Integration

7.1 Two Examples

Up to now we have been concerned with extracting information about how a function changes from the function itself. Given knowledge about an object's position, for example, we want to know the object's speed. Given information about the height of a curve we want to know its slope. We now consider problems that are, whether obviously or not, the reverse of such problems.

Example 7.1.1 An object moves in a straight line so that its speed at time \( t \) is given by \( v(t) = 3t \) in, say, cm/sec. If the object is at position 10 on the straight line when \( t = 0 \), where is the object at time \( t \)?

There are two reasonable ways to approach this problem. If \( s(t) \) is the position of the object at time \( t \), we know that \( v(t) = \frac{ds}{dt} \). Because of our knowledge of derivatives, we know therefore that \( s(t) = \frac{3}{2}t^2 + k \), and because \( s(0) = 10 \) we easily discover that \( k = 10 \), so \( s(t) = \frac{3}{2}t^2 + 10 \). For example, at \( t = 1 \) the object is at position \( 3/2 + 10 = 11.5 \). This is certainly the easiest way to deal with this problem. Not all similar problems are so easy, as we will see, the second approach to the problem is more difficult but also more general.

We start by considering how we might approximate a solution. We know that at \( t = 0 \) the object is at position 10. How might we approximate its position at, say, \( t = 1 \)? We know that the speed of the object at time \( t = 0 \) is 0; if its speed were constant then in the first second the object would not move and its position would still be 10 when \( t = 1 \). In fact, the object will not be too far from 10 at \( t = 1 \), but certainly we can do better. Let's look at the times 0.1, 0.2, 0.3, ..., 1.0, and try approximating the location of the object more useful. We can factor out a 3 and \( 1/n^2 \) to get

\[
\frac{3}{n^2}(0 + 1 + 2 + 3 + \ldots + (n - 1)),
\]

that is, \( 3/n^2 \) times the sum of the first \( n - 1 \) positive integers. Now we make use of a fact you may have run across before:

\[
1 + 2 + 3 + \ldots + k = \frac{k(k + 1)}{2}.
\]

In our case we're interested in \( k = n - 1 \), so

\[
1 + 2 + 3 + \ldots + (n - 1) = \frac{(n - 1)n}{2} = \frac{n^2 - n}{2}.
\]

This simplifies the approximate distance traveled to

\[
\frac{3}{n^2} \left( \frac{n^2 - n}{2} \right) = \frac{3n^2 - 3n}{2}.
\]

Now this is quite easy to understand: as \( n \) gets larger and larger this approximation gets closer and closer to \( 3/(2)(1 - 0) = 3/2 \), so that \( 3/2 \) is the exact distance traveled during one second, and the final position is 11.5.

So for \( t = 1 \), at least, this rather cumbersome approach gives the answer as the first approach. But really there's nothing special about \( t = 1 \); let's just call it \( t \) instead. In this case the approximate distance traveled during time interval \( i \) is

\[
3(i - 1)/n(n - 1) = 3i^2/n^2 - 3i/n^2 + 3/n.
\]

That is, speed \( 3/n \) times \( t/n \), and the total distance traveled is approximately

\[
(0 \cdot \frac{3}{n^2} + 3(1)^2/n^2 + 3(2)^2/n^2 + \ldots + 3(n - 1)^2/n^2).
\]

As before we can simplify this to

\[
\frac{3}{n^2} \left( 0 + 1 + 2 + \ldots + (n - 1) \right) = \frac{3n^2(n^2 - n)}{2} = \frac{3n^3 - 3n^2}{2}.
\]

In the limit, as \( n \) gets larger, this gets closer and closer to \( 3/2n^2 \), but the actual position of the object gets closer and closer to \( 3/2n^2 + 10 \), so that the exact position is \( 3/2n^2 + 10 \), exactly the answer given by the first approach to the problem.

Example 7.1.2 Find the area under the curve \( y = 3x \) between \( x = 0 \) and any positive value \( x \). There is here no obvious analogue to the first approach in the previous example, but the second approach works fine. (Because the function \( y = 3x \) is so simple, there is another approach that works here, but it is even more limited in potential application than is approach number one.) How might we approximate the desired area? We know how to compute areas of rectangles, so we approximate the area by rectangles. Jumping straight to the general case, suppose we divide the interval between 0 and \( x \) into \( n \) equal subintervals, and use a rectangle above each subinterval to approximate the area under the curve. There are many ways we might do this, but let's use the height of the curve at the left endpoint of the subinterval as the height of the rectangle, as in figure 7.1.1.

The height of rectangle number \( i \) is then \( 3(i - 1)/n \), the width is \( x/n \), and the area is

\[
3(1 - 1)/(n^2) + 3(3)/n^2 + \ldots + 3(n - 1)/n^2.
\]

By factoring out \( 3x^2/n \) this simplifies to

\[
\frac{3x^2}{n} \left( 0 + 1 + 2 + \ldots + (n - 1) \right) = \frac{3x^2n^2 - 3x^2}{2} = x^2 \left( 1 - \frac{1}{n} \right).
\]

As \( n \) gets larger this gets closer and closer to \( 3x^2/2 \), which must therefore be the true area under the curve.
are many, many problems that appear much different on the surface but that turn out to be the same as these problems, in the sense that when we try to approximate solutions we end up with mathematics that looks like the two examples, though of course the function involved will not always be so simple.

Even better, we now see that while the second problem did not appear to be amenable to approach one, it can in fact be solved in the same way. The reasoning is this: we know that problem one can be solved easily by finding a function whose derivative is $f$. We also know that mathematically the two problems are the same, because both can be solved by taking a limit of a sum, and the sums are identical. Therefore, we don’t really need to compute the limit of either sum because we know that we will get the same answer by computing a function with the derivative of $f$ or, which is the same thing, $3f$.

It’s true that the first problem had the added complication of the “$-1$”, and we certainly need to be able to deal with such minor variations, but that turns out to be quite simple. The lesson then is this: whenever we can solve a problem by taking the limit of a sum of a certain form, we can instead of computing the (often nasty) limit find a new function with a certain derivative.

**Exercises 7.1.**

1. Suppose an object moves in a straight line so that its speed at time $t$ is given by $v(t) = 2t^2 + 2t$, and that at $t = 1$ the object is at position $5$. Find the position of the object at $t = 2$.

2. Suppose an object moves in a straight line so that its speed at time $t$ is given by $v(t) = 3t + 2$, and that at $t = 0$ the object is at position $5$. Find the position of the object at $t = 2$.

3. By a method similar to that in example 7.1.2, find the area under $y = 2x^2$ between $x = 0$ and any positive value for $x$.

4. By a method similar to that in example 7.1.2, find the area under $y = 4x$ between $x = 0$ and any positive value for $x$.

5. By a method similar to that in example 7.1.2, find the area under $y = 4x$ between $x = 2$ and any positive value for $x$.

6. By a method similar to that in example 7.1.2, find the area under $y = 4x$ between any two positive values for $x$, say $a < b$.

7. Let $f(x) = x^2 + 3x + 2$. Approximate the area under the curve between $x = 0$ and $x = 2$ using 4 rectangles and also using 8 rectangles.

8. Let $f(x) = x^2 - 2x + 3$. Approximate the area under the curve between $x = 1$ and $x = 3$ using 4 rectangles.

### 7.2 The Fundamental Theorem of Calculus

Let’s recall the first example from the previous section. Suppose that the speed of the object is $3t$ at time $t$. How far does the object travel between time $t = a$ and time $t = b$?

We are no longer assuming that we know where the object is at time $t = 0$ or at any other time. It is certainly true that it is somewhere, so let’s suppose that at $t = 0$ the position is $k$. Then just as in the example, we know that the position of the object at any time is $3t^2/2 + k$. This means that at time $t = a$ the position is $3a^2/2 + k$ and at time $t = b$ the position is $3b^2/2 + k$. Therefore the change in position is $3b^2/2 + k - (3a^2/2 + k) = 3b^2/2 - 3a^2/2$. Notice that the $k$ drops out; this means that it doesn’t matter that we don’t know $k$, it doesn’t even matter if we use the wrong $k$, we get the correct answer. In other words, to find the change in position between time $a$ and time $b$ we can use any antiderivative of the speed function $3t$—it need not be the one antiderivative that actually gives the location of the object.

What about the second approach to this problem, in the new form? We now want to approximate the change in position between time $a$ and time $b$. We take the interval of time between $a$ and $b$, divide it into $n$ subintervals, and approximate the distance traveled during each.

The starting time of subinterval number $i$ is now $a + (i-1)(b-a)/n$, which we abbreviate as $t_i$, so that $t_0 = a$, $t_i = a + (i-1)(b-a)/n$, and so on. The speed of the object is $f(t)$, and each subinterval is $(b-a)/n = \Delta t$ seconds long. The distance traveled during subinterval number $i$ is approximately $f(t_i)\Delta t$, and the total change in distance is approximately $\sum_{i=0}^{n-1} f(t_i)\Delta t + f(t_n)\Delta t$.

The exact change in position is the limit of this sum as $n$ goes to infinity. We abbreviate this sum using sigma notation:

$$\sum_{i=0}^{n-1} f(t_i)\Delta t + f(t_n)\Delta t$$

The notation on the left side of the equal sign uses a large capital sigma, a Greek letter, and the left side is an abbreviation for the right side. The answer we seek is

$$\lim_{n \to \infty} \sum_{i=0}^{n-1} f(t_i)\Delta t$$

Since this must be the same as the answer we have already obtained, we know that

$$\lim_{n \to \infty} \sum_{i=0}^{n-1} f(t_i)\Delta t = \int_a^b f(t) \, dt$$

The significance of $3t^2/2$, into which we substitute $t = b$ and $t = a$, is of course that it is a function whose derivative is $f(t)$. As we have discussed, by the time we know that we want to compute

$$\lim_{n \to \infty} \sum_{i=0}^{n-1} f(t_i)\Delta t$$

it no longer matters what $f(t)$ stands for—it could be a speed, or the height of a curve, or something else entirely. We know that the limit can be computed by finding any function with derivative $f(t)$, substituting $a$ and $b$, and subtracting. We summarize this in a theorem. First, we introduce some new notation and terms. We write

$$\int_a^b f(t) \, dt = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(t_i)\Delta t$$

if the limit exists. That is, the left hand side means, or is an abbreviation for, the right hand side. The symbol $\int$ is called an integral sign, and the whole expression is read as “the integral of $f(t)$ from $a$ to $b$.” What we have learned is that this integral can be computed by finding a function, say $F(t)$, with the property that $F'(t) = f(t)$, and then computing $F(b) - F(a)$. The function $F(t)$ is called an antiderivative of $f(t)$. Now the theorem:

**THEOREM 7.2.1 Fundamental Theorem of Calculus** Suppose that $f(x)$ is continuous on the interval $[a, b]$. If $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

Let’s rewrite this slightly:

$$\int_a^b f(t) \, dt = F(b) - F(a).$$

We’ve replaced the variable $x$ by $t$ and by $y$. These are just different names for quantities, so the substitution doesn’t change the meaning. It does make it easier to think of the two sides of the equation as functions. The expression

$$\int_a^b f(t) \, dt$$

is a function: plug in a value for $x$, get out some other value. The expression $F(x) - F(a)$ is of course also a function, and it has a nice property:

$$\frac{d}{dx} (F(x) - F(a)) = F'(x) = f(x).$$

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since $F(a)$ is a constant and has derivative zero. In other words, by shifting our point of view slightly, we see that the odd looking function

$$G(x) = \int_a^x f(t) \, dt$$

has a derivative, and that in fact $G'(x) = f(x)$. This is really just a restatement of the Fundamental Theorem of Calculus, and indeed is often called the Fundamental Theorem of Calculus. To avoid confusion, some people call the two versions of the theorem “The Fundamental Theorem of Calculus, part I” and “The Fundamental Theorem of Calculus, part II”, although unfortunately there is no universal agreement as to which is part I and which part II. Since it really is the same theorem, differently stated, some people simply call them both “The Fundamental Theorem of Calculus.”

**THEOREM 7.2.2 Fundamental Theorem of Calculus** Suppose that $f(x)$ is continuous on the interval $[a, b]$ and let

$$G(x) = \int_a^x f(t) \, dt$$

Then $G'(x) = f(x)$.

We have not yet proved the Fundamental Theorem. In a nutshell, we gave the following argument to justify it: Suppose we want to know the value of

$$\int_a^b f(t) \, dt = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(x)\Delta t.$$

We can interpret the right hand side as the distance traveled by an object whose speed is given by $f(t)$. We know another way to compute the answer to such a problem: find the position of the object by finding an antiderivative of $f(t)$, then substitute $t = a$ and $t = b$ and subtract to find the distance traveled. This must be the answer to the original problem as well, even if $f(t)$ does not represent a speed.

What’s wrong with this? In some sense, nothing. As a practical matter it is a very convincing argument, because our understanding of the relationship between speed and distance seems to be quite solid. From the point of view of mathematics, however, it is unsatisfactory to justify a purely mathematical relationship by appealing to our understanding of the physical universe, which could, however unlikely it is in this case, be wrong. A complete proof is a bit too involved to include here, but we will indicate how it goes. First, if we can prove the second version of the Fundamental Theorem, theorem 7.2.2, then we can prove the first version from that.
Now we know that to solve certain kinds of problems, those that lead to a sum of a certain form, we “merely” find an antiderivative and substitute two values and subtract. Unfortunately, finding antiderivatives can be quite difficult. While there are a small number of rules that allow us to compute the derivative of any common function, there are no such rules for antiderivatives. There are some techniques that frequently prove useful, but we will never be able to reduce the problem to a completely mechanical process.

Because of the close relationship between an integral and an antiderivative, the integral sign is also used to mean “antiderivative.” You can tell which is intended by whether the limits of integration are included: \[ \int_a^b f(x) \, dx \]
is an ordinary integral, also called a definite integral, because it has a definite value; namely \[
\int_a^b x^2 \, dx = \frac{b^3}{3} - \frac{a^3}{3} = \frac{b^3 - a^3}{3}.
\]
We use \[ \int x^2 \, dx \]
to denote the antiderivative of \( x^2 \), also called an indefinite integral. So this is evaluated as \[ \int x^2 \, dx = \frac{x^3}{3} + C. \]
It is customary to include the constant \( C \) to indicate that there are really an infinite number of antiderivatives. We do not need this \( C \) to compute definite integrals, but in other circumstances we will need to remember that the \( C \) is there, so it is best to get into the habit of writing the \( C \). When we compute a definite integral, we first find an antiderivative and then substitute. It is convenient to first display the antiderivative and then do the substitution; we need a notation indicating that the substitution is yet to be done. A typical solution would look like this:

\[
\int_0^2 x^2 \, dx = \frac{x^3}{3} \bigg|_{0}^{2} = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}.
\]
The vertical line with subscript and superscript is used to indicate the operation “substitute and subtract” that is needed to finish the evaluation.

**Proof of Theorem 7.2.1.** We know from theorem 7.2.2 that

\[ G(x) = \int_a^x f(t) \, dt \]
is an antiderivative of \( f(x) \), and therefore any antiderivative \( F(x) \) of \( f(x) \) is of the form \( F(x) = G(x) + k \). Then

\[
F(b) - F(a) = G(b) + k - (G(a) + k) = G(b) - G(a) = \int_a^b f(t) \, dt - \int_a^a f(t) \, dt = \int_a^b f(t) \, dt.
\]

It is not hard to see that \( \int_a^a f(t) \, dt = 0 \), so this means that

\[ F(b) - F(a) = \int_a^b f(t) \, dt, \]

which is exactly what theorem 7.2.1 says. 

So the real job is to prove theorem 7.2.2. We will sketch the proof, using some facts that we do not prove. First, the following identity is true of integrals:

\[
\int_a^x f(t) \, dt = \int_a^b f(t) \, dt + \int_b^x f(t) \, dt.
\]

This can be proved directly from the definition of the integral, that is, using the limits of sums. It is quite easy to see that it must be true by thinking of either of the two applications of integrals that we have seen. It turns out that the identity is true no matter what \( c \) is, but it is easiest to think about the meaning when \( a \leq c \leq b \).

First, if \( f(t) \) represents a speed, then we know that the three integrals represent the distance traveled between time \( a \) and time \( b \); the distance traveled between time \( a \) and time \( c \); and the distance traveled between time \( c \) and time \( b \). Clearly the sum of the latter two is equal to the first of these.

Second, if \( f(t) \) represents the height of a curve, the three integrals represent the area under the curve between \( a \) and \( b \); the area under the curve between \( a \) and \( c \); and the area under the curve between \( c \) and \( b \). Again it is clear from the geometry that the first is equal to the sum of the second and third.

**Proof sketch for Theorem 7.2.2.** We want to compute \( G'(x) \), so we start with the definition of the derivative in terms of a limit:

\[
G'(x) = \lim_{\Delta x \to 0} \frac{G(x + \Delta x) - G(x)}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left( \int_x^{x + \Delta x} f(t) \, dt - \int_x^x f(t) \, dt \right)
\]

\[
= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x + \Delta x} f(t) \, dt - \int_x^x f(t) \, dt = \int_x^{x + \Delta x} f(t) \, dt.
\]

Now we need to know something about

\[
\int_x^{x + \Delta x} f(t) \, dt
\]
can be interpreted as the distance traveled by an object over a very short interval of time. Over a sufficiently short period of time, the speed of the object will not change very much, so the distance traveled will be approximately the length of time multiplied by the speed at the beginning of the interval, namely, \( dx f(x) \). Alternately, the integral may be interpreted as the area under the curve between \( x \) and \( x + \Delta x \). When \( \Delta x \) is very small, this will be very close to the area of the rectangle with base \( dx \) and height \( f(x) \); again this is \( dx f(x) \). If we accept this, we may proceed:

\[
\lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x + \Delta x} f(t) \, dt = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f(x).
\]

which is what we wanted to show.

It is still true that we are depending on an interpretation of the integral to justify the argument, but we have isolated this part of the argument into two facts that are not too hard to prove. Once the last reference to interpretation has been removed from the proofs of these facts, we will have a real proof of the Fundamental Theorem.
right to left, so that $t_0 = b$ and $t_n = a$. Then $\Delta t = t_{i+1} - t_i$ is negative and in

$$
\int_a^b v(t) \, dt = \sum_{i=1}^{n-1} v(t_i) \Delta t,
$$

the values $v(t_i)$ are negative but also $\Delta t$ is negative, so all terms are positive again. On the other hand, in

$$
\int_b^a v(t) \, dt = \sum_{i=1}^{n-1} v(t_i) \Delta t,
$$

the values $v(t_i)$ are positive but $\Delta t$ is negative, and we get a negative result:

$$
\int_b^a v(t) \, dt = \int_a^b v(t) \, dt = - \sum_{i=1}^{n-1} v(t_i) \Delta t.
$$

Finally we note one simple property of integrals:

$$
\int_a^b f(x) + g(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx.
$$

This is easy to understand once you recall that $(F(x) + G(x))' = F'(x) + G'(x)$. Hence, if $F'(x) = f(x)$ and $G'(x) = g(x)$, then

$$
\int_a^b f(x) + g(x) \, dx = \int_a^b (F(x) + G(x)) \, dx
$$

and if $a < b$ and $f(x) \leq 0$ on $[a,b]$ then

$$
\int_a^b f(x) \, dx \leq 0
$$

and in fact

$$
\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx.
$$

### Exercises 7.3.

1. An object moves so that its velocity at time $t$ is $v(t) = -9.8 t + 20$ m/s. Describe the motion of the object between $t = 0$ and $t = 5$, find the total distance traveled by the object during that time, and find the net distance traveled.

2. An object moves so that its velocity at time $t$ is $v(t) = \text{sin } t$. Set up and evaluate a single definite integral to compute the net distance traveled between $t = 0$ and $t = 2\pi$.

3. An object moves so that its velocity at time $t$ is $v(t) = 1 + \text{sin } t$ m/s. Find the net distance traveled by the object between $t = 0$ and $t = 2\pi$, and find the total distance traveled during the same period.

4. Consider the function $f(x) = (x + 2)(x + 1)(x - 1)(x - 2)$ on $[-2,2]$. Find the total area between the curve and the $x$-axis (measuring all area as positive).

5. Consider the function $f(x) = x^2 - 3x + 2$ on $[0,4]$. Find the total area between the curve and the $x$-axis (measuring all area as negative).

6. Evaluate the three integrals:

$$
A = \int_0^2 (-x^2 + 9) \, dx \quad B = \int_0^2 (-x^2 + 9) \, dx \quad C = \int_0^2 (-x^2 + 9) \, dx,
$$

and verify that $A = B + C$. 

7. Consider the curve $y = x^2$ for $0 \leq x \leq 2$. Find the area under the curve.

8. Consider the curve $y = \cos x$ for $0 \leq x \leq \pi$. Find the area under the curve.