1 Analytic Geometry

Much of the mathematics in this chapter will be review for you. However, the examples will be oriented toward applications and so will take some thought.

The $xy$ coordinate system we normally write the $x$-axis horizontally, with positive numbers to the right of the origin, and the $y$-axis vertically, with positive numbers above the origin. That is, unless stated otherwise, we take "rightward" to be the positive $x$-direction and "upward" to be the positive $y$-direction. In a purely mathematical situation, we normally choose the same scale for the $x$- and $y$-axes. For example, the line joining the origin to the point $(a, b)$ makes an angle of $45^\circ$ with the $x$-axis (and also with the $y$-axis).

In applications, often letters other than $x$ and $y$ are used, and different scales are chosen in the horizontal and vertical directions. For example, suppose you drop something from a window, and you want to study how its height above the ground changes from second to second. It is natural to let the $f$ denote the time (the number of seconds since the object was released) and to let the $h$ denote the height. For each $f$ (say, at one-second intervals) you have a corresponding height $h$. This information can be tabulated, and then plotted on the $(f, h)$ coordinate plane, as shown in Figure 1.1.1.

We use the word "quadrant" for each of the four regions into which the plane is divided by the axes: the first quadrant is where both coordinates are positive, or the "northwest" portion of the plane, and the second, third, and fourth quadrants are counted off counterclockwise, so the second quadrant is the northeast, the third is the southwest, and the fourth is the southeast.

Suppose we have two points $A$ and $B$ in the $(x, y)$-plane. We often want to know the change in $x$-coordinate (also called the "horizontal distance") in going from $A$ to $B$. This information (15%, 28%, or 33%) using mathematical terminology, and graph the tax on the $y$-axis against the taxable income on the $x$-axis.

The percentages, when converted to decimal values 0.15, 0.28, and 0.33, are the change in $x$ when the $y$ intercept and the slope, then the $x$-intercept. See Figure 1.1.1.

The graph of $y$ versus $f$ is a straight line because you are traveling at constant speed. The line passes through the points $(1, 110)$ and $(1.5, 85)$, so its slope is $m = (85 - 110)/(1.5 - 1) = -16/0.5 = -32$.

1.1 Lines

If we have two points $A(x_1, y_1)$ and $B(x_2, y_2)$, we then can draw one and only one line through both points. By the slope of this line we mean the ratio $\Delta y/\Delta x$. The slope is often denoted $m$: $m = \Delta y/\Delta x = (y_2 - y_1)/(x_2 - x_1)$. For example, the line joining the points $(1, 2)$ and $(3, 5)$ has slope $(5 - 2)/(3 - 1) = 2/2 = 1.

EXAMPLE 1.1.1 According to the 1990 U.S. federal income tax schedules, a head of household paid 15% on taxable income up to $26,050. If taxable income was between $26,050 and $40,000, then, in addition, 28% was to be paid on the amount between $26,050 and $67,200, and 33% paid on the amount over $67,200 (if any). Interpret the tax bracket and any other point $(x, y)$ on the line.

For example, if we want to find the equation of the line joining our earlier points $A(2, 1)$ and $B(3, 5)$, we can use this formula:

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

Or of course, this really just is the point-slope formula, except that we are not computing $m$ in a separate step.

The slope $m$ of a line in the form $y = mx + b$ tells us the direction in which the line is pointing. If $m$ is positive, the line goes into the 1st quadrant as you go from left to right. If $m$ is large and positive, it has a steep incline, while if $m$ is small and positive, then the line has a small angle of inclination. If $m$ is negative, the line goes into the 4th quadrant as you go from left to right. If $m$ is a large negative number (large in absolute value), then the line goes steeply downward, while if $m$ is negative but near zero, then it points only a little downward. These four possibilities are illustrated in Figure 1.1.2.

If $m = 0$, then the line is horizontal; its equation is simply $y = b$.

There is one type of line that cannot be written in the form $y = mx + b$, namely, vertical lines. A vertical line has an equation of the form $x = a$. Sometimes one says that a vertical line has an "infinite" slope.

Sometimes it is useful to find the $x$-intercept of a line $y = mx + b$. This is the $x$-value when $y = 0$. Setting $mx + b = 0$ and solving for $x$ gives: $x = -b/m$. For example, the line $y = x/2 - 3$ through the points $(4, 1)$ and $(3, 3)$ has $x$-intercept $3/2$.

EXAMPLE 1.1.2 Suppose that you are driving to Seattle at constant speed, and notice that after you have been traveling for 1 hour (i.e., $t = 1$), you pass a sign saying it is 110 miles to Seattle, and after driving another half-hour you pass a sign saying it is 85 miles to Seattle. Using the horizontal axis for the time $t$ and the vertical axis for the distance $y$ from Seattle, graph and find the equation $y = mt + b$ for your distance from Seattle. Find the slope, $y$-intercept, and $t$-intercept, and describe the practical meaning of each.

The graph of $y$ versus $t$ is a straight line because you are traveling at constant speed. The line passes through the two points $(1, 110)$ and $(1.5, 85)$, so its slope is $m = (85 - 110)/(1.5 - 1) = -32$.
110)/(1.5 − 1) = −50. The meaning of the slope is that you are traveling at 50 mph; m is negative because you are traveling toward Seattle, i.e., your distance y is decreasing. The word “velocity” is often used for m = −50, when we want to indicate direction, while the word “speed” refers to the magnitude (absolute value) of velocity, which is 50 mph. To find the equation of the line, we use the point-slope formula:

\[ y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \]

Exercises 1.1.
1. Find the equation of the line through (1, 1) and (−5, −3) in the form y = mx + b. ⇒
2. Find the equation of the line through (−1, 2) with slope 2 in the form y = mx + b. ⇒
3. Find the equation of the line through (−1, 1) and (5, −3) in the form y = mx + b. ⇒
4. Change the equation \(-2x + 2y = 0\) to the form \(y = mx + b\), graph the line, and find the y-intercept and x-intercept. ⇒
5. Change the equation \(x - y = 6\) to the form \(y = mx + b\), graph the line, and find the y-intercept and x-intercept. ⇒
6. Change the equation \(x = 2y − 1\) to the form \(y = mx + b\), graph the line, and find the y-intercept and x-intercept. ⇒
7. Change the equation \(3y = 2x\) to the form \(y = mx + b\), graph the line, and find the y-intercept and x-intercept. ⇒
8. Change the equation \(2x + 3y = 6\) to the form \(y = mx + b\), graph the line, and find the y-intercept and x-intercept. ⇒
9. Determine whether the lines \(3x + 4y = 7\) and \(2x + 4y = 5\) are parallel. ⇒
10. Suppose a triangle in the x, y-plane has vertices (−1, 0), (1, 0), and (0, 2). Find the equations of the three lines that lie along the sides of the triangle in \(y = mx + b\) form. ⇒
11. Suppose that you are driving to Seattle at constant speed. After you have been traveling for an hour you pass a sign saying it is 130 miles to Seattle, and after driving another 20 minutes you pass a sign saying it is 105 miles to Seattle. Using the horizontal axis for the time t and the vertical axis for the distance from your starting point, graph and find the equation \(y = mt + b\) for your distance from your starting point. How long does the trip to Seattle take? ⇒
12. Let \(x\) stand for temperature in degrees Celsius (centigrade), and let \(y\) stand for temperature in degrees Fahrenheit. A temperature of 0°C corresponds to 32°F, and a temperature of 100°C corresponds to 212°F. Find the equation of the line that relates temperature Fahrenheit \(y\) to temperature Celsius \(x\) in the form \(y = mx + b\). Graph the line, and find the point at which this line intersects \(y = x\). What is the practical meaning of this point? ⇒

1.2 Distance Between Two Points; Circles

Given two points \((x_1, y_1)\) and \((x_2, y_2)\), recall that their horizontal distance from one another is \(\Delta x = x_2 - x_1\) and their vertical distance from one another is \(\Delta y = y_2 - y_1\). (Actually, the word “distance” normally denoted \(\Delta x\) and \(\Delta y\) are signed distances, but this is clear from context.) The actual (positive) distance from one point to the other is the length of the hypotenuse of a right triangle with legs \(\Delta x\) and \(\Delta y\), as shown in figure 1.2.1. The Pythagorean theorem then says that the distance between the two points is the square root of the sum of the squares of the horizontal and vertical sides:

\[ \text{distance} = \sqrt{\Delta x^2 + \Delta y^2} \]

For example, the distance between points \((2, 1)\) and \((3, 3)\) is \(\sqrt{(3 - 2)^2 + (3 - 1)^2} = \sqrt{2}\).

\[ (x,y) \quad \Delta y \quad (x,y) \]

\[ (x_2,y_2) \]

Figure 1.2.1 Distance between two points, \(\Delta x\) and \(\Delta y\) positive.

As a special case of the distance formula, suppose we want to know the distance of a point \((x, y)\) to the origin. According to the distance formula, this is \(\sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}\). A point \((x, y)\) is at a distance \(r\) from the origin if and only if \(\sqrt{x^2 + y^2} = r\), or if, equivalently, if \(x^2 + y^2 = r^2\).

Suppose \((x, y)\) is any fixed point, then a point \((x, y)\) is a distance \(r\) from the point \((0, 0)\) if and only if \(\sqrt{(x - 0)^2 + (y - 0)^2} = r\), i.e., if and only if

\[ (x - 0)^2 + (y - 0)^2 = r^2 \]

This is the equation of the circle of radius \(r\) centered at the point \((0, 0)\). For example, the circle of radius 5 centered at the point \((0, 0)\) has equation \(x^2 + y^2 = 25\). If we expand this we get \(x^2 + y^2 + 12x + 36 = 25 + x^2 + y^2 + 12y + 11 = 0\), but the original form is usually more useful.

1990 Tax Rate Schedules

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<tr>
<th>Schedule X</th>
<th>Use if your filing status is</th>
<th>Single</th>
<th>Head of household</th>
</tr>
</thead>
<tbody>
<tr>
<td>If the amount on Form 1040 line 37 is over</td>
<td>But not over</td>
<td>Enter on Form 1040 line 38</td>
<td>But not over</td>
</tr>
<tr>
<td>$59,325</td>
<td>155</td>
<td>26,050</td>
<td>67,200</td>
</tr>
<tr>
<td>$118,650</td>
<td>67,200</td>
<td>49,650</td>
<td>134,950</td>
</tr>
<tr>
<td>$215,500</td>
<td>116,720</td>
<td>74,050</td>
<td>167,200</td>
</tr>
</tbody>
</table>

Exercises 1.2.
1. Find the equation of the circle of radius 3 centered at:
   a) \((0, 0)\)
   b) \((5, 6)\)
   c) \((-5, -6)\)
2. For each pair of points \((x_1, y_1)\) and \((x_2, y_2)\) find \(\Delta x\) and \(\Delta y\) in going from \(A\) to \(B\),
   a) \((0, 0)\) and \((6, 8)\)
   b) \((1, 1)\) and \((3, 3)\)
   c) \((1, 1)\) and \((3, 3)\)
3. Graph the circle \(x^2 + y^2 = 9\).
4. Graph the circle \(x^2 + y^2 = 16\).

Chapter 2 Analytic Geometry

EXAMPLE 2.1.1 Graph the circle \(x^2 + y^2 = 9\) and \(x^2 + y^2 = 16\). With a little thought we convert this to \((x - 3)^2 + (y - 2)^2 = 9\) or \((x - 3)^2 + (y - 2)^2 = 16\). Also we see that this is the circle with radius 3 and center \((1, 2)\), which is easy to graph.

1.3 Functions

A function \(y = f(x)\) is a rule for determining \(y\) when we’ve given a value of \(x\). For example, the rule \(y = x + 1\) is a function. Any line \(y = mx + b\) is called a linear function. The graph of a function looks like a curve above (or below) the x-axis, where for any value of \(x\) the rule \(y = f(x)\) tells us how far to go above (or below) the x-axis to reach the curve. Functions can be defined in various ways: by an algebraic formula or several algebraic formulas, by a graph, or by an experimentally determined table of values. (In the latter case, the table gives a bunch of points in the plane, which we might then interpolate with a smooth curve, if that makes sense.)

Given a value of \(x\), a function must give at most one value of \(y\). Thus, vertical lines are not functions. For example, the line \(x = 1\) has infinitely many values of \(y\) if \(x = 1\). It
is also true that if $x$ is any number not 1 there is no $y$ which corresponds to $x$, but that is not a problem—only multiple $y$ values is a problem.

In addition to lines, another familiar example of a function is the parabola $y = f(x) = x^2$. We can draw the graph of this function by taking various values of $x$ (say, at regular intervals) and plotting the points $(x, f(x)) = (x, x^2)$. Then connect the points with a smooth curve. (See figure 1.3.1.)

The two examples $y = f(x) = 2x + 1$ and $y = f(x) = x^2$ are both functions which can be evaluated at any value of $x$ from negative infinity to positive infinity. For many functions, however, it only makes sense to take $x$-values inside an interval or outside of some “forbidden” region. The interval of $x$-values at which we're allowed to evaluate the function is called the domain of the function.

For example, the square-root function $y = f(x) = \sqrt{x}$ is the rule which says, given an $x$-value, take the nonnegative number whose square is $x$. This rule only makes sense if $x$ is positive or zero. We say that the domain of this function is $x \geq 0$, or more formally $\{x \in \mathbb{R} | x \geq 0\}$. Alternatively, we can use interval notation, and write that the domain is $[0, \infty)$. (In interval notation, square brackets mean that the endpoint is included, and a parenthesis means that the endpoint is not included.) The fact that the domain of $y = \sqrt{x}$ is $[0, \infty)$ means that in the graph of this function (see figure 1.3.1) we have points $(x, y)$ only above $x$-values on the right side of the $x$-axis.

Another example of a function whose domain is not the entire $x$-axis is: $y = f(x) = 1/x$, the reciprocal function. We cannot substitute $x = 0$ in this formula. The function makes sense, however, for any nonzero $x$, so we take the domain to be: $\{x \in \mathbb{R} | x \neq 0\}$. The graph of this function does not have any point $(x, y)$ with $x = 0$. As $x$ gets close to 0 from either side, the graph goes off toward infinity. We call the vertical line $x = 0$ an asymptote.

The second reason why certain $x$-values are excluded from the domain of a function is that (i) we cannot divide by zero, and (ii) we cannot take the square root of a negative number. We will encounter other ways in which functions might be undefined later.

Another reason why the domain of a function might be restricted is that in a given situation the $x$-values outside of some range might have no practical meaning. For example, if $y$ is the area of a square of side $x$, then we can write $y = f(x) = x^2$. In a purely mathematical context the domain of the function $y = x^2$ is all of $\mathbb{R}$. But in the story-problem context of finding areas of squares, we restrict the domain to positive values of $x$, because a square with a negative or zero side makes no sense.

In a problem in pure mathematics, we usually take the domain to be all values of $x$ at which the formulas can be evaluated. But in a story problem there might be further restrictions on the domain because only certain values of $x$ are of interest or make practical sense.

In a story problem, often letters different from $x$ and $y$ are used. For example, the volume $V$ of a sphere is a function of the radius $r$, given by the formula $V = f(r) = \frac{4}{3}\pi r^3$. Also, letters different from $f$ may be used. For example, if $g$ is the velocity of something at time $t$, we may write $y = g(t)$ with the letter $v$ (instead of $f$) standing for the velocity function (and $t$ playing the role of $x$).

The letter playing the role of $x$ is called the independent variable, and the letter playing the role of $y$ is called the dependent variable (because its value “depends on” the value of the independent variable). In story problems, when one has to translate from English into mathematics, a crucial step is to determine what letters stand for variables.

If only words and no letters are given, then we have to decide which letters to use. Some letters are traditional. For example, almost always, $t$ stands for time.

EXAMPLE 1.3.1. An open-top box is made from an $a \times b$ rectangular piece of cardboard by cutting out a square of side $x$ from each of the four corners, and then folding the sides up and sealing them with duct tape. Find a formula for the volume $V$ of the box as a function of $x$, and find the domain of this function.

The box we get will have height $x$ and rectangular base of dimensions $a - 2x$ by $b - 2x$. Thus,

$$V = f(x) = x(a - 2x)(b - 2x).$$

Here $a$ and $b$ are constants, and $V$ is the variable that depends on $x$, i.e., $V$ is playing the role of $y$.

This formula makes mathematical sense for any $x$, but in the story problem the domain is much less. In the first place, $x$ must be positive. In the second place, it must be less than half the length of either of the sides of the cardboard. Thus, the domain is

$$\{x \in \mathbb{R} | 0 < x < \frac{1}{2}(\text{minimum of } a \text{ and } b)\}.$$
Vertical dilation. If $y$ is replaced by $by/B$ in a formula and $B > 0$, then the effect on the graph is to dilate it by a factor of $B$ in the vertical direction. As before, this is an expansion or contraction depending on whether $B$ is larger or smaller than one. Note that if we have a function $y = f(x)$, replacing $y$ by $by/B$ is equivalent to multiplying the function on the right by $B$, i.e., $y = Bf(x)$. The effect on the graph is to expand the picture away from the $x$-axis by a factor of $B$ if $B > 1$, to contract it toward the $x$-axis by a factor of $1/B$ if $0 < B < 1$, and to dilate by $|B|$ and then flip about the $x$-axis if $B$ is negative.

**EXAMPLE 1.4.2 Ellipses** A basic example of the two expansion principles is given by an ellipse of semimajor axis $a$ and semiminor axis $b$. We get such an ellipse by starting with the unit circle—the circle of radius 1 centered at the origin, the equation of which is $x^2 + y^2 = 1$—and dilating by a factor of $a$ horizontally and by a factor of $b$ vertically. To get the equation of the resulting ellipse, which crosses the $x$-axis at $\pm a$ and crosses the $y$-axis at $\pm b$, we replace $x/a$ and $y/b$ in the equation for the unit circle. This gives

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad \text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Finally, if we want to analyze a function that involves both shifts and dilations, it is usually simplest to work with the dilations first, and then the shifts. For instance, if we want to dilate a function by a factor of $A$ in the $x$-direction and then shift $C$ to the right, we do this by replacing $x$ first by $x/A$ and then by $(x - C)$ in the formula. As an example, suppose that, after dilating our unit circle with $a$ in the $x$-direction and by $b$ in the $y$-direction to get the ellipse in the last paragraph, we then wanted to shift it a distance $k$ to the right and a distance $h$ upward, so as to be centered at the point $(h, k)$. The new ellipse would have equation

$$\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1.$$

Note well that this is different than first doing shifts by $h$ and $k$ and then dilations by $a$ and $b$.

$$\left(\frac{x-a}{a}\right)^2 + \left(\frac{y-b}{b}\right)^2 = 1.$$