4 Transcendental Functions

So far we have used only algebraic functions as examples when finding derivatives, that is, functions that can be built up by the usual algebraic operations of addition, subtraction, multiplication, division, and raising to constant powers. Both in theory and practice there are other functions, called transcendental, that are very useful. Most important among these are the trigonometric functions, the inverse trigonometric functions, exponential functions, and logarithms.

4.1 Trigonometric Functions

When you first encountered the trigonometric functions it was probably in the context of “triangle trigonometry,” defining, for example, the sine of an angle as the “side opposite over the hypotenuse.” While this will still be useful in an informal way, we need to use a more expansive definition of the trigonometric functions. First an important note: while degree measure of angles is sometimes convenient because it is so familiar, it turns out to be ill-suited to mathematical calculation, so (almost) everything we do will be in terms of radian measure of angles.

The angle \( x \) is subtended by the heavy arc in the figure, that is, \( x \sim \pi / 6 \). Both coordinates of point \( A \) in this figure are negative, so the sine and cosine of \( \pi / 6 \) are both negative.

The remaining trigonometric functions can be most easily defined in terms of the sine and cosine, as usual:

\[
\begin{align*}
\tan x &= \frac{\sin x}{\cos x} \\
\cot x &= \frac{\cos x}{\sin x} \\
\sec x &= \frac{1}{\cos x} \\
\csc x &= \frac{1}{\sin x}
\end{align*}
\]

and they can also be defined as the corresponding ratios of coordinates.

Although the trigonometric functions are defined in terms of the unit circle, the unit circle diagram is not what we normally consider the graph of a trigonometric function. (The unit circle is the graph of, well, the circle.) We can easily get a qualitatively correct circle diagram is not what we normally consider the graph of a trigonometric function. The standard convention is to place the starting radius for the angle on the positive \( x \)-axis, and to measure positive angles counterclockwise around the circle. In the figure, \( x \) is the standard location of the angle \( \pi / 6 \), that is, the length of the arc from \( (1,0) \) to \( A \) is \( \pi / 6 \). The angle \( y \) in the picture is \( -\pi / 6 \), because the distance from \( (1,0) \) to \( B \) along the circle is also \( \pi / 6 \), but in a clockwise direction.

Now the fundamental trigonometric definitions are: the cosine of \( x \) and the sine of \( x \) are the first and second coordinates of the point \( A \), as indicated in the figure. The angle \( x \) shown can be viewed as an angle of a right triangle, meaning the usual triangle definitions of the sine and cosine also make sense. Since the hypotenuse of the triangle is 1, the “side opposite over hypotenuse” definition of the sine is the second coordinate of point \( A \) over 1, which is just the second coordinate; in other words, both methods give the same value for the sine.

The simple triangle definitions work only for angles that can “fit” in a right triangle, namely, angles between 0 and \( \pi / 2 \). The coordinate definitions, on the other hand, apply to any angles, as indicated in this figure:

![Unit Circle Diagram](image)

To define the radian measurement system, we consider the unit circle in the \( xy \)-plane:

![Unit Circle Diagram](image)

An angle, \( x \), at the center of the circle is associated with an arc of the circle which is said to subtend the angle. In the figure, this arc is the portion of the circle from point \((1,0)\) to point \( A \). The length of this arc is the radian measure of the angle \( x \); the fact that the radian measure is an actual geometric length is largely responsible for the usefulness of radian measure. The circumference of the unit circle is \( 2\pi \), so the radian measure of the full circular angle (that is, of the 360 degree angle) is \( 2\pi \).

While an angle with a particular measure can appear anywhere around the circle, we need a fixed, conventional location so that we can use the coordinate system to define properties of the angle. The standard convention is to place the starting radius for the angle on the positive \( x \)-axis, and to measure positive angles counterclockwise around the circle. In the figure, \( x \) is the standard location of the angle \( \pi / 6 \), that is, the length of the arc from \((1,0)\) to \( A \) is \( \pi / 6 \). The angle \( y \) in the picture is \( -\pi / 6 \), because the distance from \((1,0)\) to \( B \) along the circle is also \( \pi / 6 \), but in a clockwise direction.

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The simple triangle definitions work only for angles that can “fit” in a right triangle, namely, angles between 0 and \( \pi / 2 \). The coordinate definitions, on the other hand, apply to any angles, as indicated in this figure:

![Unit Circle Diagram](image)

4.1.1 Exercises

1. Find all values of \( \theta \) such that \( \sin(\theta) = 1 \); give your answer in radians.
2. Find all values of \( \theta \) such that \( \cos(\theta) = 1/2 \); give your answer in radians.
3. Verify the identity \( \cos^2(\theta)/2 - \sin^2(\theta) = 1 + \sin(\theta) \).
4. Verify the identity \( 2\sin(\theta) = \sin(2\theta)/2 \).
5. Verify the identity \( \sin(3\theta) - \sin(\theta) = 2\cos(2\theta)\sin(\theta) \).
6. Sketch \( y = 2\sin(x) \).
7. Sketch \( y = \sin(3x) \).
8. Sketch \( y = \sin(x - \pi) \).
9. Sketch all of the solutions of \( 2\sin(\pi/2 - \theta) = \sin(\beta) \).

11. Find all of the solutions of \( 2\sin(\pi/2 - \theta) = \sin(\beta) \) in the interval \([0, 2\pi]\).

4.2 The Derivative of \( \sin x \)

What about the derivative of the sine function? The rules for derivatives that we have are no help, since \( \sin x \) is not an algebraic function. We need to return to the definition of the derivative, set up a limit, and try to compute it. Here’s the definition:

\[
\frac{d}{dx} \sin x = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}
\]

Using some trigonometric identities, we can make a little progress on the quotient:

\[
\begin{align*}
\frac{\sin(x + \Delta x) - \sin x}{\Delta x} &= \frac{\sin x \cos \Delta x + \sin \Delta x \cos x - \sin x}{\Delta x} \\
&= \frac{\sin x \cos \Delta x - 1}{\Delta x} + \sin \Delta x \cos x.
\end{align*}
\]

Similarly, as angle \( x \) increases from 0 in the unit circle diagram, the first coordinate of the point \( A \) goes from 1 to then to \(-1\), back to 0 and back to 1, so the graph of \( y = \cos x \) must look something like this:

![Cosine Graph](image)
This isolates the difficult bits in the two limits

\[
\lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}.
\]

Here we get a little lucky: it turns out that once we know the second limit the first is quite easy. The second is quite tricky, however. Indeed, it is the hardest limit we will actually compute, and we devote a section to it.

### 4.3 A Hard Limit

We want to compute this limit:

\[
\lim_{x \to 0} \frac{\sin x}{x}.
\]

Equivalently, to make the notation a bit simpler, we can compute

\[
\lim_{h \to 0} \frac{\sin h}{h}.
\]

In the original context we need to keep \( x \) and \( \Delta x \) separate, but here it doesn’t hurt to rename \( \Delta x \) to something more convenient.

To do this we need to be quite clever, and to employ some indirect reasoning. The indirect reasoning is embodied in a theorem, frequently called the squeeze theorem.

**Theorem 4.3.1 Squeeze Theorem**

Suppose that \( g(x) \leq f(x) \leq h(x) \) for all \( x \) close to \( a \) but not equal to \( a \). If \( \lim_{x \to a} g(x) = L = \lim_{x \to a} h(x) \), then \( \lim_{x \to a} f(x) = L \).

This theorem can be proved using the official definition of limit. We won’t prove it here, but point out that it is easy to understand and believe graphically. The condition says that \( f(x) \) is trapped between \( g(x) \) below and \( h(x) \) above, and that at \( x = a \), both \( g \) and \( h \) approach the same value. This means the situation looks something like figure 4.3.1. The wiggly curve is \( x^2 \sin(1/x) \), the upper and lower curves are \( x^2 \) and \(-x^2 \). Since the sine function is always between \(-1\) and \(1\), \(-x^2 \leq x^2 \sin(1/x) \leq x^2\), and it is easy to see that \( \lim_{x \to 0} -x^2 = 0 = \lim_{x \to 0} x^2 \). It is not so easy to see directly, that is, algebraically, that \( \lim_{x \to 0} x^2 \sin(1/x) = 0 \), because the \( 1/x \) prevents us from simply plugging in \( x = 0 \). The squeeze theorem makes this “hard limit” as easy as the trivial limits involving \( x^2 \).

To do the hard limit that we want, \( \lim_{x \to 0} \sin(x)/x \), we will find two simpler functions \( g \) and \( h \) so that \( g(x) \leq \sin(x)/x \leq h(x) \), and so that \( \lim_{x \to 0} g(x) = \lim_{x \to 0} h(x) \). Not too surprisingly, this will require some trigonometry and geometry. Referring to figure 4.3.2, \( x \) is the measure of the angle in radians. Since the circle has radius \( 1 \), the coordinates of point \( A \) are \((\cos x, \sin x)\), and the area of the small triangle is \((\cos x \sin x)/2 \). This triangle is completely contained within the circular wedge-shaped region bounded by two lines and the circle from \((1,0)\) to point \( A \). Comparing the areas of the triangle and the wedge we see \((\cos x \sin x)/2 \leq x/2 \), since the area of a circular region with angle \( \theta \) and radius \( r \) is \( \theta r^2/2 \). With a little algebra this turns into \((\sin x)/x \leq 1/\cos x\), giving us the \( h \) we seek.

![Figure 4.3.1 The squeeze theorem.](image1)

To find \( g \), we note that the circular wedge is completely contained inside the larger triangle. The height of the triangle, from \((1,0)\) to point \( B \), is \( \tan x \), so comparing areas we get \( x/2 \leq (\tan x)/2 = \sin x/(2 \cos x) \). With a little algebra this becomes \( \cos x \leq (\sin x)/x \).

So now we have

\[
\cos x \leq \frac{\sin x}{x} \leq \frac{1}{\cos x}.
\]

### Exercises 4.3.

1. Compute \( \lim_{x \to 0} \frac{\sin(2x)}{x} \).
2. Compute \( \lim_{x \to 0} \frac{\sin(Tx)}{x} \).
3. Compute \( \lim_{x \to 0} \frac{\sin(4x)}{x} \).
4. Compute \( \lim_{x \to 0} \frac{\cos(x)}{x} \).
5. Compute \( \lim_{x \to 0} \frac{\sin(x) - \cos(x)}{x} \).
6. For all \( x \geq 0 \), \( 4x - 9 \leq f(x) \leq x^2 - 4x + 7 \). Find \( \lim_{x \to 1} f(x) \).
7. For all \( x \geq 0 \), \( 2x \leq g(x) \leq x^2 - 2x + 2 \). Find \( \lim_{x \to 1} g(x) \).
8. Use the Squeeze Theorem to show that \( \lim_{x \to 0} \sin^2(2/x) = 0 \).

### 4.4 The Derivative of \( \sin x \), Continued

Now we can complete the calculation of the derivative of the sine:

\[
\frac{d}{dx} \sin x = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos \Delta x - 1 + \cos \Delta x - 1}{\Delta x} = \cos x.
\]

The derivative of a function measures the slope or steepness of the function; if we examine the graphs of the sine and cosine side by side, it should be that the latter appears to accurately describe the slope of the former, and indeed this is true:

![Figure 4.3.2 Visualizing \( \sin x/x \).](image2)

Notice that where the cosine is zero the sine does appear to have a horizontal tangent line, and that the sine appears to be steepest where the cosine takes on its extreme values of \( 1 \) and \(-1 \).

Of course, now that we know the derivative of the sine, we can compute derivatives of more complicated functions involving the sine.

**Example 4.4.1** Compute the derivative of \( \sin(x^2) \).

\[
\frac{d}{dx} \sin(x^2) = \cos(x^2) \cdot 2x = 2x \cos(x^2).
\]

**Example 4.4.2** Compute the derivative of \( \sin^2(x^2 - 5x) \).

\[
\frac{d}{dx} \sin^2(x^2 - 5x) = 2 \sin(x^2 - 5x) \cos(x^2 - 5x) = 2(x^2 - 5x) \cos(x^2 - 5x) = 2(x^2 - 5x) \cos(x^2 - 5x).
\]
4.5 Derivatives of the Trigonometric Functions

All of the other trigonometric functions can be expressed in terms of the sine, and so their derivatives can easily be calculated using the rules we already have. For the cosine function we use two identities,

\[ \cos x = \sin x + \frac{\pi}{2}, \]
\[ \sin x = \cos x + \frac{\pi}{2}. \]

Now,

\[ \frac{d}{dx} \cos x = \frac{d}{dx} \sin x + \frac{\pi}{2} = - \sin x, \]
\[ \frac{d}{dx} \sin x = \frac{d}{dx} \cos x + \frac{\pi}{2} = - \cos x. \]

The derivatives of the tangent and cotangent are similar and left as exercises.

Exercises 4.5.

Find the derivatives of the following functions.

1. \( \sin x \)
2. \( \cos x \)
3. \( \tan x \)
4. \( \cot x \)
5. \( \sec x \)
6. \( \csc x \)

4.6 Exponential and Logarithmic Functions

An exponential function has the form \( a^x \), where \( a \) is a constant; examples are \( 2^x \), \( 10^x \), \( e^x \). The logarithmic functions are the inverses of the exponential functions, that is, functions that "undo" the exponential functions, just as, for example, the cube root function "undoes" the cube function: \( \sqrt[3]{x} = 2 \). Note that the original function also undoes the inverse function: \( (\sqrt[3]{x})^3 = x \).

Let \( f(x) = 2^x \). The inverse of this function is called the logarithm base 2, denoted \( \log_2(x) \) (or especially in computer science circles) \( \lg(x) \). What does this really mean? The logarithm must undo the action of the exponential function, so for example it must be that \( \lg(2^3) = 3 \) — starting with 3, the exponential function produces \( 2^3 = 8 \), and the logarithm of 8 must get us back to 3. A little thought shows that it is not a coincidence that \( \lg(2^3) \) simply gives the exponent — the exponent is the original value that we must get back to. In other words, the logarithm is the exponent. Remember this catchphrase, and what it means, and you won’t go wrong. (You do have to remember what it means. Like any good mnemonic, “the logarithm is the exponent” leaves out a lot of detail, like “Which exponent?” and “Exponent of what?”)

EXAMPLE 4.6.1 What is the value of \( \log_2(1000) \)? The “10” tells us the appropriate number to use for the base of the exponential function. The logarithm is the exponent, so the question is, what exponent \( E \) makes \( 2^E = 1000 \)? If we can find such an \( E \), then \( \log_2(1000) = \log_2(2^E) \); finding the appropriate exponent is the same as finding the logarithm. In this case, of course, it is easy: \( E = 9 \) so \( \log_2(1000) = 3 \).

Let’s review some laws of exponents and logarithms; let \( a \) be a positive number. Since \( a^m \cdot a^n = a^{m+n} \) and \( a^m/a^n = a^{m-n} \) and in general that \( a^{mn} = a^m \cdot a^n \). Since “the logarithm is the exponent,” it’s no surprise that this translates directly into a fact about the logarithm function. Here are three facts when \( x = p/q \) if we were to graph this we’d see something like this:

4.7 Derivatives of the Exponential and Logarithmic Functions

As with the sine, we don’t know anything about derivatives that allows us to compute the derivatives of the exponential and logarithmic functions without going back to basics. Let’s do a little work with the definition again:

\[ \frac{d}{dx} a^x = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \]
\[ \frac{d}{dx} a^{-x} = \lim_{h \to 0} \frac{a^{x-h} - a^x}{h} \]
\[ \frac{d}{dx} \log_a x = \frac{1}{x \ln a} \]

Exercises 4.6.

1. Expand \( \log_2(4 + 3x) \) \((x - 2)\).
2. Expand \( \log_8 \left( \frac{x}{2} - 5 + \frac{1}{x} \right) \).
3. Write \( \log_2 3x + 17 \log_3 (x - 2) - 2 \log_2 (x^2 - 4x + 1) \) as a single logarithm.
4. Solve \( \log_2(1 + \sqrt{2}) = 6 \) for \( x \).
5. Solve \( x \sqrt{x} = 8 \) for \( x \).
6. Solve \( \log_2(\log_3(x)) = 1 \) for \( x \).

4.8 Chapter 4 Transcendental Functions

14. Find all points on the graph of \( f(x) = 2 \sin(x) - \sin^2(x) \) at which the tangent line is horizontal.
15. Find an equation for the tangent line to \( f(x) = x + \pi/3 \).
16. Find an equation for the tangent line to \( x^2 + x = \pi/3 \).
17. Find an equation for the tangent line to \( \cos^2 x - \sin^2 x \) at \( x = \pi/6 \).
18. Find the points on the curve \( y = x + 2ax \) that have a horizontal tangent line.
19. Let \( C \) be a circle of radius \( r \). Let \( A \) be an arc on \( C \) subtending a central angle \( \theta \). Let \( B \) be the chord of \( C \) whose endpoints are the endpoints of \( A \). (Hence, \( B \) also subtends \( \theta \).) Let \( x \) be the length of \( A \) and let \( d \) be the length of \( B \). Sketch a diagram of the situation and compute \( \lim_{x \to d} x/d \).
There are two interesting things to note here. As in the case of the sine function we are left with a limit that involves $\Delta x$ but not $x$, which means that whatever $\lim_{\Delta x \to 0} (e^{\Delta x} - 1)/\Delta x$ is, we know that it is a number, that is, a constant. This means that $a^n$ has a remarkable property: its derivative is a constant times itself.

We earlier remarked that the hardest limit we would compute is $\lim \sin x/x = 1$; we now have a limit that is just a bit too hard to include here. In fact the hard part is to see that $\lim_{\Delta x \to 0} (e^{\Delta x} - 1)/\Delta x$ even exists—does this function really get closer and closer to some fixed value? Yes it does, but we will not prove this fact.

We can look at some examples. Consider $(2^x - 1)/x$ for some small values of $x$: 1, 0.828427124, 0.756828469, 0.72401864, 0.7083050, 0.7070877 when $x$ is 1, 1/2, 1/4, 1/8, 1/16, 1/32, respectively. It looks like this is settling in around 0.7, which turns out to be true (but the limit is not exactly 0.7). Consider next $(x^2 - 1)/x$: 2, 1.46410166, 1.26490052, 1.177621526, 1.17760771, 1.17760654, at the same values of $x$. It turns out to be true that in the limit this is about 1.1. Two examples don’t establish a pattern, but if you do more examples you will find that the limit varies directly with the value of $a$: bigger $a$, bigger limit; smaller $a$, smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between $a = 2$ and $a = 3$ the limit will be exactly 1; the value at which this happens is called $e$, so that

$$
\lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1.
$$

As you might guess from our two examples, $e$ is close to 3 than to 2, and in fact $e \approx 2.718$.

Now we see that the function $e^x$ has a truly remarkable property:

$$
\frac{d}{dx} e^x = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^x e^{\Delta x} - e^x}{\Delta x} = e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = e^x.
$$

That is, $e^x$ is its own derivative, or in other words the slope of $e^x$ is the same as its height, or the same as its second coordinate: The function $f(z) = e^z$ goes through the point $(z, e^z)$ and has slope $e^z$ there, no matter what $z$ is. It is sometimes convenient to express the function $e^x$ without an exponent, since complicated exponents can be hard to read. In such cases we use $\exp(x)$, e.g., $\exp(1 + x^2)$ instead of $e^{1+x^2}$.

4.7 Derivatives of the exponential and logarithmic functions

We have discussed this from the point of view of the graphs, which is easy to understand but is not normally considered a rigorous proof—it is too easy to be led astray by pictures that seem reasonable but that miss some hard point. It is possible to do this derivation without resorting to pictures, and indeed we will see an alternate approach soon. Note that $\ln x$ is defined only for $x > 0$. It is sometimes useful, and sometimes essential, it may be better to remember the trick, not the formula.

**EXAMPLE 4.7.1** Compute the derivative of $f(x) = 2^x$.

$$
\frac{d}{dx} 2^x = \frac{d}{dx} e^{(\ln 2)x} = (\ln 2) e^{(\ln 2)x} = (\ln 2) 2^x.
$$

**EXAMPLE 4.7.2** Compute the derivative of $f(x) = x^2 = 2^{x2}$.

$$
\frac{d}{dx} x^2 = \frac{d}{dx} e^{(\ln 2)x^2} = \left(2 \ln 2\right) e^{(\ln 2)x^2} = (2 \ln 2) 2^{x^2}.
$$

**EXAMPLE 4.7.3** Compute the derivative of $f(x) = x^x$. At first this appears to be a bit of a trick function: it is not a constant power of $x$, and it does not seem to be an exponential function, since the base is not constant. But in fact it is no harder than the previous example.

$$
\frac{d}{dx} x^x = \frac{d}{dx} e^{x \ln x} = \left(\frac{d}{dx} x\ln x\right) e^{x \ln x} = \left(1 + \ln x\right) x^x.
$$
EXAMPLE 4.7.4 Recall that we have not justified the power rule except when the exponent is a positive or negative integer. We can use the exponential function to take care of other exponents.

\[
\frac{d}{dx} x^n = \frac{d}{dx} e^{n \ln x} = (\frac{d}{dx} \ln x) e^{n \ln x} = \left( \frac{1}{x} \right) x^n = nx^{n-1}
\]

Exercises 4.7.

In 1–19, find the derivatives of the functions.

1. \(x^{-1.5}\)  
2. \(\frac{\sin x}{x}\)  
3. \((e^x)^2\)  
4. \(\sin(e^x)\)  
5. \(e^{x^2}\)  
6. \(e^{2x\sqrt{x}}\)  
7. \((1/x)^3\)  
8. \(x^2 + x^2\)  
9. \(e^{x^3}\)  
10. \(x^{-1/2}\)  
11. \(\ln(x^3 + 3x)\)  
12. \(\ln(x \cos x)\)  
13. \(\sqrt{\ln(x^2)} x^3\)  
14. \(\ln(\sin(x) + \tan(x))\)  
15. \(x^\pi + \pi x\)  
16. \(x \ln x\)  
17. \(\ln(\ln(3x))\)  
18. \(\frac{\ln(\ln x)}{\ln(4x)}\)  
19. \(x^3(x - 2)^{1/2}\)  
20. \(\frac{1}{\sqrt{4x^2 - 6x}}\)

4.8 Implicit Differentiation

As we have seen, there is a close relationship between the derivatives of \(e^x\) and \(\ln x\) because these functions are inverses. Rather than relying on pictures for our understanding, we would like to be able to exploit this relationship computationally. In fact this technique can help us find derivatives in many situations, not just when we seek the derivative of an inverse function.

Chains rule where \(y\) appears.

\[
\frac{d}{dx} x^2 + y^2 = \frac{d}{dx} \sqrt{1 + \frac{x^2}{y^2}}
\]

\[
y' = -\frac{2x}{2y} = \frac{x}{y}
\]

Now we can check that the circle of equation \(x^2 + y^2 = 1\) is tangent to the y-axis at the point \((0,1)\). At this point we will need to know whether \(y = U(x)\) or \(L(x)\). Occasionally it will turn out that we can avoid explicit use of \(U(x)\) or \(L(x)\) by the nature of the problem.

EXAMPLE 4.8.1 Find the slope of the circle \(x^2 + y^2 = 1\) at \((1,\sqrt{3})\). Since we know both the \(x\) and \(y\) coordinates of the point of interest, we do not need to explicitly recognize that this point is on \(L(x)\), and we do not need to use \(L(x)\) to compute \(y\) but could use the calculation of \(y'\) from above.

\[
y' = \frac{x}{y} - \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}
\]

It is instructive to compare this approach to others.

We might have recognized at the start that \((1,\sqrt{3})\) is on the function \(y = L(x)\) as \(\sqrt{1 - x^2}\). We could then take the derivative of \(L(x)\), using the power rule and the chain rule, to get

\[
L'(x) = \frac{1}{\sqrt{3}}(1 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{3}}
\]

Then we could compute \(L'(1) = \frac{1}{\sqrt{3}}\) by substituting \(x = 1\).

Alternatively, we could realize that the point is on \(L(x)\), but use the fact that \(y' = -x/y\).

Since the point is on \(L(x)\) we can replace \(y\) by \(L(x)\) to get

\[
y' = \frac{x}{L(x)} - \frac{x}{\sqrt{3}}
\]

without computing the derivative of \(L(x)\) explicitly. Then we substitute \(x = 1\) and get the same answer as before.

In the case of the circle it is possible to find the functions \(U(x)\) and \(L(x)\) explicitly, but there are potential advantages to using implicit differentiation anyway. In some cases it is more difficult or impossible to find an explicit formula for \(y\) and implicit differentiation is the only way to find the derivative.

EXAMPLE 4.8.2 Find the derivative of any function defined implicitly by \(y^2 + e^y = x\).

We treat \(y\) as an unspecified function and use the chain rule:

\[
\frac{d}{dx} (y^2 + e^y) = \frac{d}{dx} x
\]

\[
y' y' + e^y y' = 1
\]

\[
y'^2 e^y = 1 - 2y
\]

\[
y'^2 e^y - 1 = 2y
\]

\[
y' = \frac{1 - 2y}{2y + e^y}
\]

You might think that the step in which we solve for \(y'\) could sometimes be difficult—after all, we’re using implicit differentiation here because we can’t solve the equation \(y^2 + e^y = x\) for \(y\), so maybe after taking the derivative we get something that is hard to solve for \(y'\). In fact, this never happens. All occurrences of \(y'\) come from applying the chain rule, and whenever the chain rule is used it deposits a single \(y'\) multiplied by some other expression. So it will always be possible to group the terms containing \(y'\) together and factor out the \(y'\) just as in the previous example. If you ever get anything more difficult you have made a mistake and should fix it before trying to continue.

It is sometimes the case that a situation leads naturally to an equation that defines a function implicitly.

EXAMPLE 4.8.3 Consider all the points \((x, y)\) that have the property that the distance from \((x, y)\) to \((1, 1, y)\) plus the distance from \((x, y)\) to \((x, 2y, 2)\) is \(2\). (\(a\) is some constant.)

These points form an ellipse, which looks like a circle but is not a function but can be viewed as two functions pasted together. Because we know how to write down the distance between two points, we can write down an implicit equation for the ellipse:

\[
\sqrt{(x - 1)^2 + (y - 1)^2} + \sqrt{(x - 2)^2 + (y - 2)^2} = 2a
\]

Then we can use implicit differentiation to find the slope of the ellipse at any point, though the computation is rather messy.
of each side:

\[ \frac{\sin y}{\cos y} - \frac{\sin x}{\cos x} = \frac{\sin y}{\cos y} = \frac{\sin y}{\cos y} \]

Then we use implicit differentiation to find that

\[ \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} \]

Note that this agrees with figure 4.9.1: the graph of the arcsine has positive slope everywhere.

We can do something similar for the cosine. As with the sine, we must first truncate the cosine so that it can be inverted, as shown in figure 4.9.2. Then we use implicit differentiation to find that

\[ \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} = \frac{1}{\sqrt{1 - x^2}} \]

Exercises 4.8.

1. Find an equation for the tangent line to

\[ \frac{y^2}{x^2} = \frac{y^2}{x^2} \]

at the point \((x, y)\).

2. Find an equation for the tangent line to

\[ \frac{y^2}{x^2} + \frac{y^2}{x^2} = \frac{y^2}{x^2} \]

at the point \((x, y)\).

3. Find an equation for the tangent line to

\[ \frac{y^2}{x^2} = \frac{y^2}{x^2} \]

at the point \((x, y)\).

4. Find the equation of the tangent line to

\[ \frac{y^2}{x^2} = \frac{y^2}{x^2} \]

at the point \((x, y)\).

5. Find the equation of the tangent line to

\[ \frac{y^2}{x^2} = \frac{y^2}{x^2} \]

at the point \((x, y)\).

6. Find the equation of the tangent line to

\[ \frac{y^2}{x^2} = \frac{y^2}{x^2} \]

at the point \((x, y)\).

7. Find the equation of the tangent line to

\[ \frac{y^2}{x^2} = \frac{y^2}{x^2} \]

at the point \((x, y)\).

8. Find the equation of the tangent line to

\[ \frac{y^2}{x^2} = \frac{y^2}{x^2} \]

at the point \((x, y)\).

9. Find the equation of the tangent line to

\[ \frac{y^2}{x^2} = \frac{y^2}{x^2} \]

at the point \((x, y)\).

10. Compute \(y'\) for the ellipse of example 4.8.3.

11. If \(y = \log x\), then \(x = x\). Use implicit differentiation to find \(y'\).

12. The graph of the equation \(x^2 + xy + y^2 = 9\) is an ellipse. Find the lines tangent to this curve at the two points where it intersects the \(y\)-axis. Show that these lines are parallel.

13. Repeat the previous problem for the points at which the ellipse intersects the \(x\)-axis.

14. Find the points on the ellipse from the previous two problems where the slope is horizontal and where it is vertical.

15. Find an equation for the tangent line to \(y^2 = x^2 + x^2 \) at \((2, \sqrt{2})\). (This curve is the *envelope* of *Enneper*.)

16. Find an equation for the tangent line to \(y^2 = x^2 + x^2 \) at a point \((x_1, y_1)\) on the curve, with \(x_1 \neq 0\) and \(y_1 \neq 0\). (This curve is an *astroid*.)

17. Find an equation for the tangent line to \((x^2 + y^2)^{1/2} = x^2 + y^2 \) at a point \((x_1, y_1)\) on the curve, when \(y_1 \neq 0\). (This curve is a *lemniscate*.)

Exercises 4.9.

1. Show that the derivative of the arccos \(x\) is

\[ \frac{d}{dx} \arccos x = \frac{1}{\sqrt{1 - x^2}} \]

2. Show that the derivative of the arctan \(x\) is

\[ \frac{d}{dx} \arctan x = \frac{1}{1 + x^2} \]

Exercises 4.9.

Definition. Two curves are orthogonal if at each point of intersection, the angle between their tangent lines is \(\pi/2\). Two families of curves, \(A\) and \(B\), are orthogonal trajectories of each other if given any curve \(C\) in \(A\) and any curve \(D\) in \(B\) the curves \(C\) and \(D\) are orthogonal. For example, the family of horizontal lines in the plane is orthogonal to the family of vertical lines in the plane.

18. Show that \(x^2 - y^2 = 5\) is orthogonal to \(x^2 + y^2 = 72\). (Hint: You need to find the intersection points of the two curves and then show that the product of the derivatives at each intersection point is \(-1\).)

19. Show that \(x^2 + y^2 = 5\) is orthogonal to \(y = mx\). Conclude that the family of circles centered at the origin is an orthogonal trajectory of the family of lines that pass through the origin.

Note that there is a technical issue when \(m = 0\). The circles fail to be differentiable when they cross the \(y\)-axis. However, the circles are orthogonal to the \(x\)-axis. Explain why.

Likewise, the vertical line through the origin requires a separate argument.

20. For \(k \neq 0\) and \(c \neq 0\) show that \(y^2 - x^2 = k\) is orthogonal to \(yx = \pi/2\). In the case where \(k\) and \(c\) are both zero, the curves intersect at the origin. Are the curves \(y^2 - x^2 = 0\) and \(yx = \pi/2\) orthogonal to each other?

21. Suppose that \(m \neq 0\). Show that the family of curves \(\{y = mx + b; \ b \in \mathbb{R}\}\) is orthogonal to the family of curves \(\{y = (\pi/2 - mx) + c; \ c \in \mathbb{R}\}\).
3. The inverse of cot is usually defined so that the range of arccot is $[0, \pi]$. Sketch the graph of $y = \arccot x$. In the process you will make it clear what the domain of arcsin is. Find the derivative of the arccotangent. 

4. Show that arccot $x + \arccot x = \pi/2$.

5. Find the derivative of $\arcsin(x^2) \Rightarrow$

6. Find the derivative of $\arctan(x^2) \Rightarrow$

7. Find the derivative of $\ln(\arccos(x^2)) \Rightarrow$

8. Find the derivative of $\tan(\arcsin(x^2)) \Rightarrow$

9. Find the derivative of $\arccos(e^x) \Rightarrow$

10. Find the derivative of $\arcsin(x) + \arccot x \Rightarrow$

11. Find the derivative of $\tan(\arctan(x^2)) \Rightarrow$

4.10 Limits revisited

We have defined and used the concept of limit, primarily in our development of the derivative. Recall that $\lim_{x \to a} f(x) = L$ is true if, in a precise sense, $f(x)$ gets closer and closer to $L$ as $x$ gets closer and closer to $a$. While some limits are easy to see, others take some thought. In particular, the limits of functions that are always differentiable on their domains, since in

$$\lim_{x \to a} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

both the numerator and denominator approach zero. Typically this difficulty can be resolved when $f$ is a “nice” function and we are trying to compute a derivative. Occasionally such limits are interesting for other reasons, and the limit of a fraction in which both numerator and denominator approach zero can be difficult to analyze. Now that we have the derivative available, there is another technique that can sometimes be helpful in such circumstances.

Before we introduce the technique, we will also expand our concept of limit, in two ways. When the limit of $f(x)$ as $x$ approaches $a$ does not exist, it may be useful to note in what way it does not exist. We have already talked about one such case: one-sided limits. Another case is when “$f$ goes to infinity”. We will also occasionally want to know what happens to $f(x)$ when $x$ “goes to infinity”.

**EXAMPLE 4.10.1** What happens to $1/x$ as $x$ goes to $0$? From the right, $1/x$ gets bigger and bigger, or goes to infinity. From the left it goes to negative infinity. From the left it goes to infinity.

**EXAMPLE 4.10.2** What happens to the function $\cos(1/x)$ as $x$ goes to infinity? It seems clear that as $x$ gets larger and bigger, $1/x$ gets closer and closer to zero, so $\cos(1/x)$ should be getting closer and closer to $\cos(0) = 1$.

**4.10 Limits revisited**

First we use L'Hôpital's Rule. Since the numerator and denominator both approach zero,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the latter exists. But in fact this is an easy limit, since the denominator now approaches $−1$, so

$$\lim_{x \to a} \frac{2f(x)}{\sin x} = \frac{2f(a)}{\sin a}$$

Since $-\pi$ approaches zero as $x$ approaches $\pi$. Now

$$\lim_{x \to \pi} \frac{x - \pi}{\sin x} = \lim_{x \to \pi} \frac{x - \pi}{\sin x} = \lim_{x \to \pi} \frac{-2}{\sin x}$$

As before.

**EXAMPLE 4.10.7** Compute $\lim_{x \to \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1}$ in two ways.

As $x$ goes to infinity both the numerator and denominator go to infinity, so we may apply L'Hôpital's Rule:

$$\lim_{x \to \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = \lim_{x \to \infty} \frac{4x - 3}{2x + 47}$$

In the second quotient, it is still the case that the numerator and denominator both go to infinity, so we are allowed to use L'Hôpital's Rule again:

$$\lim_{x \to \infty} \frac{4x - 3}{2x + 47} = \lim_{x \to \infty} \frac{4}{2} = 2$$

So the original limit is $2$ as well.

Now as $x$ approaches infinity, all the quotients with some power of $x$ in the denominator approach zero, leaving $2$ in the numerator and $1$ in the denominator, so the limit again is $2$.

**EXAMPLE 4.10.8** Compute $\lim_{x \to 0} \frac{\sec x - 1}{\sin x}$ in two ways.

Both the numerator and denominator approach zero, so applying L'Hôpital's Rule:

$$\lim_{x \to 0} \frac{\sec x - 1}{\sin x} = \lim_{x \to 0} \frac{\tan x}{\cos x} = \frac{1}{1} = 1.$$
13. \( \lim_{x \to 1} \frac{1 - 1}{x} = 0 \Rightarrow \)
15. \( \lim_{x \to 1} \left( \frac{1}{2} + \frac{1}{3} \right) (x + 1) \Rightarrow \)
17. \( \lim_{x \to 1} \frac{1 + x}{1 + x^2} \Rightarrow \)
19. \( \lim_{x \to 1} \frac{2}{1 + x^2} \Rightarrow \)
21. \( \lim_{x \to 1} \frac{1 - 1}{x} = 0 \Rightarrow \)
23. \( \lim_{x \to 1} \frac{1}{x - 1} \Rightarrow \)
25. \( \lim_{x \to 1} \frac{\cos x + 1}{1} \Rightarrow \)
27. \( \lim_{x \to 1} \frac{1}{x - 1} \Rightarrow \)
29. \( \lim_{x \to 1} \frac{\sin(2x) + 1}{2} \Rightarrow \)
31. \( \lim_{x \to 1} \frac{\sin(2x) + 1}{2} \Rightarrow \)
33. \( \lim_{x \to 1} \frac{\sin(2x) + 1}{2} \Rightarrow \)
35. \( \lim_{x \to 1} \frac{\sin(2x) + 1}{2} \Rightarrow \)
37. \( \lim_{x \to 1} \frac{\sin(2x) + 1}{2} \Rightarrow \)
39. \( \lim_{x \to 1} \frac{\sin(2x) + 1}{2} \Rightarrow \)
41. \( \lim_{x \to 1} \frac{\sin(2x) + 1}{2} \Rightarrow \)
43. \( \lim_{x \to 1} \frac{\sin(2x) + 1}{2} \Rightarrow \)
45. \( \lim_{x \to 1} \frac{\sin(2x) + 1}{2} \Rightarrow \)
47. \( \lim_{x \to 1} \frac{\sin(2x) + 1}{2} \Rightarrow \)
49. \( \lim_{x \to 1} \frac{\sin(2x) + 1}{2} \Rightarrow \)

14. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)
16. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)
18. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)
20. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)
22. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)
24. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)
26. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)
28. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)
30. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)
32. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)
34. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)
36. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)
38. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)
40. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)
42. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)
44. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)
46. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)
48. \( \lim_{x \to 1} \sqrt{x} = 1 \Rightarrow \)

4.10 Limits revisited

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4.11 Hyperbolic Functions

The hyperbolic functions appear with some frequency in applications, and are quite similar in many respects to the trigonometric functions. This is a bit surprising given our initial definitions.

**DEFINITION 4.11.1.** The hyperbolic cosine is the function

\[ \cosh x = \frac{e^x + e^{-x}}{2} \]

and the hyperbolic sine is the function

\[ \sinh x = \frac{e^x - e^{-x}}{2} \]

Notice that \( \cosh x \) is even (that is, \( \cosh(-x) = \cosh(x) \)) while \( \sinh x \) is odd (\( \sinh(-x) = -\sinh(x) \)), and \( \cosh x + \sinh x = e^x \). Also, for all \( x \), \( \cosh x > 0 \), while \( \sinh x = 0 \) if and only if \( e^x - e^{-x} = 0 \), which is true precisely when \( x = 0 \).

**LEMMA 4.11.1.** The range of \( \cosh x \) is \([1, \infty)\).

**Proof.** Let \( y = \cosh x \). We solve for \( x \):

\[ y = \frac{e^x + e^{-x}}{2} \]

\[ 2y = e^x + e^{-x} \]

\[ 2y - 2 = e^x - e^{-x} \]

\[ e^x - e^{-x} = 2y - 2 \]

\[ e^x = y \pm \sqrt{y^2 - 1} \]

From the last equation, we see \( y^2 \geq 1 \), and since \( y \geq 0 \), it follows that \( y \geq 1 \).

Now suppose \( y > 1 \), so \( y = \sqrt{y^2 - 1} > 0 \). Then \( x = \ln(y + \sqrt{y^2 - 1}) \) is a real number, and \( y = \cosh x \), so \( y \) is in the range of \( \cosh x \).

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4.11.2 Geometric definitions of \( \sin, \cos, \cosh, \sinh \): \( t \) is twice the shaded area in each figure.

Given the definitions of the hyperbolic functions, finding their derivatives is straightforward. Here again we see similarities to the trigonometric functions.

**THEOREM 4.11.5**

\[ \frac{d}{dx} \cosh x = \sinh x \]

**Proof.**

\[ \frac{d}{dx} \cosh x = \frac{d}{dx} \frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x \]

Since \( \cosh x > 0 \), \( \sinh x \) is increasing and hence injective, so \( \sinh x \) has an inverse, \( \arcsinh x \). Also, \( \sinh x > 0 \) when \( x > 0 \), so \( \cosh x \) is injective on \([0, \infty)\) and has a (partial) inverse, \( \cosh x \). The other hyperbolic functions have inverses as well, though \( \arcsinh x \) is only a partial inverse. We may compute the derivatives of these functions as we have other inverse functions.

**THEOREM 4.11.6**

\[ \frac{d}{dx} \arcsinh x = \frac{1}{\sqrt{1 + x^2}} \]

**Proof.**

Let \( y = \arcsinh x \). Then \( \frac{d}{dx} \sinh y = \cosh y \cdot y' = 1 \), and so

\[ y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}} \]
The other derivatives are left to the exercises.

**Exercises 4.11.**

1. Show that the range of sinh \( x \) is all real numbers. (Hint: show that if \( y = \sinh x \) then \( x = \ln(y + \sqrt{y^2 + 1}) \)).

2. Compute the following limits:
   a. \( \lim_{x \to \infty} \cosh x \)
   b. \( \lim_{x \to \infty} \sinh x \)
   c. \( \lim_{x \to \infty} \tanh x \)
   d. \( \lim_{x \to \infty} (\cosh x - \sinh x) \)

3. Show that the range of \( \tanh x \) is \((-1, 1)\). What are the ranges of \( \coth, \sech, \) and \( \csch \)? (Use the fact that they are reciprocal functions.)

4. Prove that for every \( x, y \in \mathbb{R} \), \( \sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y \). Obtain a similar identity for \( \sinh(x - y) \).

5. Prove that for every \( x, y \in \mathbb{R} \), \( \cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y \). Obtain a similar identity for \( \cosh(x - y) \).

6. Use exercises 4 and 5 to show that \( \sinh(2x) = 2 \sinh x \cosh x \) and \( \cosh(2x) = \cosh^2 x + \sinh^2 x \) for every \( x \). Conclude also that \( (\cosh(2x) - 1)/2 = \sinh^2 x \).

7. Show that \( \frac{d}{dx} \tanh x = \sech^2 x \). Compute the derivatives of the remaining hyperbolic functions as well.

8. What are the domains of the six inverse hyperbolic functions?

9. Sketch the graphs of all six inverse hyperbolic functions.