11 More Applications of Integration

11.1 Center of Mass

Suppose a beam is 10 meters long, and that there are three weights on the beam: a 10 kilogram weight 3 meters from the left end, a 5 kilogram weight 6 meters from the left end, and a 4 kilogram weight 8 meters from the left end. Where should a fulcrum be placed so that the beam balances? Let’s assign a scale to the beam, from 0 at the left end to 10 at the right, so that we can denote locations on the beam simply as x coordinates; the weights are at x = 3, x = 6, and x = 8, as in figure 11.1.1.

<table>
<thead>
<tr>
<th>x</th>
<th>3</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>10</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

Figure 11.1.1 A beam with three masses.

Suppose to begin with that the fulcrum is placed at x = 5. What will happen? Each weight applies a force to the beam that tends to rotate it around the fulcrum; this effect is measured by a quantity called torque, proportional to the mass times the distance from the fulcrum. Of course, weights on different sides of the fulcrum rotate the beam in opposite directions. We can distinguish this by using a signed distance in the formula for torque. So with the fulcrum at 5, the torques induced by the three weights will be proportional to (3 - 5)10 = -20, (6 - 5)5 = 5, and (8 - 5)4 = 12. For the beam to balance, the sum of the torques must be zero; since the sum is -20 + 5 + 12 = -3, the beam rotates counter-clockwise, and to get the beam to balance we need to move the fulcrum to the left. To calculate exactly where the fulcrum should be, we let \( \bar{x} \) denote the location of the fulcrum when the beam is in balance. The total torque on the beam is then \( (3 - 5)10 + (6 - 5)5 + (8 - 5)4 = 92 - 19 \). Since the beam balances at \( \bar{x} \) it must be that 92 - 19 = 0 or \( \bar{x} = 92/19 \approx 4.84 \), that is, the fulcrum should be placed at \( x = 92/19 \) to balance the beam.

Now suppose that we have a beam with varying density—some portions of the beam contain more mass than other portions of the same size. We want to figure out where to put the fulcrum so that the beam balances.

EXAMPLE 11.1.1 Suppose the beam is 10 meters long and that the density is \( 1 + x \) kilograms per meter at location x on the beam. To approximate the solution, we can think of the beam as a sequence of weights “car” a beam. For example, we can think of the portion of the beam between \( x = 0 \) and \( x = 1 \) as a weight sitting at \( x = 0 \), the portion between \( x = 1 \) and \( x = 2 \) as a weight sitting at \( x = 1 \), and so on, as indicated in figure 11.1.2. We then approximate the mass of the weights by assuming that each portion of the beam has constant density. So the mass of the first weight is approximately \( m_0 = (1 + 0)1 = 1 \) kilograms, namely, \( (1 + 0) \) kilograms per meter times 1 meter. The second weight is \( m_0 = (1 + 1)1 = 2 \) kilograms, and so on to the tenth weight with \( m_9 = (1 + 9)1 = 10 \) kilograms. So in this case the total torque is

\[
(0 - 1)m_0 + (1 - 2)m_1 + \cdots + (9 - 10)m_9 = (0 - 1)(1 + 1)(1 + 2) + (0 - 2)(1 + 2)(1 + 3) + \cdots + (0 - 10)(1 + 9)(1 + 10)
\]

If we set this to zero and solve for \( \bar{x} \) we get \( \bar{x} = 6 \). In general, if we divide the beam into n portions, the mass of weight number i will be \( m_i = (1 + x_i)(x_{i+1} - x_i) = (1 + x_i)\Delta x \) and the torque induced by weight number i will be \( (x_i - \bar{x})m_i = (x_i - \bar{x})(1 + x_i)\Delta x \). The total torque is then

\[
(1 + x_0)(x_1 - \bar{x}) + (1 + x_1)(x_2 - \bar{x}) + \cdots + (1 + x_{n-1})(x_n - \bar{x}) = \sum_{i=0}^{n-1} (1 + x_i)(x_{i+1} - x_i)(x_i - \bar{x})
\]

11.2 Center of Mass

Suppose a flat plate of uniform density has the shape contained by

\[
y = x^2, \quad y = 1, \quad x = 0, \quad \text{in the first quadrant. Find the center of mass. (Since the density is constant, the center of mass depends only on the shape of the plate, not the density, or in other words, this is a purely geometric quantity. In such a case the center of mass is called the centroid.)}
\]

This is a two dimensional problem, but it can be solved as if it were two one dimensional problems: we need to find the x and y coordinates of the center of mass, \( \bar{x} \) and \( \bar{y} \), and fortunately we can do these independently. Imagine looking at the plate edge on, from below the x-axis. The plate will appear to be a beam, and the mass of a short section of

\[
M = \int_{0}^{1} x^2 \, dx = \frac{1}{3} x^3 \bigg|_{0}^{1} = \frac{1}{3} \quad \text{and} \quad M = \int_{0}^{1} y \, dy = \frac{1}{2} y^2 \bigg|_{0}^{1} = \frac{1}{2} = 60
\]

\[
\bar{y} = \frac{M_y}{M} = \frac{4776}{3} \approx 63.9
\]
11.1 Center of Mass

The “beam,” say between $x_1$ and $x_{i+1}$, is the mass of a strip of the plate between $x_1$ and $x_{i+1}$. See figure 11.1.3 showing the plate from above and as it appears edge on. Since the plate has uniform density we may as well assume that $\sigma = 1$. Then the mass of the plate between $x_1$ and $x_{i+1}$ is approximately $m_i = \sigma(1 - x_i^2)dx = (1 - x_i^2)\Delta x$. Now we can compute the moment around the $y$-axis:

$$M_y = \int_{x_1}^{x_{i+1}} x(1 - x^2) \, dx = \frac{1}{4}$$

and the total mass

$$M = \int_{x_1}^{x_{i+1}} (1 - x^2) \, dx = \frac{2}{3}$$

and finally

$$\bar{x} = \frac{3}{4} \quad \bar{y} = \frac{3}{8}$$

Next we do the same thing to find $\bar{y}$. The mass of the plate between $y_i$ and $y_{i+1}$ is approximately $m_i = \sqrt{2}dy$, so

$$M_y = \int_{y_i}^{y_{i+1}} y \sigma \sqrt{2} \, dy = \frac{2}{5}$$

and

$$\bar{y} = \frac{2}{5} + \frac{3}{7} = \frac{1}{2}$$

since the total mass $M$ is the same. The center of mass is shown in figure 11.1.3.

**EXAMPLE 11.1.4** Find the center of mass of a thin, uniform plate whose shape is the region between $y = \cos x$ and the $x$-axis between $x = -\pi/2$ and $x = \pi/2$. It is clear that $\bar{x} = 0$, but for practice let’s compute it anyway. We will need the total mass, so we compute it first:

$$M = \int_{-\pi/2}^{\pi/2} \cos x \, dx = \sin \frac{\pi^2}{\sqrt{2}} - \frac{\pi^2}{\sqrt{2}} = 2.$$

The moment around the $y$-axis is

$$M_y = \int_{-\pi/2}^{\pi/2} x \cos x \, dx = x \cos x + x \sin x \bigg|_{-\pi/2}^{\pi/2} = \frac{\pi^2}{\sqrt{2}} - \frac{\pi^2}{\sqrt{2}} = 0$$

and the moment around the $x$-axis is

$$M_x = \int_{-\pi/2}^{\pi/2} y \cdot 2 \arccos y \sqrt{2} \, dy = 2 \arccos y \sqrt{2} \left( \frac{y}{2} \right) \bigg|_0^{\sqrt{2}} = \frac{\pi^2}{\sqrt{2}} - \frac{\pi^2}{\sqrt{2}} = 0.$$

Thus

$$\bar{x} = \frac{0}{\pi} \quad \bar{y} = \frac{\pi}{8} \approx 0.393.$$

11.2 Kinetic energy; improper integrals

**Recall example 8.5.3 in which we computed the work required to lift an object from the surface of the earth to some large distance $D$. Since $F = k/x^2$ we computed**

$$\int_0^D \frac{k}{x^2} \, dx = -\frac{k}{D} + \frac{k}{1}$$

We noticed that as $D$ increases, $k/D$ decreases to zero so that the amount of work increases to $k/1$. More precisely,

$$\lim_{D \to \infty} \int_0^D \frac{k}{x^2} \, dx = \lim_{D \to \infty} -\frac{k}{D} + \frac{k}{1} = \frac{k}{1}$$

We might reasonably describe this calculation as computing the amount of work required to lift the object “to infinity,” and abbreviate the limit as

$$\lim_{D \to \infty} \int_0^D \frac{k}{x^2} \, dx = \int_{-\infty}^{1} \frac{k}{x^2} \, dx.$$

Such an integral, with a limit of infinity, is called an **improper integral**. This is a bit unfortunate, since it’s not really “improper” to do this, nor is it really “an integral”—it is an abbreviation for the limit of a particular sort of integral. Nevertheless, we’re stuck with the term, and the operation itself is perfectly legitimate. It may at first seem odd that a finite amount of work is sufficient to lift an object to “infinity,” but sometimes surprising things are nevertheless true, and this is such a case. If the value of an improper integral is a finite number, as in this example, we say that the integral converges, and if not we say that the integral diverges.

Here’s another way, perhaps even more surprising, to interpret this calculation. We know that one interpretation of

$$\int_{-\infty}^{\infty} \frac{1}{x^2} \, dx$$

is the area under $y = 1/x^2$ from $x = 1$ to infinity is finite.
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12. Does ∫∞−∞cos x dx converge or diverge? If it converges, find the value. Also find the Cauchy Principal Value, if it exists. ⇒

13. Suppose the curve y = 1/x is rotated around the x-axis generating a sort of funnel or horn shape, called Gabriel’s horn or Torricelli’s trumpet. Is this the volume of this funnel finite or infinite? If finite, compute the volume. ⇒

14. An officially sanctioned baseball must be between 142 and 149 grams. How much work, in Newton-meters, does it take to throw a ball at 90 miles per hour? At 100.9 mph? According to the Guinness Book of World Records, at http://www.baseball-almanac.com/reports/zh_gsm.shtml, "The greatest reliably recorded speed at which a baseball has been pitched is 100.9 mph by Lynn Nolan Ryan (California Angels) at Anaheim Stadium in California on August 20, 1974." ⇒

### 11.3 Probability

You perhaps have at least a rudimentary understanding of discrete probability, which measures the likelihood of an “event” when there are a finite number of possibilities. For example, when an ordinary six-sided die is rolled, the probability of getting any particular number is 1/6. In general, the probability of an event is the number of ways the event can happen divided by the number of ways that “anything” can happen.

For a slightly more complicated example, consider the case of two six-sided dice. The dice are physically distinct, which means that rolling a 2-5 is different than rolling a 5-2; each is an equally likely event out of a total of 36 ways the dice can land, so each has a probability of 1/36.

Most interesting events are not so simple. More interesting is the probability of rolling a certain sum out of the possibilities 2 through 12. It is clearly not true that all sums are equally likely. For example, when an ordinary six-sided die is rolled, the probability of getting any particular number is 1/6. The number of surfaces for the dart to land, but it is easier to explore the probabilities involved by pretending that the dart can land on any point in the usual x-y plane.

Now suppose that the initial velocity of the object, \(v_0\), is just enough to get it to infinity, that is, just enough so that the object never slows to a stop, but so that its speed decreases to zero, i.e., so that \(v = 0\). Then

\[
\omega = \sqrt{\frac{m}{r^2}} = 11181 \text{ meters per second, or about 40251 kilometers per hour. This speed is called the escape velocity.}
\]

Notice that the mass of the object, \(m\), canceled out at the last step; the escape velocity is the same for all objects. Of course, it takes considerably more energy to get a large object up to 40251 kph than a small one, so it is certainly more difficult to get a large object into deep space than a small one. Also, note that while we have computed the escape velocity for the earth, this speed would not in fact get an object “to infinity” because of the large mass in our neighborhood called the sun. Escape velocity for the sun starting at the distance of the earth from the sun is nearly 4 times the escape velocity we have calculated.

### Exercises 11.2.

1. Is the area under \(y = 1/x\) from 1 to infinity finite or infinite? If finite, compute the area. ⇒

2. Is the area under \(y = 1/x^2\) from 1 to infinity finite or infinite? If finite, compute the area. ⇒

3. Does \(\int_0^\infty x^2 + 2x - 1 dx\) converge or diverge? If it converges, find the value. ⇒

4. Does \(\int_0^\infty 1/\sqrt{x} dx\) converge or diverge? If it converges, find the value. ⇒

5. Does \(\int_0^\infty e^{-x} dx\) converge or diverge? If it converges, find the value. ⇒

6. \(\int_0^\infty (1 - 2x)^{-1/2} dx\) is an improper integral of a slightly different sort. Express it as a limit and determine whether it converges or diverges, if it converges, find the value. ⇒

7. Does \(\int_0^\infty 1/\sqrt{x} dx\) converge or diverge? If it converges, find the value. ⇒

8. Does \(\int_0^\infty x^2 e^{-x} dx\) converge or diverge? If it converges, find the value. ⇒

9. Does \(\int_0^\infty x^{-4} + x^2 dx\) converge or diverge? If it converges, find the value. ⇒

10. Does \(\int_0^\infty x dx\) converge or diverge? If it converges, find the value. Also find the Cauchy Principal Value, if it exists. ⇒

11. Does \(\int_0^\infty \sin x dx\) converge or diverge? If it converges, find the value. Also find the Cauchy Principal Value, if it exists. ⇒
DEFINITION 11.3.1 Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. If \( f(x) \geq 0 \) for every \( x \) and \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \) then \( f \) is a probability density function.

We associate a probability density function with a random variable \( X \) by stipulating that the probability that \( X \) is between \( a \) and \( b \) is \( \int_{a}^{b} f(x) \, dx \). Because of the requirement that the integral from \(-\infty\) to \( \infty \) be 1, all probabilities are less than or equal to 1, and the probability that \( X \) takes on some value between \(-\infty\) and \( \infty \) is 1, as it should be.

EXAMPLE 11.3.2 Consider again the two dice example; we can view it in a way that more resembles the probability density function approach. Consider a random variable \( X \) that takes on any real value with probabilities given by the probability density function in figure 11.3.1. The function \( f \) consists of just the top edges of the rectangles, with vertical sides drawn for clarity; the function is zero below \( 1.5 \) and above \( 12.5 \). The area of each rectangle is the probability of rolling the sum in the middle of the bottom of the rectangle, or

\[
P(n) = \frac{n}{6} e^{-n/2} 
\]

The probability of rolling a 4, 5, or 6 is

\[
P(4) = \frac{4}{6} e^{-4/2} = \frac{2}{3} e^{-2} 
\]

Of course, we could also compute probabilities that don’t make sense in the context of the dice, such as the probability that \( X \) is between 4 and 5.8.

EXAMPLE 11.3.5 The exponential distribution has probability density function

\[
f(x) = \begin{cases} 
0 & x < 0 \\
e^{-c} & x \geq 0
\end{cases}
\]

where \( c \) is a positive constant.

The mean or expected value of a random variable is quite useful, as hinted at in our discussion of dice. Recall that the mean for a discrete random variable is \( E(X) = \sum x \cdot P(x) \). In the more general context we use an integral in place of the sum.

DEFINITION 11.3.6 The mean of a random variable \( X \) with probability density function \( f \) is \( E(X) = \int_{-\infty}^{\infty} x f(x) \, dx \), provided the integral converges.

When the mean exists it is unique, since it is the result of an explicit calculation. The mean does not always exist.

The mean might look familiar; it is essentially identical to the center of mass of a one-dimensional beam, as discussed in section 11.1. The probability density function \( f \) plays the role of the physical density function, but now the “beam” has infinite length. If we consider only a finite portion of the beam, say between \( a \) and \( b \), then the center of mass is

\[
x = \frac{\int_{a}^{b} x f(x) \, dx}{\int_{a}^{b} f(x) \, dx} 
\]

The function

\[
F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t) \, dt
\]

is called the cumulative distribution function or simply (probability) distribution.

EXAMPLE 11.3.3 Suppose that \( a < b \) and

\[
f(x) = \begin{cases} 
\frac{1}{b-a} & 0 \leq x \leq b \\
0 & \text{otherwise}.
\end{cases}
\]

Then \( f(x) \) is the uniform probability density function on \([a, b]\), and the corresponding distribution is the uniform distribution on \([a, b]\).

EXAMPLE 11.3.4 Consider the function \( f(x) = e^{-x^2/2} \). What can we say about \( \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \)?

We cannot find an antiderivative of \( f \), but we can see that this integral is some finite number. Notice that \( 0 < \int_{a}^{b} f(x) \, dx = e^{b^2/2} - e^{a^2/2} \leq e^{b^2/2} \) for \( |x| > 1 \). This implies that the area under \( e^{-x^2/2} \) is less than the area under \( e^{-x^2} \), over the interval \([1, \infty)\). It is easy to compute the latter area, namely

\[
\int_{-\infty}^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}
\]

so

\[
\int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \frac{\sqrt{2\pi}}{\sqrt{e}}
\]

is some finite number smaller than \( 2/\sqrt{\pi} \). Because \( f \) is symmetric around the \( y \)-axis,

\[
\int_{-\infty}^{0} e^{-x^2/2} \, dx = \int_{0}^{\infty} e^{-x^2/2} \, dx.
\]

This means that

\[
\int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \int_{-\infty}^{0} e^{-x^2/2} \, dx + \int_{0}^{\infty} e^{-x^2/2} \, dx = A
\]

for some finite positive number \( A \). Note that if we let \( g(x) = f(x)/A \),

\[
\int_{-\infty}^{\infty} g(x) \, dx = \frac{1}{A} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \frac{1}{A} A = 1
\]

so \( g \) is a probability density function. It turns out to be very useful, and is called the standard normal probability density function or more informally the bell curve.

EXAMPLE 11.3.7 The mean of the standard normal distribution is

\[
\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx.
\]

We compute the two halves:

\[
\int_{-\infty}^{0} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx = \lim_{b \to -\infty} \int_{0}^{b} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx = \frac{1}{\sqrt{\pi}}
\]

and

\[
\int_{0}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx = \lim_{b \to \infty} \int_{0}^{b} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx = \frac{1}{\sqrt{\pi}}
\]

The sum of these is 0, which is the mean.

While the mean is very useful, it typically is not enough information to properly evaluate a situation. For example, suppose we could manufacture an 11-sided die, with the faces numbered 2 through 12 so that each face is equally likely to be down when the die is rolled. The value of a roll is the value on this lower face. Rolling the die gives the same range of values as rolling two ordinary dice, but now each value occurs with probability 1/11. The expected value of a roll is

\[
\frac{2}{11} + \frac{3}{11} + \cdots + \frac{12}{11} = 7.
\]

The mean does not distinguish the two cases, though of course they are quite different.

If \( f \) is a probability density function for a random variable \( X \), with mean \( \mu \), we would like to measure how far a “typical” value of \( X \) is from \( \mu \). One way to measure this distance
is \((X - \mu)^2\), we square the difference so as to measure all distances as positive. To get the typical such squared distance, we compute the mean. For two dice, for example, we get

\[
(2 - 7)^2 + (3 - 7)^2 + \cdots + (7 - 7)^2 + \cdots + (11 - 7)^2 + (12 - 7)^2 = 35.
\]

Because we squared the differences this does not measure the typical distance we seek; if we take the square root of this we do get such a measure, \(\sqrt{35} \approx 5.92\). Doing the computation for the strange 11-sided die we get

\[
(2 - 7)^2 + (3 - 7)^2 + \cdots + (7 - 7)^2 + \cdots + (11 - 7)^2 + (12 - 7)^2 = 350.
\]

with square root approximately 3.67. Comparing 5.92 to 3.67 tells us that the two-dice rolls clump somewhat more closely near 7 than the rolls of the weird die, which of course we already knew because these examples are quite simple.

To perform the same computation for a probability density function the sum is replaced by an integral. This is done in the computation of the mean. The expected value of the squared differences is

\[
V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx,
\]

called the variance. The square root of the variance is the standard deviation, denoted \(\sigma\).

**Example 11.3.8.** We compute the standard deviation of the normal distribution. The variance is

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} \, dx.
\]

To compute the antiderivative, use integration by parts, with \(u = x \) and \(dv = x e^{-x^2/2} \, dx\). This gives

\[
\int x^2 e^{-x^2/2} \, dx = -xe^{-x^2/2} + \int e^{-x^2/2} \, dx.
\]

We cannot do the new integral, but we know its value when the limits are \(-\infty\) to \(\infty\), from our discussion of the standard normal distribution. Thus

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2} \, dx = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2} \, dx = 0 + \frac{1}{\sqrt{2\pi}} \sqrt{\pi} = 1.
\]

The standard deviation is then \(\sqrt{1} = 1\).

### 11.3 Probability

Expect to see the number of defective chips 5 or more away from the expected 10. How about 20? Here we compute

\[
\int_{10}^{\infty} f(x) \, dx + \int_{0}^{10} f(x) \, dx \approx 0.0005.
\]

So there is only a 0.15% chance that the number of defective chips is more than 10 away from the mean; this would typically be interpreted as too suspicious to ignore—it shouldn’t happen if the process is running normally.

The big question, of course, is what level of improbability should trigger concern? It depends to some degree on the application, and in particular on the consequences of getting it wrong in one direction or the other. If we’re wrong, do we lose a little money? A lot of money? Do people die? In general, the standard choices are 5% and 1%. So what we should do is find the number of defective chips that has only, let us say, a 1% chance of occurring under normal circumstances, and use that as the relevant number. In other words, we want to know when

\[
\int_{100}^{\infty} f(x) \, dx + \int_{0}^{100} f(x) \, dx < 0.01.
\]

A bit of trial and error shows that with \(r = 8\) the value is about 0.011, and with \(r = 9\) it is about 0.004, so if the number of defective chips is 19 or more, or 1 or fewer, we should look for problems. If the number is high, we worry that the manufacturing process has a problem, or conversely that the process that tests for defective chips is not working correctly and is flagging good chips as defective. If the number is too low, we suspect that the testing procedure is broken, and is not detecting defective chips.\(\square\)

**Exercises 11.3.**

1. Verify that \(\int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \sqrt{\pi}\).
2. Show that the function in example 11.3.5 is a probability density function. Compute the mean and standard deviation. \(\Rightarrow\)
3. Compute the mean and standard deviation of the uniform distribution on \([a, b]\). (See example 11.3.5.) \(\Rightarrow\)
4. What is the expected value of one roll of a fair six-sided die? \(\Rightarrow\)
5. What is the expected sum of one roll of three fair six-sided dice? \(\Rightarrow\)
6. Let \(\mu \) and \(\sigma \) be real numbers with \(\sigma > 0\). Show that

\[
N(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x - \mu)^2}{2\sigma^2}}
\]

is a probability density function. You will not be able to compute this integral directly, use a substitution to convert the integral into the one from example 11.3.4. The function \(N\)

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**Example 11.3.9.** Here is a simple example showing how these ideas can be useful. Suppose it is known that, in the long run, 1 out of every 100 computer memory chips produced by a certain manufacturing plant is defective when the manufacturing process is running correctly. Suppose 1000 chips are selected at random and 15 of them are defective. This is more than the ‘expected’ number (10), but is it so many that we should suspect that something has gone wrong in the manufacturing process? We are interested in the probability that various numbers of defective chips arise; the probability distribution is discrete: there can only be a whole number of defective chips. But (under reasonable assumptions) the distribution is very close to a normal distribution, namely this one:

\[
f(x) = \frac{1}{\sqrt{2\pi \cdot 1000 \cdot 0.01}} e^{-\frac{(x - 10)^2}{2(1000 \cdot 0.01)}}
\]

which is pictured in figure 11.3.3 (recall that \(\exp(x) = e^x\)).

**Figure 11.3.3 Normal density function for the defective chips example.**

Now how do we measure how unlikely it is that under normal circumstances we would see 15 defective chips? We can’t compute the probability of exactly 15 defective chips, as this would be \(\int_{14.5}^{15.5} f(x) \, dx = 0\). We could compute \(\int_{14.5}^{15.5} f(x) \, dx \approx 0.036\); this means there is only a 3.6% chance that the number of defective chips is 15. (We cannot compute these integrals exactly; computer software has been used to approximate the integral values in this discussion.) But this is misleading: \(\int_{10.5}^{11.5} f(x) \, dx \approx 0.126\), which is larger, certainly, but still small, even for the “most likely” outcome. The most useful question, in most circumstances, is this: how likely is it that the number of defective chips is “far from” the mean? For example, how likely, or unlikely, is it that the number of defective chips is different by 5 or more from the expected value of 10? This is the probability that the number of defective chips is less than 5 or larger than 15, namely

\[
\int_{-\infty}^{5} f(x) \, dx + \int_{15}^{\infty} f(x) \, dx \approx 0.11.
\]

So there is an 11% chance that this happens—not large, but not tiny. Hence the 15 defective chips do not appear to be cause for alarm: about one time in nine we would

### 11.4 Arc Length

Here is another geometric application of the integral: find the length of a portion of a curve. As usual, we need to think about how we might approximate the length, and turn the approximation into an integral.

We already know how to compute one simple arc length, that of a line segment. If the endpoints are \((x_0, y_0)\) and \((x_1, y_1)\) then the length of the segment is the distance between the points, \(\sqrt{(x_1-x_0)^2 + (y_1-y_0)^2}\), from the Pythagorean theorem, as illustrated in figure 11.4.1.

**Figure 11.4.1 The length of a line segment.**
Figure 11.4.2 Approximating arc length with line segments.

Now if the graph of \( f \) is "nice" (say, differentiable) it appears that we can approximate the length of a portion of the curve with line segments, and that as the number of segments increases, and their lengths decrease, the sum of the lengths of the line segments will approach the true arc length; see figure 11.4.2.

Now we need to write a formula for the sum of the lengths of the line segments, in a form that we know becomes an integral in the limit. So we suppose we have divided the interval \([a, b]\) into \( n \) subintervals as usual, each with length \( \Delta x = (b - a)/n \), and endpoints \( a = x_0, x_1, x_2, \ldots, x_n = b \). The length of a typical line segment, joining \((x_i, f(x_i))\) to \((x_{i+1}, f(x_{i+1}))\), is \( \sqrt{\Delta x^2 + (f(x_{i+1}) - f(x_i))^2} \). By the Mean Value Theorem (6.5.2), there is a number \( t_i \) in \((x_i, x_{i+1})\) such that \( f'(t_i) \Delta x = f(x_{i+1}) - f(x_i) \), so the length of the line segment can be written as

\[
\sqrt{(\Delta x)^2 + [f'(t_i)]^2 \Delta x^2} = \sqrt{1 + [f'(t_i)]^2} \Delta x.
\]

The arc length is then

\[
\lim_{n \to \infty} \sum_{i=0}^{n-1} \sqrt{1 + [f'(t_i)]^2} \Delta x = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx.
\]

Note that the sum looks a bit different than others we have encountered, because the expression contains \( t_i \), instead of \( x_i \). In the past we have always used left endpoints (namely, \( x_i \)) to get a representative value of \( f \) on \([x_i, x_{i+1}]\); now we are using a different point, but the principle is the same.

To summarize, to compute the length of a curve on the interval \([a, b]\), we compute the integral

\[
\int_a^b \sqrt{1 + [f'(x)]^2} \, dx.
\]

Unfortunately, integrals of this form are typically difficult or impossible to compute exactly, because usually none of our methods for finding antiderivatives will work. In practice this means that the integral will usually have to be approximated.

As usual, the question is: how might we approximate the surface area? For a surface obtained by rotating a curve around an axis, we can take a polygonal approximation to the curve, as in the last section, and rotate it around the same axis. This gives a surface composed of many "truncated cones," a truncated cone is called a frustum of a cone. Figure 11.5.1 illustrates this approximation.

Figure 11.5.1 Approximating a surface (left) by portions of cones (right).

So we need to be able to compute the area of a frustum of a cone. Since the frustum can be formed by removing a small cone from the top of a larger one, we can compute the desired area if we know the surface area of a cone. Suppose a right circular cone has base radius \( r \) and slant height \( h \). If we cut the cone from the vertex to the base circle and flatten it out, we obtain a sector of a circle with radius \( h \) and arc length \( 2\pi r \), as in figure 11.5.2. The angle at the center, in radians, is \( 2\pi r/h \), and the area of the cone is equal to the area of the sector of the circle. Let \( A \) be the area of the sector; since the area of the entire circle is \( \pi r^2 \), we have

\[
\frac{A}{\pi r^2} = \frac{2\pi r/h}{\pi r} = \frac{2}{\pi} h.
\]

Now suppose we have a frustum of a cone with slant height \( h \) and radii \( r_0 \) and \( r_1 \), as in figure 11.5.3. The area of the entire cone is \( \pi r_1 h_0 + h \), and the area of the small cone is \( \pi r_0 h_0 \), thus, the area of the frustum is \( \pi r_1 (h_0 + h) - \pi r_0 h_0 = \pi [(r_1 - r_0) h_0 + r_1 h] \). By

EXAMPLE 11.4.1 Let \( f(x) = \sqrt{x^2 - x} \), the upper half circle of radius \( r \). The arc length of this curve is half of the circumference, namely \( rt \). Let’s compute this with the arc length formula. The derivative \( f’(x) = -x/\sqrt{x^2 - x} \) so the integral is

\[
\int_a^b \sqrt{1 + [f'(x)]^2} \, dx = \int_a^b \sqrt{1 + \left(\frac{-x}{\sqrt{x^2 - x}}\right)^2} \, dx = \int_a^b \sqrt{1 + \left(\frac{-x}{x}\right)^2} \, dx = \int_a^b \sqrt{1 + \left(\frac{-1}{1}\right)^2} \, dx = \int_a^b 0 \, dx.
\]

Using a trigonometric substitution, we find the antiderivative, namely \( \arcsin(x/r) \). Notice that the integral is improper at both endpoints, as the function \( 1/(r^2 - x^2) \) is undefined when \( x = \pm r \). So we need to compute

\[
\lim_{D \to 0^+} \int_D^b \frac{1}{\sqrt{r^2 - x^2}} \, dx = \lim_{D \to 0^-} \int_D^b \frac{1}{\sqrt{r^2 - x^2}} \, dx.
\]

This is not difficult, and has value \( \pi r \), so the original integral, with the extra \( \pi r \) in the front, has value \( 2\pi r \) as expected.

Exercises 11.4.
1. Find the arc length of \( f(x) = x^{1/3} \) on \([0, 2]\).
2. Find the arc length of \( f(x) = x^2/8 - \ln x \) on \([1, 2]\).
3. Find the arc length of \( f(x) = (1/3)(x^2 + 3)^{3/2} \) on the interval \([0, a]\).
4. Find the arc length of \( f(x) = \ln(x^2 + 3) \) on the interval \([r/4, r/3]\).
5. Let \( a > 0 \). Show that the length of \( y = \cosh x \) on \([0, a]\) is equal to \( \int_0^a \cosh x \, dx \).
6. Find the arc length of \( f(x) = \cos x \) on \([0, 2\pi]\).
7. Set up the integral to find the arc length of \( f(x) \) on the interval \([0, x]\); do not evaluate the integral. If you have access to appropriate software, approximate the value of the integral.
8. Set up the integral to find the arc length of \( y = \arctan x \) on the interval \([2, 3]\); do not evaluate the integral. If you have access to appropriate software, approximate the value of the integral.
9. Find the arc length of \( y = x^2 \) on the interval \([0, 1]\). (This can be done exactly; it is a bit tricky and a bit long.)

11.5 Surface Area
Another geometric question that arises naturally is: "What is the surface area of a volume?" For example, what is the surface area of a sphere? More advanced techniques are required to approach this question in general, but we can compute the areas of some volumes generated by revolution.

Figure 11.5.2 The area of a cone.

Figure 11.5.3 The area of a frustum.
curve is rotated around the x-axis, it forms a frustum of a cone. The area is
\[
2\pi rh - 2\pi \frac{f(x_i) + f(x_{i+1})}{2} \sqrt{1 + (f'(x_i))^2} \Delta x.
\]
Here \(\sqrt{1 + (f'(x_i))^2} \Delta x\) is the length of the line segment, as we found in the previous section. Assuming \(f\) is a continuous function, there must be some \(x_i\) in \([x_i, x_{i+1}]\) such that \((f(x_i) + f(x_{i+1}))/2 = f(x'_i)\), so the approximation for the surface area is
\[
\sum_{i=0}^{n-1} 2\pi f(x_i) \Delta x \approx 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} \, dx.
\]
EXAMPLE 11.5.1 We compute the surface area of a sphere of radius \(r\). The sphere can be obtained by rotating the graph of \(f(x) = \sqrt{r^2 - x^2}\) about the x-axis. The derivative
\[
f'(x) = -x/\sqrt{r^2 - x^2},
\]
so the surface area is given by
\[
A = 2\pi \int_a^b \sqrt{r^2 - x^2} \sqrt{1 + (f'(x))^2} \, dx
\]
\[
= 2\pi \int_a^b \sqrt{r^2 - x^2} \, dx = 2\pi \int_a^b x \, dx = 2\pi \int_a^b 1 \, dx = 4\pi r^2.
\]
EXAMPLE 11.5.2 Consider the ellipse with equation \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\). Generalize the preceding result: rotate the ellipse given by \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) about the x-axis and find the surface area of the resulting ellipsoid. You should consider two cases, when \(a > b\) and when \(a < b\). Compare to the area of a sphere.

11.5 Surface Area

7. Consider the ellipse with equation \(x^2/4 + y^2 = 1\). If the ellipse is rotated around the x-axis it forms an ellipsoid. Compute the surface area.

8. Generalize the preceding result: rotate the ellipse given by \(x^2/a^2 + y^2/b^2 = 1\) about the x-axis and find the surface area of the resulting ellipsoid. You should consider two cases, when \(a > b\) and when \(a < b\). Compare to the area of a sphere.