4

Trigonometric Functions

So far we have used only algebraic functions as examples when finding derivatives, that is, functions that can be built up by the usual algebraic operations of addition, subtraction, multiplication, division, and raising to constant powers. Both in theory and practice there are other functions, called transcendental, that are very useful. Most important among these are the trigonometric functions, the inverse trigonometric functions, exponential functions, and logarithms.

4.1 Trigonometric Functions

When you first encountered the trigonometric functions it was probably in the context of "triangle trigonometry," defining, for example, the sine of an angle as the "side opposite the hypotenuse." While this will still be useful in an informal way, we need to use a more expansive definition of the trigonometric functions. First an important note: while degree measure of angles is sometimes convenient because it is so familiar, it turns out to be ill-suited to mathematical calculation, so (almost) everything we do will be in terms of radian measure of angles.

An angle, \( x \), at the center of the circle is associated with an arc of the circle which is said to subtend the angle. In the figure, this arc is the portion of the circle from point \((1,0)\) to point \(A\). The length of this arc is the radian measure of the angle \(x\); the fact that the radian measure is an actual geometric length is largely responsible for the usefulness of radian measure. The circumference of the unit circle is \(2\pi = 2\pi(1) = 2\pi\), so the radian measure of the full circular angle (that is, of the 360 degree angle) is \(2\pi\).

While an angle with a particular measure can appear anywhere around the circle, we need a fixed, conventional location so that we can use the coordinate system to define properties of the angle. The standard convention is to place the starting radius for the angle on the positive \(x\)-axis, and to measure positive angles counterclockwise around the circle. In the figure, \(x\) is the standard location of the angle \(\pi/6\), that is, the length of the arc from \((1,0)\) to \(A\) is \(\pi/6\). The angle \(y\) in the picture is \(-x/6\), because the distance from \((1,0)\) to \(B\) along the circle is also \(x/6\), but in a clockwise direction.

Now the fundamental trigonometric definitions are: the cosine of \(x\) and the sine of \(x\) are the first and second coordinates of the point \(A\), as indicated in the figure. The angle \(x\) shown can be viewed as an angle of a right triangle, meaning the usual triangle definitions of the sine and cosine also make sense. Since the hypotenuse of the triangle is 1, the "side opposite over hypotenuse" definition of the sine is the second coordinate of point \(A\) over 1, which is just the second coordinate; in other words, both methods give the same value for the sine.

The simple triangle definitions work only for angles that can "fit" in a right triangle, namely, angles between 0 and \(\pi/2\). The coordinate definitions, on the other hand, apply to any angles, as indicated in this figure:

\[
\begin{align*}
\cos x &= \frac{1}{4y}, \\
\sin x &= \frac{y}{4x}, \\
\tan x &= \frac{\sin x}{\cos x}, \\
\cot x &= \frac{\cos x}{\sin x}, \\
\sec x &= \frac{1}{\cos x}, \\
\csc x &= \frac{1}{\sin x},
\end{align*}
\]

and they can also be defined as the corresponding ratios of coordinates.

Although the trigonometric functions are defined in terms of the unit circle diagram, the sine function, \(y = \sin x\). As \(x\) increases from 0 in the unit circle diagram, the second coordinate of the point \(A\) goes from 1 to a maximum of 1, then back to 0, then to a minimum of \(-1\), then back to 0, and then it obviously repeats itself. So the graph of \(y = \sin x\) must look something like this:

Exercises 4.1.

Some useful trigonometric identities are in appendix B.

1. Find all values of \(\theta\) such that \(\sin(\theta) = -1\), give your answer in radians.
2. Find all values of \(\theta\) such that \(\cos(\theta) = 1/2\), give your answer in radians.
3. Use an angle sum identity to compute \(\cos(\pi/12)\).
4. Use an angle sum identity to compute \(\sin(5\pi/12)\).
5. Verify the identity \(\cos^2(t)/(1 - \sin(t)) = 1 + \sin(t)\).
6. Verify the identity \(2\cos(2\theta) = \cos(\theta)\cos(\theta)\).
7. Verify the identity \(\sin(3\theta) - \sin(\theta) = 2\cos(2\theta)\sin(\theta)\).
8. Sketch \(y = 2\sin(x)\).
9. Sketch \(y = \sin(2x)\).
10. Sketch \(y = \sin(x - \pi)\).
11. Find all of the solutions of \(2\sin(t) - 1 - \sin^2(t) = 0\) in the interval \([0, 2\pi]\).

4.2 The Derivative of \(\sin x\)

What about the derivative of the sine function? The rules for derivatives that we have are no help, since \(\sin x\) is not an algebraic function. We need to return to the definition of the derivative, set up a limit, and try to compute it. Here’s the definition:

\[
\frac{\Delta y}{\Delta x}
\]

\[
\frac{d}{dx} \sin x = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}
\]

Using some trigonometric identities, we can make a little progress on the quotient:

\[
\frac{\sin(x + \Delta x) - \sin x}{\Delta x}
\]

\[
\frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}
\]

\[
= \sin x \cos \Delta x + \cos x - \sin x - \sin x \cos \Delta x
\]

\[
= \sin x \cos \Delta x - \frac{\sin \Delta x}{\Delta x}.
\]
This isolates the difficult bits in the two limits
\[
\lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}.
\]
Here we get a little lucky: it turns out that once we know the second limit the first is quite easy. The second is quite tricky, however. Indeed, it is the hardest limit we will actually compute, and we devote a section to it.

### 4.3 A HARD LIMIT

We want to compute this limit:
\[
\lim_{x \to 0} \frac{\sin x}{x}.
\]
Equivalently, to make the notation a bit simpler, we can compute
\[
\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}.
\]
In the original context we need to keep \(x\) and \(\Delta x\) separate, but here it doesn’t harm to rename \(\Delta x\) to something more convenient.

To do this we need to be quite clever, and to employ some indirect reasoning. The indirect reasoning is embodied in a theorem, frequently called the squeeze theorem.

**THEOREM 4.3.1 Squeeze Theorem.** Suppose that \(g(x) \leq f(x) \leq h(x)\) for all \(x\) close to \(a\) but not equal to \(a\). If \(\lim_{x \to a} g(x) = L = \lim_{x \to a} h(x)\), then \(\lim_{x \to a} f(x) = L\).

This theorem can be proved using the official definition of limit. We won’t prove it here, but point out that it is easy to understand and believe graphically. The condition says that \(f(x)\) is trapped between \(g(x)\) below and \(h(x)\) above, and that at \(x = a\), both \(g\) and \(h\) approach the same value. This means the situation looks something like figure 4.3.1.

The wiggly curve is \(x \sin(x)/x\), the upper and lower curves are \(x\) and \(-x^2\). Since the sine function is always between \(-1\) and \(1\), \(-x^2 \leq x^2 \sin(x)/x \leq x^2\), and it is easy to see that \(\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0\). It is not so easy to see directly, that is algebraically, that \(\lim_{x \to 0} x^2 \sin(x)/x = 0\), because the \(x\) in the numerator is not being cancelled. The squeeze theorem makes this “hard limit” as easy as the trivial limits involving \(x^2\).

To do the hard limit that we want, \(\lim_{x \to 0} \sin(x)/x\), we will find two simpler functions \(g\) and \(h\) so that \(g(x) \leq \sin(x)/x \leq h(x)\), and so that \(\lim_{x \to 0} g(x) = \lim_{x \to 0} h(x)\). Not too surprisingly, this will require some trigonometry and geometry. Referring to figure 4.3.2, \(x\) is the measure of the angle in radians. Since the circle has radius 1, the coordinates of point \(A\) are \((\cos x, \sin x)\), and the area of the triangle is \(\cos x \sin x/2\). This triangle is completely contained within the circular wedge-shaped region bordered by two lines and the circle from \((1, 0)\) to point \(A\). Comparing the areas of the triangle and the wedge we see \(\cos x \sin x/2 \leq x/2\), since the area of a circular region with angle \(\pi/2\) and radius \(r\) is \(\pi r^2/2\). With a little algebra this turns into \(\sin x/x \leq 1/\cos x\), giving us the \(h\) we seek.

Finally, the two limits \(\lim_{x \to 0} \cos x\) and \(\lim_{x \to 0} 1/\cos x\) are easy, because \(\cos 0 = 1\). By the squeeze theorem, \(\lim_{x \to 0} \sin(x)/x = 1\) as well.

Before we can complete the calculation of the derivative of the sine, we need one other limit:
\[
\lim_{x \to 0} \frac{\cos x - 1}{x}.
\]
This limit is just as hard as \(\sin x/x\), but closely related to it, so that we don’t have to do a similar calculation; instead we can do a bit of trickier algebra.

\[
\lim_{x \to 0} \frac{\cos x - 1}{x} = \lim_{x \to 0} \frac{\cos x - 1}{x} \frac{x + 1}{x + 1} = \lim_{x \to 0} \frac{-\sin^2 x}{x + 1} = \lim_{x \to 0} \frac{-\sin x \sin x}{x + 1} = \lim_{x \to 0} \frac{-\sin x \sin x}{x \cos x + 1}.
\]

To compute the desired limit it is sufficient to compute the limits of the two final fractions, as \(x\) goes to \(0\). The first of these is the hard limit we’ve just done, namely \(1\). The second turns out to be simple, because the denominator presents no problem:
\[
\lim_{x \to 0} \frac{\sin x}{x} = \frac{\sin 0}{0} = 0.
\]
Thus,
\[
\lim_{x \to 0} \frac{\cos x - 1}{x} = 0.
\]

### Exercises 4.3

1. Compute \(\lim_{x \to 0} \sin(5x)/x\) \(\Rightarrow\)
2. Compute \(\lim_{x \to 0} \sin(T(x))/x\) \(\Rightarrow\)
3. Compute \(\lim_{x \to 0} \cos(4x)/x\) \(\Rightarrow\)
4. Compute \(\lim_{x \to 0} \frac{\cos(x)}{\sin(x)} \Rightarrow\)
5. Compute \(\lim_{x \to 0} \cos(x) - x \Rightarrow\)
6. For all \(x \geq 0\), \(4x - 9 \leq f(x) \leq x^2 - 4x + 7\). Find \(\lim_{x \to 0} f(x) \Rightarrow\)
7. For all \(x \geq 0\), \(2x \leq g(x) \leq x^2 - 4x + 2\). Find \(\lim_{x \to 0} g(x) \Rightarrow\)
8. Use the Squeeze Theorem to show that \(\lim_{x \to 0} x \cos(2/x) = 0\).
4.5 Derivatives of the Trigonometric Functions

All of the other trigonometric functions can be expressed in terms of the sine, and so their derivatives can easily be calculated using the rules we already have. For the cosine we need to use two identities,
\[ \cos x = \sin(x + \frac{\pi}{2}), \]
\[ \sin x = \cos(x - \frac{\pi}{2}). \]

Now:
\[ \frac{d}{dx} \cos x = \frac{d}{dx} \sin(x + \frac{\pi}{2}) = \cos(x + \frac{\pi}{2}) = \cos x, \]
\[ \frac{d}{dx} \sin x = \frac{d}{dx} \cos(x - \frac{\pi}{2}) = -\cos(x - \frac{\pi}{2}) = -\cos x. \]

The derivatives of the cotangent and cosecant are similar and left as exercises.

4.6 Exponential and Logarithmic Functions

from the example: \( \log_2(3) = 5 \), \( \log_2(5) = 3 \), \( \log_2(8) = 8 \). So \( \log_2(a^3b^5) = \log_2(a^3) + \log_2(b^5). \) Now let's make this a bit more general. Suppose \( A \) and \( B \) are two numbers, \( A = a^n \) and \( B = a^m \). Then \( \log_2(AB) = \log_2(a^n a^m) = \log_2(a^{n+m}) = x + y = \log_2(a) + \log_2(b). \)

Now consider \( a^x = b \). Then \( \frac{d}{dx} a^x = \ln a \cdot a^x \). Again it's clear that more generally \( (a^x)' = a^{x+1} \), and again this gives us a fact about logarithms. If \( A = a^x \) then \( A' = (a^x)' = a^{x+1} \), so \( \log_2(A) = x = \log_2(a^x) \) the exponential can be "pulled out in front.”

We’ve cheated a bit in the previous two paragraphs. It is obvious that \( a^0 = a \cdot a \cdot \ldots \cdot a \) and the result of that the exponential function’s properties. You need first to understand that since it’s not "obvious" what \( 2^0 \) should mean, we are really free to make it mean whatever we want, so long as we keep the behavior that is obvious, namely, when \( x \) is a positive integer. What else do we want to be true about \( 2^x \)? We want the properties of the previous two paragraphs to be true for all exponents. \( 2^{x+y} = 2^x \cdot 2^y \) and \( (2^x)^y = 2^{xy} \).

After the positive integers, the next easiest number to understand is \( 0 \). \( 2^0 = 1 \). You have probably learned this fact in the past, why is it true? It’s true precisely because we want \( 2^0 = 1 \) to be true about the function \( 2^x \). We need it to be true that \( 2^0 = 2^0 \cdot 2^x \), and this only works if \( 2^0 = 1 \). The same argument implies that \( a^0 = 1 \) for any \( a \).

The next easiest set of numbers to understand is the negative integers: for example, \( 2^{-1} = \frac{1}{2} \). We know that whatever \( 2^{-1} \) means it must be that \( 2^{-1} = 2^{1/2} = 2^{-1/2} \), which means that \( 2^{1/2} \) must be \( 1/2 \). In fact, by the same argument, once we know what \( 2^x \) means for some value of \( x \), \( 2 \cdot 2^x \) must be \( 2^{1/2} \) and more generally \( a^{-x} = \frac{1}{a^x} \).

Next, consider an exponent \( 1/q \). For which \( q \) is a positive integer. We want \( \sqrt{2} = 2^{1/2} \) to be true about the function \( 2^x \). This means a \( q \) is the qth root of 2, \( 2^{1/q} = \sqrt{2} \). This is all we need to understand that \( 2^{1/2} = (2^{1/4})^2 = (\sqrt{2})^2 \) and \( 2^{1/3} = (2^{1/9})^3 = (\sqrt[3]{2})^3 \).

What’s left is the hard part, what does \( 2^x \) mean when \( x \) cannot be written as a fraction, like \( x = \sqrt{2} \) or \( x = \pi/4 \)? What we know so far is how to assign meaning to \( 2^x \)

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14. Find all points on the graph of \( f(x) = 2 \sin x - 3 \sin^2 x \) at which the tangent line is horizontal.
15. Find an equation for the tangent line to \( f(x) = 2 \sin x - 3 \sin^2 x \) at \( x = \pi/3 \).
16. Find an equation for the tangent line to \( f(x) = 2 \sin x - 3 \sin^2 x \) at \( x = \pi/6 \).
17. Find the equation for the tangent line to \( f(x) = \cos^2 x - 3 \sin^2 x \) at \( x = \pi/6 \).
18. Find the points on the curve \( y = x^2 - 2 \cos x \) that have a horizontal tangent line. 
19. Let \( C \) be a circle of radius \( r \). Let \( A \) be an arc on \( C \) subtending a central angle \( \theta \). Let \( B \) be the chord of \( C \) whose endpoints are the endpoints of \( A \). (Hence, \( B \) also subtends \( \theta \).) Let \( x \) be the length of \( A \) and let \( d \) be the length of \( B \). Sketch a diagram of the situation and compute \( \lim_{x \to 0} x/d \).

4.6 Exponential and Logarithmic Functions

An exponential function has the form \( a^x \), where \( a \) is a constant; examples are \( 2^x \), \( 10^x \), \( e^x \).

The logarithmic functions are the inverses of the exponential functions, that is, functions that "undo" the exponential functions, just as, for example, the cube root function "undoes" the cube function. \( \sqrt[3]{x} = 2 \). Note that the original function also undoes the inverse function:
\[ \left( x^{1/3} \right)^3 = x \]

Let \( f(x) = 2^x \). The inverse of this function is called the logarithm base 2, denoted \( \log_2(x) \) (or especially in computer science circles) \( \lg(x) \). What does this really mean? The logarithm must undo the action of the exponential function, so for example it must be that \( \log_2(8) = 3 \); starting with 3, the exponential function produces 2 for 3, and the logarithm of 8 must get us back to 3.

It's a little thought shows that it is not a coincidence that \( \log_2(2^3) = 3 \) simply gives the exponent—the original value that we must get back to. In other words, the logarithm is the exponent. Remember this catchphrase, and what it means, and you won't go wrong. (You do have to remember what it means. Like any good mnemonic, “the logarithm is the exponent” leaves out a lot of detail, like “Which exponent?” and “Exponent of what?”)

EXAMPLE 4.6.1 What is the value of \( \log_2(1000) \)? The "100" tells us the appropriate number to use for the base of the exponential function. The logarithm is the exponent, so the question is, what exponent \( E \) makes \( 10^E = 1000 \)? If we can find such an \( E \), then \( \log_2(1000) = \log_2(10^E) = E \); finding the appropriate exponent is the same as finding the logarithm. In this case, of course, it is easy: \( E = 3 \) so \( \log_2(1000) = 3 \).

Let’s review some laws of exponents and logarithms, let \( a \) be a positive number. Since \( a^m \cdot a^n = a^{m+n} \) and \( a^m/a^n = a^{m-n} \), and in general that \( a^{m+n} = a^m \cdot a^n \). Since “the logarithm is the exponent,” it’s no surprise that this translates directly into a fact about the logarithm function. Here are three facts:

4.7 Derivatives of the Exponential and Logarithmic Functions

As with the sine, we don’t know anything about derivatives that allows us to compute the derivatives of the exponential and logarithmic functions without going back to basics. Let’s do a little work with the definition again:

\[ \frac{d}{dx} a^x = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{a^{\Delta x} (a^{\Delta x} - 1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} = a \]
There are two interesting things to note here: As in the case of the sine function we are left with a limit that involves \(\Delta x\) but not \(x\), which means that whatever \(\lim_{\Delta x \to 0} (a^{\Delta x} - 1)/\Delta x\) we know that it is a number, that is, a constant. This means that \(a^x\) has a remarkable property: its derivative is a constant times itself.

We earlier remarked that the hardest limit we would compute is \(\lim \sin x/x\) = 1; we now have a limit that is just a bit too hard to include here. In fact the hard part is to see that \(\lim (a^{\Delta x} - 1)/\Delta x\) even exists—does this function really get closer and closer to some fixed value? Yes it does, but we will not prove this fact.

We can look at some examples. Consider \((2^x - 1)/x\) for some small values of \(x\): 1, 0.828427124, 0.756828469, 0.724061864, 0.70878051, 0.70780777 when \(x\) is 1, 1/2, 1/4, 1/8, 1/16, 1/32, respectively. It looks like this is settling in around 0.7, which turns out to be true (but the limit is not exactly 0.7). Consider now \((1/x/x)\) = 1. 1.643160616, 1.26490692, 1.177621526, 1.17670854, at the same values of \(x\). It turns out to be true that in the limit this is about 1.1. Two examples don’t establish a pattern, but if you do more examples you will find that the limit varies directly with the value of \(a\): bigger \(a\), bigger limit; smaller \(a\), smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between \(a = 2\) and \(a = 3\) the limit will be exactly 1; the value at which this happens is called \(e\), so that

\[
\lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1.
\]

As you might guess from our two examples, \(e\) is closer to 3 than to 2, and in fact \(e \approx 2.718\).

Now we see that the function \(e^x\) has a truly remarkable property:

\[
\frac{d}{dx} e^x = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{\Delta x} e^x - e^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} e^x = e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = e^x.
\]

That is, \(e^x\) is its own derivative, or in other words the slope of \(e^x\) is the same as its height, or the same as its second coordinate. The function \(f(z) = e^z\) goes through the point \((z, e^z)\) and has slope \(e^z\) there, no matter what \(z\) is. It is sometimes convenient to express the function \(e^z\) without an exponent, since complicated exponents can be hard to read. In such cases we use \(\exp(x)\), e.g., \(\exp(1 + x^2)\) instead of \(e^{1+x^2}\).

### 4.7 Derivatives of the exponential and logarithmic functions

We have discussed this from the point of view of the graphs, which is easy to understand but is not normally considered a rigorous proof—it is too easy to be led astray by pictures that seem reasonable but that miss some hard point. It is possible to do this derivation without resorting to pictures, and indeed we will see an alternate approach soon. We earlier remarked that the hardest limit we would compute is \(\lim \sin x/x\) = 1; we now have a limit that is just a bit too hard to include here. In fact the hard part is to see that \(\lim (a^{\Delta x} - 1)/\Delta x\) even exists—does this function really get closer and closer to some fixed value? Yes it does, but we will not prove this fact.

We can look at some examples. Consider \((2^x - 1)/x\) for some small values of \(x\): 1, 0.828427124, 0.756828469, 0.724061864, 0.70878051, 0.70780777 when \(x\) is 1, 1/2, 1/4, 1/8, 1/16, 1/32, respectively. It looks like this is settling in around 0.7, which turns out to be true (but the limit is not exactly 0.7). Consider now \((1/x/x)\) = 1. 1.643160616, 1.26490692, 1.177621526, 1.17670854, at the same values of \(x\). It turns out to be true that in the limit this is about 1.1. Two examples don’t establish a pattern, but if you do more examples you will find that the limit varies directly with the value of \(a\): bigger \(a\), bigger limit; smaller \(a\), smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between \(a = 2\) and \(a = 3\) the limit will be exactly 1; the value at which this happens is called \(e\), so that

\[
\lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1.
\]

As you might guess from our two examples, \(e\) is closer to 3 than to 2, and in fact \(e \approx 2.718\).

Now we see that the function \(e^x\) has a truly remarkable property:

\[
\frac{d}{dx} e^x = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{\Delta x} e^x - e^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} e^x = e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = e^x.
\]

That is, \(e^x\) is its own derivative, or in other words the slope of \(e^x\) is the same as its height, or the same as its second coordinate. The function \(f(z) = e^z\) goes through the point \((z, e^z)\) and has slope \(e^z\) there, no matter what \(z\) is. It is sometimes convenient to express the function \(e^z\) without an exponent, since complicated exponents can be hard to read. In such cases we use \(\exp(x)\), e.g., \(\exp(1 + x^2)\) instead of \(e^{1+x^2}\).
EXAMPLE 7.4.4 Recall that we have not justified the power rule except when the exponent is a positive or negative integer. We can use the exponential function to take care of other exponents:

\[
\frac{d}{dx} e^{\ln x} = \left(\frac{d}{dx} \ln x\right) x^{\ln x} = \left(\frac{1}{x}\right) x = x^{\ln x} - 1
\]

Exercises 4.7.

In 1-19, find the derivatives of the functions.

1. \(1^{-x}\)
2. \(\ln x\)
3. \((x^3)^{1/2}\)
4. \(\sin x^3\)
5. \(e^{x^3}\)
6. \(x^2 + 2\)
7. \(x^{1/2}\)
8. \(x^2 + 3x + 1\)
9. \((1/2)^{x^2}\)
10. \(\ln(x^2 + 3x)\)
11. \(\sqrt[3]{x^2 + 3x}\)
12. \(\ln(x^2 + 3x)\)
13. \(\sin(x^2 + 3x)\)
14. \(\cos(x^2 + 3x)\)
15. \(\sin(x^2 + 3x)\)
16. \(x^2 + 3x\)
17. \(\ln(\ln(3x))\)
18. \(\ln(\ln(x^2))\)
19. \(x^2 + x^3\)
20. Find the value of \(a\) so that the tangent line to \(y = \ln(x)\) at \(x = a\) is a line through the origin.
21. If \(y = \ln(x^2 + 2)\) compute \(f'(x^2 + 2)\).

4.8 Implicit Differentiation

As we have seen, there is a close relationship between the derivatives of \(e^x\) and \(\ln x\) because these functions are inverses. Rather than relying on pictures for our understanding, we would like to be able to exploit this relationship computationally. In fact this technique can help us find derivatives in many situations, not just when we seek the derivative of an inverse function.

4.8 Implicit Differentiation

chain rule where \(y\) appears.

\[
\frac{d}{dx} y^2 = \frac{d}{dx} (x^2 + 2y) \implies \frac{d}{dx} y^2 - 2y \frac{dy}{dx} = 2x
\]

Now we have an expression for \(y'\), but it contains \(y\) as well as \(x\). This means that if we want to compute \(y'\) for some particular value of \(x\) we'll have to know or compute \(y\) at that value of \(x\) as well. It is at this point that we will need to know whether \(y = U(x)\) or \(L(x)\). Occasionally it will turn out that we can avoid exploiting the use of \(U(x)\) or \(L(x)\) by the nature of the problem.

EXAMPLE 4.8.1 Find the slope of the circle \(x^2 + y^2 = a^2\) at the point \((1, \sqrt{3})\). Since we know both the \(x\) and \(y\) coordinates of the point of interest, we do not need to explicitly recognize that this point is on \(U(x)\), and we do not need to use \(L(x)\) to compute \(y'\) but we could. Using the calculation of \(y'\) from above,

\[
y' = -\frac{x}{\sqrt{3}} = -\frac{1}{\sqrt{3}} = \frac{-1}{3}
\]

It is instructive to compare this approach to others.

We might have recognized at the start that \((1, \sqrt{3})\) is on the function \(y = U(x) = -\sqrt{x^2 - 1}\). We could then take the derivative of \(L(x)\), using the power rule and the chain rule, to get

\[
L'(x) = \frac{1}{2}\left(1-x^2\right)^{-3/2}(-2x) = \frac{-x}{\sqrt{1-x^2}}
\]

Then we could compute \(U'(1) = 1/(\sqrt{x})\) by substituting \(x = 1\).

Alternatively, we could realize that the point is on \(U(x)\), but use the fact that \(y' = -x/y\).

Since the point is on \(L(x)\) we can replace \(y' = y'/L(x)\) to get

\[
y' = \frac{x}{\sqrt{1-x^2}}
\]

without computing the derivative of \(L(x)\) explicitly. Then we substitute \(x = 1\) and get the same answer as before.

In the case of the circle it is possible to find the functions \(U(x)\) and \(L(x)\) explicitly, but there are potential advantages to using implicit differentiation anyway. In some cases it is more difficult or impossible to find an explicit formula for \(y\) and implicit differentiation is the only way to find the derivative.

EXAMPLE 4.8.2 Find the derivative of any function defined implicitly by \(y^2 + e^x = x\). We treat \(y\) as an unspecified function and use the chain rule:

\[
\frac{d}{dx} (y^2 + e^x) = \frac{d}{dx} x
\]

\[
y' y^2 + e^x y' = 1 - 2y y' = 1 - 2y
\]

If \(y' = 1 - 2y\) then \(y' x^2 + e^x\) is true, and we can think of \(y' x^2 + e^x\) as defining both \(U(x)\) and \(L(x)\).

Now we can take the derivative of both sides as before, remembering that \(y\) is not simply a variable but a function—in this case, \(y\) is either \(U(x)\) or \(L(x)\) but we're not yet specifying which one. When we take the derivative we just have to remember to apply the
of each side:
\[ \cos y = \frac{m}{y_n} \]
\[ y' = \frac{m}{y_n} \]
\[ y' = \frac{m}{\sqrt{1-(m/n)(n-1)}} = \frac{m}{\sqrt{1+(m/n)(n-1)}} \]

Exercises 4.8.
In exercises 1–8, find a formula for the derivative \( y' \) at the point \((x, y)\):
1. \( y' = 1 + x^2 \Rightarrow \)
2. \( x^2 + xy + y^2 = 7 \Rightarrow \)
3. \( x^2 + xy = y^2 + y^2x^2 \Rightarrow \)
4. \( \cos x \sin y = 1 \Rightarrow \)
5. \( \sqrt{2} \cdot \sqrt{2} = 9 \Rightarrow \)
6. \( \tan(x/y) = x + y \Rightarrow \)
7. \( \sin(x + y) = xy \Rightarrow \)
8. \( \frac{1}{x} - \frac{1}{y} = 7 \Rightarrow \)

9. A hyperbola passing through \((6, 6)\) consists of all points whose distance from the origin is a constant more than its distance from the point \((5, 2)\). Find the slope of the tangent line to the hyperbola at \((8, 6)\).
10. Compute \( y' \) for the ellipse of example 4.8.3.

11. If \( y = \log_x \) then \( a_x = x \). Use implicit differentiation to find \( y' \).
12. The graph of the equation \( x^2 - xy + y^2 = 9 \) is an ellipse. Find the lines tangent to this curve at the two points where it intersects the \( y \)-axis. Show that these lines are parallel.
13. Repeat the previous problem for the points at which the ellipse intersects the \( x \)-axis.
14. Find the points on the ellipse from the previous two problems where the slope is horizontal and where it is vertical.
15. Find an equation for the tangent line to \( x^2 + y^2 = 9 \) at \((2, \sqrt{7})\). (This curve is the kampyle of Eudosmus) \( \Rightarrow \)
16. An equation for the tangent line to \( x^2 + y^2 = 9 \) at a point \((x, y)\) on the curve, with \( x \neq 0 \) and \( y \neq 0 \). (This curve is an astroid) \( \Rightarrow \)
17. Find an equation for the tangent line to \( x^2 + y^2 = 9 \) at a point \((x, y)\) on the curve, with \( x \neq 0 \), \(-1, 1\). (This curve is a lemniscate) \( \Rightarrow \)

4.9 Inverse Trigonometric Functions

The trigonometric functions frequently arise in problems, and often it is necessary to invert the functions, for example, to find an angle with a specified sine. Of course, there are many angles with the same sine, so the sine function doesn’t actually have an inverse that reliably “undoes” the sine function. If you know that \( \sin x = 0.5 \), you can’t reverse this to discover \( x \), that is, you can’t solve for \( x \), as there are infinitely many angles with sine 0.5. Nevertheless, it is useful to have something like an inverse to the sine, however imperfect. The usual approach is to pick out some collection of angles that produces all possible values of the sine exactly once. If we “discard” all other angles, the resulting function does have a proper inverse.

The sine takes on all values between \(-1 \) and 1 exactly once on the interval \([\pi/2, \pi]\). If we truncate the sine, keeping only the interval \([\pi/2, \pi]\), as shown in figure 4.9.1, then this truncated sine has an inverse function. We call this the inverse sine or the arcsine, and write \( y = \arcsin(x) \).

Recall that a function and its inverse undo each other in either order, for example, \( \sqrt{2}x = x \) and \( \sqrt{2}x \). This does not work with the sine and the “inverse sine” because the inverse sine is the inverse of the truncated sine function, not the real sine function. It is true that \( \sin(\arcsin(y)) = x \), that is, the sine undoes the arcsine. It is not true that the arcsine undoes the sine, for example, \( \sin(\pi/6) = 1/2 \) and \( \arcsin(1/2) = \pi/6 \), so doing first the sine then the arcsine does not get us back where we started. This is because 5\pi/6

Exercises 4.9.
1. Show that the derivative of \( \arccos x \) is \( \frac{1}{\sqrt{1-x^2}} \)
2. Show that the derivative of \( \arctan x \) is \( \frac{1}{1+x^2} \)
3. The inverse of cot is usually defined so that the range of arccot is \((0, \pi)\). Sketch the graph of \(y = \arccot x\). In the process you will make it clear what the domain of arccot is. Find the derivative of the arccotangent. 
4. Show that \(\arccot x + \arctan x = \pi/2\).
5. Find the derivative of \(\arccot(x^3)\). 
6. Find the derivative of \(\arctan(e^x)\). 
7. Find the derivative of \(\arctan(e^{-x})\). 
8. Find the derivative of \(\arccot(e^x)\). 
9. Find the derivative of \(\arccot x\). 
10. Find the derivative of \(\arccos(\arctan x)\). 
11. Find the derivative of \(\ln(\arccot(x^3))\).

### 4.10 Limits revisited

We have defined and used the concept of limit, primarily in our development of the derivative. Recall that \(\lim_{x \to a} f(x) = L\) is true if, in a precise sense, \(f(x)\) gets closer and closer to \(L\) as \(x\) gets closer and closer to \(a\). While some limits are easy to see, others take some ingenuity; in particular, the limits that are always difficult are on their face, since in

\[
\lim_{x \to 0} \frac{f(x + 2x) - f(x)}{x} = \lim_{x \to 0} \frac{f(x + 2x) - f(x)}{2x}
\]

both the numerator and denominator approach zero. Typically this difficulty can be resolved when \(f\) is a “nice” function and we are trying to compute a derivative. Occasionally such limits are interesting for other reasons, and the limit of a fraction in which both numerator and denominator approach zero can be difficult to analyze. Now that we have the derivative available, there is another technique that can sometimes be helpful in such circumstances.

Before we introduce the technique, we will also expand our concept of limit, in two ways. When the limit of \(f(x)\) as \(x\) approaches \(a\) does not exist, it may be useful to note in what way it does not exist. We have already talked about one such case: one-sided limits. Another case is when \(f\) goes to infinity. We will also occasionally want to know what happens to \(f\) when \(x\) “goes to infinity”.

#### EXAMPLE 4.10.1

What happens to \(1/x\) as \(x\) goes to 0? From the right, \(1/x\) gets bigger and bigger, or goes to infinity. From the left it goes to negative infinity.

#### EXAMPLE 4.10.2

What happens to the function \(\cos(1/x)\) as \(x\) goes to infinity? It seems clear that as \(x\) gets larger and larger, \(1/x\) gets closer and closer to zero, so \(\cos(1/x)\) should be getting closer and closer to \(\cos(0) = 1\).

#### 4.10 Limits revisited

First we use L'Hôpital’s Rule. Since the numerator and denominator both approach zero,

\[
\lim_{x \to 0} \frac{2x^2 - 3x + 7}{x^2 + 4x^2 + 1} = \lim_{x \to 0} \frac{4x - 3}{8x + 4}
\]

provided the latter exists. But in fact this is an easy limit, since the denominator now approaches \(-1\), so

\[
\frac{2x^2 - 3x + 7}{x^2 + 4x^2 + 1} = \frac{2x^2}{x^2(1 + 4)}
\]

and that

\[
\lim_{x \to 0} \frac{2x^2}{x^2} = \lim_{x \to 0} 2 = 2
\]

since \(2x^2\) approaches zero as \(x\) approaches \(0\). Now

\[
\lim_{x \to 0} \frac{2x^2 - 3x + 7}{x^2 + 4x^2 + 1} = \frac{4x - 3}{8x + 4}
\]

as before.

#### EXAMPLE 4.10.7

Compute \(\lim_{x \to \infty} \frac{x^2 - 3x + 7}{x^2 + 4x^2 + 1}\) in two ways.

As \(x\) goes to infinity both the numerator and denominator go to infinity, so we may apply L'Hôpital’s Rule:

\[
\lim_{x \to \infty} \frac{2x^2 - 3x + 7}{x^2 + 4x^2 + 1} = \lim_{x \to \infty} \frac{4x - 3}{8x + 4}
\]

In the second quotient, it is still the case that the numerator and denominator both go to infinity, so we are allowed to use L'Hôpital’s Rule again:

\[
\lim_{x \to \infty} \frac{4x - 3}{8x + 4} = \lim_{x \to \infty} \frac{4}{8} = \frac{1}{2}
\]

The original limit is \(1/2\).

#### EXAMPLE 4.10.8

Compute \(\lim_{x \to \infty} \frac{\sin x}{x}\).

Both the numerator and denominator approach zero, so applying L'Hôpital’s Rule:

\[
\lim_{x \to \infty} \frac{\sin x}{x} = \lim_{x \to \infty} \frac{\cos x}{1} = 0
\]

#### Exercises 4.10

Compute the limits.

1. \(\lim_{x \to \infty} \frac{\cos x - 1}{x}\)
2. \(\lim_{x \to \infty} \frac{e^x}{x^2}\)
3. \(\lim_{x \to \infty} \sqrt{x^2 + x^2 - \sqrt{x^2 - x}}\)
4. \(\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}\)
5. \(\lim_{x \to \infty} \frac{x^2 + 1}{x + 1}\)
6. \(\lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{x}\)
7. \(\lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{x}\)
8. \(\lim_{x \to \infty} \frac{\ln x}{x + 1}\)
9. \(\lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{x}\)
10. \(\lim_{x \to \infty} \frac{\sqrt{x^3 + 1}}{x}\)
11. \(\lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{\sqrt{x^2 + 1}}\)

As with ordinary limits, these concepts can be made precise. Roughly, we want \(\lim_{x \to \infty} f(x) = \infty\) to mean that we can make \(f(x)\) arbitrarily large by making \(x\) close enough to \(a\), and \(\lim_{x \to \infty} f(x) = L\) should mean we can make \(f(x)\) close as we want to \(L\) by making \(x\) large enough. Compare this definition to the definition of limit in section 2.3, definition 2.3.2.

#### DEFINITION 4.10.3

If \(f\) is a function, we say that \(\lim_{x \to \infty} f(x) = \infty\) if for every \(N > 0\) there is a \(\delta > 0\) such that whenever \(x - \delta < \delta, f(x) > N\). We can extend this in the obvious ways to define \(\lim_{x \to \infty} f(x) = -\infty, \lim_{x \to \infty} f(x) = \pm \infty,\) and \(\lim_{x \to \infty} f(x) = \pm \infty\).

#### DEFINITION 4.10.4

Limit at infinity

If \(f\) is a function, we say that \(\lim_{x \to \infty} f(x) = L\) if for every \(\epsilon > 0\) there is an \(N > 0\) so that whenever \(x > N, |f(x) - L| < \epsilon\). We may similarly define \(\lim_{x \to \infty} f(x) = L\), and using the idea of the previous definition, we may define \(\lim_{x \to \infty} f(x) = \pm \infty\).

We include these definitions for completeness, but we will not explore them in detail. Suffice it to say that such limits behave in much the same way that ordinary limits do; in particular there are some analogs of theorem 2.3.6.

Now consider this limit:

\[
\lim_{x \to \infty} \frac{x^2}{\sin x}
\]

As \(x\) approaches \(\pi\), both the numerator and denominator approach zero, so it is not obvious what, if anything, the quotient approaches. We can often compute such limits by application of the following theorem.

#### THEOREM 4.10.5

L'Hôpital’s Rule

For ‘sufficiently nice’ functions \(f(x)\) and \(g(x)\), if \(\lim_{x \to \infty} f(x) = 0\) or \(\lim_{x \to \infty} g(x) = \pm \infty\) or both \(f(x) \to \pm \infty\) and \(g(x) \to \pm \infty\), and if \(\lim_{x \to \infty} \frac{f(x)}{g(x)}\) exists, then \(\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}\). This remains true if \(f \to \pm \infty\) or \(g \to \pm \infty\)

This theorem is somewhat difficult to prove, in part because it incorporates so many different possibilities, so we will not prove it here. We also will not need to worry about the precise definition of ‘sufficiently nice’, as the functions we encounter will be suitable.

#### EXAMPLE 4.10.6

Compute \(\lim_{x \to \infty} \frac{x^2 + 7}{\sin x}\) in two ways.

### 100 Chapter 4 Transcendental Functions
4.11 Hyperbolic Functions

DEFINITION 4.11.3 The other hyperbolic functions are

\[
\begin{align*}
tanh x &= \frac{\sinh x}{\cosh x} \\
coth x &= \frac{\cosh x}{\sinh x} \\
sech x &= \frac{1}{\cosh x} \\
\text{csch} x &= \frac{1}{\sinh x} \\
\text{coth} x &= \frac{1}{\tanh x}
\end{align*}
\]

The domain of coth and csch is \( x \neq 0 \) while the domain of the other hyperbolic functions is all real numbers. Graphs are shown in figure 4.11.1.

![Figure 4.11.1](image)

Certainly the hyperbolic functions do not closely resemble the trigonometric functions graphically. But they do have analogous properties, beginning with the following identity.

THEOREM 4.11.4 For all \( x \in \mathbb{R} \), \( \cosh^2 x - \sinh^2 x = 1 \).

Proof. The proof is a straightforward computation:

\[
\cosh^2 x - \sinh^2 x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = \frac{e^{2x} + 2e^x + 2 - e^{2x} - 2e^x + 2 - 2e^{2x}}{4} = 1.
\]

This immediately gives two additional identities:

\[
1 - \tanh^2 x = \text{sech}^2 x \quad \text{and} \quad \coth^2 x - 1 = \text{csch}^2 x.
\]

The identity of the theorem also helps to provide a geometric motivation. Recall that the graph of \( x^2 - y^2 = 1 \) is a hyperbola with asymptotes \( x = \pm y \) whose \( x \)-intercepts are \( \pm 1 \). If \( (x, y) \) is a point on the right half of the hyperbola, and if we let \( x = \cosh t \), then \( y = \pm \sqrt{x^2 - 1} = \pm \sqrt{\cosh^2 x - 1} = \pm \sinh t \). So for some suitable \( t \), \( \cosh t \) and \( \sinh t \) are the coordinates of a typical point on the hyperbola. In fact, it turns out that \( t \) is twice the area shown in the first graph of figure 4.11.2. Even this is analogous to trigonometry; \( \cos t \) and \( \sin t \) are the coordinates of a typical point on the unit circle, and \( t \) is twice the area shown in the second graph of figure 4.11.2.

![Figure 4.11.2](image)

Given the definitions of the hyperbolic functions, finding their derivatives is straightforward. Here again we see similarities to the trigonometric functions.

THEOREM 4.11.5 \( \frac{d}{dx} \cosh x = \sinh x \) and \( \frac{d}{dx} \sinh x = \cosh x \).

Proof. \( \frac{d}{dx} \cosh x = \frac{d}{dx} \frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x \), and \( \frac{d}{dx} \sinh x = \frac{d}{dx} \frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x \).

Since \( \cosh x > 0 \), \( \sinh x \) is increasing and hence injective, so \( \sinh x \) has an inverse, \( \text{arsinh} x \). Also, \( \text{arsinh} x > 0 \) when \( x > 0 \), so \( \text{arsinh} x \) is injective on \( (0, \infty) \) and has a (partial) inverse, \( \text{arcosh} x \). The other hyperbolic functions have inverses as well, though \( \text{arcosh} x \) is only a partial inverse. We may compute the derivatives of these functions as we have other inverse functions.

THEOREM 4.11.6 \( \frac{d}{dx} \text{arcsinh} x = \frac{1}{\sqrt{x^2 + 1}} \).

Proof. Let \( y = \text{arsinh} x \), so \( \sinh y = x \). Then \( \frac{d}{dx} \sinh y = \cosh y \cdot y' = 1 \), and so

\[
y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{x^2 + 1}}.
\]
The other derivatives are left to the exercises.

Exercises 4.11.

1. Show that the range of \( \sinh x \) is all real numbers. (Hint: show that if \( y = \sinh x \) then \( x = \ln(y + \sqrt{y^2 + 1}) \).)

2. Compute the following limits:
   a. \( \lim_{x \to \infty} \cosh x \)
   b. \( \lim_{x \to \infty} \sinh x \)
   c. \( \lim_{x \to \infty} \tanh x \)
   d. \( \lim_{x \to \infty} (\cosh x - \sinh x) \)

3. Show that the range of \( \tanh x \) is \((-1, 1)\). What are the ranges of \( \coth, \sech, \) and \( \csch \)? (Use the fact that they are reciprocal functions.)

4. Prove that for every \( x, y \in \mathbb{R} \), \( \sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y \). Obtain a similar identity for \( \sinh(x - y) \).

5. Prove that for every \( x, y \in \mathbb{R} \), \( \cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y \). Obtain a similar identity for \( \cosh(x - y) \).

6. Use exercises 4 and 5 to show that \( \sinh(2x) = 2 \sinh x \cosh x \) and \( \cosh(2x) = \cosh^2 x + \sinh^2 x \) for every \( x \). Conclude also that \( (\cosh(2x) - 1)/2 = \sinh^2 x \).

7. Show that \( \frac{d}{dx}(\tanh x) = \sech^2 x \). Compute the derivatives of the remaining hyperbolic functions as well.

8. What are the domains of the six inverse hyperbolic functions?

9. Sketch the graphs of all six inverse hyperbolic functions.