13 Vector Functions

13.1 Space Curves

We have already seen that a convenient way to describe a line in three dimensions is to provide a vector that “points to” every point on the line as a parameter \( t \) varies, like:

\[
\{1, 2, 3\} + t(1, -2, 2) = (1 + t, 2 - 2t, 3 + 2t).
\]

Except that this gives a particularly simple geometric object, there is nothing special about the individual functions of \( t \) that make up the coordinates of this vector—any vector with a parameter, like \((f(t), g(t), h(t))\), will describe some curve in three dimensions as \( t \) varies through all possible values.

**Example 13.1.1** Describe the curves \((\cos t, \sin t, 0)\), \((\cos t, \sin t, t)\), and \((\cos t, \sin t, 2t)\).

As \( t \) varies, the first two coordinates in all three functions trace out the points on the unit circle, starting with \((1, 0)\) when \( t = 0 \) and proceeding counterclockwise around the circle as \( t \) increases. In the first case, the \( z \) coordinate is always 0, so this describes precisely the unit circle in the \( x-y \) plane. In the second case, the \( x \) and \( y \) coordinates still describe a circle, but now the \( z \) coordinate varies, so that the height of the curve matches the value of \( t \). When \( t = \pi \), for example, the resulting vector is \((-1, 0, \pi)\). A list of thought should convince you that the result is a helix. In the third vector, the \( z \) coordinate varies twice as fast as the parameter \( t \), so we get a stretched out helix. Both are shown in figure 13.1.1.

On the left is the first helix, shown for \( t \) between 0 and 4\( \pi \); on the right is the second helix, shown for \( t \) between 0 and 2\( \pi \). Both start and end at the same point, but the first helix takes two full “turns” to get there, because its \( z \) coordinate grows more slowly.

**Exercises 13.1.**

1. Describe the curve \( r = (\sin t, \cos t) \).
2. Describe the curve \( r = (t \cos t, t \sin t) \).

![Figure 13.1.1](image)

13.2 Calculus with vector functions

A vector function \( r(t) = (f(t), g(t), h(t)) \) is a function of one variable—that is, there is only one “input” value. What makes vector functions more complicated than the functions \( y = f(x) \) that we studied in the first part of this book is of course that the “output” values are now three-dimensional vectors instead of simply numbers. It is natural to wonder if there is a corresponding notion of derivative for vector functions. In the simpler case of a vector function, the derivative of the form \( \langle f(t), g(t), h(t) \rangle \) is called a vector function; it is a function from the real numbers \( R \) to the set of all three-dimensional vectors. We can alternately think of it as three separate functions, \( x = f(t) \), \( y = g(t) \), and \( z = h(t) \), that describe points in space. In this case we usually refer to the set of equations as parametric equations for the curve, just as for a line. While the parameter \( t \) in a vector function might represent any one of a number of physical quantities, or be simply a “pure number”, it is often convenient and useful to think of \( t \) as representing time. The vector function then tells you where in space a particular object is at any time.

Vector functions can be difficult to understand, that is, difficult to picture. When available, computer software can be very helpful. When working by hand, one useful approach is to consider the “projections” of the curve onto the three standard coordinate planes. We have already done this in part; in example 13.1.1 we noted that all three curves project to a circle in the \( x-y \) plane, since \((\cos t, \sin t, 2t)\) is a two-dimensional vector function for the unit circle.

**Example 13.1.2** Graph the projections of \((\cos t, \sin t, 2t)\) onto the \( x-z \) plane and the \( y-z \) plane. The two-dimensional vector function for the projection onto the \( x-z \) plane is \((\cos t, 2t)\), or in parametric form, \((x = \cos t, z = 2t)\). By eliminating \( t \) we get the equation \( x = \cos z/2 \), the familiar curve shown on the left in figure 13.1.2. For the projection onto the \( y-z \) plane, we start with the vector function \((\sin 2t, 2t)\), which is the same as \( y = \sin t, z = 2t \). Eliminating \( t \) gives \( y = \sin(z/2) \), as shown on the right in figure 13.1.2.

**Exercises 13.2.**

1. Describe the curve \( r = (\sin t, \cos t, \cos t) \).
2. Describe the curve \( r = (t \cos t, t \sin t, t) \).

![Figure 13.1.2](image)

13.3 Vector Functions of a Function

A vector function \( y = s(t) \), in which \( t \) represents time and \( s(t) \) is position on a line, we have seen that the derivative \( s'(t) \) represents velocity; we might hope that in a similar way the derivative of a vector function would tell us something about the velocity of an object moving in three dimensions.

One way to approach the question of the derivative for vector functions is to write down an expression that is analogous to the derivative we already understand, and see if we can make sense of it. This gives us:

\[
\frac{dr}{dt} = \lim_{\Delta t \to 0} \frac{r(t + \Delta t) - r(t)}{\Delta t}.
\]

If we say that what we mean by the limit of a vector is the vector of the individual coordinate limits. So starting with a familiar expression for what appears to be a derivative, we find that we can make good computational sense out of it—but what does it actually mean?

We know how to interpret \( r'(t) \) and \( r(t) \)—they are vectors that point to locations in space; if \( t \) is time, we can think of these points as positions of a moving object at times that are \( \Delta t \) apart. We also know what \( \Delta r = r(t + \Delta t) - r(t) \) means—it is a vector that points from the head of \( r(t) \) to the head of \( r(t + \Delta t) \), assuming both have their tails at the origin. So when \( \Delta t \) is small, \( \Delta r \) is a tiny vector pointing from one point on the path of the object to a nearby point. As \( \Delta t \) gets close to 0, this vector points in a direction that is closer and closer to the direction in which the object is moving; geometrically, it approaches a vector tangent to the path of the object at a particular point.

![Figure 13.3.1](image)
of $\Delta s$ so that in the limit it doesn’t disappear. Thus the limiting vector $(f'(t),y'(t),z'(t))$ will (usually) be a good, non-zero vector that is tangent to the curve. What about the length of this vector? It’s nice that we’ve kept it away from zero, but what does it measure, if anything? Consider the length of one of the vectors that approaches the tangent vector:

$$\frac{|x(t + \Delta t) - x(t)|}{\Delta t}$$

The numerator is the length of the vector that points from one position of the object to a “neighboring” position; this length is approximately the distance traveled by the object between times $t$ and $t + \Delta t$. Dividing this distance by the length of time it takes to travel that distance gives the average speed. As $\Delta t$ approaches zero, this average speed approaches the actual, instantaneous speed of the object at time $t$.

So by performing an “obvious” calculation to get something that looks like the derivative of $y(t)$, we get precisely what we would want from such a derivative: the vector $r'(t)$ is useful—it is a vector tangent to the curve.

**EXAMPLE 13.2.1** We have seen that $r = (\cos t, \sin t, t)$ is a helix. We compute $r' = (-\sin t, \cos t, 1)$, and $|r'| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$. So thinking of this as a description of a moving object, its speed is always $\sqrt{2}$; see figure 13.2.2.

![Figure 13.2.2 A tangent vector on the helix. (AP)](image)

**EXAMPLE 13.2.2** The velocity vector for $(\cos t, \sin t, \cos t)$ is $(-\sin t, \cos t, -\sin t)$. As before, the first two coordinates mean that from above this curve looks like a circle. The $z$ coordinate is now also periodic, so that as the object moves around the curve its height oscillates up and down. In fact it turns out that the curve is a tilted ellipse, as shown in figure 13.2.3.

![Figure 13.2.3 The ellipse $r = (\cos t, \sin t, \cos t)$. (AP)](image)

**EXAMPLE 13.2.3** The velocity vector for $(\cos t, \sin t, \cos 2t)$ is $(-\sin t, \cos t, -2\sin 2t)$. The $z$ coordinate is now oscillating twice as fast as in the previous example, so the graph is not surprising; see figure 13.2.4.

![Figure 13.2.4 $r = (\cos t, \sin t, \cos 2t)$. (AP)](image)

**EXAMPLE 13.2.4** Find the angle between the curves $[t, 1-t, 3+t^2]$ and $[3-t, 1-t, 3+t^2]$ where they meet.

**THEOREM 13.2.5** Suppose $r(t)$ and $s(t)$ are differentiable functions, $f(t)$ is a differentiable function, and $a$ is a real number.

$$\begin{align*}
a. \quad \frac{d}{dt}ar(t) &= ar'(t) \\
b. \quad \frac{d}{dt}(r(t) + s(t)) &= r'(t) + s'(t) \\
c. \quad \frac{d}{dt}(f(r(t)) &= f'(r(t))r'(t) \\
d. \quad \frac{d}{dt}(r(t) \cdot s(t)) &= r'(t) \cdot s(t) + r(t) \cdot s'(t) \\
e. \quad \frac{d}{dt}(r(t) \times s(t)) &= r'(t) \times s(t) + r(t) \times s'(t) \\
f. \quad \frac{d}{dt}(r(t)f(t)) &= r'(t)f(t) + r(t)f'(t) \\
\end{align*}$$

Note that because the cross product is not commutative you must remember to do the three cross products in formula (e) in the correct order.
Now that we know how to make sense of $r'$, we immediately know what an antiderivative must be, namely
\[
\int r(t) \, dt = \left( \int f(t) \, dt, \int g(t) \, dt, \int h(t) \, dt \right),
\]
if $r = (f(t), g(t), h(t))$. What about definite integrals? Suppose that $v(t)$ gives the velocity of an object at time $t$. Then $v(t)\Delta t$ is a vector that approximates the displacement of the object over the time $\Delta t$: $v(t)\Delta t$ points in the direction of travel, and $\|v(t)\|\Delta t$ is the speed of the object times $\Delta t$, which is approximately the distance traveled. Thus, if we sum up many such tiny vectors:
\[
\sum_{i=1}^{n-1} v(t_i) \Delta t
\]
we get an approximation to the displacement vector over the time interval $[t_k, t_{k+1}]$. If we take the limit we get the exact value of the displacement vector:
\[
\lim_{n \to \infty} \sum_{i=1}^{n-1} v(t_i) \Delta t = \int_{t_k}^{t_{k+1}} v(t) \, dt = r(t_{k+1}) - r(t_k).
\]
Denote $r(t_k)$ by $r_k$. Then given the velocity vector we can compute the vector function $r$ giving the location of the object:
\[
r(t) = r_0 + \int_{t_0}^{t} v(u) \, du.
\]

**EXAMPLE 13.2.7** An object moves with velocity vector $\langle \cos t, \sin t, \cos t \rangle$, starting at $(1, 1, 1)$ at time $0$. Find the function $r$ giving its location.

\[
r(t) = (1, 1, 1) + \int_0^t (\cos u, \sin u, \cos u) \, du
\]
\[
= (1, 1, 1) + (\sin u - \cos u, \cos u, \sin u)\bigg|_0^t
\]
\[
= (1, 1, 1) + (\sin t - \cos t, \cos t - \sin t, \sin t - \cos t)
\]
\[
= (1 + \sin t, 1 + \cos t, 1 + \sin t)
\]

See figure 13.2.6.

### 13.3 Arc length and curvature

Sometimes it is useful to compute the length of a curve in space: for example, if the curve represents the path of a moving object, the length of the curve between two points may be the distance traveled by the object between those times.

Recall that if the curve is given by the vector function $r$ then the vector $\Delta s = r(t + \Delta t) - r(t)$ points from one position on the curve to another, as depicted in figure 13.2.1. If the points are close together, the length of $\Delta s$ is close to the length of the curve between the two points. If we add up the lengths of many such tiny vectors, placed head to tail along a segment of the curve, we get an approximation to the length of the curve over that segment. In the limit, as usual, this sum turns into an integral that computes precisely the length of the curve. First note that
\[
|\Delta s| = \left| \frac{\Delta s}{\Delta t} \right| \Delta t = \left| \frac{\Delta r}{\Delta t} \right| \Delta t,
\]

when $\Delta t$ is small. Then the length of the curve between $r(a)$ and $r(b)$ is
\[
\lim_{n \to \infty} \sum_{i=1}^{n-1} |\Delta s| = \lim_{n \to \infty} \sum_{i=1}^{n-1} \left| \frac{\Delta r}{\Delta t} \right| \Delta t = \lim_{n \to \infty} \sum_{i=1}^{n-1} |r'(t_i)| \Delta t = \int_a^b |r'(t)| \, dt.
\]
EXAMPLE 13.3.4 Suppose \( r(t) = (\cos t, \sin t, t) \). We know that this curve is a helix. The distance along the helix from \((1,0,0)\) to \((\cos t, \sin t, t)\) is

\[
s = \int_0^t |r'(u)| \, du = \int_0^t \sqrt{\cos^2 u + \sin^2 u + 1} \, du = \int_0^t \sqrt{2} \, du = \sqrt{2} t.
\]

Thus, the value of \( t \) that gets us distance \( s \) along the helix is \( t = s/\sqrt{2} \), and so the same curve is given by \( \tilde{r}(s) = (\cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2}) \).

In general, if we have a vector function \( r(t) \), to convert it to a vector function in terms of arc length we compute

\[
s = \int_0^t |r'(u)| \, du = f(t),
\]

solve \( s = f(t) \) for \( t \), get \( t = g(s) \), and substitute this back into \( r(t) \) to get \( \tilde{r}(s) = r(g(s)) \).

Suppose that \( t \) is time. By the Fundamental Theorem of Calculus, if we start with arc length

\[
s(t) = \int_0^t |r'(u)| \, du
\]

and take the derivative, we get

\[
x'(s) = r'(g(s)).
\]

Here \( x'(t) \) is the rate at which the arc length is changing, and we have seen that \( |x'(s)| \) is the speed of a moving object; these are of course the same.

Suppose that \( r(s) \) is given in terms of arc length; what is \( |x'(s)|^2 \)? It is the rate at which arc length is changing relative to arc length; it must be \( 1 \) in the case of a helix, for example, the arc length parameterization is \((\cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2})\); the derivative is \(-\sin(s/\sqrt{2}), \cos(s/\sqrt{2}), 1/\sqrt{2}\), and the length of this is

\[
\sqrt{-\frac{\sin^2(s/\sqrt{2})}{2} + \frac{\cos^2(s/\sqrt{2})}{2}} = \sqrt{1 - \frac{1}{2}} = 1
\]

So in general, \( x' \) is a unit tangent vector.

Given a curve \( r(t) \), we would like to be able to measure, at various points, how sharply curved it is. Clearly this is related to how “fast” a tangent vector is changing direction, so a first guess might be that we can measure curvature with \(|x'(s)| \). A little thought shows that this is flawed, if we think of \( t \) as time, for example, we could be tracing out the curve more or less quickly as time passes. The second derivative \( x''(t) \) incorporates this notion of time, so it depends not only on the geometric properties of the curve but on how quickly we move along the curve.

EXAMPLE 13.3.5 Consider \( r(t) = (\cos t, \sin t, 0) \) and \( s(t) = (\cos 2t, \sin 2t, 0) \). Both of these vector functions represent the unit circle in the \( x\)-y plane, but if \( t \) is interpreted as time, the second describes an object moving twice as fast as the first. Computing the second derivatives, we find \( x''(t) = -1, \|x''(t)\| = 1 \).

To remove the dependence on time, we use the arc length parameterization. If \( r(s) \) is given by \( r(s) \), then the first derivative \( x'(s) \) is a unit vector, that is, \( x'(s) = T(s) \). We now compute the second derivative \( x''(s) = -T(s) \) and use \( T(s) \) as the “official” measure of curvature, usually denoted \( \kappa \).

EXAMPLE 13.3.6 We have seen that the arc length parameterization of a particular helix is \( r(s) = (\cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2}) \). Computing the second derivative gives \( x''(s) = (-\cos(s/\sqrt{2}), -\sin(s/\sqrt{2}), 0) \) with length \( 1/2 \).

What if we set a curve as a vector function \( r(t) \), where \( t \) is not arc length? We have seen that arc length can be difficult to compute; furthermore, we do not need to convert to the arc length parameterization to compute curvature. Instead, let us imagine that we have done this, so we have found \( t = g(s) \) and then formed \( r(s) = r(g(s)) \).

The first derivative \( x'(s) \) is a unit tangent vector, so it is the same as the unit tangent vector \( T(t) = T(g(s)) \). Taking the derivative of this we get

\[
\frac{d}{ds} T(g(s)) = T'(g(s)) \frac{ds}{dt} \frac{dT}{ds} = \frac{dT}{ds}.
\]

The curvature is the length of this vector:

\[
\kappa = \left\| \frac{dT}{ds} \right\| = \frac{dT}{ds} \cdot \frac{dT}{ds}.
\]

Recall that we have seen that \( ds/dt = \|x'(t)\| \). Thus we can compute the curvature by computing only derivatives with respect to \( t \); we do not need to do the conversion to arc length.

EXAMPLE 13.3.7 Returning to the helix, suppose we start with the parameterization \( r(t) = (\cos t, \sin t) \). Then \( r(t) = (\sin(t), -\cos(t)) / \sqrt{2} \), and \( T(t) = \frac{(-\cos t, \sin t) / \sqrt{2}}{\sqrt{1/2}} \). Then \( T(t) = (-\cos t, \sin t) / \sqrt{2} \) and \( T'(t) = 1 / \sqrt{2} \). Finally, \( \kappa = 1 / \sqrt{2} \), as before.

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Computing the length of this vector and dividing by \( |x'(t)|^3 \) is still a bit tedious. With the aid of a computer we get

\[
\kappa = \frac{\sqrt{18 + 16 \cos \theta + 17}}{(-16 \cos \theta + 2 \cos 2 \theta + 1)^{3/2}}
\]

Graphing this we get

![Graph of \( \kappa \) vs. \( \theta \)]

Compare this to figure 13.2.4—you may want to load the Java applet there so that you can see it from different angles. The highest curvature occurs where the curve has its highest and lowest points, and indeed in the picture these appear to be the most sharply curved portions of the curve, while the curve is almost a straight line midway between those points.

Let’s see why this alternate formula is correct. Starting with the definition of \( T \), \( x' = |x'| T \) so by the product rule \( x'' = |x'| T + |x'|' T \). Then by Theorem 12.4.2 the cross product is

\[
x' \times x'' = |x'| T \times (|x'| T + |x'|' T)
\]

and

\[
x' \times x'' = |x'| T \times (|x'| T + |x'|' T)
\]

because \( T \times T = 0 \), since \( T \) is parallel to itself. Then

\[
|x' \times x''| = |x'| T \times T = |x'| T
\]

using exercise 8 in section 13.2 to see that \( \theta = \pi/2 \). Dividing both sides by \( |x'| \) then gives the desired formula.

We used the fact here that \( T' \) is perpendicular to \( T \); the vector \( N = -T' / |T'| \) is thus a unit vector perpendicular to \( T \), called the unit normal to the curve. Occasionally of use is the unit binormal \( B = T \times N \), a unit vector perpendicular to both \( T \) and \( N \).
13.4 Motion along a curve

We have already seen that if \( t \) is time and an object’s location is given by \( \mathbf{r}(t) \), then the derivative \( \mathbf{r}'(t) \) is the velocity vector \( \mathbf{v}(t) \). Just as \( \mathbf{v}(t) \) is a vector describing how \( \mathbf{r}(t) \) changes, so is \( \mathbf{v}'(t) \) a vector describing how \( \mathbf{v}(t) \) changes, namely, \( \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) \) is the acceleration vector.

**Example 13.4.1** Suppose \( \mathbf{r}(t) = (\cos t, \sin t, 1) \). Then \( \mathbf{v}(t) = (-\sin t, \cos t, 0) \) and \( \mathbf{a}(t) = (\cos t, -\sin t, 0) \). This describes the motion of an object traveling on a circle of radius 1, with constant \( z \) coordinate 1. The velocity vector is of course tangent to the curve; note that \( \mathbf{a} \cdot \mathbf{v} = 0 \), so \( \mathbf{v} \) and \( \mathbf{a} \) are perpendicular. In fact, it is not hard to see that a points from the location of the object to the center of the circular path at \( (0, 0, 1) \).

Recall that the unit tangent vector is given by \( \mathbf{T}(t) = \mathbf{v}(t)/\|\mathbf{v}(t)\| \), so \( \mathbf{v} = \|\mathbf{v}\| \mathbf{T} \). If we take the derivative of both sides of this equation we get
\[
\mathbf{a} = \frac{\mathbf{v}'(t)}{\|\mathbf{v}(t)\|} + \frac{\mathbf{v}(t)\mathbf{T}'(t)}{\|\mathbf{v}(t)\|}.
\]  
(13.4.1)

Also recall the definition of the curvature, \( \kappa = \|\mathbf{T}'(t)\|/\|\mathbf{v}(t)\| \). Finally, recall that we defined the unit normal vector as \( \mathbf{N} = \mathbf{T}'/\|\mathbf{T}'\| \), so \( \mathbf{T}' = \|\mathbf{N}\| \mathbf{N} \). Substituting into equation 13.4.1 we get
\[
\mathbf{a} = \\frac{\mathbf{v}'(t)}{\|\mathbf{v}(t)\|} + \frac{\mathbf{v}(t)\mathbf{T}'(t)}{\|\mathbf{v}(t)\|} \mathbf{N}.
\]  
(13.4.2)

The quantity \( \mathbf{v}'(t) \) is the speed of the object, often written as \( v(t) \); \( \mathbf{v}'(t) \) is the rate at which the speed is changing, or the scalar acceleration of the object, \( a(t) \). Rewriting equation 13.4.2 with these gives
\[
\mathbf{a} = \frac{\mathbf{v}'(t)}{\|\mathbf{v}(t)\|} + \kappa \mathbf{v}(t) \mathbf{N} = \mathbf{a}_T + \kappa \mathbf{v} \mathbf{N}.
\]

\( \mathbf{a}_T \) is the tangential component of acceleration and \( \mathbf{a}_N \) is the normal component of acceleration. We have already seen that \( \mathbf{a}_T \) measures how the speed is changing; if you are riding in a vehicle with large \( \mathbf{a}_T \) you will feel a force pulling you into your seat. The other component, \( \mathbf{a}_N \), measures how sharply your direction is changing with respect to time. So it naturally is related to how sharply the path is curved, measured by \( \kappa \), and also to how fast you are going. Because \( \mathbf{a}_N \) includes \( \kappa \), note that the effect of speed is magnified; doubling your speed around a curve quadruples the value of \( \mathbf{a}_N \). You feel the effect of this as a force pushing you toward the outside of the curve, the “centrifugal force.”

In practice, if want \( \mathbf{a}_N \) we would use the formula for \( \kappa \):
\[
\kappa = \frac{\|\mathbf{v}' \times \mathbf{v}\|}{\|\mathbf{v}\|^3} = \frac{\|\mathbf{v}' \times \mathbf{v}\|}{\|\mathbf{v}\|^3}.
\]

To compute \( \mathbf{a}_T \) we can project it onto \( \mathbf{v} \):
\[
\mathbf{a}_T = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} = \frac{\mathbf{v}' \cdot \mathbf{v}'}{\|\mathbf{v}\|}.
\]

**Example 13.4.2** Suppose \( \mathbf{r} = (t, t^3, t^5) \). Compute \( \mathbf{v}, \mathbf{a}, \mathbf{a}_T \), and \( \mathbf{a}_N \).

Taking derivatives we get \( \mathbf{v} = (1, 3t^2, 5t^4) \) and \( \mathbf{a} = (0, 6t, 20t^3) \). Then
\[
\mathbf{a}_T = \frac{4t + 18t^3}{\sqrt{1 + 4t^2 + 9t^4 + 9t^6}} \text{ and } \mathbf{a}_N = \frac{\sqrt{1 + 4t^2 + 9t^4 + 9t^6}}{\sqrt{1 + 4t^2 + 9t^4 + 9t^6}}.
\]

**Exercises 13.4.**

1. Let \( \mathbf{r} = (\cos t, \sin t, t) \). Compute \( \mathbf{v}, \mathbf{a}, \mathbf{a}_T, \) and \( \mathbf{a}_N \).
2. Let \( \mathbf{r} = (\cos t, t^2) \). Compute \( \mathbf{v}, \mathbf{a}, \mathbf{a}_T, \) and \( \mathbf{a}_N \).
3. Let \( \mathbf{r} = (\cos t, \sin t, t) \). Compute \( \mathbf{v}, \mathbf{a}, \mathbf{a}_T, \) and \( \mathbf{a}_N \).
4. Let \( \mathbf{r} = (t^2, 2t - 3, t^3 - 3t) \). Compute \( \mathbf{v}, \mathbf{a}, \mathbf{a}_T, \) and \( \mathbf{a}_N \).
5. Let \( \mathbf{r} = (t^2, 2t, t^3) \). Compute \( \mathbf{v}, \mathbf{a}, \mathbf{a}_T, \) and \( \mathbf{a}_N \).
6. Suppose an object moves so that its acceleration is given by \( \mathbf{a} = (-3\cos t, -2\sin t, 0) \). At time \( t = 0 \) the object is at \( (3, 0, 0) \) and its velocity vector is \( (0, 2, 0) \). Find \( \mathbf{v}(t) \) and \( \mathbf{r}(t) \) for the object.
7. Suppose an object moves so that its acceleration is given by \( \mathbf{a} = (-3\cos t, -2\sin t, 0) \). At time \( t = 0 \) the object is at \( (3, 0, 0) \) and its velocity vector is \( (0, 2, 1) \). Find \( \mathbf{v}(t) \) and \( \mathbf{r}(t) \) for the object.
8. Suppose an object moves so that its acceleration is given by \( \mathbf{a} = (-3\cos t, -2\sin t, 0) \). At time \( t = 0 \) the object is at \( (3, 0, 0) \) and its velocity vector is \( (0, 2, 1) \). Find \( \mathbf{v}(t) \) and \( \mathbf{r}(t) \) for the object.
9. Suppose an object moves so that its acceleration is given by \( \mathbf{a} = (-3\cos t, -2\sin t, 0) \). At time \( t = 0 \) the object is at \( (3, 0, 0) \) and its velocity vector is \( (0, 2, 1) \). Find \( \mathbf{v}(t) \) and \( \mathbf{r}(t) \) for the object.