12

Polar Coordinates, Parametric Equations

12.1 Polar Coordinates

Coordinate systems are tools that let us use algebraic methods to understand geometry. While the **rectangular** (also called **Cartesian**) coordinates that we have been using are the most common, some problems are easier to analyze in alternate coordinate systems.

A coordinate system is a scheme that allows us to identify any point in the plane or in three-dimensional space by a set of numbers. In rectangular coordinates these numbers are interpreted, roughly speaking, as the lengths of the sides of a rectangle. In **polar coordinates** a point in the plane is identified by a pair of numbers (r,θ). The number θ measures the angle between the positive x-axis and a ray that goes through the point, as shown in figure 12.1.1; the number r measures the distance from the origin to the point. Figure 12.1.1 shows the point with rectangular coordinates $(1,\sqrt{3})$ and polar coordinates $(2,\pi/3)$, 2 units from the origin and $\pi/3$ radians from the positive x-axis.

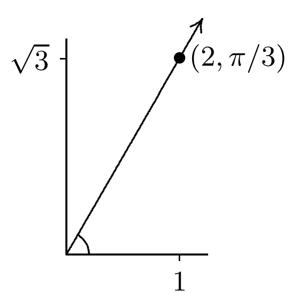

Figure 12.1.1 Polar coordinates of the point $(1,\sqrt{3})$.

Just as we describe curves in the plane using equations involving x and y, so can we describe curves using equations involving r and θ. Most common are equations of the form $r = f(\theta)$.

EXAMPLE 12.1.1 Graph the curve given by $r = 2$. All points with $r = 2$ are at distance 2 from the origin, so $r = 2$ describes the circle of radius 2 with center at the origin. □

EXAMPLE 12.1.2 Graph the curve given by $r = 1 + \cos\theta$. We first consider $y = 1 + \cos x$, as in figure 12.1.2. As θ goes through the values in $[0, 2\pi]$, the value of r tracks the value of y, forming the "cardioid" shape of figure 12.1.2. For example, when $\theta = \pi/2$, $r = 1 + \cos(\pi/2) = 1$, so we graph the point at distance 1 from the origin along the positive y-axis, which is at an angle of $\pi/2$ from the positive x-axis. When $\theta = 7\pi/4$, $r = 1 + \cos(7\pi/4) = 1 + \sqrt{2}/2 \approx 1.71$, and the corresponding point appears in the fourth quadrant. This illustrates one of the potential benefits of using polar coordinates: the equation for this curve in rectangular coordinates would be quite complicated. □

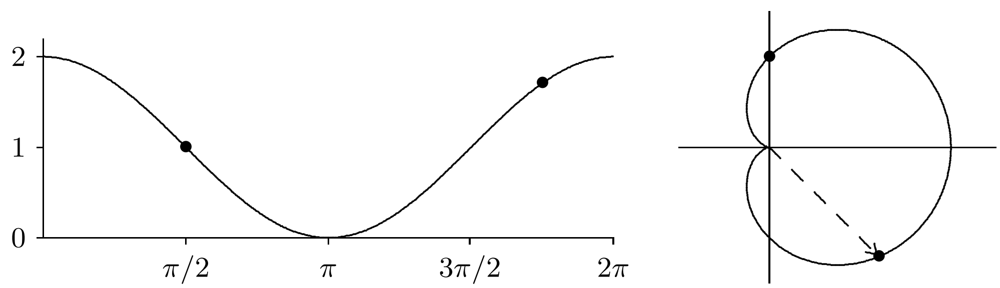

Figure 12.1.2 A cardioid: $y = 1 + \cos x$ on the left, $r = 1 + \cos\theta$ on the right.

Each point in the plane is associated with exactly one pair of numbers in the rectangular coordinate system; each point is associated with an infinite number of pairs in polar coordinates. In the cardioid example, we considered only the range $0 \le \theta \le 2\pi$, and already there was a duplicate: $(2, 0)$ and $(2, 2\pi)$ are the same point. Indeed, every value of θ outside the interval $[0, 2\pi)$ duplicates a point on the curve $r = 1 + \cos\theta$ when $0 \le \theta < 2\pi$. We can even make sense of polar coordinates like $(-2, \pi/4)$: go to the direction $\pi/4$ and then move a distance 2 in the opposite direction; see figure 12.1.3. As usual, a negative angle θ means an angle measured clockwise from the positive x-axis. The point in figure 12.1.3 also has coordinates $(2, 5\pi/4)$ and $(2, -3\pi/4)$.

The relationship between rectangular and polar coordinates is quite easy to understand. The point with polar coordinates (r, θ) has rectangular coordinates $x = r\cos\theta$ and $y = r\sin\theta$; this follows immediately from the definition of the sine and cosine functions. Using figure 12.1.3 as an example, the point shown has rectangular coordinates

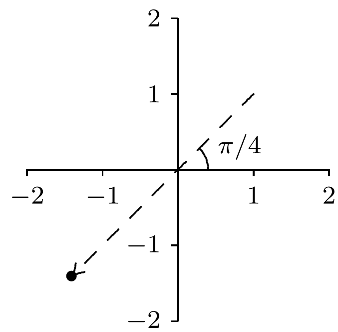

Figure 12.1.3 The point $(-2, \pi/4) = (2, 5\pi/4) = (2, -3\pi/4)$ in polar coordinates.

$x = (-2)\cos(\pi/4) = -\sqrt{2} \approx 1.4142$ and $y = (-2)\sin(\pi/4) = -\sqrt{2}$. This makes it very easy to convert equations from rectangular to polar coordinates.

EXAMPLE 12.1.3 Find the equation of the line $y = 3x + 2$ in polar coordinates. We merely substitute: $r \sin\theta = 3r\cos\theta + 2$, or $r = \dfrac{2}{\sin\theta - 3\cos\theta}$. □

EXAMPLE 12.1.4 Find the equation of the circle $(x - 1/2)^2 + y^2 = 1/4$ in polar coordinates. Again substituting: $(r\cos\theta - 1/2)^2 + r^2\sin^2\theta = 1/4$. A bit of algebra turns this into $r = \cos(t)$. You should try plotting a few (r, θ) values to convince yourself that this makes sense. □

EXAMPLE 12.1.5 Graph the polar equation $r = \theta$. Here the distance from the origin exactly matches the angle, so a bit of thought makes it clear that when $\theta \geq 0$ we get the spiral of Archimedes in figure 12.1.4. When $\theta < 0$, r is also negative, and so the full graph is the right hand picture in the figure. □

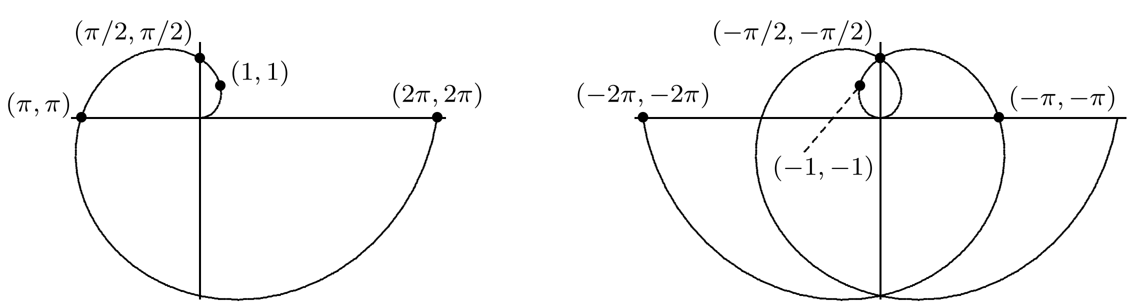

Figure 12.1.4 The spiral of Archimedes and the full graph of $r = \theta$.

Converting polar equations to rectangular equations can be somewhat trickier, and graphing polar equations directly is also not always easy.

EXAMPLE 12.1.6 Graph $r = 2\sin\theta$. Because the sine is periodic, we know that we will get the entire curve for values of θ in $[0, 2\pi)$. As θ runs from 0 to $\pi/2$, r increases from 0 to 2. Then as θ continues to π, r decreases again to 0. When θ runs from π to 2π, r is negative, and it is not hard to see that the first part of the curve is simply traced out again, so in fact we get the whole curve for values of θ in $[0, \pi)$. Thus, the curve looks something like figure 12.1.5. Now, this suggests that the curve could possibly be a circle, and if it is, it would have to be the circle $x^2 + (y-1)^2 = 1$. Having made this guess, we can easily check it. First we substitute for x and y to get $(r\cos\theta)^2 + (r\sin\theta - 1)^2 = 1$; expanding and simplifying does indeed turn this into $r = 2\sin\theta$. □

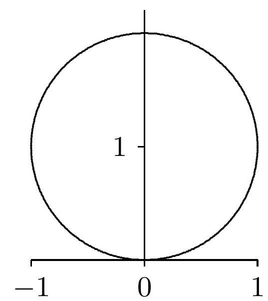

Figure 12.1.5 Graph of $r = 2\sin\theta$.

Exercises 12.1.

1. Plot these polar coordinate points on one graph: $(2, \pi/3)$, $(-3, \pi/2)$, $(-2, -\pi/4)$, $(1/2, \pi)$, $(1, 4\pi/3)$, $(0, 3\pi/2)$.

Find an equation in polar coordinates that has the same graph as the given equation in rectangular coordinates.

2. $y = 3x \Rightarrow$

3. $y = -4 \Rightarrow$

4. $xy^2 = 1 \Rightarrow$

5. $x^2 + y^2 = 5 \Rightarrow$

6. $y = x^3 \Rightarrow$

7. $y = \sin x \Rightarrow$

8. $y = 5x + 2 \Rightarrow$

9. $x = 2 \Rightarrow$

10. $y = x^2 + 1 \Rightarrow$

11. $y = 3x^2 - 2x \Rightarrow$

12. $y = x^2 + y^2 \Rightarrow$

Sketch the curve.

13. $r = \cos\theta$

14. $r = \sin(\theta + \pi/4)$

15. $r = -\sec\theta$

16. $r = \theta/2$, $\theta \geq 0$

17. $r = 1 + \theta^1/\pi^2$

18. $r = \cot\theta\csc\theta$

19. $r = \dfrac{1}{\sin\theta + \cos\theta}$

20. $r^2 = -2\sec\theta\csc\theta$

In the exercises below, find an equation in rectangular coordinates that has the same graph as the given equation in polar coordinates.

21. $r = \sin(3\theta) \Rightarrow$

22. $r = \sin^2\theta \Rightarrow$

23. $r = \sec\theta\csc\theta \Rightarrow$

24. $r = \tan\theta \Rightarrow$

12.2 Slopes in polar coordinates

When we describe a curve using polar coordinates, it is still a curve in the x-y plane. We would like to be able to compute slopes and areas for these curves using polar coordinates.

We have seen that $x = r\cos\theta$ and $y = r\sin\theta$ describe the relationship between polar and rectangular coordinates. If in turn we are interested in a curve given by $r = f(\theta)$, then we can write $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$, describing x and y in terms of θ alone. The first of these equations describes θ implicitly in terms of x, so using the chain rule we may compute

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx}.$$

Since $d\theta/dx = 1/(dx/d\theta)$, we can instead compute

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}.$$

EXAMPLE 12.2.1 Find the points at which the curve given by $r = 1 + \cos\theta$ has a vertical or horizontal tangent line. Since this function has period 2π, we may restrict our attention to the interval $[0, 2\pi)$ or $(-\pi, \pi]$, as convenience dictates. First, we compute the slope:

$$\frac{dy}{dx} = \frac{(1+\cos\theta)\cos\theta - \sin\theta\sin\theta}{-(1+\cos\theta)\sin\theta - \sin\theta\cos\theta} = \frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\sin\theta\cos\theta}.$$

This fraction is zero when the numerator is zero (and the denominator is not zero). The numerator is $2\cos^2\theta + \cos\theta - 1$ so by the quadratic formula

$$\cos\theta = \frac{-1 \pm \sqrt{1 + 4\cdot 2}}{4} = -1 \quad \text{or} \quad \frac{1}{2}.$$

This means θ is π or $\pm\pi/3$. However, when $\theta = \pi$, the denominator is also 0, so we cannot conclude that the tangent line is horizontal.

Setting the denominator to zero we get

$$\begin{aligned} -\sin\theta - 2\sin\theta\cos\theta &= 0 \\ \sin\theta(1 + 2\cos\theta) &= 0, \end{aligned}$$

so either $\sin\theta = 0$ or $\cos\theta = -1/2$. The first is true when θ is 0 or π, the second when θ is $2\pi/3$ or $4\pi/3$. However, as above, when $\theta = \pi$, the numerator is also 0, so we cannot conclude that the tangent line is vertical. Figure 12.2.1 shows points corresponding to θ equal to 0, $\pm\pi/3$, $2\pi/3$ and $4\pi/3$ on the graph of the function. Note that when $\theta = \pi$ the curve hits the origin and does not have a tangent line. □

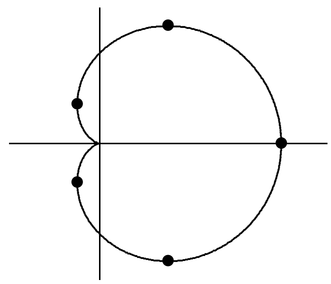

Figure 12.2.1 Points of vertical and horizontal tangency for $r = 1 + \cos\theta$.

We know that the second derivative $f''(x)$ is useful in describing functions, namely, in describing concavity. We can compute $f''(x)$ in terms of polar coordinates as well. We already know how to write $dy/dx = y'$ in terms of θ, then

$$\frac{d}{dx}\frac{dy}{dx} = \frac{dy'}{dx} = \frac{dy'}{d\theta}\frac{d\theta}{dx} = \frac{dy'/d\theta}{dx/d\theta}.$$

EXAMPLE 12.2.2 We find the second derivative for the cardioid $r = 1 + \cos\theta$:

$$\begin{aligned}\frac{d}{d\theta}\frac{\cos\theta + \cos^2\theta - \sin^2\theta}{-\sin\theta - 2\sin\theta\cos\theta} \cdot \frac{1}{dx/d\theta} = \cdots &= \frac{3(1+\cos\theta)}{(\sin\theta + 2\sin\theta\cos\theta)^2} \cdot \frac{1}{-(\sin\theta + 2\sin\theta\cos\theta)} \\ &= \frac{-3(1+\cos\theta)}{(\sin\theta + 2\sin\theta\cos\theta)^3}.\end{aligned}$$

The ellipsis here represents rather a substantial amount of algebra. We know from above that the cardioid has horizontal tangents at $\pm\pi/3$; substituting these values into the second derivative we get $y''(\pi/3) = -\sqrt{3}/2$ and $y''(-\pi/3) = \sqrt{3}/2$, indicating concave down and concave up respectively. This agrees with the graph of the function. □

Exercises 12.2.

Compute $y' = dy/dx$ and $y'' = d^2y/dx^2$.

1. $r = \theta \Rightarrow$

2. $r = 1 + \sin\theta \Rightarrow$

3. $r = \cos\theta \Rightarrow$

4. $r = \sin\theta \Rightarrow$

5. $r = \sec\theta \Rightarrow$

6. $r = \sin(2\theta) \Rightarrow$

Sketch the curves over the interval $[0, 2\pi]$ unless otherwise stated.

7. $r = \sin\theta + \cos\theta$

8. $r = 2 + 2\sin\theta$

9. $r = \dfrac{3}{2} + \sin\theta$

10. $r = 2 + \cos\theta$

11. $r = \dfrac{1}{2} + \cos\theta$

12. $r = \cos(\theta/2), 0 \le \theta \le 4\pi$

13. $r = \sin(\theta/3), 0 \le \theta \le 6\pi$

14. $r = \sin^2\theta$

15. $r = 1 + \cos^2(2\theta)$

16. $r = \sin^2(3\theta)$

17. $r = \tan\theta$

18. $r = \sec(\theta/2), 0 \le \theta \le 4\pi$

19. $r = 1 + \sec\theta$

20. $r = \dfrac{1}{1 - \cos\theta}$

21. $r = \dfrac{1}{1 + \sin\theta}$

22. $r = \cot(2\theta)$

23. $r = \pi/\theta, 0 \le \theta \le \infty$

24. $r = 1 + \pi/\theta, 0 \le \theta \le \infty$

25. $r = \sqrt{\pi/\theta}, 0 \le \theta \le \infty$

12.3 Areas in polar coordinates

We can use the equation of a curve in polar coordinates to compute some areas bounded by such curves. The basic approach is the same as with any application of integration: find an approximation that approaches the true value. For areas in rectangular coordinates, we approximated the region using rectangles; in polar coordinates, we use sectors of circles, as depicted in figure 12.3.1. Recall that the area of a sector of a circle is $\alpha r^2/2$, where α is the angle subtended by the sector. If the curve is given by $r = f(\theta)$, and the angle subtended by a small sector is $\Delta\theta$, the area is $(\Delta\theta)(f(\theta))^2/2$. Thus we approximate the total area as

$$\sum_{i=0}^{n-1} \frac{1}{2} f(\theta_i)^2 \;\Delta\theta.$$

In the limit this becomes

$$\int_a^b \frac{1}{2} f(\theta)^2 \; d\theta.$$

EXAMPLE 12.3.1 We find the area inside the cardioid $r = 1 + \cos\theta$.

$$\int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2 \; d\theta = \frac{1}{2}\int_0^{2\pi} 1+2\cos\theta+\cos^2\theta \; d\theta = \frac{1}{2}\left.\left(\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\right|_0^{2\pi} = \frac{3\pi}{2}.$$

□

EXAMPLE 12.3.2 We find the area between the circles $r = 2$ and $r = 4\sin\theta$, as shown in figure 12.3.2. The two curves intersect where $2 = 4\sin\theta$, or $\sin\theta = 1/2$, so $\theta = \pi/6$ or

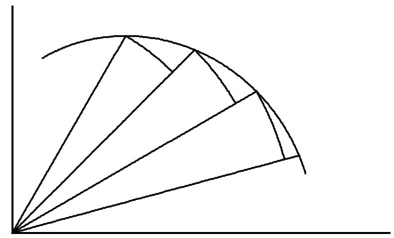

Figure 12.3.1 Approximating area by sectors of circles.

$5\pi/6$. The area we want is then

$$\frac{1}{2}\int_{\pi/6}^{5\pi/6} 16\sin^2\theta - 4\,d\theta = \frac{4}{3}\pi + 2\sqrt{3}.$$

$\square$

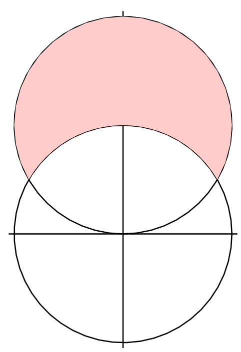

Figure 12.3.2 An area between curves.

This example makes the process appear more straightforward than it is. Because points have many different representations in polar coordinates, it is not always so easy to identify points of intersection.

EXAMPLE 12.3.3 We find the shaded area in the first graph of figure 12.3.3 as the difference of the other two shaded areas. The cardioid is $r = 1 + \sin\theta$ and the circle is $r = 3\sin\theta$. We attempt to find points of intersection:

$$\begin{aligned} 1 + \sin\theta &= 3\sin\theta \\ 1 &= 2\sin\theta \\ 1/2 &= \sin\theta. \end{aligned}$$

This has solutions $\theta = \pi/6$ and $5\pi/6$; $\pi/6$ corresponds to the intersection in the first quadrant that we need. Note that no solution of this equation corresponds to the intersection point at the origin, but fortunately that one is obvious. The cardioid goes through the origin when $\theta = -\pi/2$; the circle goes through the origin at multiples of π, starting with 0.

Now the larger region has area

$$\frac{1}{2}\int_{-\pi/2}^{\pi/6}(1+\sin\theta)^2\,d\theta = \frac{\pi}{2} - \frac{9}{16}\sqrt{3}$$

and the smaller has area

$$\frac{1}{2}\int_{0}^{\pi/6}(3\sin\theta)^2\,d\theta = \frac{3\pi}{8} - \frac{9}{16}\sqrt{3}$$

so the area we seek is $\pi/8$. □

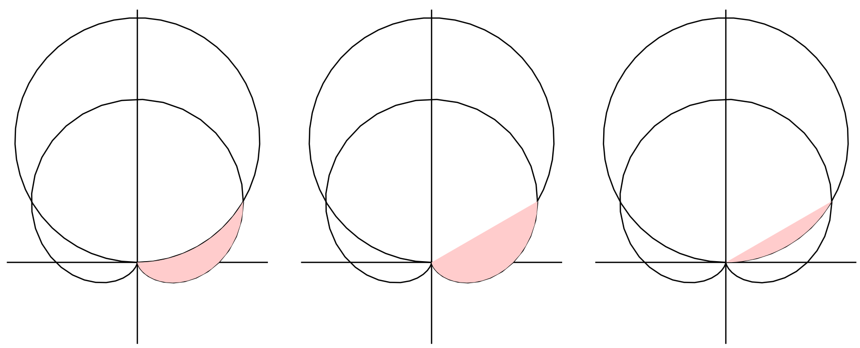

Figure 12.3.3 An area between curves.

Exercises 12.3.

Find the area enclosed by the curve.

1. $r = \sqrt{\sin\theta} \Rightarrow$

2. $r = 2 + \cos\theta \Rightarrow$

3. $r = \sec\theta, \pi/6 \le \theta \le \pi/3 \Rightarrow$

4. $r = \cos\theta, 0 \le \theta \le \pi/3 \Rightarrow$

5. $r = 2a\cos\theta, a > 0 \Rightarrow$

6. $r = 4 + 3\sin\theta \Rightarrow$

7. Find the area inside the loop formed by $r = \tan(\theta/2)$. $\Rightarrow$

8. Find the area inside one loop of $r = \cos(3\theta)$. $\Rightarrow$

9. Find the area inside one loop of $r = \sin^2\theta$. $\Rightarrow$

10. Find the area inside the small loop of $r = (1/2) + \cos\theta$. $\Rightarrow$

11. Find the area inside $r = (1/2) + \cos\theta$, including the area inside the small loop. $\Rightarrow$

12. Find the area inside one loop of $r^2 = \cos(2\theta)$. $\Rightarrow$

13. Find the area enclosed by $r = \tan\theta$ and $r = \dfrac{\csc\theta}{\sqrt{2}}$. $\Rightarrow$

14. Find the area inside $r = 2\cos\theta$ and outside $r = 1$. ⇒

15. Find the area inside $r = 2\sin\theta$ and above the line $r = (3/2)\csc\theta$. ⇒

16. Find the area inside $r = \theta$, $0 \le \theta \le 2\pi$. ⇒

17. Find the area inside $r = \sqrt{\theta}$, $0 \le \theta \le 2\pi$. ⇒

18. Find the area inside both $r = \sqrt{3}\cos\theta$ and $r = \sin\theta$. ⇒

19. Find the area inside both $r = 1 - \cos\theta$ and $r = \cos\theta$. ⇒

20. The center of a circle of radius 1 is on the circumference of a circle of radius 2. Find the area of the region inside both circles. ⇒

21. Find the shaded area in figure 12.3.4. The curve is $r = \theta$, $0 \le \theta \le 3\pi$. ⇒

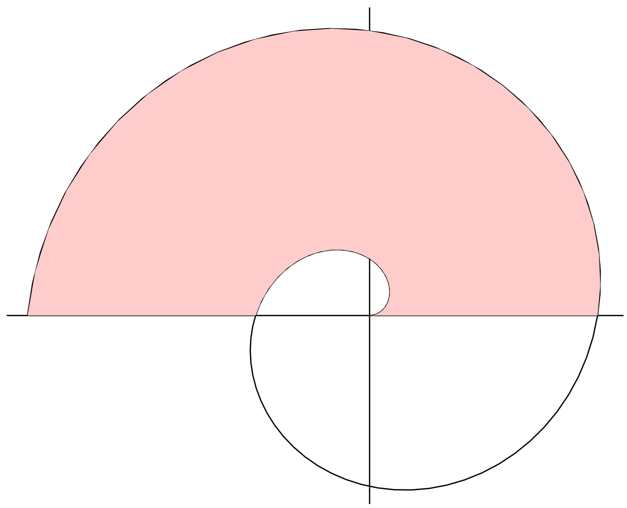

Figure 12.3.4 An area bounded by the spiral of Archimedes.

12.4 Parametric Equations

When we computed the derivative dy/dx using polar coordinates, we used the expressions $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$. These two equations completely specify the curve, though the form $r = f(\theta)$ is simpler. The expanded form has the virtue that it can easily be generalized to describe a wider range of curves than can be specified in rectangular or polar coordinates.

Suppose $f(t)$ and $g(t)$ are functions. Then the equations $x = f(t)$ and $y = g(t)$ describe a curve in the plane. In the case of the polar coordinates equations, the variable t is replaced by θ which has a natural geometric interpretation. But t in general is simply an arbitrary variable, often called in this case a **parameter**, and this method of specifying a curve is known as **parametric equations**. One important interpretation of t is *time*. In this interpretation, the equations $x = f(t)$ and $y = g(t)$ give the position of an object at time t.

EXAMPLE 12.4.1 Describe the path of an object that moves so that its position at time t is given by $x = \cos t$, $y = \cos^2 t$. We see immediately that $y = x^2$, so the path lies on this parabola. The path is not the entire parabola, however, since $x = \cos t$ is always between -1 and 1. It is now easy to see that the object oscillates back and forth on the parabola between the endpoints $(1, 1)$ and $(-1, 1)$, and is at point $(1, 1)$ at time $t = 0$. $\square$

It is sometimes quite easy to describe a complicated path in parametric equations when rectangular and polar coordinate expressions are difficult or impossible to devise.

EXAMPLE 12.4.2 A wheel of radius 1 rolls along a straight line, say the x-axis. A point on the rim of the wheel will trace out a curve, called a cycloid. Assume the point starts at the origin; find parametric equations for the curve.

Figure 12.4.1 illustrates the generation of the curve (click on the AP link to see an animation). The wheel is shown at its starting point, and again after it has rolled through about 490 degrees. We take as our parameter t the angle through which the wheel has turned, measured as shown clockwise from the line connecting the center of the wheel to the ground. Because the radius is 1, the center of the wheel has coordinates $(t, 1)$. We seek to write the coordinates of the point on the rim as $(t + \Delta x, 1 + \Delta y)$, where Δx and Δy are as shown in figure 12.4.2. These values are nearly the sine and cosine of the angle t, from the unit circle definition of sine and cosine. However, some care is required because we are measuring t from a nonstandard starting line and in a clockwise direction, as opposed to the usual counterclockwise direction. A bit of thought reveals that $\Delta x = -\sin t$ and $\Delta y = -\cos t$. Thus the parametric equations for the cycloid are $x = t - \sin t$, $y = 1 - \cos t$. $\square$

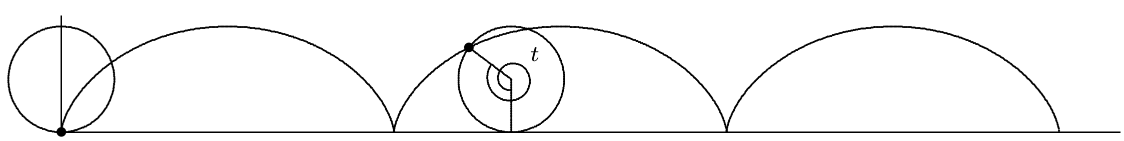

Figure 12.4.1 A cycloid. (AP)

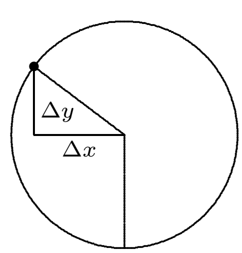

Figure 12.4.2 The wheel.

Exercises 12.4.

1. What curve is described by $x = t^2$, $y = t^4$? If t is interpreted as time, describe how the object moves on the curve.

2. What curve is described by $x = 3\cos t$, $y = 3\sin t$? If t is interpreted as time, describe how the object moves on the curve.

3. What curve is described by $x = 3\cos t$, $y = 2\sin t$? If t is interpreted as time, describe how the object moves on the curve.

4. What curve is described by $x = 3\sin t$, $y = 3\cos t$? If t is interpreted as time, describe how the object moves on the curve.

5. Sketch the curve described by $x = t^3 - t$, $y = t^2$. If t is interpreted as time, describe how the object moves on the curve.

6. A wheel of radius 1 rolls along a straight line, say the x-axis. A point P is located halfway between the center of the wheel and the rim; assume P starts at the point $(0, 1/2)$. As the wheel rolls, P traces a curve; find parametric equations for the curve. ⇒

7. A wheel of radius 1 rolls around the outside of a circle of radius 3. A point P on the rim of the wheel traces out a curve called a **hypercycloid**, as indicated in figure 12.4.3. Assuming P starts at the point $(3, 0)$, find parametric equations for the curve. ⇒

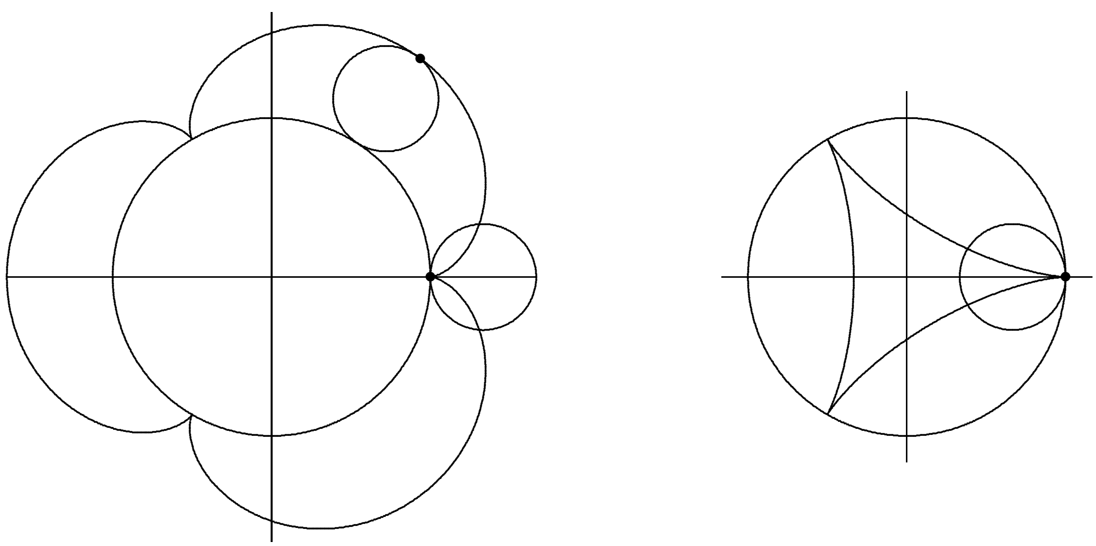

Figure 12.4.3 A hypercycloid and a hypocycloid.

8. A wheel of radius 1 rolls around the inside of a circle of radius 3. A point P on the rim of the wheel traces out a curve called a **hypocycloid**, as indicated in figure 12.4.3. Assuming P starts at the point $(3, 0)$, find parametric equations for the curve. ⇒

9. An **involute** of a circle is formed as follows: Imagine that a long (that is, infinite) string is wound tightly around a circle, and that you grasp the end of the string and begin to unwind it, keeping the string taut. The end of the string traces out the involute. Find parametric equations for this curve, using a circle of radius 1, and assuming that the string unwinds counter-clockwise and the end of the string is initially at $(1, 0)$. Figure 12.4.4 shows part of the curve; the dotted lines represent the string at a few different times. ⇒

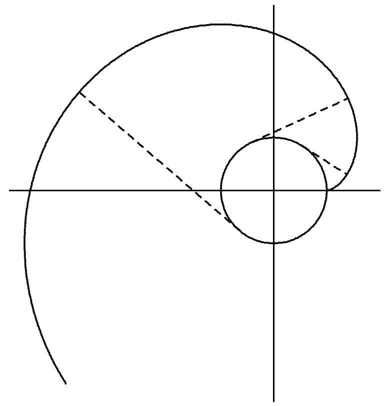

Figure 12.4.4 An involute of a circle.

12.5 Calculus with Parametric Equations

We have already seen how to compute slopes of curves given by parametric equations—it is how we computed slopes in polar coordinates.

EXAMPLE 12.5.1 Find the slope of the cycloid $x = t - \sin t$, $y = 1 - \cos t$. We compute $x' = 1 - \cos t$, $y' = \sin t$, so

$$\frac{dy}{dx} = \frac{\sin t}{1 - \cos t}.$$

Note that when t is an odd multiple of π, like π or 3π, this is $(0/2) = 0$, so there is a horizontal tangent line, in agreement with figure 12.4.1. At even multiples of π, the fraction is $0/0$, which is undefined. The figure shows that there is no tangent line at such points. □

Areas can be a bit trickier with parametric equations, depending on the curve and the area desired. We can potentially compute areas between the curve and the x-axis quite easily.

EXAMPLE 12.5.2 Find the area under one arch of the cycloid $x = t - \sin t$, $y = 1 - \cos t$. We would like to compute

$$\int_0^{2\pi} y\,dx,$$

but we do not know y in terms of x. However, the parametric equations allow us to make a substitution: use $y = 1 - \cos t$ to replace y, and compute $dx = (1 - \cos t)\,dt$. Then the integral becomes

$$\int_0^{2\pi} (1 - \cos t)(1 - \cos t)\,dt = 3\pi.$$

Note that we need to convert the original x limits to t limits using $x = t - \sin t$. When $x = 0$, $t = \sin t$, which happens only when $t = 0$. Likewise, when $x = 2\pi$, $t - 2\pi = \sin t$ and $t = 2\pi$. Alternately, because we understand how the cycloid is produced, we can see directly that one arch is generated by $0 \le t \le 2\pi$. In general, of course, the t limits will be different than the x limits. $\square$

This technique will allow us to compute some quite interesting areas, as illustrated by the exercises.

As a final example, we see how to compute the length of a curve given by parametric equations. Section 11.4 investigates arc length for functions given as y in terms of x, and develops the formula for length:

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Using some properties of derivatives, including the chain rule, we can convert this to use parametric equations $x = f(t)$, $y = g(t)$:

$$\begin{aligned}\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 \left(\frac{dy}{dx}\right)^2}\, \frac{dt}{dx}\, dx \\ &= \int_u^v \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_u^v \sqrt{(f'(t))^2 + (g'(t))^2}\, dt.\end{aligned}$$

Here u and v are the t limits corresponding to the x limits a and b.

EXAMPLE 12.5.3 Find the length of one arch of the cycloid. From $x = t - \sin t$, $y = 1 - \cos t$, we get the derivatives $f' = 1 - \cos t$ and $g' = \sin t$, so the length is

$$\int_0^{2\pi} \sqrt{(1 - \cos t)^2 + \sin^2 t}\, dt = \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt.$$

Now we use the formula $\sin^2(t/2) = (1 - \cos(t))/2$ or $4\sin^2(t/2) = 2 - 2\cos t$ to get

$$\int_0^{2\pi} \sqrt{4\sin^2(t/2)}\, dt.$$

Since $0 \le t \le 2\pi$, $\sin(t/2) \ge 0$, so we can rewrite this as

$$\int_0^{2\pi} 2\sin(t/2)\, dt = 8.$$

$\square$

Exercises 12.5.

1. Consider the curve of exercise 6 in section 12.4. Find all values of t for which the curve has a horizontal tangent line. ⇒

2. Consider the curve of exercise 6 in section 12.4. Find the area under one arch of the curve. ⇒

3. Consider the curve of exercise 6 in section 12.4. Set up an integral for the length of one arch of the curve. ⇒

4. Consider the hypercycloid of exercise 7 in section 12.4. Find all points at which the curve has a horizontal tangent line. ⇒

5. Consider the hypercycloid of exercise 7 in section 12.4. Find the area between the large circle and one arch of the curve. ⇒

6. Consider the hypercycloid of exercise 7 in section 12.4. Find the length of one arch of the curve. ⇒

7. Consider the hypocycloid of exercise 8 in section 12.4. Find the area inside the curve. ⇒

8. Consider the hypocycloid of exercise 8 in section 12.4. Find the length of one arch of the curve. ⇒

9. Recall the involute of a circle from exercise 9 in section 12.4. Find the point in the first quadrant in figure 12.4.4 at which the tangent line is vertical. ⇒

10. Recall the involute of a circle from exercise 9 in section 12.4. Instead of an infinite string, suppose we have a string of length π attached to the unit circle at $(-1,0)$, and initially laid around the top of the circle with its end at $(1,0)$. If we grasp the end of the string and begin to unwind it, we get a piece of the involute, until the string is vertical. If we then keep the string taut and continue to rotate it counter-clockwise, the end traces out a semi-circle with center at $(-1,0)$, until the string is vertical again. Continuing, the end of the string traces out the mirror image of the initial portion of the curve; see figure 12.5.1. Find the area of the region inside this curve and outside the unit circle. ⇒

11. Find the length of the curve from the previous exercise, shown in figure 12.5.1. ⇒

12. Find the length of the spiral of Archimedes (figure 12.3.4) for $0 \le \theta \le 2\pi$. ⇒

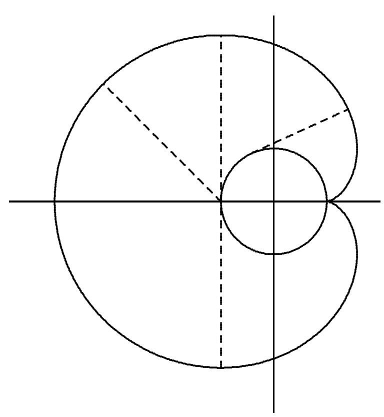

Figure 12.5.1 A region formed by the end of a string.