9

Applications of Integration

9.1 Area between Curves

We have seen how integration can be used to find an area between a curve and the x-axis. With very little change we can find some areas between curves; indeed, the area between a curve and the x-axis may be interpreted as the area between the curve and a second “curve” with equation \( y = 0 \). In the simplest of cases, the idea is quite easy to understand.

**EXAMPLE 9.1.1** Find the area below \( f(x) = -x^2 + 4x + 1 \) and above \( g(x) = -x^2 + 7x^2 - 10x + 3 \) over the interval \( 1 \leq x \leq 2 \). These are the same curves as before but lowered by 2. In figure 9.1.1 we show the two curves together, with the desired area shaded, then \( f \) alone with the area under \( f \) shaded, and then \( g \) alone with the area under \( g \) shaded.

![Figure 9.1.1 Area between curves as a difference of areas.](image)

**EXAMPLE 9.1.2** Find the area below \( f(x) = -x^2 + 4x + 1 \) and above \( g(x) = -x^2 + 7x^2 - 10x + 3 \) over the interval \( 1 \leq x \leq 2 \), these are the same curves as before but lowered by 2. In figure 9.1.1 we show the two curves together, with the desired area shaded, then \( f \) alone with the area under \( f \) shaded, and then \( g \) alone with the area under \( g \) shaded.

![Figure 9.1.2 Approximating area between curves with rectangles.](image)

**EXAMPLE 9.1.3** Find the area between \( f(x) = -x^2 + 4x \) and \( g(x) = x^2 - 6x + 5 \) over the interval \( 0 \leq x \leq 1 \); the curves are shown in figure 9.1.4. Generally, we should interpret “area” in the usual sense, as a necessarily positive quantity. Since the two curves cross, we need to compute two areas and add them. First we find the intersection point of the curves:

\[
-x^2 + 4x = x^2 - 6x + 5 \\
0 = 2x^2 - 10x + 5 \\
x = \frac{10 \pm \sqrt{100 - 40}}{4} = 5 \pm \sqrt{10}/2.
\]

The intersection point we want is \( x = a = (5 - \sqrt{10})/2 \). Then the total area is

\[
\int_0^2 x^2 - 6x + 5 - (-x^2 + 4x) \, dx + \int_a^1 -x^2 + 4x - (x^2 - 6x + 5) \, dx \\
= \int_0^2 x^2 - 10x + 5 \, dx + \int_a^1 -2x^2 + 10x - 5 \, dx \\
= \left[ \frac{2x^3}{3} - 5x^2 + 5x \right]_0^2 + \left[ \frac{2x^3}{3} + 5x^2 - 5x \right]_a^1 \\
= \frac{52}{3} + 5\sqrt{10}/2.
\]

after a bit of simplification.

![Figure 9.1.3 Area between curves.](image)

![Figure 9.1.4 Area between curves that cross.](image)

**EXAMPLE 9.1.4** Find the area between \( f(x) = -x^2 + 4x \) and \( g(x) = x^2 - 6x + 5 \), the curves are shown in figure 9.1.5. Here we are not given a specific interval, so it must
be the case that there is a “natural” region involved. Since the curves are both parabolas, the only reasonable interpretation is the region between the two intersection points, which we found in the previous example:

\[
\frac{5 + \sqrt{15}}{2}
\]

If we let \(a = (5 - \sqrt{15})/2\) and \(b = (5 + \sqrt{15})/2\), the total area is

\[
\int_a^b (x^2 + 4x - (x^2 - 6x + 5)) \, dx = \int_a^b 10x - 5 \, dx
\]

\[
= \left[ \frac{5x^2}{2} + 5x^2 - 5x \right]_a^b
\]

\[
= 5\sqrt{\frac{15}{}}
\]

after a bit of simplification.

![Figure 9.1.5 Area bounded by two curves.](image)

### Exercises 9.1.
Find the area bounded by the curves.

1. \(y = x^2 - 4x\) and \(y = x^2\) (the part to the right of the y-axis) ⇒
2. \(y = x^3\) and \(x = y^2\) ⇒
3. \(x = 1 - y^2\) and \(y = -x - 1\) ⇒
4. \(x = 3y - y^2\) and \(x + y = 3\) ⇒
5. \(y = \cos(\pi/2)\) and \(y = 1 - x^2\) (in the first quadrant) ⇒
6. \(y = \sin(\pi/2)\) and \(y = x\) (in the first quadrant) ⇒
7. \(y = \sqrt{x} - y = x^2\) ⇒
8. \(y = \sqrt{x} + y = x^2 + 1\), \(0 \leq x \leq 4\) ⇒
9. \(x = 0\) and \(x = 25 - y^2\) ⇒
10. \(y = \sin x\cos x\) and \(y = \sin x\), \(0 \leq x \leq \pi\) ⇒

### 9.2 Distance, Velocity, Acceleration

**Example 9.2.1** Suppose an object is acted upon by a constant force \(F\). Find \(v(t)\) and \(a(t)\). By Newton’s law \(F = ma\), so the acceleration is \(F/m\), where \(m\) is the mass of the object. Then we first have

\[
v(t) = v(t_0) + \int_{t_0}^t F/m \, dt = v_0 + F/m (t - t_0)
\]

using the usual convention \(v_0 = v(t_0)\). Then

\[
a(t) = a(t_0) + \int_{t_0}^t \left( -\frac{F}{m} \right) \, dt = a_0 + v_0(t - t_0) + \frac{F}{m}(t - t_0)^2/2
\]

For instance, when \(F/m = -g\) is the constant of gravitational acceleration, then this is the falling body formula (if we neglect air resistance) familiar from elementary physics:

\[
v(t) = v_0 + gt - \frac{g}{2}t^2
\]

or in the common case that \(v_0 = 0\),

\[
s_0 + vt - \frac{g}{2}t^2.
\]

Recall that the integral of the velocity function gives the net distance traveled, that is, the displacement. If you want to know the total distance traveled, you must find out where the velocity function crosses the t-axis, integrate separately over the time intervals when \(v(t)\) is positive and when \(v(t)\) is negative, and add up the absolute values of the different integrals. For example, if an object is thrown straight upward at 19.6 m/sec, its velocity function is \(v(t) = -9.8t + 19.6\), using \(g = 9.8\) m/sec\(^2\) for the force of gravity. This is a straight line which is positive for \(t < 2\) and negative for \(t > 2\). The net distance traveled in the first 4 seconds is thus

\[
\int_0^4 (-9.8t + 19.6) \, dt = 0
\]

while the total distance traveled in the first 4 seconds is

\[
\int_0^2 (-9.8t + 19.6) \, dt + \left| \int_2^4 (-9.8t + 19.6) \, dt \right| = 19.6 + | -19.6 | = 39.2
\]

meters, 19.6 meters up and 19.6 meters down.

### Exercises 9.2.
For each velocity function find both the net distance and the total distance traveled during the indicated time interval (graph \(v(t)\) to determine when it’s positive and when it’s negative):

1. \(v = \cos(\pi t/2), 0 \leq t \leq 2.5\) ⇒
2. \(v = -9.8t + 49, 0 \leq t \leq 10\) ⇒
3. \(v = (t - 3)(t - 1), 0 \leq t \leq 5\) ⇒
4. \(v = \sin(\pi t/3) - t, 0 \leq t \leq 1\) ⇒
5. An object is shot upwards from ground level with an initial velocity of 2 meters per second; it is subject only to the force of gravity (no air resistance). Find its maximum altitude and the time at which it hits the ground.
6. An object is shot upwards from ground level with an initial velocity of 3 meters per second; it is subject only to the force of gravity (no air resistance). Find its maximum altitude and the time at which it hits the ground.
We have seen how to compute certain areas by using integration; some volumes may also be computed by evaluating an integral. Generally, the volumes that we can compute this way have cross-sections that are easy to describe.

### Example 9.3.1
Find the volume of a pyramid with a square base that is 20 meters tall and 20 meters on a side at the base. As with most of our applications of integration, we begin by asking how we might approximate the volume. Since we can easily compute the volume of a rectangular prism (that is, a “box”), we will use some boxes to approximate the volume of the pyramid, as shown in figure 9.3.1: on the left is a cross-sectional view, on the right is a 3D view of part of the pyramid with some of the boxes used to approximate the volume.

Each box has volume of the form \( \Delta V = \Delta y \cdot \text{base area} \). Unfortunately, there are two variables here; fortunately, we can write \( x \) in terms of \( y \): \( x = 10 - y/2 \) or \( x = 10 - y_0/2 \). Then the total volume is approximately

\[
\sum_{i=1}^{n} \pi (y_i/2)^2 \Delta y
\]

and in the limit we get the volume as the value of an integral:

\[
\int_{y_0}^{y_1} \pi (y/2)^2 \, dy = \int_{y_0}^{y_1} \frac{\pi}{15} y^3 \, dy = \frac{\pi}{15} \left( \frac{y_1^4}{4} - \frac{y_0^4}{4} \right) = \frac{\pi}{60} \left( 2000 - 8000 \right) = \frac{1000 \pi}{3}.
\]

As you may know, the volume of a pyramid is \( (1/3)(\text{height})(\text{area of base}) = (1/3)(20)(400) \), which agrees with our answer.

### Example 9.3.2
The base of a solid is the region between \( f(x) = x^2 - 1 \) and \( g(x) = -x^2 + 1 \), and its cross-sections perpendicular to the \( x \)-axis are equilateral triangles, as indicated in figure 9.3.2. The solid has been truncated to show a triangular cross-section above \( x = 1/2 \). Find the volume of the solid.

The volume of the pyramid, as shown in figure 9.3.1: on the left is a cross-sectional view, on the right is a 3D view of part of the pyramid with some of the boxes used to approximate the volume.

Each box has volume of the form \( (2x)(2x)\Delta y \). Unfortunately, there are two variables here; fortunately, we can write \( x \) in terms of \( y \): \( x = 10 - y/2 \) or \( x = 10 - y_0/2 \). Then the total volume is approximately

\[
\sum_{i=1}^{n} \pi (10 - y_i/2)^2 \Delta y
\]

and in the limit we get the volume as the value of an integral:

\[
\int_{y_0}^{y_1} \pi (10 - y/2)^2 \, dy = \int_{y_0}^{y_1} \frac{\pi}{3} (10^3 - 200y^2 + 800y - 800) \, dy
\]

As you may know, the volume of a pyramid is \( (1/3)(\text{height})(\text{area of base}) = (1/3)(20)(400) \), which agrees with our answer.

### Example 9.3.3
Find the volume of a right circular cone with base radius 10 and height 20. (A right circular cone is one with a circular base and with the tip of the cone directly over the center of the base.) We can view this cone as produced by the rotation of the line \( y = x/2 \) rotated about the \( x \)-axis, as indicated in figure 9.3.3.

At a particular point on the \( x \)-axis, say \( x_i \), the radius of the resulting cone is the \( y \)-coordinate of the corresponding point on the line, namely \( y_i = x_i/2 \). Thus the total volume is approximately

\[
\sum_{i=1}^{n} \pi (x_i/2)^2 \Delta x
\]

and the exact volume is

\[
\int_{0}^{20} \frac{x^2}{4} \, dx = \frac{2000}{3} - \frac{2000}{3} = \frac{2000}{3}
\]

### Example 9.3.4
Find the volume of the object generated when the area between \( y = x^2 \) and \( y = x \) is rotated around the \( x \)-axis. This solid has a “hole” in the middle; we can compute the volume by subtracting the volume of the hole from the volume enclosed by the outer surface of the solid. In figure 9.3.4 we show the region that is rotated, the resulting solid with the front half cut away, the cone that forms the outer surface, the horn-shaped hole, and a cross-section perpendicular to the \( x \)-axis.

We have already computed the volume of a cone; in this case it is \( \pi/3 \). At a particular value of \( x \), say \( x_i \), the cross-section of the horn is a circle with radius \( x_i^2 \), so the volume of the horn is

\[
\int_{0}^{x_i} \pi x^2 \, dx = \frac{\pi x_i^4}{4} - \frac{\pi x_i^4}{4} = \frac{\pi x_i^4}{2},
\]

so the desired volume is \( \pi/3 - \pi/3 = 2\pi/15 \).

As with the area between curves, there is an alternate approach that computes the desired volume “all at once” by approximating the volume of the actual solid. We can approximate the volume of a slice of the solid with a washer-shaped volume, as indicated in figure 9.3.5.

The volume of such a washer is the area of the face times the thickness. The thickness, as usual, is \( \Delta x \), while the area of the face is the area of the outer circle minus the area of the inner circle.

\[
\text{Volume} = \pi \left( r_2^2 - r_1^2 \right) \Delta x
\]

where \( r_2 \) and \( r_1 \) are the outer and inner radii of the washer, respectively.
Figure 9.3.5 Solid with a hole, showing the outer cone and the shape to be removed to form the hole. (AP)

the inner circle, say \( R^2 - r^2 \). If the present example, at a particular \( x_i \), the radius \( R \) is \( x_i \) and \( r = x_i^2 \). Hence, the whole volume is

\[
\int_0^1 \pi x^2 - \pi x^4 \, dx = \pi \left( \frac{1}{5} - \frac{1}{7} \right) = \frac{2\pi}{15}
\]

Of course, what we have done here is exactly the same calculation as before, except we "kinds" of typical rectangles: those that go from the line to the parabola and those that resulting solid by the method we have used so far. The problem is that there are two

Exercises 9.3.

1. Verify that \( \int_0^1 (1 + \sqrt{y})^2 - (1 - \sqrt{y})^2 \, dy + \pi \int_0^1 (1 + \sqrt{y})^2 - (y - 1)^2 \, dy = \frac{8}{3} + \frac{65}{6} \pi = \frac{27}{2} \pi \).

2. Verify that \( \int_0^1 2x(x + 1 - (x - 1)^2) \, dx = \frac{25}{3} \pi \).

3. Verify that \( \int_0^1 \pi (1 - x)^2 \, dx = \frac{8}{3} \pi \).

4. Verify that \( \int_0^1 2\pi \sqrt{1 - y} \, dy = \frac{8}{3} \pi \).

5. Use integration to find the volume of the solid obtained by revolving the region bounded by \( x + y = 2 \) and the \( x \) and \( y \) axes around the \( x \)-axis.

6. Find the volume of the solid obtained by revolving the region bounded by \( y = x - x^2 \) and the \( x \)-axis around the \( y \)-axis.

7. Find the volume of the solid obtained by revolving the region bounded by \( y = \sqrt{\frac{1}{2}x} \) between \( x = 0 \) and \( x = 2 \), the \( y \)-axis, and the line \( y = 1 \) around the \( x \)-axis.

8. Let \( S \) be the region of the \( y \)-plane bounded above by the curve \( x^2 = \frac{1}{2}y \) and below by the line \( y = 1 \), on the left by the line \( x = 2 \), and on the right by the line \( x = 4 \). Find the volume of the solid obtained by rotating \( S \) around (a) the \( x \)-axis, (b) the line \( y = 1 \), (c) the \( y \)-axis, (d) the line \( x = 2 \).

9. The equation \( x^2 + y^2 + z^2 = 4 \) describes an ellipse. Find the volume of the solid obtained by rotating the ellipse around the \( x \)-axis and also around the \( y \)-axis. These solids are called ellipsoids: one is vaguely rugby-ball-shaped, one is sort of flying-saucer shaped, or perhaps squashed-beach-ball-shaped.

Figure 9.3.7 Ellipsoids.

10. Use integration to compute the volume of a sphere of radius \( r \). You should of course get the well-known formula \( 4\pi r^3/3 \).

11. A hemispherical bowl of radius \( r \) contains water to a depth \( h \). Find the volume of water in the bowl.

12. The base of a tetrahedron (a triangular pyramid) of height \( h \) is an equilateral triangle of side \( a \). Its cross-sections perpendicular to an altitude are equilateral triangles. Express its volume \( V \) in terms of \( h \) and \( s \). Verify that your answer is \( (1/3)(\text{area of base})(\text{height}) \).

13. The base of a solid is the region between \( f(x) = \cos x \) and \( g(x) = -\cos x, -\pi/2 \leq x \leq \pi/2 \), and its cross-sections perpendicular to the \( x \)-axis are squares. Find the volume of the solid.

9.4 Average Value of a Function

The average of some finite set of values is a familiar concept. For example, the class scores on a quiz are 10, 9, 10, 8, 7, 5, 7, 6, 4, 2, 7, 8, then the average score is the sum of these numbers divided by the size of the class:

\[
\text{average score} = \frac{10 + 9 + 10 + 8 + 7 + 5 + 7 + 6 + 4 + 2 + 7 + 8}{12} = 6.83
\]

Suppose that between \( t = 0 \) and \( t = 1 \) the speed of an object is \( \sin(t) \). What is the average speed of the object over that time? The question sounds as if it must make sense, yet we can’t merely add up some number of speeds and divide, since the speed is changing continuously over the time interval.

To make sense of “average” in this context, we fall back on the idea of approximation. Consider the speed of the object at tenth of a second intervals: \( \sin(0.1\pi), \sin(0.2\pi), \sin(0.3\pi), \ldots, \sin(0.9\pi) \). The average speed “should” be fairly close to the average of these ten speeds:

\[
\frac{1}{10} \sum_{i=1}^{10} \sin(i\pi/10) \approx \frac{1}{10} \cdot 0.63 = 0.063
\]

If, of course, we compute more speeds at more times, the average of these speeds should be closer to the “real” average. If we take the average of \( n \) speeds at equally spaced times, we get:

\[
\frac{1}{n} \sum_{i=1}^{n} \sin(i\pi/n)
\]

Here the individual times are \( t_i = i/n \), so rewriting slightly we have

This is almost the sort of sum that we know turns into an integral; what’s apparently missing is \( \Delta t \) — but in fact, \( \Delta t = 1/n \), the length of each subinterval. So rewriting again:

\[
\frac{1}{n} \sum_{i=1}^{n} \sin(t_i) \Delta t = \frac{1}{n} \sum_{i=1}^{n} \sin(t_i),
\]

Now this has exactly the right form, so that in the limit we get

\[
\text{average speed} = \int_0^1 \sin(t) \, dt = \left[ -\cos(t) \right]_0^1 = -\cos(1) + \cos(0) = \frac{2}{\pi} \approx 0.3366 = 0.64.
\]

It’s not entirely obvious from this one simple example how to compute such an average in general. Let’s look at a somewhat more complicated case. Suppose that the velocity
of an object is \(16t^2 + 5\) feet per second. What is the average velocity between \(t = 1\) and \(t = 3\)? Again we set up an approximation to the average: 

\[
\frac{1}{n} \sum_{i=0}^{n-1} (16t^2_i + 5),
\]

where the values \(t_i\) are evenly spaced times between 1 and 3. Once again we are “missing” \(\Delta t\), and this time \(1/n\) is not the correct value. What is \(\Delta t\) in general? It is the length of a subinterval; in this case we take the interval \([1, 3]\) and divide it into \(n\) subintervals, so each has length \((3 - 1)/n = 2/n = \Delta t\). Now with the usual “multiply and divide by the same thing” trick we can rewrite the sum:

\[
\frac{1}{n} \sum_{i=0}^{n-1} (16t^2_i + 5) = \frac{1}{n} \sum_{i=0}^{n-1} (16t^2_i + 5) \cdot \frac{1}{n} - \frac{1}{n} \sum_{i=0}^{n-1} \Delta t.
\]

In the limit this becomes

\[
\frac{1}{n} \int_1^3 16t^2 + 5 \, dt = \frac{1}{2} \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \cdot 5280 = 1760.
\]

Does this seem reasonable? Let’s picture it: in figure 9.4.1 the velocity function together with the horizontal line \(y = 223/3 \approx 74.3\). Certainly the height of the horizontal line looks at least plausible for the average height of the curve.

Here’s another way to interpret “average” that may make our computation appear even more reasonable. The object of our example goes a certain distance between \(t = 1\) and \(t = 3\). If instead the object were to travel at the average speed over the same time, it should go the same distance. At an average speed of 223/3 feet per second for two seconds the object would go 446/3 feet. How far does it actually go? We know how to compute this:

\[
\frac{1}{3} \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \cdot \frac{1}{2} \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \cdot \frac{1}{2} \cdot 5280 = \frac{1}{3} \cdot 2640 = 880
\]

and \(t = 3\). If instead the object were to travel at the average speed over the same time, it should go the same distance. At an average speed of 223/3 feet per second for two seconds the object would go 446/3 feet. How far does it actually go? We know how to compute this:

\[
\frac{1}{3} \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \cdot \frac{1}{2} \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \cdot \frac{1}{2} \cdot 5280 = \frac{1}{3} \cdot 2640 = 880
\]

EXAMPLE 9.5.1 How much work is done in lifting a 10 pound weight from the surface of the earth to an orbit \(200\) miles above the earth? This is the same problem as before in different units, and we are not specifying a value for \(G\) or \(m\) or \(r_b\). In fact, get the answer by any method you can think of. Note that if we assume the force due to gravity is 10 pounds over the whole distance we will have \(W = 10 \cdot 16t^2 + 5 \cdot 200 = 16t^2 + 5 \cdot 3200\). On the other hand, if we assume the force is constant, we get a much simpler answer. The work is:

\[
W = \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \cdot 2640 = 880
\]

EXAMPLE 9.5.2 How much work is done in lifting a 10 pound weight from the surface of the earth to an orbit \(200\) miles above the earth? This is the same problem as before in different units, and we are not specifying a value for \(G\) or \(m\) or \(r_b\). In fact, get the answer by any method you can think of. Note that if we assume the force due to gravity is 10 pounds over the whole distance we will have \(W = 10 \cdot 16t^2 + 5 \cdot 200 = 16t^2 + 5 \cdot 3200\). On the other hand, if we assume the force is constant, we get a much simpler answer. The work is:

\[
W = \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \cdot 2640 = 880
\]

and \(t = 3\). If instead the object were to travel at the average speed over the same time, it should go the same distance. At an average speed of 223/3 feet per second for two seconds the object would go 446/3 feet. How far does it actually go? We know how to compute this:

\[
\frac{1}{3} \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \cdot \frac{1}{2} \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \cdot \frac{1}{2} \cdot 5280 = \frac{1}{3} \cdot 2640 = 880
\]

and \(t = 3\). If instead the object were to travel at the average speed over the same time, it should go the same distance. At an average speed of 223/3 feet per second for two seconds the object would go 446/3 feet. How far does it actually go? We know how to compute this:

\[
\frac{1}{3} \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \cdot \frac{1}{2} \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \cdot \frac{1}{2} \cdot 5280 = \frac{1}{3} \cdot 2640 = 880
\]

and \(t = 3\). If instead the object were to travel at the average speed over the same time, it should go the same distance. At an average speed of 223/3 feet per second for two seconds the object would go 446/3 feet. How far does it actually go? We know how to compute this:

\[
\frac{1}{3} \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \cdot \frac{1}{2} \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \cdot \frac{1}{2} \cdot 5280 = \frac{1}{3} \cdot 2640 = 880
\]

and \(t = 3\). If instead the object were to travel at the average speed over the same time, it should go the same distance. At an average speed of 223/3 feet per second for two seconds the object would go 446/3 feet. How far does it actually go? We know how to compute this:

\[
\frac{1}{3} \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \cdot \frac{1}{2} \int_1^3 16t^2 + 5 \, dt = \frac{1}{3} \cdot \frac{1}{2} \cdot 5280 = \frac{1}{3} \cdot 2640 = 880
\]
EXAMPLE 9.5.5 Suppose a force is applied that compresses the spring to length $0$. The spring has been stretched or compressed: according to Hooke’s Law the magnitude of this force is proportional to the distance the spring has been stretched or compressed; this effect is quantified by the spring constant $k$. If the spring is either stretched or compressed the spring provides an opposing force; this is the basis for the work completed by compressing or stretching a spring. To lift the bucket $10$ meters by raising the cable $10$ meters? (The top half of the cable ends in the circular cross-section through the tank has radius $r$ = 1 meter. If the depth of the water is $5$ meters, how much work is required to pump all the water out the top of the tank?)

9.5 Center of Mass

Suppose a beam is $10$ meters long and that there are three weights on the beam: a $10$ kilogram weight $3$ meters from the left end, a $5$ kilogram weight $6$ meters from the left end, and a $4$ kilogram weight $8$ meters from the left end. Where should a fulcrum be placed so that the beam balances? Let’s assign a scale to the beam, from $0$ at the left end to $10$ at the right, so that we can denote locations on the beam simply as $x$ coordinates; the weights are at $x = 3$, $x = 6$, and $x = 8$, as in figure 9.6.1.

\[ 0 = m_1 x_1 + m_2 x_2 + m_3 x_3 \]

\[ x_1 = 3, \quad x_2 = 6, \quad x_3 = 8 \]

9.6 Center of Mass

Suppose a beam is $10$ meters long and that there are three weights on the beam: a $10$ kilogram weight $3$ meters from the left end, a $5$ kilogram weight $6$ meters from the left end, and a $4$ kilogram weight $8$ meters from the left end. Where should a fulcrum be placed so that the beam balances? Let’s assign a scale to the beam, from $0$ at the left end to $10$ at the right, so that we can denote locations on the beam simply as $x$ coordinates; the weights are at $x = 3$, $x = 6$, and $x = 8$, as in figure 9.6.1.

\[ 0 = m_1 x_1 + m_2 x_2 + m_3 x_3 \]

\[ x_1 = 3, \quad x_2 = 6, \quad x_3 = 8 \]

9.6 Center of Mass

Suppose a beam is $10$ meters long and that there are three weights on the beam: a $10$ kilogram weight $3$ meters from the left end, a $5$ kilogram weight $6$ meters from the left end, and a $4$ kilogram weight $8$ meters from the left end. Where should a fulcrum be placed so that the beam balances? Let’s assign a scale to the beam, from $0$ at the left end to $10$ at the right, so that we can denote locations on the beam simply as $x$ coordinates; the weights are at $x = 3$, $x = 6$, and $x = 8$, as in figure 9.6.1.

\[ 0 = m_1 x_1 + m_2 x_2 + m_3 x_3 \]

\[ x_1 = 3, \quad x_2 = 6, \quad x_3 = 8 \]
If we set this equal to zero and solve for $x$ we get an approximation to the balance point of the beam:

$$0 = \sum_{i=0}^{n-1} x_i (1 + x_i) \Delta x - \frac{2}{3} \sum_{i=0}^{n-1} (1 + x_i) \Delta x$$

$$\bar{x} = \frac{\sum_{i=0}^{n-1} x_i (1 + x_i) \Delta x}{\sum_{i=0}^{n-1} (1 + x_i) \Delta x}$$

The denominator of this fraction has a very familiar formulation. Consider one term of the sum in the denominator: $(1 + x_i) \Delta x$. This is the density times a short length, $\Delta x$, which in other words is approximately the mass of the beam between $x_i$ and $x_{i+1}$.

When we add these up we get approximately the mass of the beam.

Now each of the sums in the fraction has the right form to turn into an integral, which in turn gives us the exact value of $\bar{x}$:

$$\bar{x} = \frac{\int_0^1 x (1 + x) \, dx}{\int_0^1 (1 + x) \, dx}$$

The numerator of this fraction is called the moment of the system around zero:

$$\int_0^1 x (1 + x) \, dx = \int_0^1 x + 2 \, dx = \frac{1150}{3}$$

and the denominator is the mass of the beam:

$$\int_0^1 (1 + x) \, dx = 60$$

and the balance point, officially called the center of mass, is

$$\bar{x} = \frac{1150}{3} \times \frac{1}{60} = \frac{115}{18} \approx 6.89$$

Figure 9.6.3 Center of mass for a two dimensional plate.

of the “beam”, say between $x_i$ and $x_{i+1}$, is the mass of a strip of the plate between $x_i$ and $x_{i+1}$. See figure 9.6.3 showing the plate from above and as it appears edge on. Since the plate has uniform density we may as well assume that $\sigma = 1$. Then the mass of the plate between $x_i$ and $x_{i+1}$ is approximately $m_i = \sigma (1 - x_i^2) \Delta x = (1 - x_i^2) \Delta x$. Now we can compute the moment around the $y$-axis:

$$M_y = \int_0^1 x (1 - x^2) \, dx = \frac{1}{4}$$

and the total mass

$$M = \int_0^1 (1 - x^2) \, dx = \frac{2}{3}$$

and finally

$$\bar{y} = \frac{1.3}{2} \times \frac{3}{5} = \frac{9}{10}$$

Next we do the same thing to find $\bar{y}$. The mass of the plate between $y_i$ and $y_{i+1}$ is approximately $m_i = \sqrt{2} \Delta y$, so

$$M_y = \int_{y_i}^{y_{i+1}} y \sqrt{2} \, dy = \frac{2}{5}$$

and

$$\bar{y} = \frac{2 \times \frac{3}{5}}{\frac{2}{5}} = \frac{3}{2}$$

since the total mass $M$ is the same. The center of mass is shown in figure 9.6.3.

Exercise 9.6.4 Find the center of mass of a thin, uniform plate whose shape is the region between $y = \cos x$ and the $x$-axis between $x = -\pi/2$ and $x = \pi/2$. It is clear that $\bar{x} = 0$, but for practice let’s compute it anyway. We will need the total mass, so we compute it first:

$$M = \int_{-\pi/2}^{\pi/2} \cos x \, dx = \sin x \bigg|_{-\pi/2}^{\pi/2} = 2$$

The moment around the $y$-axis is

$$M_y = \int_{-\pi/2}^{\pi/2} x \cos x \, dx = \cos x + x \sin x \bigg|_{-\pi/2}^{\pi/2} = 0$$

and the moment around the $x$-axis is

$$M_x = \int_{-\pi/2}^{\pi/2} y \cdot 2 \arccos y \, dy = \sqrt{2} \arccos y \cdot \frac{\sqrt{2} \sqrt{1 - y^2}}{2} + \arccos y \bigg|_{-\pi/2}^{\pi/2} = \frac{\pi}{2}$$

Thus

$$\bar{x} = 0 \quad \bar{y} = \frac{\pi}{8} \approx 0.393$$

218 Chapter 9 Applications of Integration
9.7 Kinetic energy; improper integrals

Recall example 9.5.3 in which we computed the work required to lift an object from the surface of the earth to some large distance \( D \) away. Since \( F = k/r^2 \) we computed

\[
\int_0^D \frac{k}{r^2} dr = \frac{k}{r} \bigg|_0^D = \frac{k}{D} - \frac{k}{0}.
\]

We noticed that as \( D \) increases, \( k/D \) decreases to zero so that the amount of work increases to \( k/\infty \). More precisely,

\[
\lim_{D \to \infty} \int_0^D \frac{k}{r^2} dr = \lim_{D \to \infty} \frac{k}{r} \bigg|_0^D = \frac{k}{D} - \frac{k}{0}.
\]

We might reasonably describe this calculation as computing the amount of work required to lift the object "to infinity," and abbreviate the limit as

\[
\lim_{D \to \infty} \int_0^D \frac{k}{r^2} dr = \int_0^\infty \frac{k}{r^2} dr.
\]

Such an integral, with a limit of infinity, is called an improper integral. This is a bit unfortunate, since it's not really "improper" to do this, nor is it really "can integral"—it is an abbreviation for the limit of a particular sort of integral. Nevertheless, we're stuck with the term, and the operation itself is perfectly legitimate. It may at first seem odd that a function is the area under a curve, but since the area increases as \( D \) increases, this is such a case. If the value of an improper integral is a finite number, as in this example, we say that the integral converges, and if not we say that the integral diverges.

Here's another way, perhaps even more surprising, to interpret this calculation. We know that one interpretation of

\[
\int_0^D \frac{1}{D} dx
\]

is the area under \( y = 1/x^2 \) from \( x = 1 \) to \( x = D \). Of course, as \( D \) increases this area increases. But since

\[
\int_0^D \frac{1}{D} dx = \int_0^1 \frac{1}{D} dx + \int_1^D \frac{1}{D} dx
\]

while the area increases, it never exceeds 1, that is

\[
\int_0^D \frac{1}{D} dx = 1.
\]

The area of the infinite region under \( y = 1/x^2 \) from \( x = 1 \) to infinity is finite.

9.7 Kinetic energy; improper integrals 221

"downward." This makes the work \( W \) negative when it should be positive, so typically the work in this case is defined as

\[
W = -\int_{x_0}^{x_1} F dx.
\]

Also, by Newton's Law, \( F = ma(t) \). This means that

\[
W = \int_{x_0}^{x_1} ma(t) dx.
\]

Unfortunately this integral is a bit problematic: \( a(t) \) is in terms of \( t \), while the limits and the "dx" are in terms of \( x \). But \( x \) and \( t \) are certainly related here: \( x = x(t) \) is the function that gives the position of the object at time \( t \), so \( v = v(t) = dx/dt = x'(t) \) is its velocity and \( a(t) = v'(t) = x''(t) \). We can use \( x = x(t) \) as a substitution to convert the integral from "dx" to "dt" in the usual way, with a bit of cleverness along the way:

\[
dv dx = x'(t) dt = a(t) dt = a(t) \frac{dx}{dt} dt
\]

Substituting in the integral:

\[
W = -\int_{x_0}^{x_1} ma(t) dx = \int_{x_0}^{x_1} mv dx = \int_{x_0}^{x_1} m \frac{1}{2} \left( x(t) \right)^2 dt
\]

You may recall seeing the expression \( mv^2/2 \) in a physics course—it is called the kinetic energy of the object. We have shown here that the work done in moving the object from one place to another is the same as the change in kinetic energy.

We know that the work required to move an object from the surface of the earth to infinity is

\[
W = \int_0^\infty \frac{k}{r^3} dr = \frac{k}{2}.
\]

At the surface of the earth the acceleration due to gravity is approximately 9.8 meters per second squared, so the force on an object of mass \( m \) is \( F = 9.8m \). The radius of the earth is approximately 6378.1 kilometers or 6378100 meters. Since the force due to gravity obeys an inverse square law, \( F = k/r^2 \) and \( 9.8m = k/6378100^2 \), \( k = 62505380 \) and \( W = 62505380 \).
12. Does \( \int_{-\infty}^{\infty} e^x \, dx \) converge or diverge? If it converges, find the value. Also find the Canny Principal Value, if it exists.

13. Suppose the curve \( y = 1/x \) is rotated around the \( z \)-axis generating a sort of funnel or horn shape, called Gabriel's horn or Torricelli's trumpet. Is the volume of this funnel from \( z = 1 \) to infinity finite or infinite? If finite, compute the volume.

14. An officially sanctioned baseball must be between 142 and 149 grams. How much work, in Newton-meters, does it take to throw a ball at 60 miles per hour? At 90 mph? At 100.9 mph? (According to the Guinness Book of World Records, at \text{http://www.baseball-almanac.com/records/sh_gsin.shtml} the greatest reliably recorded speed at which a baseball has been pitched is 106.9 mph by Lynn Nolan Ryan (California Angels) at Anaheim Stadium in California on August 20, 1974.)

9.8 Probability

You perhaps have at least a rudimentary understanding of discrete probability, which measures the likelihood of an “event” when there are a finite number of possibilities. For example, when an ordinary six-sided die is rolled, the probability of getting any particular number is \( 1/6 \). In general, the probability of an event is the number of ways the event can happen divided by the number of ways that “anything” can happen.

For a slightly more complicated example, consider the case of two six-sided dice. The dice are physically distinct, which means that rolling a 2-5 is different than rolling a 5-2; each is an equally likely event out of a total of 36 ways the dice can land, so each has a probability of \( 1/36 \).

Most interesting events are not so simple. More interesting is the probability of rolling a certain sum out of the possibilities 2 through 12. It is clearly not true that all sums are equally likely: the only way to roll a 2 is to roll 1-1, while there are many ways to roll a 7. Because the number of possibilities is quite small, and because a pattern quickly becomes evident, it is easy to see that the probabilities of the various sums are:

\[
\begin{align*}
P(2) &= P(12) = 1/36 \\
P(3) &= P(11) = 2/36 \\
P(4) &= P(10) = 3/36 \\
P(5) &= P(9) = 4/36 \\
P(6) &= P(8) = 5/36 \\
P(7) &= 6/36
\end{align*}
\]

Here we use \( P(n) \) to mean “the probability of rolling an \( n \).” Since we have correctly accounted for all possibilities, the sum of all those probabilities is \( 36/36 = 1 \); the probability that the sum is one of 2 through 12 is 1, because there are no other possibilities.

**DEFINITION 9.8.1** Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. If \( f(x) \geq 0 \) for every \( x \) and \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \), then \( f \) is a probability density function.

We associate a probability density function with a random variable \( X \) by stipulating that the probability that \( X \) is between \( a \) and \( b \) is \( \int_a^b f(x) \, dx \). Because of the requirement that the integral from \( -\infty \) to \( \infty \) be 1, all probabilities are less than or equal to 1, and the probability that \( X \) takes on some value between \( -\infty \) and \( \infty \) as it should be.

**EXAMPLE 9.8.2** Consider again the two dice example; we can view it in a way that more resembles the probability density function approach. Consider a random variable \( X \) that takes on any real value with probabilities given by the probability density function in figure 9.8.1. The function \( f \) consists of just the top edges of the rectangles, with vertical sides drawn for clarity; the function is zero below 1.5 and above 12.5. The area of each rectangle is the probability of rolling the sum in the middle of the bottom of the rectangle, or

\[
P(n) = \frac{n^{1/2}}{36} \int_{x=n-1/2}^{x=n+1/2} f(x) \, dx.
\]

The probability of rolling a 4, 5, or 6 is

\[
P(n) = \frac{n^{1/2}}{12} f(x) \, dx.
\]

Of course, we could also compute probabilities that don’t make sense in the context of the dice, such as the probability that \( X \) is between 4 and 5.8.

![A probability density function for two dice.](image)

224 Chapter 9 Applications of Integration

The study of probability is concerned with more difficult questions as well; for example, suppose the two dice are rolled many times. On the average, what sum will come up? In the language of probability, this average is called the expected value of the sum. This is at first a little misleading, as it does not tell us what to “expect” when the two dice are rolled, but what we expect the long term average will be.

Suppose that two dice are rolled 36 million times. Based on the probabilities, we would expect about 1 million rolls to be 2, about 2 million to be 3, and so on, with a roll of 7 topping the list at about 6 million. The sum of all rolls would be 1 million times 2 plus 2 million times 3, and so on, and dividing by 36 million we would get the average:

\[
\begin{align*}
\bar{x} &= (2 \cdot 10^6 + 3 \cdot 10^6 + \ldots + 7 \cdot 6 \cdot 10^6 + \ldots + 12 \cdot 10^6) / 36 \\
&= 2 \cdot (1 \cdot 10^6 + 3 \cdot 10^6 + \ldots + 7 \cdot 6 \cdot 10^6 + \ldots + 12 \cdot 10^6) / 36 \\
&= 2 \cdot (10^6 + 3 \cdot 10^6 + \ldots + 7 \cdot 6 \cdot 10^6 + \ldots + 12 \cdot 10^6) / 36
\end{align*}
\]

There is nothing special about the 36 million in this calculation. No matter what the number of rolls, once we simplify the average, we get the same \( \sum_{i=2}^{12} iP(i) \). While the actual average value of a large number of rolls will not be exactly 7, the average should be close to 7 when the number of rolls is large. Turning this around, if the average is not close to 7, we should suspect that the dice are not fair.

A variable, say \( X \), that can take certain values, each with a corresponding probability, is called a random variable; in the example above, the random variable was the sum of the two dice. If the possible values for \( X \) are \( x_1, x_2, \ldots, x_n \), then the expected value of the random variable is \( E(X) = \sum_{i=1}^{n} x_i P(x_i) \). The expected value is also called the mean.

When the number of possible values for \( X \) is finite, we say that \( X \) is a discrete random variable. In many applications of probability, the number of possible values of a random variable is very large, perhaps even infinite. To deal with the infinite case we need a different approach, and since there is a sum involved, it should not be wholly surprising that integration turns out to be a useful tool. It then turns out that even when the number of possibilities is large but finite, it is frequently easier to pretend that the number is infinite. Suppose, for example, that a dart is thrown at a dart board. Since the dart board consists of a finite number of atoms, there are in some sense only a finite number of places for the dart to land, but it is easier to explore the probabilities involved by pretending that the dart can land on any point in the usual \( x,y \) plane.
We have shown that $A$ is some finite number without computing it; we cannot compute it with the techniques we have available. By using some techniques from multivariable calculus, it can be shown that $A = \sqrt{2\pi}$. 

EXAMPLE 9.8.5 The exponential distribution has probability density function

$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x \geq 0 \end{cases}$$

where $e$ is a positive constant.

The mean or expected value of a random variable is quite useful, as hinted at in our discussion of dice. Recall that the mean for a discrete random variable is $E(X) = \sum \mu_i P(x_i)$. In the more general context we use an integral in place of the sum.

**DEFINITION 9.8.6** The mean of a random variable $X$ with probability density function $f$ is $\mu = E(X) = \int_{-\infty}^{\infty} x f(x) \, dx$, provided the integral converges.

When the mean exists, it is unique, since it is the result of an explicit calculation. The mean does not always exist.

The mean might look familiar; it is essentially identical to the center of mass of a one-dimensional beam, as discussed in section 9.6. The probability density function $f$ plays the role of the physical density function, but now the "beam" has infinite length. If we consider only a finite portion of the beam, say between $a$ and $b$, then the center of mass is

$$\mu(a,b) = \frac{\int_a^b x f(x) \, dx}{\int_a^b f(x) \, dx}$$

is $(X - \mu)^2$; we square the differences so as to measure all distances as positive. To get the typical such squared distance, we compute the mean. For two dice, for example, we get

$$(2 - 7)^2 + (3 - 7)^2 / 11 + (7 - 7)^2 / 11 + \cdots + (11 - 7)^2 / 11 = 35 / 11 = 3.18.$$  

Because we squared the differences this does not directly measure the typical distance we seek; if we take the square root of this we get such a measure, $\sqrt{35 / 11} \approx 2.42$. Doing the computation for the strange 11-sided die we get

$$(2 - 7)^2 + (3 - 7)^2 / 11 + \cdots + (11 - 7)^2 / 11 + (12 - 7)^2 / 11 = 10.$$  

with square root approximately 3.16. Comparing 2.42 to 3.16 tells us that the two-dice rolls clump somewhat more closely near 7 than the rolls of the weird die, which of course we already knew because these examples are quite simple.

To perform the same computation for a probability density function the sum is replaced by an integral, just as in the computation of the mean. The expected value of the squared distances is

$$V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx,$$

called the variance. The square root of the variance is the standard deviation, denoted $\sigma$.

**EXAMPLE 9.8.8** We compute the standard deviation of the normal standard distribution. The variance is

$$\sqrt{\int_{-\infty}^{\infty} x^2 e^{-x^2 / 2} \, dx}.$$  

To compute the antiderivative, use integration by parts, with $u = x$ and $dv = xe^{-x^2 / 2} \, dx$. This gives

$$\int x^2 e^{-x^2 / 2} \, dx = -xe^{-x^2 / 2} + \int e^{-x^2 / 2} \, dx.$$  

We cannot do the new integral, but we know its value when the limits are $-\infty$ to $\infty$, from our discussion of the standard normal distribution. Thus

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2 / 2} \, dx = \frac{1}{\sqrt{2\pi}} \left( -xe^{-x^2 / 2} \right)_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2 / 2} \, dx = 0 + \frac{1}{\sqrt{2\pi}} \sqrt{2\pi} = 1.$$  

The standard deviation is then $\sqrt{V(X)} = 1$. 

If we extend the beam to infinity, we get

$$\mu = \int_{-\infty}^{\infty} x f(x) \, dx = \int_{-\infty}^{\infty} x f(x) \, dx = E(X),$$

because $\int_{-\infty}^{\infty} f(x) \, dx = 1$. In the center of mass interpretation, this integral is the total mass of the beam, which is always 1 when $f$ is a probability density function.

**EXAMPLE 9.8.7** The mean of the standard normal distribution is

$$\mu = \int_{-\infty}^{\infty} x e^{-x^2 / 2} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2 / 2} \, dx,$$

we compute the two halves:

$$\int_{0}^{\infty} e^{-x^2 / 2} \, dx = \lim_{x \to \infty} \int_{0}^{x} e^{-x^2 / 2} \, dx = \frac{1}{\sqrt{2\pi}}$$

and

$$\int_{-\infty}^{0} e^{-x^2 / 2} \, dx = \lim_{x \to \infty} \int_{-x}^{0} e^{-x^2 / 2} \, dx = 0.$$  

The sum of these is 0, which is the mean.

While the mean is very useful, it typically is not enough information to properly evaluate a situation. For example, suppose we could manufacture an 11-sided die, with the faces numbered 2 through 12 so that each face is equally likely to be down when the die is rolled. The value of a roll is the value on this lower face. Rolling the die gives the same range of values as rolling two ordinary dice, but now each value occurs with probability 1/11. The expected value of a roll is

$$2 \cdot \frac{2}{11} + 3 \cdot \frac{3}{11} + \cdots + 12 \cdot \frac{12}{11} = 7.$$  

The mean does not distinguish the two cases, though of course they are quite different.

If $f$ is a probability density function for a random variable $X$, with mean $\mu$, we would like to measure how far a "typical" value of $X$ is from $\mu$. One way to measure this distance
9.8 Probability

231

9.9 Arc Length

233

The probability density function of the normal distribution with mean \( \mu \) and standard deviation \( \sigma \). Show that the mean of the normal distribution is \( \mu \) and the standard deviation is \( \sigma \).

7. Let

\[
f(x) = \begin{cases} 
1 & x \ge 1 \\
0 & x < 1 
\end{cases}
\]

Show that \( f \) is a probability density function, and that the distribution has no mean.

8. Let

\[
f(x) = \begin{cases} 
1 & x \le 1 \\
0 & 0 < x < 2 \\
\frac{1}{2} & 0 
\end{cases}
\]

Show that \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \). Is \( f \) a probability density function? Justify your answer.

9. If you have access to appropriate software, find \( r \) so that

\[
\int_{-r}^{r} f(x) \, dx = 0.05,
\]

using the function of example 9.8.9. Discuss the impact of using this new value of \( r \) to decide whether to investigate the chip manufacturing process.

9.9 Arc Length

Here is another geometric application of the integral: find the length of a portion of a curve. As usual, we need to think about how we might approximate the length, and turn the approximation into an integral.

We already know how to compute one simple arc length, that of a line segment. If the endpoints are \((x_0, y_0)\) and \((x_1, y_1)\) then the length of the segment is the distance between the points, \(\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}\), from the Pythagorean theorem, as illustrated in figure 9.9.1.

Figure 9.9.1 The length of a line segment.

9.9 Arc Length

Approximating arc length with line segments.

Now if the graph of \( f \) is “nice” (say, differentiable) it appears that we can approximate the length of a portion of the curve with line segments, and that as the number of segments increases, and their lengths decrease, the sum of the lengths of the line segments will approach the true arc length; see figure 9.9.2.

Now we need to write a formula for the sum of the lengths of the line segments, in a form that we know becomes an integral in the limit. So we suppose we have divided the interval \([a, b]\) into \( n \) subintervals as usual, each with length \( \Delta x = (b - a)/n \), and endpoints \( a = x_0, x_1, x_2, \ldots, x_n = b \). The length of a typical line segment, joining \((x_i, f(x_i))\) to \((x_{i+1}, f(x_{i+1}))\), is \(\sqrt{(\Delta x)^2 + (f(x_{i+1}) - f(x_i))^2}\). By the Mean Value Theorem (6.5.2), there is a number \( t_i \) in \((x_i, x_{i+1})\) such that \(f(t_i)\Delta x = f(x_{i+1}) - f(x_i)\), so the length of the line segment can be written as

\[
\sqrt{(\Delta x)^2 + (f(t_i))^2} \Delta x = \sqrt{1 + (f'(t_i))^2} \Delta x.
\]

The arc length is then

\[
\lim_{n \to \infty} \sum_{i=0}^{n-1} \sqrt{1 + (f'(t_i))^2} \Delta x = \int_a^b \sqrt{1 + (f'(x))^2} \, dx.
\]

Note that the sum looks a bit different than others we have encountered, because the approximation contains a \( t_i \) instead of an \( x \). In the past we have always used left endpoints (namely, \( x_i \)) to get a representative value of \( f \) on \([x_i, x_{i+1}]\); now we are using a different point, but the principle is the same.

To summarize, to compute the length of a curve on the interval \([a, b]\), we compute the integral

\[
\int_a^b \sqrt{1 + (f'(x))^2} \, dx.
\]

Unfortunately, integrals of this form are typically difficult or impossible to compute exactly, because usually none of our methods for finding antiderivatives will work. In practice this means that the integral will usually have to be approximated.
As usual, the question is: how might we approximate the surface area? For a surface obtained by rotating a curve around an axis, we can take a polygonal approximation to the curve, as in the last section, and rotate it around the same axis. This gives a surface composed of many “truncated cones,” a truncated cone is called a frustum of a cone. Figure 9.10.1 illustrates this approximation.

Figure 9.10.1 Approximating a surface (left) by portions of cones (right).

So we need to be able to compute the area of a frustum of a cone. Since the frustum can be formed by removing a small cone from the top of a larger one, we can compute the desired area if we know the surface area of a cone. Suppose a right circular cone has base radius \( r \) and slant height \( h \). If we cut the cone from the vertex to the base circle and flatten it out, we obtain a sector of a circle with radius \( h \) and arc length \( 2\pi r \), as in figure 9.10.2. The angle at the center, in radians, is \( 2\pi r/h \), and the area of the cone is equal to the area of the sector of the circle. Let \( A \) be the area of the sector, since the area of the entire circle is \( \pi r^2 \), we have

\[
\frac{A}{\pi r^2} = \frac{2\pi r}{h}
\]

Now suppose we have a frustum of a cone with slant height \( h \) and radii \( r_0 \) and \( r_1 \), as in figure 9.10.3. The area of the entire cone is \( \pi r_0 h_0 \), and the area of the small cone is \( \pi r_0 h_0 \); thus, the area of the frustum is \( \pi r_1(h_1 - h_0) = \pi r_0 h_1 + \pi r_1 h_0 \). By similar triangles,

\[
\frac{h_0}{r_0} = \frac{h_0 + h}{r_1}
\]

With a bit of algebra this becomes \( r_1 - r_0 h_0 = r_0 h \); substitution into the area gives

\[
\pi((r_0 h_0 + r_1 h) = \pi(r_0 h + r_1 h) = \pi h(r_0 + r_1) = 2\pi \frac{r_0 + r_1}{2} h = 2\pi h.
\]

The final form is particularly easy to remember, with \( r \) equal to the average of \( r_0 \) and \( r_1 \), as it is also the formula for the area of a cylinder. (Think of a cylinder of radius \( r \) and height \( h \) as the frustum of a cone of infinite height.)

Figure 9.10.3 The area of a frustum.

Now we are ready to approximate the area of a surface of revolution. On one subinterval, the situation is as shown in figure 9.10.4. When the line joining two points on the curve is rotated around the \( x \)-axis, it forms a frustum of a cone. The area is

\[
2\pi r h = 2 \pi f(x) \sqrt{1 + [f'(x)]^2} \approx 2 \pi f(x) \Delta s,
\]

Here \( \sqrt{1 + [f'(x)]^2} \Delta s \) is the length of the line segment, as we found in the previous section. Assuming \( f \) is a continuous function, there must be some \( x_i^* \) in \([x_i, x_{i+1}]\) such that \( f(x_i^*) + f(x_{i+1})/2 = f(x_i^*) \), so the approximation for the surface area is

\[
\sum_{i=1}^{n} 2 \pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2} \Delta x.
\]

This is not quite the sum of areas we have seen before, because it contains two different values in the interval \([x_i, x_{i+1}]\), namely \( x_i^* \) and \( t_i \). Nevertheless, using more advanced techniques than we have available here, it turns out that

\[
\lim_{n \to \infty} \sum_{i=1}^{n} 2 \pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2} \Delta x = \int_a^b 2 \pi f(x) \sqrt{1 + [f'(x)]^2} dx
\]

is the surface area we seek. (Roughly speaking, this is because while \( x_i^* \) and \( t_i \) are distinct values in \([x_i, x_{i+1}]\), they get closer and closer to each other as the length of the interval shrinks.)

Figure 9.10.4 One subinterval.

EXAMPLE 9.10.1 We compute the surface area of a sphere of radius \( r \). The sphere can be obtained by rotating the graph of \( f(x) = \sqrt{r^2 - x^2} \) about the \( x \)-axis. The derivative of \( f(x) \) is \( -x / \sqrt{r^2 - x^2} \), so the surface area is given by

\[
A = 2\pi \int_a^b \sqrt{1 + \left(\frac{x}{\sqrt{r^2 - x^2}}\right)^2} dx
\]

\[
= 2\pi \int_a^b \sqrt{1 + \frac{x^2}{r^2 - x^2}} dx
\]

\[
= 2\pi \int_a^b \frac{1}{\sqrt{r^2}} dx = 2\pi \int_a^b 1 dx = 4\pi r^2
\]

If the curve is rotated around the \( y \)-axis, the formula is nearly identical, because the length of the line segment we use to approximate a portion of the curve doesn’t change. Instead of the radius \( f(x_i^*) \), we use the new radius \( t_i = (x_i + x_{i+1})/2 \), and the surface area integral becomes

\[
\int_a^b 2\pi x \sqrt{1 + [f'(x)]^2} dx.
\]

EXAMPLE 9.10.2 Compute the area of the surface formed when \( f(x) = x^2 \) between 0 and 2 is rotated around the \( y \)-axis.

We compute \( f'(x) = 2x \), and then

\[
2\pi \int_a^b x \sqrt{1 + [f'(x)]^2} dx = \pi (17^{1/2} - 1),
\]

by a simple substitution.

Exercises 9.10.

1. Compute the area of the surface formed when \( f(x) = 2x \) between -1 and 0 is rotated around the \( x \)-axis.

2. Compute the surface area of example 9.10.2 by rotating \( f(x) = \sqrt{x} \) around the \( x \)-axis.

3. Compute the area of the surface formed when \( f(x) = x^2 \) between 0 and 1 is rotated around the \( x \)-axis.

4. Compute the area of the surface formed when \( f(x) = 2 - \frac{1}{3}x \) between 0 and 1 is rotated around the \( y \)-axis.

5. Consider the surface obtained by rotating the graph of \( f(x) = x/2 \) between 1 and 3 around the \( x \)-axis. This surface is called Gabriel’s horn or Torricelli’s trumpet. In exercise 13 in section 9.7 we saw that Gabriel’s horn has finite volume. Show that Gabriel’s horn has infinite surface area.

6. Consider the circle \((x - 2)^2 + y^2 = 1\). Sketch the surface obtained by rotating this circle about the \( y \)-axis. (The surface is called a torus.) What is the surface area?

7. Consider the ellipse with equation \( \frac{x^2}{4} + y^2 = 1 \). If the ellipse is rotated around the \( x \)-axis it forms an ellipsoid. Compute the surface area. 

8. Generalize the preceding result: rotate the ellipse given by \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) about the \( x \)-axis and find the surface area of the resulting ellipsoid. You should consider two cases, when \( a > b \) and when \( a < b \). Compare to the area of a sphere.