end{align*}
$$

Subtracting the first two, we get $2y - 2x = \lambda (2y - 2x)$, so either $\lambda = 1$ or $x = y$. If $\lambda = 1$ then $\mu = 0$, so $z = 0$ and the last two equations are:

$$
\begin{align*}
x^2 &= 2 + y^2 \\
1 &= x + y + z.
\end{align*}
$$

Solving these gives $x = 1, y = 0, x = 0, y = 1$, so the points of interest are $(1,0,0)$ and $(0,1,0)$, which are both distance 1 from the origin. If $x = y$, the fourth equation is $2x^2 = 1$, giving $x = y = \pm 1/\sqrt{2}$, and from the fifth equation we get $z = 1 - \pm 1/\sqrt{2}$. The distance from the origin to $(1/\sqrt{2}, 1/\sqrt{2}, -1 + 1/\sqrt{2})$ is $\sqrt{1/2 + 1/2} \approx 1.08$ and the distance from the origin to $(-1/\sqrt{2}, -1/\sqrt{2}, 1 - 1/\sqrt{2})$ is $\sqrt{1/2 + 2/2} = 2/\sqrt{2}$. Thus, the points $(1,0,0)$ and $(0,1,0)$ are closest to the origin and $(1/\sqrt{2}, 1/\sqrt{2}, 1 - 1/\sqrt{2})$ is farthest from the origin. This applet shows the cylinder, the plane, the four points of interest, and the origin.

**Exercises 14.8.**

1. A tin-sided rectangular box is to hold 1/2 cubic meter; what shape should the box be to minimize surface area?

2. The post office will accept packages whose combined length and girth are at most 130 inches (girth is the maximum distance around the package perpendicular to the length). What is the largest volume that can be sent in a rectangular box?

3. The bottom of a rectangular box costs twice as much per unit area as the sides and top. Find the shape for a given volume that will minimize cost.

4. Using Lagrange multipliers, find the shortest distance from the point $(x_0, y_0, z_0)$ to the plane $ax + by + cz = d$. Find all points on the surface $xyz + z^2 + 1 = 0$ that are closest to the origin.

5. The material for the bottom of an aquarium costs half as much as the high strength glass for the four sides. Find the shape of the cheapest aquarium that holds a given volume $V$.

6. The plane $x - y + z = 2$ intersects the cylinder $x^2 + y^2 = 4$ in an ellipse. Find the points on the ellipse closest to and farthest from the origin.

7. Find three positive numbers whose sum is 48 and whose product is as large as possible.

8. Find three positive numbers whose sum is 48 and whose product is as large as possible.

9. Find all points on the plane $x + y + z = 5$ in the first octant at which $f(x,y,z) = xyz^2$ has a maximum value.