11

More Applications of Integration

11.1 Center of Mass

Suppose a beam is 10 meters long, and that there are three weights on the beam: a 10 kilogram weight 3 meters from the left end, a 5 kilogram weight 6 meters from the left end, and a 4 kilogram weight 8 meters from the left end. Where should a fulcrum be placed so that the beam balances? Let's assign a scale to the beam, from 0 at the left end to 10 at the right, so that we can denote locations on the beam simply as x coordinates; the weights are at $x = 3$, $x = 6$, and $x = 8$, as in figure 11.1.1.

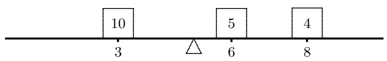

Figure 11.1.1 A beam with three masses.

Suppose to begin with that the fulcrum is placed at $x = 5$. What will happen? Each weight applies a force to the beam that tends to rotate it around the fulcrum; this effect is measured by a quantity called **torque**, proportional to the mass times the distance from the fulcrum. Of course, weights on different sides of the fulcrum rotate the beam in opposite directions. We can distinguish this by using a signed distance in the formula for torque. So with the fulcrum at 5, the torques induced by the three weights will be proportional to $(3-5)10 = -20$, $(6-5)5 = 5$, and $(8-5)4 = 12$. For the beam to balance, the sum of the torques must be zero; since the sum is $-20+5+12 = -3$, the beam rotates counter-clockwise, and to get the beam to balance we need to move the

fulcrum to the left. To calculate exactly where the fulcrum should be, we let $\bar{x}$ denote the location of the fulcrum when the beam is in balance. The total torque on the beam is then $(3-\bar{x})10 + (6-\bar{x})5 + (8-\bar{x})4 = 92 - 19\bar{x}$. Since the beam balances at $\bar{x}$ it must be that $92 - 19\bar{x} = 0$ or $\bar{x} = 92/19 \approx 4.84$, that is, the fulcrum should be placed at $x = 92/19$ to balance the beam.

Now suppose that we have a beam with varying density—some portions of the beam contain more mass than other portions of the same size. We want to figure out where to put the fulcrum so that the beam balances.

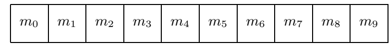

Figure 11.1.2 A solid beam.

EXAMPLE 11.1.1 Suppose the beam is 10 meters long and that the density is $1 + x$ kilograms per meter at location x on the beam. To approximate the solution, we can think of the beam as a sequence of weights "on" a beam. For example, we can think of the portion of the beam between $x = 0$ and $x = 1$ as a weight sitting at $x = 0$, the portion between $x = 1$ and $x = 2$ as a weight sitting at $x = 1$, and so on, as indicated in figure 11.1.2. We then approximate the mass of the weights by assuming that each portion of the beam has constant density. So the mass of the first weight is approximately $m_0 = (1+0)1 = 1$ kilograms, namely, $(1+0)$ kilograms per meter times 1 meter. The second weight is $m_1 = (1+1)1 = 2$ kilograms, and so on to the tenth weight with $m_9 = (1+9)1 = 10$ kilograms. So in this case the total torque is

$$(0-\bar{x})m_0 + (1-\bar{x})m_1 + \cdots + (9-\bar{x})m_9 = (0-\bar{x})1 + (1-\bar{x})2 + \cdots + (9-\bar{x})10.$$

If we set this to zero and solve for $\bar{x}$ we get $\bar{x} = 6$. In general, if we divide the beam into n portions, the mass of weight number i will be $m_i = (1+x_i)(x_{i+1} - x_i) = (1+x_i)\Delta x$ and the torque induced by weight number i will be $(x_i - \bar{x})m_i = (x_i - \bar{x})(1+x_i)\Delta x$. The total torque is then

$$\begin{aligned}
&(x_0-\bar{x})(1+x_0)\Delta x + (x_1-\bar{x})(1+x_1)\Delta x + \cdots + (x_{n-1}-\bar{x})(1+x_{n-1})\Delta x \\
&\qquad = \sum_{i=0}^{n-1} x_i(1+x_i)\Delta x - \sum_{i=0}^{n-1} \bar{x}(1+x_i)\Delta x \\
&\qquad = \sum_{i=0}^{n-1} x_i(1+x_i)\Delta x - \bar{x}\sum_{i=0}^{n-1} (1+x_i)\Delta x.
\end{aligned}$$

If we set this equal to zero and solve for $\bar{x}$ we get an approximation to the balance point of the beam:

$$\begin{aligned}
0 &= \sum_{i=0}^{n-1} x_i(1+x_i)\Delta x - \bar{x}\sum_{i=0}^{n-1} (1+x_i)\Delta x \\
\bar{x}\sum_{i=0}^{n-1} (1+x_i)\Delta x &= \sum_{i=0}^{n-1} x_i(1+x_i)\Delta x \\
\bar{x} &= \frac{\sum_{i=0}^{n-1} x_i(1+x_i)\Delta x}{\sum_{i=0}^{n-1} (1+x_i)\Delta x}.
\end{aligned}$$

The denominator of this fraction has a very familiar interpretation. Consider one term of the sum in the denominator: $(1+x_i)\Delta x$. This is the density near x_i times a short length, Δx, which in other words is approximately the mass of the beam between x_i and x_{i+1}. When we add these up we get approximately the mass of the beam.

Now each of the sums in the fraction has the right form to turn into an integral, which in turn gives us the exact value of $\bar{x}$:

$$\bar{x} = \frac{\int_0^{10} x(1+x)\,dx}{\int_0^{10} (1+x)\,dx}.$$

The numerator of this fraction is called the **moment** of the system around zero:

$$\int_0^{10} x(1+x)\,dx = \int_0^{10} x + x^2\,dx = \frac{1150}{3},$$

and the denominator is the mass of the beam:

$$\int_0^{10} (1+x)\,dx = 60,$$

and the balance point, officially called the **center of mass**, is

$$\bar{x} = \frac{1150}{3}\frac{1}{60} = \frac{115}{18} \approx 6.39.$$

□

It should be apparent that there was nothing special about the density function $\sigma(x) = 1 + x$ or the length of the beam, or even that the left end of the beam is at the origin. In general, if the density of the beam is $\sigma(x)$ and the beam covers the interval $[a, b]$, the moment of the beam around zero is

$$M_0 = \int_a^b x\sigma(x)\,dx$$

and the total mass of the beam is

$$M = \int_a^b \sigma(x)\,dx$$

and the center of mass is at

$$\bar{x} = \frac{M_0}{M}.$$

EXAMPLE 11.1.2 Suppose a beam lies on the x-axis between 20 and 30, and has density function $\sigma(x) = x - 19$. Find the center of mass. This is the same as the previous example except that the beam has been moved. Note that the density at the left end is $20 - 19 = 1$ and at the right end is $30 - 19 = 11$, as before. Hence the center of mass must be at approximately $20 + 6.39 = 26.39$. Let's see how the calculation works out.

$$\begin{aligned}
M_0 &= \int_{20}^{30} x(x-19)\,dx = \int_{20}^{30} x^2 - 19x\,dx = \left.\frac{x^3}{3} - \frac{19x^2}{2}\right|_{20}^{30} = \frac{4750}{3} \\
M &= \int_{20}^{30} x - 19\,dx = \left.\frac{x^2}{2} - 19x\right|_{20}^{30} = 60 \\
\frac{M_0}{M} &= \frac{4750}{3}\frac{1}{60} = \frac{475}{18} \approx 26.39.
\end{aligned}$$

□

EXAMPLE 11.1.3 Suppose a flat plate of uniform density has the shape contained by $y = x^2$, $y = 1$, and $x = 0$, in the first quadrant. Find the center of mass. (Since the density is constant, the center of mass depends only on the shape of the plate, not the density, or in other words, this is a purely geometric quantity. In such a case the center of mass is called the **centroid**.)

This is a two dimensional problem, but it can be solved as if it were two one dimensional problems: we need to find the x and y coordinates of the center of mass, $\bar{x}$ and $\bar{y}$, and fortunately we can do these independently. Imagine looking at the plate edge on, from below the x-axis. The plate will appear to be a beam, and the mass of a short section of

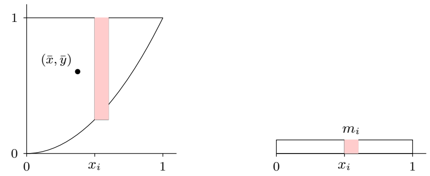

Figure 11.1.3 Center of mass for a two dimensional plate.

the "beam", say between x_i and x_{i+1}, is the mass of a strip of the plate between x_i and x_{i+1}. See figure 11.1.3 showing the plate from above and as it appears edge on. Since the plate has uniform density we may as well assume that $\sigma = 1$. Then the mass of the plate between x_i and x_{i+1} is approximately $m_i = \sigma(1-x_i^2)\Delta x = (1-x_i^2)\Delta x$. Now we can compute the moment around the y-axis:

$$M_y = \int_0^1 x(1-x^2)\,dx = \frac{1}{4}$$

and the total mass

$$M = \int_0^1 (1-x^2)\,dx = \frac{2}{3}$$

and finally

$$\bar x = \frac{1}{4}\frac{3}{2} = \frac{3}{8}.$$

Next we do the same thing to find $\bar y$. The mass of the plate between y_i and y_{i+1} is approximately $n_i = \sqrt{y}\Delta y$, so

$$M_x = \int_0^1 y\sqrt{y}\,dy = \frac{2}{5}$$

and

$$\bar y = \frac{2}{5}\frac{3}{2} = \frac{3}{5},$$

since the total mass M is the same. The center of mass is shown in figure 11.1.3. $\square$

EXAMPLE 11.1.4 Find the center of mass of a thin, uniform plate whose shape is the region between $y = \cos x$ and the x-axis between $x = -\pi/2$ and $x = \pi/2$. It is clear

that $\bar x = 0$, but for practice let's compute it anyway. We will need the total mass, so we compute it first:

$$M = \int_{-\pi/2}^{\pi/2} \cos x\,dx = \sin x\Big|_{-\pi/2}^{\pi/2} = 2.$$

The moment around the y-axis is

$$M_y = \int_{-\pi/2}^{\pi/2} x\cos x\,dx = \cos x + x\sin x\Big|_{-\pi/2}^{\pi/2} = 0$$

and the moment around the x-axis is

$$M_x = \int_0^1 y\cdot 2\arccos y\,dy = y^2\arccos y - \frac{y\sqrt{1-y^2}}{2} + \frac{\arcsin y}{2}\Bigg|_0^1 = \frac{\pi}{4}.$$

Thus

$$\bar x = \frac{0}{2}, \quad \bar y = \frac{\pi}{8} \approx 0.393.$$

$\square$

Exercises 11.1.

1. A beam 10 meters long has density $\sigma(x) = x^2$ at distance x from the left end of the beam. Find the center of mass $\bar x$. ⇒
2. A beam 10 meters long has density $\sigma(x) = \sin(\pi x/10)$ at distance x from the left end of the beam. Find the center of mass $\bar x$. ⇒
3. A beam 4 meters long has density $\sigma(x) = x^3$ at distance x from the left end of the beam. Find the center of mass $\bar x$. ⇒
4. Verify that $\displaystyle\int 2x\arccos x\,dx = x^2\arccos x - \frac{x\sqrt{1-x^2}}{2} + \frac{\arcsin x}{2} + C$.
5. A thin plate lies in the region between $y = x^2$ and the x-axis between $x = 1$ and $x = 2$. Find the centroid. ⇒
6. A thin plate fills the upper half of the unit circle $x^2 + y^2 = 1$. Find the centroid. ⇒
7. A thin plate lies in the region contained by $y = x$ and $y = x^2$. Find the centroid. ⇒
8. A thin plate lies in the region contained by $y = 4 - x^2$ and the x-axis. Find the centroid. ⇒
9. A thin plate lies in the region contained by $y = x^{1/3}$ and the x-axis between $x = 0$ and $x = 1$. Find the centroid. ⇒
10. A thin plate lies in the region contained by $\sqrt{x} + \sqrt{y} = 1$ and the axes in the first quadrant. Find the centroid. ⇒
11. A thin plate lies in the region between the circle $x^2 + y^2 = 4$ and the circle $x^2 + y^2 = 1$, above the x-axis. Find the centroid. ⇒
12. A thin plate lies in the region between the circle $x^2 + y^2 = 4$ and the circle $x^2 + y^2 = 1$ in the first quadrant. Find the centroid. ⇒
13. A thin plate lies in the region between the circle $x^2 + y^2 = 25$ and the circle $x^2 + y^2 = 16$ above the x-axis. Find the centroid. ⇒

11.2 Kinetic energy; improper integrals

Recall example 8.5.3 in which we computed the work required to lift an object from the surface of the earth to some large distance D away. Since $F = k/x^2$ we computed

$$\int_{r_0}^{D} \frac{k}{x^2}\,dx = -\frac{k}{D} + \frac{k}{r_0}.$$

We noticed that as D increases, k/D decreases to zero so that the amount of work increases to k/r_0. More precisely,

$$\lim_{D\to\infty}\int_{r_0}^{D} \frac{k}{x^2}\,dx = \lim_{D\to\infty} -\frac{k}{D} + \frac{k}{r_0} = \frac{k}{r_0}.$$

We might reasonably describe this calculation as computing the amount of work required to lift the object "to infinity," and abbreviate the limit as

$$\lim_{D\to\infty}\int_{r_0}^{D} \frac{k}{x^2}\,dx = \int_{r_0}^{\infty} \frac{k}{x^2}\,dx.$$

Such an integral, with a limit of infinity, is called an **improper integral**. This is a bit unfortunate, since it's not really "improper" to do this, nor is it really "an integral"—it is an abbreviation for the limit of a particular sort of integral. Nevertheless, we're stuck with the term, and the operation itself is perfectly legitimate. It may at first seem odd that a finite amount of work is sufficient to lift an object to "infinity", but sometimes surprising things are nevertheless true, and this is such a case. If the value of an improper integral is a finite number, as in this example, we say that the integral **converges**, and if not we say that the integral **diverges**.

Here's another way, perhaps even more surprising, to interpret this calculation. We know that one interpretation of

$$\int_1^D \frac{1}{x^2}\,dx$$

is the area under $y = 1/x^2$ from $x = 1$ to $x = D$. Of course, as D increases this area increases. But since

$$\int_1^D \frac{1}{x^2}\,dx = -\frac{1}{D} + \frac{1}{1},$$

while the area increases, it never exceeds 1, that is

$$\int_1^\infty \frac{1}{x^2}\,dx = 1.$$

The area of the infinite region under $y = 1/x^2$ from $x = 1$ to infinity is finite.

Consider a slightly different sort of improper integral: $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$. There are two ways we might try to compute this. First, we could break it up into two more familiar integrals:

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx.$$

Now we do these as before:

$$\int_{-\infty}^{0} xe^{-x^2}\,dx = \lim_{D\to\infty} -\frac{e^{-x^2}}{2}\Bigg|_{-D}^{0} = -\frac{1}{2},$$

and

$$\int_{0}^{\infty} xe^{-x^2}\,dx = \lim_{D\to\infty} -\frac{e^{-x^2}}{2}\Bigg|_{0}^{D} = \frac{1}{2},$$

so

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0.$$

Alternately, we might try

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \lim_{D\to\infty}\int_{-D}^{D} xe^{-x^2}\,dx = \lim_{D\to\infty} -\frac{e^{-x^2}}{2}\Bigg|_{-D}^{D} = \lim_{D\to\infty} -\frac{e^{-D^2}}{2} + \frac{e^{-D^2}}{2} = 0.$$

So we get the same answer either way. This does not always happen; sometimes the second approach gives a finite number, while the first approach does not; the exercises provide examples. In general, we interpret the integral $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ according to the first method: both integrals $\displaystyle\int_{-\infty}^{a} f(x)\,dx$ and $\displaystyle\int_{a}^{\infty} f(x)\,dx$ must converge for the original integral to converge. The second approach does turn out to be useful; when $\displaystyle\lim_{D\to\infty}\int_{-D}^{D} f(x)\,dx = L$, and L is finite, then L is called the **Cauchy Principal Value** of $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$.

Here's a more concrete application of these ideas. We know that in general

$$W = \int_{x_0}^{x_1} F\,dx$$

is the work done against the force F in moving from x_0 to x_1. In the case that F is the force of gravity exerted by the earth, it is customary to make $F < 0$ since the force is "downward." This makes the work W negative when it should be positive, so typically the work in this case is defined as

$$W = -\int_{x_0}^{x_1} F\,dx.$$

Also, by Newton's Law, $F = ma(t)$. This means that

$$W = -\int_{x_0}^{x_1} ma(t)\,dx.$$

Unfortunately this integral is a bit problematic: $a(t)$ is in terms of t, while the limits and the "dx" are in terms of x. But x and t are certainly related here: $x = x(t)$ is the function that gives the position of the object at time t, so $v = v(t) = dx/dt = x'(t)$ is its velocity and $a(t) = v'(t) = x''(t)$. We can use $v = x'(t)$ as a substitution to convert the integral from "dx" to "dv" in the usual way, with a bit of cleverness along the way:

$$\begin{aligned} dv &= x''(t)\,dt = a(t)\,dt = a(t)\frac{dt}{dx}\,dx \\ \frac{dx}{dt}\,dv &= a(t)\,dx \\ v\,dv &= a(t)\,dx. \end{aligned}$$

Substituting in the integral:

$$W = -\int_{x_0}^{x_1} ma(t)\,dx = -\int_{v_0}^{v_1} mv\,dv = -\left.\frac{mv^2}{2}\right|_{v_0}^{v_1} = -\frac{mv_1^2}{2} + \frac{mv_0^2}{2}.$$

You may recall seeing the expression $mv^2/2$ in a physics course—it is called the **kinetic energy** of the object. We have shown here that the work done in moving the object from one place to another is the same as the change in kinetic energy.

We know that the work required to move an object from the surface of the earth to infinity is

$$W = \int_{r_0}^{\infty} \frac{k}{r^2}\,dr = \frac{k}{r_0}.$$

At the surface of the earth the acceleration due to gravity is approximately 9.8 meters per second squared, so the force on an object of mass m is $F = 9.8m$. The radius of the earth is approximately 6378.1 kilometers or 6378100 meters. Since the force due to gravity obeys an inverse square law, $F = k/r^2$ and $9.8m = k/6378100^2$, $k = 398665564178000m$ and $W = 62505380m$.

Now suppose that the initial velocity of the object, v_0, is just enough to get it to infinity, that is, just enough so that the object never slows to a stop, but so that its speed decreases to zero, i.e., so that $v_1 = 0$. Then

$$62505380m = W = -\frac{mv_1^2}{2} + \frac{mv_0^2}{2} = \frac{mv_0^2}{2}$$

so

$$v_0 = \sqrt{125010760} \approx 11181 \quad \text{meters per second},$$

or about 40251 kilometers per hour. This speed is called the **escape velocity**. Notice that the mass of the object, m, canceled out at the last step; the escape velocity is the same for all objects. Of course, it takes considerably more energy to get a large object up to 40251 kph than a small one, so it is certainly more difficult to get a large object into deep space than a small one. Also, note that while we have computed the escape velocity for the earth, this speed would not in fact get an object "to infinity" because of the large mass in our neighborhood called the sun. Escape velocity for the sun *starting at the distance of the earth from the sun* is nearly 4 times the escape velocity we have calculated.

Exercises 11.2.

1. Is the area under $y = 1/x$ from 1 to infinity finite or infinite? If finite, compute the area. ⇒

2. Is the area under $y = 1/x^3$ from 1 to infinity finite or infinite? If finite, compute the area. ⇒

3. Does $\int_0^{\infty} x^2 + 2x - 1\,dx$ converge or diverge? If it converges, find the value. ⇒

4. Does $\int_1^{\infty} 1/\sqrt{x}\,dx$ converge or diverge? If it converges, find the value. ⇒

5. Does $\int_0^{\infty} e^{-x}\,dx$ converge or diverge? If it converges, find the value. ⇒

6. $\int_0^{1/2} (2x-1)^{-3}\,dx$ is an improper integral of a slightly different sort. Express it as a limit and determine whether it converges or diverges; if it converges, find the value. ⇒

7. Does $\int_0^{1} 1/\sqrt{x}\,dx$ converge or diverge? If it converges, find the value. ⇒

8. Does $\int_0^{\pi/2} \sec^2 x\,dx$ converge or diverge? If it converges, find the value. ⇒

9. Does $\int_{-\infty}^{\infty} \frac{x^2}{4+x^6}\,dx$ converge or diverge? If it converges, find the value. ⇒

10. Does $\int_{-\infty}^{\infty} x\,dx$ converge or diverge? If it converges, find the value. Also find the Cauchy Principal Value, if it exists. ⇒

11. Does $\int_{-\infty}^{\infty} \sin x\,dx$ converge or diverge? If it converges, find the value. Also find the Cauchy Principal Value, if it exists. ⇒

12. Does $\int_{-\infty}^{\infty} \cos x\,dx$ converge or diverge? If it converges, find the value. Also find the Cauchy Principal Value, if it exists. ⇒

13. Suppose the curve $y = 1/x$ is rotated around the x-axis generating a sort of funnel or horn shape, called **Gabriel's horn** or **Toricelli's trumpet**. Is the volume of this funnel from $x = 1$ to infinity finite or infinite? If finite, compute the volume. ⇒

14. An officially sanctioned baseball must be between 142 and 149 grams. How much work, in Newton-meters, does it take to throw a ball at 80 miles per hour? At 90 mph? At 100.9 mph? (According to the Guinness Book of World Records, at `http://www.baseball-almanac.com/recbooks/rb_guin.shtml`, "The greatest reliably recorded speed at which a baseball has been pitched is 100.9 mph by Lynn Nolan Ryan (California Angels) at Anaheim Stadium in California on August 20, 1974.") ⇒

11.3 Probability

You perhaps have at least a rudimentary understanding of **discrete probability**, which measures the likelihood of an "event" when there are a finite number of possibilities. For example, when an ordinary six-sided die is rolled, the probability of getting any particular number is 1/6. In general, the probability of an event is the number of ways the event can happen divided by the number of ways that "anything" can happen.

For a slightly more complicated example, consider the case of two six-sided dice. The dice are physically distinct, which means that rolling a 2–5 is different than rolling a 5–2; each is an equally likely event out of a total of 36 ways the dice can land, so each has a probability of 1/36.

Most interesting events are not so simple. More interesting is the probability of rolling a certain sum out of the possibilities 2 through 12. It is clearly not true that all sums are equally likely: the only way to roll a 2 is to roll 1–1, while there are many ways to roll a 7. Because the number of possibilities is quite small, and because a pattern quickly becomes evident, it is easy to see that the probabilities of the various sums are:

$$\begin{aligned} P(2) = P(12) &= 1/36 \\ P(3) = P(11) &= 2/36 \\ P(4) = P(10) &= 3/36 \\ P(5) = P(9) &= 4/36 \\ P(6) = P(8) &= 5/36 \\ P(7) &= 6/36 \end{aligned}$$

Here we use $P(n)$ to mean "the probability of rolling an n." Since we have correctly accounted for all possibilities, the sum of all these probabilities is $36/36 = 1$; the probability that the sum is one of 2 through 12 is 1, because there are no other possibilities.

The study of probability is concerned with more difficult questions as well; for example, suppose the two dice are rolled many times. On the average, what sum will come up? In the language of probability, this average is called the **expected value** of the sum. This is at first a little misleading, as it does not tell us what to "expect" when the two dice are rolled, but what we expect the long term average will be.

Suppose that two dice are rolled 36 million times. Based on the probabilities, we would expect about 1 million rolls to be 2, about 2 million to be 3, and so on, with a roll of 7 topping the list at about 6 million. The sum of all rolls would be 1 million times 2 plus 2 million times 3, and so on, and dividing by 36 million we would get the average:

$$\begin{aligned} \bar{x} &= (2\cdot 10^6 + 3(2\cdot 10^6) + \cdots + 7(6\cdot 10^6) + \cdots + 12\cdot 10^6)\frac{1}{36\cdot 10^6} \\ &= 2\frac{10^6}{36\cdot 10^6} + 3\frac{2\cdot 10^6}{36\cdot 10^6} + \cdots + 7\frac{6\cdot 10^6}{36\cdot 10^6} + \cdots + 12\frac{10^6}{36\cdot 10^6} \\ &= 2P(2) + 3P(3) + \cdots + 7P(7) + \cdots + 12P(12) \\ &= \sum_{i=2}^{12} iP(i) = 7. \end{aligned}$$

There is nothing special about the 36 million in this calculation. No matter what the number of rolls, once we simplify the average, we get the same $\sum_{i=2}^{12} iP(i)$. While the actual average value of a large number of rolls will not be exactly 7, the average should be close to 7 when the number of rolls is large. Turning this around, if the average is not close to 7, we should suspect that the dice are not fair.

A variable, say X, that can take certain values, each with a corresponding probability, is called a **random variable**; in the example above, the random variable was the sum of the two dice. If the possible values for X are $x_1, x_2, \ldots, x_n$, then the expected value of the random variable is $E(X) = \sum_{i=1}^{n} x_i P(x_i)$. The expected value is also called the **mean**.

When the number of possible values for X is finite, we say that X is a discrete random variable. In many applications of probability, the number of possible values of a random variable is very large, perhaps even infinite. To deal with the infinite case we need a different approach, and since there is a sum involved, it should not be wholly surprising that integration turns out to be a useful tool. It then turns out that even when the number of possibilities is large but finite, it is frequently easier to pretend that the number is infinite. Suppose, for example, that a dart is thrown at a dart board. Since the dart board consists of a finite number of atoms, there are in some sense only a finite number of places for the dart to land, but it is easier to explore the probabilities involved by pretending that the dart can land on any point in the usual x-y plane.

DEFINITION 11.3.1 Let $f:\mathbb{R}\to\mathbb{R}$ be a function. If $f(x)\ge 0$ for every x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ then f is a **probability density function**. $\square$

We associate a probability density function with a random variable X by stipulating that the probability that X is between a and b is $\int_a^b f(x)\,dx$. Because of the requirement that the integral from $-\infty$ to ∞ be 1, all probabilities are less than or equal to 1, and the probability that X takes on some value between $-\infty$ and ∞ is 1, as it should be.

EXAMPLE 11.3.2 Consider again the two dice example; we can view it in a way that more resembles the probability density function approach. Consider a random variable X that takes on any real value with probabilities given by the probability density function in figure 11.3.1. The function f consists of just the top edges of the rectangles, with vertical sides drawn for clarity; the function is zero below 1.5 and above 12.5. The area of each rectangle is the probability of rolling the sum in the middle of the bottom of the rectangle, or

$$P(n) = \int_{n-1/2}^{n+1/2} f(x)\,dx.$$

The probability of rolling a 4, 5, or 6 is

$$P(n) = \int_{7/2}^{13/2} f(x)\,dx.$$

Of course, we could also compute probabilities that don't make sense in the context of the dice, such as the probability that X is between 4 and 5.8. $\square$

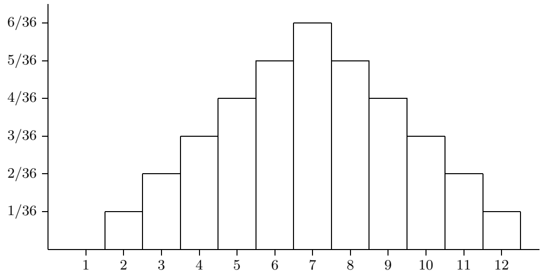

Figure 11.3.1 A probability density function for two dice.

The function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$$

is called the **cumulative distribution function** or simply (probability) distribution.

EXAMPLE 11.3.3 Suppose that $a<b$ and

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a\le x\le b \\ 0 & \text{otherwise.} \end{cases}$$

Then $f(x)$ is the **uniform probability density function** on $[a,b]$. and the corresponding distribution is the **uniform distribution** on $[a,b]$. $\square$

EXAMPLE 11.3.4 Consider the function $f(x)=e^{-x^2/2}$. What can we say about

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx?$$

We cannot find an antiderivative of f, but we can see that this integral is some finite number. Notice that $0<f(x)=e^{-x^2/2}\le e^{-x/2}$ for $|x|>1$. This implies that the area under $e^{-x^2/2}$ is less than the area under $e^{-x/2}$, over the interval $[1,\infty)$. It is easy to compute the latter area, namely

$$\int_1^{\infty} e^{-x/2}\,dx = \frac{2}{\sqrt{e}},$$

so

$$\int_1^{\infty} e^{-x^2/2}\,dx$$

is some finite number smaller than $2/\sqrt{e}$. Because f is symmetric around the y-axis,

$$\int_{-\infty}^{-1} e^{-x^2/2}\,dx = \int_1^{\infty} e^{-x^2/2}\,dx.$$

This means that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \int_{-\infty}^{-1} e^{-x^2/2}\,dx + \int_{-1}^{1} e^{-x^2/2}\,dx + \int_1^{\infty} e^{-x^2/2}\,dx = A$$

for some finite positive number A. Now if we let $g(x)=f(x)/A$,

$$\int_{-\infty}^{\infty} g(x)\,dx = \frac{1}{A}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \frac{1}{A}A = 1,$$

so g is a probability density function. It turns out to be very useful, and is called the **standard normal probability density function** or more informally the **bell curve**,

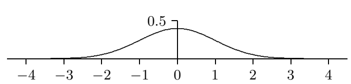

Figure 11.3.2 The bell curve.

giving rise to the **standard normal distribution**. See figure 11.3.2 for the graph of the bell curve. $\square$

We have shown that A is some finite number without computing it; we cannot compute it with the techniques we have available. By using some techniques from multivariable calculus, it can be shown that $A=\sqrt{2\pi}$.

EXAMPLE 11.3.5 The **exponential distribution** has probability density function

$$f(x) = \begin{cases} 0 & x<0 \\ ce^{-cx} & x\ge 0 \end{cases}$$

where c is a positive constant. $\square$

The mean or expected value of a random variable is quite useful, as hinted at in our discussion of dice. Recall that the mean for a discrete random variable is $E(X)=\sum_{i=1}^{n} x_i P(x_i)$. In the more general context we use an integral in place of the sum.

DEFINITION 11.3.6 The **mean** of a random variable X with probability density function f is $\mu = E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$, provided the integral converges. $\square$

When the mean exists it is unique, since it is the result of an explicit calculation. The mean does not always exist.

The mean might look familiar; it is essentially identical to the center of mass of a one-dimensional beam, as discussed in section 11.1. The probability density function f plays the role of the physical density function, but now the "beam" has infinite length. If we consider only a finite portion of the beam, say between a and b, then the center of mass is

$$\bar{x} = \frac{\int_a^b xf(x)\,dx}{\int_a^b f(x)\,dx}.$$

If we extend the beam to infinity, we get

$$\bar{x} = \frac{\int_{-\infty}^{\infty} xf(x)\,dx}{\int_{-\infty}^{\infty} f(x)\,dx} = \int_{-\infty}^{\infty} xf(x)\,dx = E(X),$$

because $\int_{-\infty}^{\infty} f(x)\,dx = 1$. In the center of mass interpretation, this integral is the total mass of the beam, which is always 1 when f is a probability density function.

EXAMPLE 11.3.7 The mean of the standard normal distribution is

$$\int_{-\infty}^{\infty} x\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx.$$

We compute the two halves:

$$\int_{-\infty}^{0} x\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx = \lim_{D\to-\infty} -\frac{e^{-x^2/2}}{\sqrt{2\pi}}\bigg|_D^0 = -\frac{1}{\sqrt{2\pi}}$$

and

$$\int_0^{\infty} x\frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx = \lim_{D\to\infty} -\frac{e^{-x^2/2}}{\sqrt{2\pi}}\bigg|_0^D = \frac{1}{\sqrt{2\pi}}.$$

The sum of these is 0, which is the mean. $\square$

While the mean is very useful, it typically is not enough information to properly evaluate a situation. For example, suppose we could manufacture an 11-sided die, with the faces numbered 2 through 12 so that each face is equally likely to be down when the die is rolled. The value of a roll is the value on this lower face. Rolling the die gives the same range of values as rolling two ordinary dice, but now each value occurs with probability 1/11. The expected value of a roll is

$$\frac{2}{11}+\frac{3}{11}+\cdots+\frac{12}{11} = 7.$$

The mean does not distinguish the two cases, though of course they are quite different.

If f is a probability density function for a random variable X, with mean μ, we would like to measure how far a "typical" value of X is from μ. One way to measure this distance is $(X-\mu)^2$; we square the difference so as to measure all distances as positive. To get the typical such squared distance, we compute the mean. For two dice, for example, we get

$$(2-7)^2\frac{1}{36}+(3-7)^2\frac{2}{36}+\cdots+(7-7)^2\frac{6}{36}+\cdots(11-7)^2\frac{2}{36}+(12-7)^2\frac{1}{36}=\frac{35}{36}.$$

Because we squared the differences this does not directly measure the typical distance we seek; if we take the square root of this we do get such a measure, $\sqrt{35/36}\approx 2.42$. Doing the computation for the strange 11-sided die we get

$$(2-7)^2\frac{1}{11}+(3-7)^2\frac{1}{11}+\cdots+(7-7)^2\frac{1}{11}+\cdots(11-7)^2\frac{1}{11}+(12-7)^2\frac{1}{11}=10,$$

with square root approximately 3.16. Comparing 2.42 to 3.16 tells us that the two-dice rolls clump somewhat more closely near 7 than the rolls of the weird die, which of course we already knew because these examples are quite simple.

To perform the same computation for a probability density function the sum is replaced by an integral, just as in the computation of the mean. The expected value of the squared distances is

$$V(X)=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx,$$

called the **variance**. The square root of the variance is the **standard deviation**, denoted σ.

EXAMPLE 11.3.8 We compute the standard deviation of the standard normal distribution. The variance is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}x^2e^{-x^2/2}\,dx.$$

To compute the antiderivative, use integration by parts, with $u=x$ and $dv=xe^{-x^2/2}\,dx$. This gives

$$\int x^2e^{-x^2/2}\,dx=-xe^{-x^2/2}+\int e^{-x^2/2}\,dx.$$

We cannot do the new integral, but we know its value when the limits are $-\infty$ to ∞, from our discussion of the standard normal distribution. Thus

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}x^2e^{-x^2/2}\,dx=-\frac{1}{\sqrt{2\pi}}xe^{-x^2/2}\bigg|_{-\infty}^{\infty}+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-x^2/2}\,dx=0+\frac{1}{\sqrt{2\pi}}\sqrt{2\pi}=1.$$

The standard deviation is then $\sqrt{1}=1$. □

EXAMPLE 11.3.9 Here is a simple example showing how these ideas can be useful. Suppose it is known that, in the long run, 1 out of every 100 computer memory chips produced by a certain manufacturing plant is defective when the manufacturing process is running correctly. Suppose 1000 chips are selected at random and 15 of them are defective. This is more than the 'expected' number (10), but is it so many that we should suspect that something has gone wrong in the manufacturing process? We are interested in the probability that various numbers of defective chips arise; the probability distribution is discrete: there can only be a whole number of defective chips. But (under reasonable assumptions) the distribution is very close to a normal distribution, namely this one:

$$f(x)=\frac{1}{\sqrt{2\pi}\sqrt{1000(.01)(.99)}}\exp\left(\frac{-(x-10)^2}{2(1000)(.01)(.99)}\right),$$

which is pictured in figure 11.3.3 (recall that $\exp(x)=e^x$).

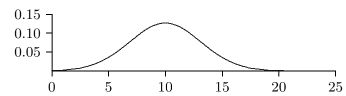

Figure 11.3.3 Normal density function for the defective chips example.

Now how do we measure how unlikely it is that under normal circumstances we would see 15 defective chips? We can't compute the probability of exactly 15 defective chips, as this would be $\int_{15}^{15}f(x)\,dx=0$. We could compute $\int_{14.5}^{15.5}f(x)\,dx\approx 0.036$; this means there is only a 3.6% chance that the number of defective chips is 15. (We cannot compute these integrals exactly; computer software has been used to approximate the integral values in this discussion.) But this is misleading: $\int_{9.5}^{10.5}f(x)\,dx\approx 0.126$, which is larger, certainly, but still small, even for the "most likely" outcome. The most useful question, in most circumstances, is this: how likely is it that the number of defective chips is "far from" the mean? For example, how likely, or unlikely, is it that the number of defective chips is different by 5 or more from the expected value of 10? This is the probability that the number of defective chips is less than 5 or larger than 15, namely

$$\int_{-\infty}^{5}f(x)\,dx+\int_{15}^{\infty}f(x)\,dx\approx 0.11.$$

So there is an 11% chance that this happens—not large, but not tiny. Hence the 15 defective chips does not appear to be cause for alarm: about one time in nine we would

expect to see the number of defective chips 5 or more away from the expected 10. How about 20? Here we compute

$$\int_{-\infty}^{0}f(x)\,dx+\int_{20}^{\infty}f(x)\,dx\approx 0.0015.$$

So there is only a 0.15% chance that the number of defective chips is more than 10 away from the mean; this would typically be interpreted as too suspicious to ignore—it shouldn't happen if the process is running normally.

The big question, of course, is what level of improbability should trigger concern? It depends to some degree on the application, and in particular on the consequences of getting it wrong in one direction or the other. If we're wrong, do we lose a little money? A lot of money? Do people die? In general, the standard choices are 5% and 1%. So what we should do is find the number of defective chips that has only, let us say, a 1% chance of occurring under normal circumstances, and use that as the relevant number. In other words, we want to know when

$$\int_{-\infty}^{10-r}f(x)\,dx+\int_{10+r}^{\infty}f(x)\,dx<0.01.$$

A bit of trial and error shows that with $r=8$ the value is about 0.011, and with $r=9$ it is about 0.004, so if the number of defective chips is 19 or more, or 1 or fewer, we should look for problems. If the number is high, we worry that the manufacturing process has a problem, or conceivably that the process that tests for defective chips is not working correctly and is flagging good chips as defective. If the number is too low, we suspect that the testing procedure is broken, and is not detecting defective chips. □

Exercises 11.3.

1. Verify that $\int_1^{\infty}e^{-x/2}\,dx=2/\sqrt{e}$.

2. Show that the function in example 11.3.5 is a probability density function. Compute the mean and standard deviation. ⇒

3. Compute the mean and standard deviation of the uniform distribution on $[a,b]$. (See example 11.3.3.) ⇒

4. What is the expected value of one roll of a fair six-sided die? ⇒

5. What is the expected sum of one roll of three fair six-sided dice? ⇒

6. Let μ and σ be real numbers with $\sigma>0$. Show that

$$N(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

is a probability density function. You will not be able to compute this integral directly; use a substitution to convert the integral into the one from example 11.3.4. The function N

is the probability density function of the **normal distribution** with mean μ and standard deviation σ. Show that the mean of the normal distribution is μ and the standard deviation is σ.

7. Let

$$f(x)=\begin{cases}\frac{1}{x^2} & x\ge 1\\ 0 & x<1\end{cases}$$

Show that f is a probability density function, and that the distribution has no mean.

8. Let

$$f(x)=\begin{cases}x & -1\le x\le 1\\ 1 & 1<x\le 2\\ 0 & \text{otherwise.}\end{cases}$$

Show that $\int_{-\infty}^{\infty}f(x)\,dx=1$. Is f a probability density function? Justify your answer.

9. If you have access to appropriate software, find r so that

$$\int_{-\infty}^{10-r}f(x)\,dx+\int_{10+r}^{\infty}f(x)\,dx\approx 0.05,$$

using the function of example 11.3.9. Discuss the impact of using this new value of r to decide whether to investigate the chip manufacturing process. ⇒

11.4 Arc Length

Here is another geometric application of the integral: find the length of a portion of a curve. As usual, we need to think about how we might approximate the length, and turn the approximation into an integral.

We already know how to compute one simple arc length, that of a line segment. If the endpoints are $P_0(x_0,y_0)$ and $P_1(x_1,y_1)$ then the length of the segment is the distance between the points, $\sqrt{(x_1-x_0)^2+(y_1-y_0)^2}$, from the Pythagorean theorem, as illustrated in figure 11.4.1.

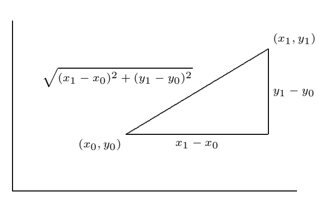

Figure 11.4.1 The length of a line segment.

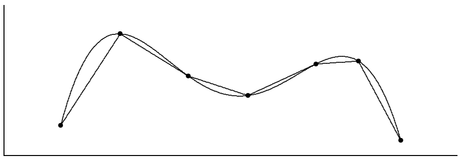

Figure 11.4.2 Approximating arc length with line segments.

Now if the graph of f is "nice" (say, differentiable) it appears that we can approximate the length of a portion of the curve with line segments, and that as the number of segments increases, and their lengths decrease, the sum of the lengths of the line segments will approach the true arc length; see figure 11.4.2.

Now we need to write a formula for the sum of the lengths of the line segments, in a form that we know becomes an integral in the limit. So we suppose we have divided the interval $[a,b]$ into n subintervals as usual, each with length $\Delta x = (b-a)/n$, and endpoints $a = x_0, x_1, x_2, \ldots, x_n = b$. The length of a typical line segment, joining $(x_i, f(x_i))$ to $(x_{i+1}, f(x_{i+1}))$, is $\sqrt{(\Delta x)^2 + (f(x_{i+1}) - f(x_i))^2}$. By the Mean Value Theorem (6.5.2), there is a number t_i in (x_i, x_{i+1}) such that $f'(t_i)\Delta x = f(x_{i+1}) - f(x_i)$, so the length of the line segment can be written as

$$\sqrt{(\Delta x)^2 + (f'(t_i))^2 \Delta x^2} = \sqrt{1 + (f'(t_i))^2}\,\Delta x.$$

The arc length is then

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \sqrt{1 + (f'(t_i))^2}\,\Delta x = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

Note that the sum looks a bit different than others we have encountered, because the approximation contains a t_i instead of an x_i. In the past we have always used left endpoints (namely, x_i) to get a representative value of f on $[x_i, x_{i+1}]$; now we are using a different point, but the principle is the same.

To summarize, to compute the length of a curve on the interval $[a,b]$, we compute the integral

$$\int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

Unfortunately, integrals of this form are typically difficult or impossible to compute exactly, because usually none of our methods for finding antiderivatives will work. In practice this means that the integral will usually have to be approximated.

EXAMPLE 11.4.1 Let $f(x) = \sqrt{r^2 - x^2}$, the upper half circle of radius r. The length of this curve is half the circumference, namely πr. Let's compute this with the arc length formula. The derivative f' is $-x/\sqrt{r^2 - x^2}$ so the integral is

$$\int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\,dx = \int_{-r}^{r} \sqrt{\frac{r^2}{r^2 - x^2}}\,dx = r\int_{-r}^{r} \sqrt{\frac{1}{r^2 - x^2}}\,dx.$$

Using a trigonometric substitution, we find the antiderivative, namely $\arcsin(x/r)$. Notice that the integral is improper at both endpoints, as the function $\sqrt{1/(r^2 - x^2)}$ is undefined when $x = \pm r$. So we need to compute

$$\lim_{D\to -r^+} \int_D^0 \sqrt{\frac{1}{r^2 - x^2}}\,dx + \lim_{D\to r^-} \int_0^D \sqrt{\frac{1}{r^2 - x^2}}\,dx.$$

This is not difficult, and has value π, so the original integral, with the extra r in front, has value πr as expected. □

Exercises 11.4.

1. Find the arc length of $f(x) = x^{3/2}$ on $[0,2]$. ⇒

2. Find the arc length of $f(x) = x^2/8 - \ln x$ on $[1,2]$. ⇒

3. Find the arc length of $f(x) = (1/3)(x^2 + 2)^{3/2}$ on the interval $[0,a]$. ⇒

4. Find the arc length of $f(x) = \ln(\sin x)$ on the interval $[\pi/4, \pi/3]$. ⇒

5. Let $a > 0$. Show that the length of $y = \cosh x$ on $[0,a]$ is equal to $\displaystyle\int_0^a \cosh x\,dx$.

6. Find the arc length of $f(x) = \cosh x$ on $[0, \ln 2]$. ⇒

7. Set up the integral to find the arc length of $\sin x$ on the interval $[0,\pi]$; do not evaluate the integral. If you have access to appropriate software, approximate the value of the integral. ⇒

8. Set up the integral to find the arc length of $y = xe^{-x}$ on the interval $[2,3]$; do not evaluate the integral. If you have access to appropriate software, approximate the value of the integral. ⇒

9. Find the arc length of $y = e^x$ on the interval $[0,1]$. (This can be done exactly; it is a bit tricky and a bit long.) ⇒

11.5 Surface Area

Another geometric question that arises naturally is: "What is the surface area of a volume?" For example, what is the surface area of a sphere? More advanced techniques are required to approach this question in general, but we can compute the areas of some volumes generated by revolution.

As usual, the question is: how might we approximate the surface area? For a surface obtained by rotating a curve around an axis, we can take a polygonal approximation to the curve, as in the last section, and rotate it around the same axis. This gives a surface composed of many "truncated cones;" a truncated cone is called a **frustum** of a cone. Figure 11.5.1 illustrates this approximation.

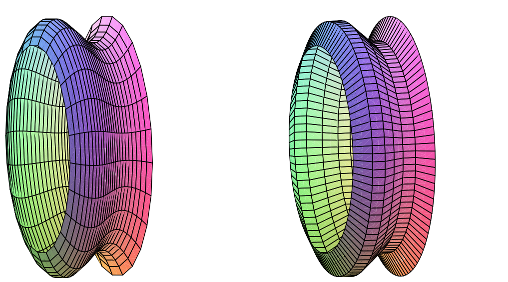

Figure 11.5.1 Approximating a surface (left) by portions of cones (right).

So we need to be able to compute the area of a frustum of a cone. Since the frustum can be formed by removing a small cone from the top of a larger one, we can compute the desired area if we know the surface area of a cone. Suppose a right circular cone has base radius r and slant height h. If we cut the cone from the vertex to the base circle and flatten it out, we obtain a sector of a circle with radius h and arc length $2\pi r$, as in figure 11.5.2. The angle at the center, in radians, is then $2\pi r/h$, and the area of the cone is equal to the area of the sector of the circle. Let A be the area of the sector; since the area of the entire circle is πh^2, we have

$$\frac{A}{\pi h^2} = \frac{2\pi r/h}{2\pi}$$
$$A = \pi r h.$$

Now suppose we have a frustum of a cone with slant height h and radii r_0 and r_1, as in figure 11.5.3. The area of the entire cone is $\pi r_1(h_0 + h)$, and the area of the small cone is $\pi r_0 h_0$; thus, the area of the frustum is $\pi r_1(h_0 + h) - \pi r_0 h_0 = \pi((r_1 - r_0)h_0 + r_1 h)$. By

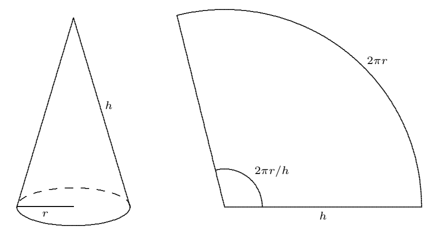

Figure 11.5.2 The area of a cone.

similar triangles,

$$\frac{h_0}{r_0} = \frac{h_0 + h}{r_1}.$$

With a bit of algebra this becomes $(r_1 - r_0)h_0 = r_0 h$; substitution into the area gives

$$\pi((r_1 - r_0)h_0 + r_1 h) = \pi(r_0 h + r_1 h) = \pi h(r_0 + r_1) = 2\pi\frac{r_0 + r_1}{2}h = 2\pi r h.$$

The final form is particularly easy to remember, with r equal to the average of r_0 and r_1, as it is also the formula for the area of a cylinder. (Think of a cylinder of radius r and height h as the frustum of a cone of infinite height.)

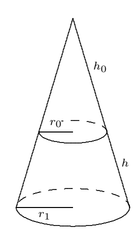

Figure 11.5.3 The area of a frustum.

Now we are ready to approximate the area of a surface of revolution. On one subinterval, the situation is as shown in figure 11.5.4. When the line joining two points on the curve is rotated around the x-axis, it forms a frustum of a cone. The area is

$$2\pi rh = 2\pi\frac{f(x_i)+f(x_{i+1})}{2}\sqrt{1+(f'(t_i))^2}\,\Delta x.$$

Here $\sqrt{1+(f'(t_i))^2}\,\Delta x$ is the length of the line segment, as we found in the previous section. Assuming f is a continuous function, there must be some x_i^* in $[x_i, x_{i+1}]$ such that $(f(x_i)+f(x_{i+1}))/2 = f(x_i^*)$, so the approximation for the surface area is

$$\sum_{i=0}^{n-1} 2\pi f(x_i^*)\sqrt{1+(f'(t_i))^2}\,\Delta x.$$

This is not quite the sort of sum we have seen before, as it contains two different values in the interval $[x_i, x_{i+1}]$, namely x_i^* and t_i. Nevertheless, using more advanced techniques than we have available here, it turns out that

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} 2\pi f(x_i^*)\sqrt{1+(f'(t_i))^2}\,\Delta x = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

is the surface area we seek. (Roughly speaking, this is because while x_i^* and t_i are distinct values in $[x_i, x_{i+1}]$, they get closer and closer to each other as the length of the interval shrinks.)

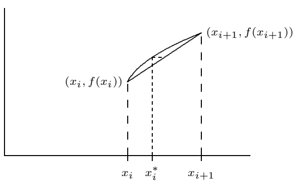

Figure 11.5.4 One subinterval.

EXAMPLE 11.5.1 We compute the surface area of a sphere of radius r. The sphere can be obtained by rotating the graph of $f(x) = \sqrt{r^2-x^2}$ about the x-axis. The derivative

f' is $-x/\sqrt{r^2-x^2}$, so the surface area is given by

$$\begin{aligned}A &= 2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx\\ &= 2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{\frac{r^2}{r^2-x^2}}\,dx\\ &= 2\pi\int_{-r}^{r} r\,dx = 2\pi r\int_{-r}^{r} 1\,dx = 4\pi r^2\end{aligned}$$

□

If the curve is rotated around the y axis, the formula is nearly identical, because the length of the line segment we use to approximate a portion of the curve doesn't change. Instead of the radius $f(x_i^*)$, we use the new radius $\bar{x}_i = (x_i + x_{i+1})/2$, and the surface area integral becomes

$$\int_a^b 2\pi x\sqrt{1+(f'(x))^2}\,dx.$$

EXAMPLE 11.5.2 Compute the area of the surface formed when $f(x) = x^2$ between 0 and 2 is rotated around the y-axis.

We compute $f'(x) = 2x$, and then

$$2\pi\int_0^2 x\sqrt{1+4x^2}\,dx = \frac{\pi}{6}(17^{3/2}-1),$$

by a simple substitution. □

Exercises 11.5.

1. Compute the area of the surface formed when $f(x) = 2\sqrt{1-x}$ between -1 and 0 is rotated around the x-axis. ⇒
2. Compute the surface area of example 11.5.2 by rotating $f(x) = \sqrt{x}$ around the x-axis.
3. Compute the area of the surface formed when $f(x) = x^3$ between 1 and 3 is rotated around the x-axis. ⇒
4. Compute the area of the surface formed when $f(x) = 2+\cosh(x)$ between 0 and 1 is rotated around the x-axis. ⇒
5. Consider the surface obtained by rotating the graph of $f(x) = 1/x$, $x \ge 1$, around the x-axis. This surface is called **Gabriel's horn** or **Toricelli's trumpet**. In exercise 13 in section 11.2 we saw that Gabriel's horn has finite volume. Show that Gabriel's horn has infinite surface area.
6. Consider the circle $(x-2)^2 + y^2 = 1$. Sketch the surface obtained by rotating this circle about the y-axis. (The surface is called a **torus**.) What is the surface area? ⇒

7. Consider the ellipse with equation $x^2/4 + y^2 = 1$. If the ellipse is rotated around the x-axis it forms an **ellipsoid**. Compute the surface area. ⇒
8. Generalize the preceding result: rotate the ellipse given by $x^2/a^2 + y^2/b^2 = 1$ about the x-axis and find the surface area of the resulting ellipsoid. You should consider two cases, when $a > b$ and when $a < b$. Compare to the area of a sphere. ⇒