4

Transcendental Functions

So far we have used only algebraic functions as examples when finding derivatives, that is, functions that can be built up by the usual algebraic operations of addition, subtraction, multiplication, division, and raising to constant powers. Both in theory and practice there are other functions, called transcendental, that are very useful. Most important among these are the trigonometric functions, the inverse trigonometric functions, exponential functions, and logarithms.

4.1 Trigonometric Functions

When you first encountered the trigonometric functions it was probably in the context of "triangle trigonometry," defining, for example, the sine of an angle as the "side opposite over the hypotenuse." While this will still be useful in an informal way, we need to use a more expansive definition of the trigonometric functions. First an important note: while degree measure of angles is sometimes convenient because it is so familiar, it turns out to be ill-suited to mathematical calculation, so (almost) everything we do will be in terms of radian measure of angles.

The angle $x$ is subtended by the heavy arc in the figure, that is, $x \approx 7\pi/6$. Both coordinates of point $A$ in this figure are negative, so the sine and cosine of $7\pi/6$ are both negative.

The remaining trigonometric functions can be most easily defined in terms of the sine and cosine, as usual:

\[
\tan x = \frac{\sin x}{\cos x}, \\
\cot x = \frac{\cos x}{\sin x}, \\
\sec x = \frac{1}{\cos x}, \\
\csc x = \frac{1}{\sin x},
\]

and they can also be defined as the corresponding ratios of coordinates.

Although the trigonometric functions are defined in terms of the unit circle, the unit circle diagram is not what we normally consider the graph of a trigonometric function. (The unit circle is the graph of, well, the circle.) We can easily get a qualitatively correct idea of the graphs of the trigonometric functions from the unit circle diagram. Consider the sine function, $y = \sin x$. As $x$ increases from 0 in the unit circle diagram, the second coordinate of the point $A$ goes from 1 to a maximum of 1, then back to 0, then to a minimum of $-1$, then back to 0, and then it obviously repeats itself. So the graph of $y = \sin x$ must look something like this:

An angle, $x$, at the center of the circle is associated with an arc of the circle which is said to subtend the angle. In the figure, this arc is the portion of the circle from point $(1, 0)$ to point $A$. The length of this arc is the radian measure of the angle $x$; the fact that the radian measure is an actual geometric length is largely responsible for the usefulness of radian measure. The circumference of the unit circle is $2\pi r = 2\pi (1) = 2\pi$, so the radian measure of the full circular angle (that is, of the 360 degree angle) is $2\pi$.

While an angle with a particular measure can appear anywhere around the circle, we need a fixed, conventional location so that we can use the coordinate system to define properties of the angle. The standard convention is to place the starting radius for the angle on the positive $x$-axis, and to measure positive angles counterclockwise around the circle. In the figure, $x$ is the standard location of the angle $\pi/6$, that is, the length of the arc from $(1, 0)$ to $A$ is $\pi/6$. The angle $y$ in the picture is $-\pi/6$, because the distance from $(1, 0)$ to $B$ along the circle is also $\pi/6$, but in a clockwise direction.

Now the fundamental trigonometric definitions are: the cosine of $x$ and the sine of $x$ are the first and second coordinates of the point $A$, as indicated in the figure. The angle $x$ shown can be viewed as an angle of a right triangle, meaning the usual triangle definitions of the sine and cosine also make sense. Since the hypotenuse of the triangle is 1, the "side opposite over hypotenuse" definition of the sine is the second coordinate of point $A$ over 1, which is just the second coordinate; in other words, both methods give the same value for the sine.

The simple triangle definitions work only for angles that can "fit" in a right triangle, namely, angles between 0 and $\pi/2$. The coordinate definitions, on the other hand, apply to any angles, as indicated in this figure:

![Unit Circle Diagram](image)

Similarly, as angle $x$ increases from 0 in the unit circle diagram, the first coordinate of the point $A$ goes from 1 to a maximum of 1, then back to 0, then to a minimum of $-1$, then back to 0, and then it obviously repeats itself. So the graph of $y = \cos x$ must look something like this:

\[ \begin{align*}
\Delta x &> 0 \quad \Rightarrow \quad \frac{\sin(x + \Delta x) - \sin x}{\Delta x} \\
\Delta x &< 0 \quad \Rightarrow \quad \frac{\sin(x + \Delta x) - \sin x}{\Delta x} \\
\end{align*} \]

Using some trigonometric identities, we can make a little progress on the quotient:

\[ \frac{\sin(x + \Delta x) - \sin x}{\Delta x} = \frac{\sin x \cos \Delta x + \sin \Delta x \cos x - \sin x}{\Delta x} = \sin x \frac{\cos \Delta x - 1}{\Delta x} + \cos x \frac{\sin \Delta x}{\Delta x}. \]
This isolates the difficult bits in the two limits
\[ \lim_{\Delta x \to 0} \cos \Delta x - 1 \quad \text{and} \quad \lim_{\Delta x \to 0} \sin \Delta x. \]
Here we get a little lucky: it turns out that once we know the second limit the first is quite easy. The second is quite tricky, however. Indeed, it is the hardest limit we will actually compute, and we devote a section to it.

### 4.3 A hard limit

We want to compute this limit:
\[ \lim_{x \to 0} \frac{\sin x}{x}. \]
Equivalently, to make the notation a bit simpler, we can compute
\[ \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}. \]
In the original context we need to keep \( x \) and \( \Delta x \) separate, but here it doesn't hurt to rename \( \Delta x \) to something more convenient.

To do this we need to be quite clever, and to employ some indirect reasoning. The indirect reasoning is embodied in a theorem, frequently called the squeeze theorem.

**THEOREM 4.3.1 Squeeze Theorem.** Suppose that \( g(x) \leq f(x) \leq h(x) \) for all \( x \) close to \( a \) but not equal to \( a \). If \( \lim_{x \to a} g(x) = L = \lim_{x \to a} h(x) \), then \( \lim_{x \to a} f(x) = L \).

This theorem can be proved using the official definition of limit. We won't prove it here, but point out that it is easy to understand and believe graphically. The condition says that \( f(x) \) is trapped between \( g(x) \) and \( h(x) \) above and \( x \) below, and that as \( x \to a \), both \( g(x) \) and \( h(x) \) approach the same value. This means the situation looks something like figure 4.3.1. The wiggly curve is \( x^2 \sin(1/x) \), the upper and lower curves are \( x^2 \) and \( -x^2 \). Since the sine function is always between \( -1 \) and \( 1 \), \( x^2 \leq x^2 \sin(x/x) \leq x^2 \), and it is easy to see that \( \lim_{x \to 0} -x^2 = 0 = \lim_{x \to 0} x^2 \). It is not so easy to see directly, that is algebraically, that \( \lim_{x \to 0} x^2 \sin(\pi/x) = 0 \). This is the extreme values theorem. To do this we need to find two such functions \( g(x) \) and \( h(x) \), and so that \( \lim_{x \to 0} g(x) = \lim_{x \to 0} h(x) \). Of course, now that we know the derivative of the sine, we can compute derivatives of more complicated functions involving the sine.

#### Exercises 4.3.

1. Compute \( \lim_{x \to 0} \frac{\sin(x) - x}{x^3} \).
2. Compute \( \lim_{x \to 0} \frac{x^2}{\sin(x) - x} \).
3. Compute \( \lim_{x \to 0} \frac{\sin(3x)}{3x} \).
4. Compute \( \lim_{x \to 0} \frac{\tan(x)}{x} \).
5. Compute \( \lim_{x \to 0} \frac{\cos(x) - 1}{x^2} \).
6. For all \( x \geq 0 \), \( 4x - 9 \leq f(x) \leq x^2 - 4x + 7 \). Find \( \lim_{x \to 0} f(x) \).
7. For all \( x \geq 0 \), \( 2x \leq g(x) \leq x^2 - 2x \). Find \( \lim_{x \to 0} g(x) \).
8. Use the Squeeze Theorem to show that \( \lim_{x \to 0} x^3 \cos(1/x) = 0 \).

#### 4.4 The derivative of \( \sin x \), continued

Now we can complete the calculation of the derivative of the sine:
\[ \frac{d}{dx} \sin x = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} + \cos x \sin \Delta x = \cos x. \]
The derivative of a function measures the slope or steepness of the function; if we examine the graphs of the sine and cosine side by side, it should be that the latter appears to accurately describe the slope of the former, and indeed this is true:

\[ \frac{d}{dx} \cos x = -\sin x. \]

Notice that where the cosine is zero the sine does appear to have a horizontal tangent line, and that the sine appears to be steepest where the cosine takes on its extreme values of \( 1 \) and \( -1 \).

Of course, now that we know the derivative of the sine, we can compute derivatives of more complicated functions involving the sine.

**EXAMPLE 4.4.1** Compute the derivative of \( \sin^2 x \).
\[ \frac{d}{dx} \sin^2 x = 2 \sin x \cos x = 2x \cos^2 x. \]

**EXAMPLE 4.4.2** Compute the derivative of \( \sin^3 (x^2 - 5x) \).
\[ \frac{d}{dx} \sin^3 (x^2 - 5x) = 3 \sin^2 (x^2 - 5x) \cos (x^2 - 5x) \cdot 2x - 2 \cos (x^2 - 5x) = 6x \sin^2 (x^2 - 5x) \cos (x^2 - 5x). \]
4.5 Derivatives of the Trigonometric Functions

All of the other trigonometric functions can be expressed in terms of the sine, and so their derivatives can easily be calculated using the rules we already have. For the cosine we need to use two identities,
\[ \cos x = \sin(x + \frac{\pi}{2}), \quad \sin x = -\cos(x + \frac{\pi}{2}). \]

Now:
\[ \frac{d}{dx} \cos x = \frac{d}{dx} \sin(x + \frac{\pi}{2}) = \cos(x + \frac{\pi}{2}) = 1 - \sin x. \]
\[ \frac{d}{dx} \sin x = \frac{d}{dx} \cos(x + \frac{\pi}{2}) = -\cos(x + \frac{\pi}{2}) = \sec^2 x. \]
\[ \frac{d}{dx} \sec x = \frac{d}{dx} (\cos x)^{-1} = -1(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x} = \sec x \tan x. \]

The derivatives of the cotangent and cosecant are similar and left as exercises.

Exercises 4.5.
Find the derivatives of the following functions.
1. \( \sin(x) \) \( \Rightarrow \)
2. \( \sqrt{\tan x} \) \( \Rightarrow \)
3. \( \frac{1}{\sin x} \) \( \Rightarrow \)
4. \( \cot x \) \( \Rightarrow \)
5. \( \sqrt{\sin x} \) \( \Rightarrow \)

4.6 Exponential and Logarithmic Functions

An exponential function has the form \( a^x \), where \( a \) is a constant; examples are \( 2^x, 10^x, e^x \).

The logarithmic functions are the inverses of the exponential functions, that is, functions that "undo" the exponential functions, just as, for example, the cube root function "undos" the cubic function: \( \sqrt[3]{x} \).

Let \( f(x) = 2^x \). The inverse of this function is called the logarithm base 2, denoted \( \log_2(x) \) (or especially in computer science circles) \( \ln(x) \). What does this really mean? The logarithm must undo the action of the exponential function, so for example it must be that \( \log_2(2^3) = 3 \) starting with 3, the exponential function produces \( 2^3 = 8 \), and the logarithm of 8 must get us back to 3. A little thought shows that it is not a coincidence that \( \log_2(2^3) \) simply gives the exponent—the original value that we must get back to. In other words, the logarithm is the exponent. Remember this catchphrase, and what it means, and you won’t go wrong. (You do have to remember what it means. Like any good mnemonic, “the logarithm is the exponent” leaves out a lot of detail, like “Which exponent?” and “Exponent of what?”)

**EXAMPLE 4.6.1** What is the value of \( \log_2(1000) \)? The “100” tells us the appropriate number to use for the base of the exponential function. The logarithm is the exponent, so the question is, what exponent \( E \) makes \( 2^E = 1000 \)? If we can find such an \( E \), then \( \log_2(1000) = \log_2(2^E) = E \); finding the appropriate exponent is the same as finding the logarithm. In this case, of course, it is easy: \( E = 3 \) so \( \log_2(1000) = 3 \).

Let’s review some laws of exponents and logarithms, let \( a \) be a positive number. Since \( a^0 = a \cdot a = 0 \) and \( a^1 = a \cdot a = 1 \), it’s clear that \( a^{x+y} = a^{x} \cdot a^{y} \) and \( a^{x-y} = a^{x} / a^{y} \), and in general that \( a^{xy} = a^{x+y} \). Since “the logarithm is the exponent,” it’s no surprise that this translates directly into a fact about the logarithm function. Here are three facts from the example: \( \log_2(2^5) = 5 \), \( \log_2(2^{-3}) = -3 \), and \( \log_2(2^0) = 0 \). The inverse of this function is called the logarithm base 2, denoted \( \log_2(x) \).

\[ \frac{d}{dx} a^x = \lim_{\Delta x \to 0} \frac{a^x(a^\Delta x - a^x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{a^x(a^\Delta x - 1)}{\Delta x} = \lim_{\Delta x \to 0} a^{x+1}a^{\Delta x} = a^x \lim_{\Delta x \to 0} a^{\Delta x} = a^x \lim_{x \to 0} a^{x} = a^x. \]

Exercises 4.6.
1. Expand \( \log_2((x + 45) - (x - 2)) \).
2. Expand \( \log_3(x - 3) + \log_3(x + 3) \).
3. Write \( \log_3(2x^3 - 6x^2 + 8) \) as a single logarithm.
4. Solve \( \log_2(1 + \sqrt{2}) = 6 \) for \( x \).
5. Solve \( x^2 = 9 \) for \( x \).
6. Solve \( \log_2(\log_2(x)) = 1 \) for \( x \).

4.7 Derivatives of the Exponential and Logarithmic Functions

As with the sine, we don’t know anything about derivatives that allows us to compute the derivatives of the exponential and logarithmic functions without going back to basics. Let’s do a little work with the definition again:
\[ \frac{d}{dx} e^x = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{x}(e^{\Delta x} - 1)}{\Delta x} = \lim_{\Delta x \to 0} e^{x+1}e^{\Delta x} = e^x \lim_{x \to 0} e^{\Delta x} = e^x \lim_{x \to 0} 1 = e^x. \]
4.7 Derivatives of the exponential and logarithmic functions

There are two interesting things to note here: As in the case of the sine function we are left with a limit that involves \( \Delta x \) but not \( x \), which means that whatever \( \lim_{\Delta x \to 0} (a^{\Delta x} - 1)/\Delta x \) is, we know that it is a number, that is, a constant. This means that \( a^x \) has a remarkable property: its derivative is a constant times itself.

We earlier remarked that the hardest limit we would compute is \( \lim_{\Delta x \to 0} \sin x/\Delta x = 1 \); we now have a limit that is just a bit too hard to include here. In fact the hard part is to see that \( \lim_{\Delta x \to 0} (a^{\Delta x} - 1)/\Delta x \) even exists—does this function really get closer and closer to some fixed value? Yes it does, but we will not prove this fact.

We can look at some examples. Consider \((2^x - 1)/x\) for some small values of \( x \): 1, 0.828427124, 0.756828469, 0.724061864, 0.708789551, 0.707807777 when \( x \) is 1, 1/2, 1/4, 1/8, 1/16, 1/32, respectively. It looks like this is settling in around 0.7, which turns out to be true (but the limit is not exactly 0.7). Consider next \((2^x - 1)/x^2\): 1.0426187, 1.46410166, 1.26396502, 1.17720773, 1.1760854, at the same values of \( x \). It turns out to be true that in the limit this is about 1.1. Two examples don’t establish a pattern, but if you do more examples you will find that the limit varies directly with the value of \( x \): bigger \( x \), bigger limit; smaller \( x \), smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between \( a = 2 \) and \( a = 3 \) the limit will be exactly 1; the value at which this happens is called \( e \), so that

\[
\lim_{\Delta x \to 0} \frac{e^\Delta x - 1}{\Delta x} = 1.
\]

As you might guess from our two examples, \( e \) is closer to 3 than to 2, and in fact \( e \approx 2.718 \).

Now we see that the function \( e^x \) has a truly remarkable property:

\[
\frac{d}{dx} e^x = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^x(e^{\Delta x} - 1)}{\Delta x} = e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = e^x.
\]

That is, \( e^x \) is its own derivative, or in other words the slope of \( e^x \) is the same as its height, or the same as its second coordinate. The function \( f(z) = e^z \) goes through the point \((z, e^z)\) and has slope \( e^z \) there, no matter what \( z \) is. It is sometimes convenient to express the function \( e^x \) without an exponent, since complicated exponents can be hard to read. In such cases we use \( \exp(x) \), e.g., \( \exp(1 + x^2) \) instead of \( e^{1+x^2} \).

88 Chapter 4 Transcendental Functions

What about the logarithm function? This too is hard, but as the cosine function was easier to do once the sine was done, so the logarithm is easier to do now that we know the derivative of the exponential function. Let’s start with \( \log_a x \), which as you probably know is often abbreviated \( \ln x \) and called the “natural logarithm” function.

Consider the relationship between the two functions, namely, that they are inverses, that one “undoes” the other. Graphically this means that they have the same graph except that one is “flipped” or “reflected” through the line \( y = x \), as shown in figure 4.7.1.

This means that the slopes of these two functions are closely related as well: For example, the slope of \( e^x \) is \( e \) at \( x = 1 \); at the corresponding point on the \( \ln x \) curve, the slope must be \( 1/e \), because the “rise” and the “run” have been interchanged. Since the slope of \( e^x \) is \( e \) at the point \((1, e)\), the slope of \( \ln(x) \) is \( 1/e \) at the point \((e, 1)\).

Figure 4.7.1. The exponential and logarithm functions.

Figure 4.7.2. Slope of the exponential and logarithm functions.

More generally, we know that the slope of \( e^z \) is \( e^z \) at the point \((z, e^z)\), so the slope of \( \ln(z) \) is \( 1/e^z \) at \((e^z, z)\), as indicated in figure 4.7.2. In other words, the slope of \( \ln x \) is the reciprocal of the first coordinate at any point: this means that the slope of \( \ln x \) at \((x, \ln x)\) is \( 1/x \). The upshot is:

\[
\frac{d}{dx} \ln x = \frac{1}{x}.
\]

88 Chapter 4 Transcendental Functions

we can replace \( \log_e e \) to get

\[
\frac{d}{dx} \log_e x = \frac{1}{\ln a}.
\]

You may if you wish memorize the formulas:

\[
\frac{d}{dx} a^x = (\ln a)a^x \quad \text{and} \quad \frac{d}{dx} \log_a x = \frac{1}{\ln a}.
\]

Because the “trick” \( a = e^{\ln a} \) is often useful, and sometimes essential, it may be better to remember the trick, not the formula.

EXAMPLE 4.7.1 Compute the derivative of \( f(x) = 2^x \).

\[
\frac{d}{dx} 2^x = \frac{d}{dx} (e^{\ln 2})^x = \frac{d}{dx} e^{x \ln 2} = (\ln 2)e^{x \ln 2} = (\ln 2)a^x.
\]

EXAMPLE 4.7.2 Compute the derivative of \( f(x) = 2^x = 2^{x^2} \).

\[
\frac{d}{dx} 2^{x^2} = \frac{d}{dx} (e^{\ln 2})^{x^2} = \frac{d}{dx} e^{x^2 \ln 2} = (2x)e^{x^2 \ln 2} = (2x)e^{x^2} = (2x)2^{x^2}.
\]

EXAMPLE 4.7.3 Compute the derivative of \( f(x) = x^x \). At first this appears to be a bad function: it is not a constant power of \( x \), and it does not seem to be an exponential function, since the base is not constant. But in fact it is no harder than the previous example.

\[
\frac{d}{dx} x^x = \frac{d}{dx} e^{x \ln x} = \frac{d}{dx} e^{(x + \ln x)x} = (\ln x + 1)x^x.
\]
EXAMPLE 4.7.4 Recall that we have not justified the power rule except when the exponent is a positive or negative integer. We can use the exponential function to take care of other exponents.

\[
\frac{d}{dx} e^{\ln x} = \frac{d}{dx} x = x^{\prime} = 1 \quad \text{and} \quad e^{\ln x} = x^{1} = x.
\]

Then using the chain rule on the right hand side:

\[
1 = \left( \frac{d}{dx} x \right)^{\prime} = y^{\prime} x^{\prime}.
\]

We will begin by illustrating the technique to find what we already know, the derivative of \( \ln x \). Let’s write \( y = \ln x \) and then \( x = e^{y} \), that is, \( x = e^{y} \). We say that this equation defines the function \( y = \ln x \) implicitly because while it is not an explicit expression \( y = \ldots \) it is true that if \( x = e^{y} \) then \( y \) is in fact the natural logarithm function.

Now, for the time being, pretend that all we know of \( x = e^{y} \) what can we say about derivatives? We can take the derivative of both sides of the equation:

\[
\frac{d}{dx} x = \frac{d}{dx} e^{y}.
\]

4.8 Implicit Differentiation

As we have seen, there is a close relationship between the derivatives of \( e^{y} \) and \( \ln x \) because these functions are inverses. Rather than relying on pictures for our understanding, we would like to be able to exploit this relationship computationally. In fact this technique can help us find derivatives in many situations, not just when we seek the derivative of an inverse function.

Exercises 4.7.

In 1–19, find the derivatives of the functions.

1. \( 3^{x} \) ⇒
2. \( \tan x \) ⇒
3. \( (e^{x})^{y} \) ⇒
4. \( \sin(e^{x}) \) ⇒
5. \( e^{\sin x} \) ⇒
6. \( x^{\tan x} \) ⇒
7. \( x^{\pi} \) ⇒
8. \( x^{2} + 2 \) ⇒
9. \( \ln(x^{2} + 3x) \) ⇒
10. \( e^{\ln(x)} \) ⇒
11. \( \ln(x^{2} + 3x) \) ⇒
12. \( \ln(x^{1}) \) ⇒
13. \( \sqrt{\ln(x^{2})} \) ⇒
14. \( \ln(\sin(x) \times \tan(x)) \) ⇒
15. \( e^{\tan(x)} \) ⇒
16. \( x^{\ln x} \) ⇒
17. \( \ln(\ln(x)) \) ⇒
18. \( \ln(\ln(x^{2})) \) ⇒
19. \( x^{2} + 3x = 1 \) ⇒

20. Find the value of \( a \) so that the tangent line to \( y = \ln(x) \) at \( x = a \) is a line through the origin.

21. If \( f(x) = \ln(x^{2} + 2) \) compute \( f'(x^{1/2}) \).

4.8 Implicit Differentiation

As we have seen, there is a close relationship between the derivatives of \( e^{y} \) and \( \ln x \) because these functions are inverses. Rather than relying on pictures for our understanding, we would like to be able to exploit this relationship computationally. In fact this technique can help us find derivatives in many situations, not just when we seek the derivative of an inverse function.

4.8 Implicit Differentiation

Chain rule where \( y \) appears.

\[
\frac{d}{dx} x = \frac{d}{dx} (x^{2} + y^{2})
\]

\[
0 = 2x + 2yy' \quad \text{and} \quad y' = -\frac{x}{y}.
\]

Now we have an expression for \( y' \), but it contains \( y \) as well as \( x \). This means that if we want to compute \( y' \) for some particular value of \( x \) we’ll have to know or compute \( y \) at that value of \( x \) as well. It is at this point that we will need to know whether \( y = U(x) \) or \( U(x) \). Occasionally it will turn out that we can avoid explicit use of \( \frac{d}{dx} U(x) \) or \( \frac{d}{dx} L(x) \) by the nature of the problem.

EXAMPLE 4.8.1 Find the slope of the circle \( 4 = x^{2} + y^{2} \) at the point \((1, -√3)\). Since we know both the \( x \) and \( y \) coordinates of the point of interest, we do not need to explicitly recognize that this point is on \( L(x) \), and we do not need to use \( L(x) \) to compute \( y \) — but we could. Using the calculation of \( y' \) from above,

\[
y' = \frac{y}{x} = -\frac{1}{\sqrt{3}}.
\]

It is instructive to compare this approach to others.

We might have recognized at the start that \((1, -√3)\) is on the function \( y = L(x) = -\sqrt{1 - x^{2}} \). We could then take the derivative of \( L(x) \), using the power rule and the chain rule, to get

\[
L'(x) = -\frac{1}{2}(1 - x^{2})^{-1/2}(-2x) = \frac{x}{\sqrt{1 - x^{2}}}
\]

Then we could compute \( L'(1) = 1/\sqrt{2} \) by substituting \( x = 1 \).

Alternately, we could realize that the point is on \( L(x) \), but use the fact that \( y' = -x/y \).

Since the point is on \( L(x) \) we can replace \( y \) by \( L(x) \) to get

\[
y' = \frac{x}{L(x)} = \frac{x}{\sqrt{1 - x^{2}}}
\]

without computing the derivative of \( L(x) \) explicitly. Then we substitute \( x = 1 \) and get the same answer as before.

In the case of the circle it is possible to find the functions \( U(x) \) and \( L(x) \) explicitly, but there are potential advantages to using implicit differentiation anyway. In some cases it is more difficult or impossible to find an explicit formula for \( y \) and implicit differentiation is the only way to find the derivative.

EXAMPLE 4.8.2 Find the derivative of any function defined implicitly by \( yx^{2} + cx^{\pi} = x \).

We treat \( y \) as an unspecified function and use the chain rule:

\[
\frac{d}{dx} (yx^{2} + cx^{\pi}) = \frac{d}{dx} x
\]

\[
y'x^{2} + 2yx' + c\pi x^{\pi - 1} = 1
\]

\[
y'x^{2} + 2yx' = 1 - c\pi x^{\pi - 1}
\]

\[
y' = \frac{1}{2} - \frac{c\pi}{2} x^{\pi - 1}
\]

You might think that the step in which we solve for \( y' \) could sometimes be difficult — after all, we’re using implicit differentiation here because we can’t solve the equation \( yx^{2} + cx^{\pi} = x \) for \( y \), so maybe after taking the derivative we get something that is hard to solve for \( y' \). In fact, this never happens. All occurrences \( y' \) come from applying the chain rule, and whenever the chain rule is used it deposits a single \( y' \) multiplied by some other expression. So it will always be possible to group the terms containing \( y' \) together and factor out the \( y' \) just as in the previous example. If you ever get anything more difficult you have made a mistake and should fix it before trying to continue.

It is sometimes the case that a situation lends naturally to an equation that defines a function implicitly.

EXAMPLE 4.8.3 Consider all the points \((x, y)\) that have the property that the distance from \((x, y)\) to \((1, y1)\) plus the distance from \((x, y)\) to \((x2, y2)\) is \(2a\) (\(a\) is some constant).

These points form an ellipse, which like a circle is not a function but can viewed as two functions pasted together. Because we know how to write down the distance between two points, we can write down an implicit equation for the ellipse:

\[
\sqrt{(x - x1)^{2} + (y - y1)^{2}} + \sqrt{(x - x2)^{2} + (y - y2)^{2}} = 2a
\]

Then we can use implicit differentiation to find the slope of the ellipse at any point, though the computation is rather messy.

EXAMPLE 4.8.4 We have already justified the power rule by using the exponential function, but we could also do it for rational exponents by using implicit differentiation. Suppose that \( y = x^{m/n} \), where \( m \) and \( n \) are positive integers. We can write this implicitly as \( y^{n} = x^{m} \), then we justified the power rule for integers, we can take the derivative
of each side:

\[ \tan^{p-1} y = \frac{m^{p-1}}{n^{p-1}} \]

\[ y' = \frac{m}{n} \tan^{p-1} \]

\[ y' = \frac{m}{n} \cot^{p-1} \]

\[ y' = \frac{m}{n} \csc^{p-1} \]

\[ y' = \frac{m}{n} \sec^{p-1} \]

\[ y' = \frac{m}{n} \cos^{p-1} \]

\[ y' = \frac{m}{n} \sin^{p-1} \]

\[ y' = \frac{m}{n} \sin^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \cos^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \tan^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \cot^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \csc^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sec^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sinh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \cosh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \tanh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \coth^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \csch^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sech^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sinh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \cosh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \tanh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \coth^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \csch^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sech^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sinh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \cosh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \tanh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \coth^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \csch^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sech^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sinh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \cosh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \tanh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \coth^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \csch^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sech^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sinh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \cosh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \tanh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \coth^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \csch^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sech^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sinh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \cosh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \tanh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \coth^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \csch^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sech^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sinh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \cosh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \tanh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \coth^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \csch^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sech^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sinh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \cosh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \tanh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \coth^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \csch^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sech^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sinh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \cosh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \tanh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \coth^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \csch^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sech^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sinh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \cosh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \tanh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \coth^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \csch^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sech^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sinh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \cosh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \tanh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \coth^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \csch^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sech^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sinh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \cosh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \tanh^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \coth^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \csch^{-1} \left( \frac{m}{n} \right) \]

\[ y' = \frac{m}{n} \sech^{-1} \left( \frac{m}{n} \right) \]
4.10 Limits revisited

3. The inverse of cot is usually defined so that the range of arccot is \((0, \pi)\). Sketch the graph of \(y = \text{arccot} x\). In the process you will make it clear what the domain of arccot is. Find the derivative of the arccotangent. 

4. Show that \(\text{arccot} x + \text{arccot} \sqrt{2} = \pi/2\).

5. Find the derivative of \(\text{arcsec} x\).

6. Find the derivative of \(\text{arcsec}^{-1} x\).

7. Find the derivative of \(\text{arcsec}(x^2)\).

8. Find the derivative of \(\text{arcsec}(x^3)\).

9. Find the derivative of \(\text{arcsec}^{-1} x^2\).

10. Find the derivative of \(\text{arcsec} x + \text{arcsec} x + \text{arcsec} x\).

11. Find the derivative of \(\text{ln}(\text{arcsec}(x^3))\).

4.10 Limits revisited

We have defined and used the concept of limit, primarily in our development of the derivative. Recall that \(\lim_{x \to a} f(x) = L\) is true if, in a precise sense, \(f(x)\) gets closer and closer to \(L\) as \(x\) gets closer and closer to \(a\). While some limits are easy to see, others take some thought; in particular, the limits that are always difficult on their face, since in

\[
\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}
\]

both the numerator and denominator approach zero. Typically this difficulty can be resolved when \(f\) is a "nice" function and we are trying to compute a derivative. Occasionally such limits are interesting for other reasons, and the limit of a fraction in which both numerator and denominator approach zero can be difficult to analyze. Now that we have the derivative available, there is another technique that can sometimes be helpful in such circumstances.

Before we introduce the technique, we will also expand our concept of limit, in two ways. When the limit of \(f(x)\) as \(x\) approaches \(a\) does not exist, it may be useful to note in what way it does not exist. We have already talked about one such case: one-sided limits. Another case is when \(f\) goes to infinity. We will also occasionally want to know what happens to \(f\) when \(x\) "goes to infinity".

**EXAMPLE 4.10.1** What happens to \(1/x\) as \(x\) goes to 0? From the right, \(1/x\) gets bigger and bigger, or goes to infinity. From the left it goes to negative infinity. Suffice it to say that such limits behave in much the same way that ordinary limits do; in particular there are some analogs of theorem 2.3.6.

Now consider this limit:

\[
\lim_{x \to \infty} \frac{1}{x^2}
\]

As \(x\) approaches \(\pi\), both the numerator and denominator approach zero, so it is not obvious what, if anything, the quotient approaches. We can often compute such limits by application of the following theorem.

**THEOREM 4.10.5** L'Hôpital's Rule

For "sufficiently nice" functions \(f(x)\) and \(g(x)\), if \(\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)\) or both \(\lim_{x \to a} f(x) = \pm \infty\) and \(\lim_{x \to a} g(x) = \pm \infty\), and if \(\lim_{x \to a} \frac{f'(x)}{g'(x)}\) exists, then \(\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}\). This remains true if "\(x \to a\)" is replaced by "\(x \to -\infty\)" or "\(x \to +\infty\)."

This theorem is somewhat difficult to prove, in part because it incorporates so many different possibilities, so we will not prove it here. We also will not need to worry about the precise definition of "sufficiently nice", as the functions we encounter will be suitable.

**EXAMPLE 4.10.6** Compute \(\lim_{x \to 0} \frac{x^3}{\sin x}\) in two ways.

**EXAMPLE 4.10.8** Compute \(\lim_{x \to 0} \frac{\sec x - 1}{\sin x}\).

Both the numerator and denominator approach zero, so applying L'Hôpital's Rule:

\[
\lim_{x \to 0} \frac{\sec x - 1}{\sin x} = \lim_{x \to 0} \frac{\sec x \tan x}{\cos x} \to \frac{0}{1} = 0.
\]

**EXAMPLE 4.10.9** Compute \(\lim_{x \to 0} x \ln x\).

This doesn't appear to be suitable for L'Hôpital's Rule, but it also is not "obvious". As \(x\) approaches zero, \(x\) in \(x \ln x\) goes to \(-\infty\), so the product looks like (something very small) (something very large and negative). But this could be anything: it depends on how small and how large. For example, consider \((x^2)(1/x), (x)/(1/x),\) and \((x/1)^2\). As \(x\) approaches zero, each of these is (something very small) (something very large), yet the limits are respectively zero, 1, and \(\infty\).

We can in fact turn this into a L'Hôpital's Rule problem:

\[
x \ln x = \lim_{x \to 0} \frac{x}{1/x} \to \frac{0}{1/0} = 0.
\]

As \(x\) approaches zero, both the numerator and denominator approach zero (one \(-\infty\) and one \(+\infty\)), but only the sign is important. Using L'Hôpital's Rule:

\[
\lim_{x \to 0} x \ln x = \lim_{x \to 0} \frac{x}{1/x} = \lim_{x \to 0} x - x^2 = 0.
\]

One way to interpret this is that since \(\lim_{x \to 0} x = 0\), the \(x\) approaches zero much faster than the \(\ln x\) approaches \(-\infty\).

**Exercises 4.10.**

1. Compute the limits.
   1. \(\lim_{x \to 0} \frac{\cos x - 1}{\sin x}\)
   2. \(\lim_{x \to 0} \frac{x^2}{x}\)
   3. \(\lim_{x \to 0} \frac{\sqrt{x^2 + x} - \sqrt{x^2 + x - x}}{x - 2x}\)
   4. \(\lim_{x \to 0} \frac{1 - e^x}{x}\)
   5. \(\lim_{x \to 0} \frac{\sin x}{x}\)
   6. \(\lim_{x \to 0} \frac{x^2}{x^2 - x}\)
   7. \(\lim_{x \to 0} \frac{\sqrt{x^2 - 1}}{x}\)
   8. \(\lim_{x \to 0} \frac{\ln (1 + x) - 1}{x}\)
   9. \(\lim_{x \to 0} \frac{\sqrt[3]{x} - 1}{x}\)
   10. \(\lim_{x \to 0} \frac{x}{1 - x^2 + 1}\)
   11. \(\lim_{x \to 0} \frac{\sqrt[3]{x} + 1}{x}\)
   12. \(\lim_{x \to 0} \frac{\sqrt[4]{x} - \sqrt[2]{x}}{x}\)
The domain of coth and csch is $\mathbb{R} \setminus \{0\}$.

The proof is a straightforward computation:

$$
\lim_{x \to \pm \infty} \frac{x}{\ln(1 + 2)} = \pm \infty
$$

This immediately gives two additional identities:

$$
1 - \tanh^2 x = \text{sech}^2 x \quad \text{and} \quad \coth^2 x - 1 = \text{csch}^2 x.
$$

The identity of the theorem also helps to provide a geometric motivation. Recall that the graph of $x^2 - y^2 = 1$ is a hyperbola with asymptotes $x = \pm y$ whose $x$-intercepts are $\pm 1$. If $(x, y)$ is a point on the right half of the hyperbola, and if we let $x = \cosh t$, then $y = \pm \sqrt{x^2 - 1} = \pm \cosh t = \pm \sinh t$. So for some suitable $t$, coth $t$ and sinh $t$ are the coordinates of a typical point on the hyperbola. In fact, it turns out that $t$ is twice the area shown in the first graph of figure 11.2. Even this is analogous to trigonometry: cos and sin are the coordinates of a typical point on the unit circle, and $t$ is twice the area shown in the second graph of figure 11.2.

Figure 11.1.1 The hyperbolic functions: $\cosh$, $\sinh$, $\tanh$, $\text{sech}$, $\text{coth}$.

Certainly the hyperbolic functions do not closely resemble the trigonometric functions graphically. But they do have analogous properties, beginning with the following identity.

**Theorem 11.14.** For all $x$ in $\mathbb{R}$, $\cosh^2 x - \sinh^2 x = 1$.

**Proof.** The proof is a straightforward computation:

$$
\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x} - e^{2x} - 2 - 4e^{-2x}}{4} = \frac{4}{4} = 1.
$$

This immediately gives two additional identities:

$$
1 - \tanh^2 x = \text{sech}^2 x \quad \text{and} \quad \coth^2 x - 1 = \text{csch}^2 x.
$$

The identity of the theorem also helps to provide a geometric motivation. Recall that the graph of $x^2 - y^2 = 1$ is a hyperbola with asymptotes $x = \pm y$ whose $x$-intercepts are $\pm 1$. If $(x, y)$ is a point on the right half of the hyperbola, and if we let $x = \cosh t$, then $y = \pm \sqrt{x^2 - 1} = \pm \cosh t = \pm \sinh t$. So for some suitable $t$, coth $t$ and sinh $t$ are the coordinates of a typical point on the hyperbola. In fact, it turns out that $t$ is twice the area shown in the first graph of figure 11.2. Even this is analogous to trigonometry: cos and sin are the coordinates of a typical point on the unit circle, and $t$ is twice the area shown in the second graph of figure 11.2.

Figure 11.1.2 Geometric definitions of $\sin$, $\cos$, $\cosh$, $\sinh$: $t$ is twice the shaded area in each figure.

Given the definitions of the hyperbolic functions, finding their derivatives is straightforward. Here again we see similarities to the trigonometric functions.

**Theorem 11.15.** $\frac{d}{dt} \cosh x = \sinh x$ and $\frac{d}{dt} \sinh x = \cosh x$.

**Proof.**

$$
\frac{d}{dt} \cosh x = \frac{d}{dt} \left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x,
$$

and

$$
\frac{d}{dt} \sinh x = \frac{d}{dt} \left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x.
$$

Since $\cosh x > 0$, sinh $x$ is increasing and hence injective, so sinh $x$ has an inverse, arsinh $x$. Also, sinh $x > 0$ when $x > 0$, so cosh $x$ is injective on $[0, \infty)$ and has a (partial) inverse, arccosh $x$. The other hyperbolic functions have inverses as well, though arccosh $x$ is only a partial inverse. We may compute the derivatives of these functions as we have other inverse functions.

**Theorem 11.16.** $\frac{d}{dx} \text{arcsinh} x = \frac{1}{\sqrt{1 + x^2}}$.

**Proof.** Let $y = \text{arcsinh} x$, so sinh $y = x$. Then $\frac{d}{dx} \sinh y = \cosh y \cdot y' = 1$, and so

$$
y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}.
$$
The other derivatives are left to the exercises.

**Exercises 4.11.**

1. Show that the range of \( \sinh x \) is all real numbers. (Hint: show that if \( y = \sinh x \) then \( x = \ln(y + \sqrt{y^2 + 1}) \).)

2. Compute the following limits:
   a. \( \lim_{x \to \infty} \cosh x \)
   b. \( \lim_{x \to \infty} \sinh x \)
   c. \( \lim_{x \to \infty} \tanh x \)
   d. \( \lim_{x \to \infty} (\cosh x - \sinh x) \)

3. Show that the range of \( \tanh x \) is \((-1, 1)\). What are the ranges of \( \coth \), \( \text{sech} \), and \( \text{csch} \)? (Use the fact that they are reciprocal functions.)

4. Prove that for every \( x, y \in \mathbb{R} \), \( \sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y \). Obtain a similar identity for \( \sinh(x - y) \).

5. Prove that for every \( x, y \in \mathbb{R} \), \( \cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y \). Obtain a similar identity for \( \cosh(x - y) \).

6. Use exercises 4 and 5 to show that \( \sinh(2x) = 2 \sinh x \cosh x \) and \( \cosh(2x) = \cosh^2 x + \sinh^2 x \) for every \( x \). Conclude also that \( \frac{\cosh(2x) - 1}{2} = \sinh^2 x \).

7. Show that \( \frac{d}{dx} (\tanh x) = \text{sech}^2 x \). Compute the derivatives of the remaining hyperbolic functions as well.

8. What are the domains of the six inverse hyperbolic functions?

9. Sketch the graphs of all six inverse hyperbolic functions.