8
Applications of Integration

8.1 Area between Curves

We have seen how integration can be used to find an area between a curve and the z-axis. With very little change we can find some areas between curves; indeed, the area between a curve and the z-axis may be interpreted as the area between the curve and a second "curve" with equation \( y = 0 \). In the simplest of cases, the idea is quite easy to understand.

**EXAMPLE 8.1.1** Find the area below \( f(x) = -x^2 + 4x + 3 \) and above \( g(x) = -x^3 + 7x^2 - 10x + 5 \) over the interval \( 1 \leq x \leq 2 \). In figure 8.1.1 we show the two curves together, with the desired area shaded, then \( f \) alone with the area under \( f \) shaded, and then \( g \) alone with the area under \( g \) shaded.

![Figure 8.1.1 Area between curves as a difference of areas.](image)

**EXAMPLE 8.1.2** Find the area below \( f(x) = -x^2 + 4x + 1 \) and above \( g(x) = -x^3 + 7x^2 - 10x + 3 \) over the interval \( 1 \leq x \leq 2 \); these are the same curves as before but lowered by 2. In figure 8.1.3 we show the two curves together. Note that the lower curve now dips below the \( x \)-axis. This makes it somewhat tricky to view the desired area as a big area minus a smaller area, but it is just as easy as before to think of approximating the area by rectangles. The height of a typical rectangle will still be \( f(x) - g(x) \), even if \( g(x) \) is negative. Thus the area is

\[
\int_{1}^{2} (-x^2 + 4x + 1 - (-x^3 + 7x^2 - 10x + 3)) \, dx = \int_{1}^{2} (x^3 - 8x^2 + 14x - 2) \, dx.
\]

This is of course the same integral as before, because the region between the curves is identical to the former region—it has just been moved down by 2.

![Figure 8.1.2 Approximating area between curves with rectangles.](image)

**EXAMPLE 8.1.3** Find the area between \( f(x) = -x^2 + 4x \) and \( g(x) = x^3 - 6x + 5 \) over the interval \( 0 \leq x \leq 1 \); the curves are shown in figure 8.1.4. Generally we should interpret "area" in the usual sense, as a necessarily positive quantity. Since the two curves cross, we need to compute two areas and add them. First we find the intersection point of the curves:

\[
-x^2 + 4x = x^3 - 6x + 5
\]

\[
0 = x^3 - x^2 - 6x - 5
\]

\[
x = \frac{10 \pm \sqrt{100 - 4 \cdot 1 \cdot 5}}{2}
\]

The intersection point we want is \( x = 0 = (5 - \sqrt{15})/2 \). Then the total area is

\[
\int_{0}^{1} x^2 - 6x + 5 - (-x^2 + 4x) \, dx + \int_{0}^{1} -x^2 + 4x - (x^3 - 6x + 5) \, dx
\]

\[
= \int_{0}^{1} 2x^2 - 10x + 5 \, dx + \int_{0}^{1} -2x^2 + 10x - 5 \, dx
\]

\[
= -52 + 15 \sqrt{15}
\]

after a bit of simplification.

![Figure 8.1.3 Area between curves.](image)

**EXAMPLE 8.1.4** Find the area between \( f(x) = -x^2 + 4x \) and \( g(x) = x^3 - 6x + 5 \); the curves are shown in figure 8.1.5. Here we are not given a specific interval, so it must
be the case that there is a “natural” region involved. Since the curves are both parabolas, the only reasonable interpretation is the region between the two intersection points, which we found in the previous example:

$$5 + v \sqrt{5}$$

If we let $$a = (5 - \sqrt{5})/2$$ and $$b = (5 + \sqrt{5})/2$$, the total area is

$$\int_a^b x^2 + 4x - (x^2 - 6x + 5) \, dx = \int_a^b -2x^2 + 10x - 5 \, dx$$

$$= -2x^3 + 5x^2 - 5x \bigg|_a^b$$

$$= 5v \sqrt{5}.$$  

after a bit of simplification.

![Figure 8.1.5](image)

**Exercises 8.1.**

Find the area bounded by the curves.

1. $$y = x^2 - x$$ and $$y = x$$ (the part to the right of the y-axis) ⇒
2. $$x = y^2$$ and $$x = y^2$$ ⇒
3. $$x = 1 - y^2$$ and $$y = -x - 1$$ ⇒
4. $$x = 3y - y^2$$ and $$x + y = 3 ⇒$$
5. $$y = \cos(\pi x/2)$$ and $$y = 1 - x^2$$ (in the first quadrant) ⇒
6. $$y = \sin(\pi x/3)$$ and $$y = x$$ (in the first quadrant) ⇒
7. $$y = \sqrt{7}$$ and $$y = x^2 ⇒$$
8. $$y = \sqrt{7}$$ and $$y = \sqrt{7}x, 0 \leq x \leq 4 ⇒$$
9. $$x = 0$$ and $$x = 25 - y^2 ⇒$$
10. $$y = \sin x \cos x$$ and $$y = \sin x, 0 \leq x \leq π ⇒$$

**8.2 Distance, Velocity, Acceleration**

We next recall a general principle that will later be applied to distance-velocity-acceleration problems, among other things. If $$F(u)$$ is an anti-derivative of $$f(u)$$, then $$\int_a^b f(u) \, du = F(b) - F(a)$$. Suppose that we want to let the upper limit of integration vary, i.e., we replace $$b$$ by some variable $$x$$. We think of $$a$$ as a fixed starting value $$x_0$$. In this new notation the last equation (after adding $$F(a)$$ to both sides) becomes:

$$F(x) = F(x_0) + \int_a^x f(u) \, du.$$  

(Here $$x$$ is the variable of integration, called a “dummy variable,” since it is not the variable in the function $$F(x)$$. In general, it is not a good idea to use the same letter as a variable of integration and as a limit of integration. That is, $$\int_a^x f(x) \, dx$$ is bad notation, and can lead to errors and confusion.)

An important application of this principle occurs when we are interested in the position of an object at time $$t$$ (say, on the x-axis) and we know its position at time $$t_0$$. Let $$s(t)$$ denote the position of the object at time $$t$$ (its distance from a reference point, such as the origin on the x-axis). Then the net change in position between $$t_0$$ and $$t$$ is $$s(t) - s(t_0)$$. Since $$s(t)$$ is an anti-derivative of the velocity function $$v(t)$$, we can write

$$s(t) = s(t_0) + \int_{t_0}^t v(u) \, du.$$  

Similarly, since the velocity is an anti-derivative of the acceleration function $$a(t)$$, we have

$$v(t) = v(t_0) + \int_{t_0}^t a(u) \, du.$$  

**EXAMPLE 8.2.1**

Suppose an object is acted upon by a constant force $$F$$. Find $$v(t)$$ and $$s(t)$$. By Newton’s law $$F = ma$$, so the acceleration is $$F/m$$, where $$m$$ is the mass of the object. Then we first have

$$v(t) = v(t_0) + \int_{t_0}^t F/m \, du = v_0 + \frac{F}{m}(t - t_0).$$

using the usual convention $$v_0 = v(t_0)$$. Then

$$s(t) = s(t_0) + \int_{t_0}^t \left( v_0 + \frac{F}{m}(u - t_0) \right) \, du = s_0 + \left( v_0(u - t_0) + \frac{F}{2m}(u - t_0)^2 \right) \bigg|_{t_0}^t$$

$$= s_0 + v_0(t - t_0) + \frac{F}{2m}(t - t_0)^2.$$  

For instance, when $$F/m = -g$$ is the constant of gravitational acceleration, then this is the falling body formula (if we neglect air resistance) familiar from elementary physics:

$$s_0 + v_0(t - t_0) - \frac{1}{2}g(t - t_0)^2,$$

or in the common case that $$t_0 = 0$$,

$$s_0 + v_0t - \frac{1}{2}gt^2.$$  

Recall that the integral of the velocity function gives the net distance traveled, that is, the displacement. If you want to know the total distance traveled, you must find out where the velocity function crosses the t-axis, integrate separately over the time intervals when $$v(t)$$ is positive and when $$v(t)$$ is negative, and add up the absolute values of the different integrals. For example, if an object is thrown straight upward at 19.6 m/sec, its velocity function is $$v(t) = -9.8t + 19.6$$, using $$g = 9.8$$ m/sec$$^2$$ for the force of gravity. This is a straight line which is positive for $$t < 2$$ and negative for $$t > 2$$. The net distance traveled in the first 4 seconds is thus

$$\int_{0}^{2} (-9.8t + 19.6) \, dt = 0,$$

while the total distance traveled in the first 4 seconds is

$$\int_{0}^{2} (-9.8t + 19.6) \, dt + \int_{2}^{4} (-9.8t + 19.6) \, dt = 19.6 + \left| -19.6 \right| = 39.2$$

meters. 19.6 meters up and 19.6 meters down.

**Exercises 8.2.**

For each velocity function find both the net distance and the total distance traveled during the indicated time interval (graph v(t) to determine when it’s positive and when it’s negative):

1. $$v = \cos(\pi t), 0 \leq t \leq 2.5 \Rightarrow$$
2. $$v = -9.8t + 49, 0 \leq t \leq 10 \Rightarrow$$
3. $$v = 3(t - (1 - t), 0 \leq t \leq 5 \Rightarrow$$
4. $$v = \sin(\pi t)/2 - 1, 0 \leq t \leq 1 \Rightarrow$$
5. An object is shot upwards from ground level with an initial velocity of 2 meters per second; it is subject only to the force of gravity (no air resistance). Find its maximum altitude and the time at which it hits the ground. ⇒
6. An object is shot upwards from ground level with an initial velocity of 3 meters per second; it is subject only to the force of gravity (no air resistance). Find its maximum altitude and the time at which it hits the ground. ⇒
8.3 Volume

We have seen how to compute certain areas by using integration; some volumes may also be computed by evaluating an integral. Generally, the volumes that we can compute this way have cross-sections that are easy to describe.

**EXAMPLE 8.3.1** Find the volume of a pyramid with a square base that is 20 meters tall and 20 meters on a side at the base. As with most of our applications of integration, we begin by asking how we might approximate the volume. Since we can easily compute the volume of a rectangular prism (that is, a “box”), we will use some boxes to approximate the volume of a thin “slab” is then

\[ (1 - x^2)\sqrt{1 - x^2}\Delta x. \]

Thus the total volume is

\[ \int_0^1 \sqrt{1 - x^2}^2 dx = \frac{16}{\pi}. \]

One easy way to get “nice” cross-sections is by rotating a plane figure around a line. For example, in figure 8.3.3 we see a plane region under a curve and between two vertical lines; then the result of rotating this around the x-axis, and a typical circular cross-section.

**Figure 8.3.3** A solid of rotation. (AP)

Of course a real “slice” of this figure will not have straight sides, but we can approximate the volume of the slice by a cylinder or disk with circular top and bottom and straight sides; the volume of this disk will have the form \( \pi x^2 \Delta x. \) As long as we can write \( r \) in terms of \( x \) we can compute the volume by an integral.

**EXAMPLE 8.3.3** Find the volume of a right circular cone with base radius 10 and height 20. (A right circular cone is one with a circular base and with the tip of the cone directly over the center of the base.) We can view this cone as produced by the rotation of the line \( y = 10 - \frac{x}{2} \) rotated about the \( x \)-axis, as indicated in figure 8.3.4.

At a particular point on the \( x \)-axis, say \( x_i \), the radius of the resulting cone is the \( y \)-coordinate of the corresponding point on the line, namely \( y_i = x_i/2 \). Thus the total volume is approximately

\[ \frac{1}{2} \pi \sum_{i=0}^{n} y_i^2 \Delta x \]

and the exact volume is

\[ \int_0^{20} \pi x^2/4 \, dx = \frac{\pi}{4} \left( \frac{20^3}{3} \right) = \frac{2000\pi}{3}. \]

The volume of the pyramid, as shown in figure 8.3.1: on the left is a cross-sectional view, on the right is a 3D view of part of the pyramid with some of the boxes used to approximate the volume.

Each box has volume of the form \( (2x_i)/2x_i \Delta y \). Unfortunately, there are two variables here; fortunately, we can write \( x \) in terms of \( y \): \( x = 20 - y/2 \) or \( x = 10 - y/2 \). Then the total volume is approximately

\[ \int_0^{20} 4(10 - y/2)^2 \, dy = \int_0^{20} (20 - y)^2 \, dy = \frac{8000}{3}, \]

as you may know, the volume of a pyramid is \((1/3)\text{(height)(area of base)}\) = \((1/3)(20)(400)\), which agrees with our answer.

**EXAMPLE 8.3.2** The base of a solid is the region between \( f(x) = x^2 - 1 \) and \( g(x) = -x^2 + 1 \), and its cross-sections perpendicular to the \( x \)-axis are equilateral triangles, as indicated in figure 8.3.2. The solid has been truncated to show a triangular cross-section above \( x = 1/2 \). Find the volume of the solid.

**Figure 8.3.2** Solid with equilateral triangles as cross-sections. (AP)

A cross-section at a value \( x \) on the \( x \)-axis is a triangle with base \( 2(1 - x^2) \) and height \( \sqrt{3}(1 - x^2) \), so the area of the cross-section is

\[ \frac{1}{2}\text{(base)(height)} = (1 - x^2)\sqrt{3}(1 - x^2), \]

Note that we can instead do the calculation with a generic height and radius:

\[ \int_0^1 \pi x^2 \, dx = \pi \frac{1^3}{3} - \pi \frac{1^3}{3} = \frac{\pi}{3}, \]

giving us the usual formula for the volume of a cone.

**EXAMPLE 8.3.4** Find the volume of the object generated when the area between \( y = x^2 \) and \( y = x \) is rotated around the \( x \)-axis. This solid has a “hole” in the middle; we can compute the volume by subtracting the volume of the hole from the volume enclosed by the outer surface of the solid. In figure 8.3.5 we show the region that is rotated, the resulting solid with the front half cut away, the cone that forms the outer surface, the horn-shaped hole, and a cross-section perpendicular to the \( x \)-axis.

We have already computed the volume of a cone; in this case it is \( \pi/3 \). At a particular value of \( x \), say \( x_i \), the cross-section of the horn is a circle with radius \( x_i^2 \), so the volume of the horn is

\[ \int_0^{1} \pi x^2 \, dx = \int_0^{1} \pi x^4 \, dx = \frac{\pi}{5}, \]

so the desired volume is \( \pi/3 - \pi/5 = 2\pi/15 \).

As with the area between curves, there is an alternate approach that computes the desired volume “all at once” by approximating the volume of the actual solid. We can approximate the volume of a slice of the solid with a washer-shaped volume, as indicated in figure 8.3.5.

The volume of such a washer is the area of the face times the thickness. The thickness, as usual, is \( \Delta x \); while the area of the face is the area of the outer circle minus the area of the inner circle.

\[ \pi x^2 \Delta x - \pi (x - \Delta x)^2 \Delta x = \pi (2x - x^2) \Delta x. \]

The desired volume is \( \pi/3 - \pi/5 = 2\pi/15 \).
the inner circle, say $R^2 - r^2$. In the present example, at a particular $x_i$, the radius $R$ is $x_i$ and $r$ is $x_i^2$. Hence, the whole volume is
\[
\int_0^1 \pi x^2 - \pi x^4 \, dx = \pi \left( \frac{x^3}{3} - \frac{x^5}{5} \right)_0^1 = \frac{2\pi}{15}.
\]

Of course, what we have done here is exactly the same calculation as before, except we have in effect recomputed the volume of the outer cone.

Suppose the region between $f(x) = x + 1$ and $g(x) = (x - 1)^2$ is rotated around the $y$-axis; see figure 8.3.6. It is possible, but inconvenient, to compute the volume of the resulting solid by the method we have used so far. The problem is that there are two “kinds” of typical rectangles: those that go from the line to the parabola and those that touch the parabola on both ends. To compute the volume using this approach, we need to break the problem into two parts and compute two integrals:
\[
\pi \int_0^1 (1 + \sqrt{y})^2 - (1 - \sqrt{y})^2 \, dy + \pi \int_0^1 (1 + \sqrt{y})^2 - (y - 1)^2 \, dy = \frac{27}{4}. 
\]

If instead we consider a typical vertical rectangle, but still rotate around the $y$-axis, we get a “thin” shell instead of a “thick” washer. If we add up the volume of such thin shells we will get an approximation to the true volume. What is the volume of such a shell? Consider the shell at $x$. Imagine that we cut the shell vertically in one place and “unroll” it into a thin, flat sheet. This sheet will be almost a rectangular prism that is $\Delta x$ thick, $f(x) - g(x)$ tall, and $2\pi x_i$ wide (namely, the circumference of the shell before it was unrolled). The volume will then be approximately the volume of a rectangular prism with these dimensions: $2\pi x_i f(x) - g(x)) \Delta x$. If we add these up and take the limit as usual, we get the integral
\[
\int_0^3 2\pi x f(x) - g(x) \, dx = \frac{27}{4}. 
\]

Not only does this accomplish the task with only one integral, the integral is somewhat easier than those in the previous calculation. Things are not always so neat, but it is often the case that one of the two methods will be simpler than the other, so it is worth considering both before starting to do calculations.

### Exercises 8.3

1. Verify that $\int_0^1 (1 + \sqrt{y})^2 - (1 - \sqrt{y})^2 \, dy + \pi \int_0^1 (1 + \sqrt{y})^2 - (y - 1)^2 \, dy = \frac{27}{4}$.

2. Verify that $\int_0^3 2\pi x f(x) - g(x) \, dx = \frac{27}{4}$.

3. Verify that $\int_0^3 2\pi x f(x) - g(x) \, dy = \frac{27}{4}$.

4. Use integration to find the volume of the solid obtained by revolving the region bounded by $x + y = 2$ and the $x$ and $y$ axes around the $x$-axis.

5. Use integration to find the volume of the solid obtained by revolving the region bounded by $y = x - x^2$ and the $x$- and $y$-axes around the $x$-axis.

6. Find the volume of the solid obtained by revolving the region bounded by $y = x - x^2$ and the $x$- and $y$-axes around the $x$-axis.

7. Find the volume of the solid obtained by revolving the region bounded by $y = \sqrt{\sin x}$ between $x = 0$ and $x = \pi$, the $y$-axis, and the line $y = 1$ around the $x$-axis.

8. Let $S$ be the region of the $x'y'$-plane bounded above by the curve $x^2 + y^2 = 64$, below by the line $y = 1$, on the left by the line $x = 2$, and on the right by the line $x = 4$. Find the volume of the solid obtained by rotating $S$ around the $x$-axis.

9. The equation $x^2 + 2x + 1 = 1$ describes an ellipse. Find the volume of the solid obtained by rotating the ellipse around the $x$-axis and also around the $y$-axis. These solids are called ellipsoids; one is vaguely rugby-ball shaped, one is sort of flying-saucer shaped, or perhaps squashed-bowls-ball-shaped.

10. Use integration to compute the volume of a sphere of radius $r$. You should of course get the well-known formula $4\pi r^3/3$.

11. A hemispheric bowl of radius $r$ contains water to a depth $h$. Find the volume of water in the bowl.

12. The base of a tetrahedron (a triangular pyramid) of height $h$ is an equilateral triangle of side $a$. Its cross-sections perpendicular to an altitude are equilateral triangles. Emprise its volume $V$ as an integral, and find a formula for $V$ in terms of $h$ and $s$. Verify that your answer is $(1/3)(\text{area of base})(\text{height}).$
Now this has exactly the right form, so that in the limit we get
\[
\text{average} = \int_0^1 \frac{\sin(\pi t)}{\pi} \, dt = -\frac{\cos(\pi)}{\pi} + \frac{\cos(0)}{\pi} - \frac{2}{\pi} = -0.64.
\]

Of course, this is exactly what we computed before, but we didn’t need to rely on a particular interpretation of the function. If we interpret \(\sin(\pi t)\) as the height of the function, we interpret the result as the average height of \(\sin(\pi t)\) over \([0, 1]\).

It’s not entirely obvious from this one simple example how to compute such an average in general. Let’s look at a somewhat more complicated case. Suppose that the function is \(16t^2 + 5\). What is the average between \(t = 1\) and \(t = 3^2\)? Again we set up an approximation to the average:
\[
\frac{1}{n} \sum_{i=1}^{n-1} 16b_i^2 + 5,
\]
where the values \(t_i\) are evenly spaced between 1 and 3. Once again we are “missing” \(\Delta t\), and this time \(1/n\) is not the correct value. What is \(\Delta t\) in general? It is the length of a subinterval; in this case we take the interval \([1, 3]\) and divide it into \(n\) subintervals, so each has length \((3 - 1)/n = 2/n = \Delta t\). Now with the usual “multiply and divide by the same thing” trick we can rewrite the sum:
\[
\frac{1}{n} \sum_{i=1}^{n-1} 16b_i^2 + 5 = \frac{1}{n} \sum_{i=1}^{n-1} (16b_i^2 + 5) - \frac{1}{2} \sum_{i=1}^{n-1} (16b_i^2 + 5)\Delta t.
\]
In the limit this becomes
\[
\int_1^3 16t^2 + 5 \, dt = \frac{144}{3} + \frac{223}{3}.
\]
Does this seem reasonable? Let’s picture it: in figure 8.4.1 is the function \(16\), the surface of the earth is 10 pounds, so 10 = \(\int_0^1 f(x) \, dx\). Notice that if we assume the force due to gravity is 10 pounds over the whole distance we have a large number of atoms of water that must be lifted different distances to get to the top of the tank. Fortunately, we don’t really have to deal with individual atoms—we can consider all the atoms at a given depth together.

To approximate the work, we can divide the water in the tank into horizontal sections, approximate the volume of water in a section by a thin disk, and compute the amount of work required to lift each disk to the top of the tank. As usual, we take the limit as the sections get thinner and thinner to get the total work.

At depth \(h\) the circular cross-section through the tank has radius \(r = (10 - h)/7\), by similar triangles, and area \(\pi(10-h)^2/25\). A section of the tank at depth \(h\) thus has volume approximately \(\pi(10-h)^2/25\Delta h\) and so contains \(\pi(10-h)^2/25\Delta h\) kilograms of water,

8.5 Work

A fundamental concept in classical physics is work: If an object is moved in a straight line against a force \(F\) for a distance \(s\) the work done is \(W = Fs\).

EXAMPLE 8.5.1 How much work is done in lifting a 10 pound weight vertically a distance of 5 feet? The force due to gravity on a 10 pound weight is 10 pounds at the surface of the earth, and it does not change appreciably over 5 feet. The work done is \(W = 10 \times 5 = 50\) foot-pounds. In reality few situations are so simple. The force might not be constant over the range of motion, as in the next example.

EXAMPLE 8.5.2 How much work is done in lifting a 10 pound weight from the surface of the earth to an orbit 100 miles above the surface? Over 100 miles the force due to gravity does change significantly, so we need to take this into account. The force exerted on a 10 pound weight of mass \(m = 226.79\) kilograms at the earth is \(F = gm = \frac{mg}{r^2}\), and in fact will approach the value of \(F = \frac{mg}{r^2}\) as \(r\) becomes large. In short, with a finite amount of work, namely \(W = \frac{mg}{r^2} \times r^2 = m\) foot-pounds, the work done by gravity is the product of the weight of the object and the height that it is lifted.

To calculate the work required to lift the object from 0 to 100 miles above the surface of the earth we have to integrate the force function \(F\) over the interval \([0, 100]\). The work done is
\[
W = \int_0^{100} \frac{mg}{r^2} \, dr = -\frac{m}{r} \bigg|_0^{100} = -\frac{m}{100} + \frac{m}{0} = \pi.
\]

Using \(m = 226.79\) kilograms, this result gives the work required to lift the object from the surface of the earth to an orbit 100 miles above the surface.

At the surface of the earth, the acceleration due to gravity is approximately 9.8 meters per second squared, so the force is \(F = \frac{mg}{r^2} = \frac{226.79}{(6378.1)^2}\) newtons. The radius of the earth is approximately 6378.1 kilometers or 6378100 meters. Now the problem proceeds as before. From \(F = k/r^2\) we compute \(k = 6.250538000 \times 10^3\) Newton-meters. As \(D\) increases \(W\) of course gets larger, since the quantity being subtracted, \(-k/D\), gets smaller. But note that the work \(W\) will never exceed \(6.250538000 \times 10^{-3}\) foot-pounds in this example.

The work done in lifting a 10 kilogram object from the surface of the earth to a distance \(D\) from the center of the earth is
\[
W = \int_0^D \frac{k}{r^2} \, dr = -\frac{k}{r} \bigg|_0^D = -\frac{k}{D} + \frac{k}{0} = \pi.
\]

EXAMPLE 8.5.3 How much work is done in lifting a 10 kilogram object from the surface of the earth to a distance \(D\) from the center of the earth? This is the same problem as before in different units, and we are not specifying a value for \(D\). As before
\[
W = \int_0^D \frac{k}{r^2} \, dr = -\frac{k}{r} \bigg|_0^D = -\frac{k}{D} + \frac{k}{0} = \pi.
\]

While “weight in pounds” is a measure of force, “weight in kilograms” is a measure of mass. To convert to force we need to use Newton’s law \(F = ma\). At the surface of the earth the acceleration due to gravity is approximately 9.8 meters per second squared, so the force is \(F = 10 \times 9.8 = 98\) N. The units here are “kilogram-meters per second squared” or “kg m/s²”, also known as a Newton (N), so \(F = 98 N\). The radius of the earth is approximately 6378.1 kilometers or 6378100 meters. Now the problem proceeds as before. From \(F = k/r^2\) we compute \(k = 6.250538000 \times 10^3\) foot-pounds. Then the work is
\[
W = \int_0^D \frac{k}{r^2} \, dr = \int_0^D \frac{6.250538000 \times 10^3}{r^2} \, dr.
\]

Above the surface of the earth the force of gravity is the weight of the object. To lift the object from the surface of the earth to a distance \(D\) above the surface, we need to consider the additional gravitational force due to the distance \(D\) above the surface.

In summary, to compute the average value of \(f(x)\) over \([a, b]\), compute the integral of \(f\) over the interval and divide by the length of the interval:
\[
\text{average} = \frac{1}{b-a} \int_a^b f(x) \, dx.
\]

Exercises 8.4.

1. Find the average height of \(\cos x\) over the intervals \([0, \pi/2], [-\pi/2, \pi/2], [0, 2\pi] \Rightarrow\).
2. Find the average height of \(x^3\) over the interval \([-2, 2] \Rightarrow\).
3. Find the average height of \(2x^2\) over the interval \([1, 4] \Rightarrow\).
4. Find the average height of \(\sqrt{x^2}\) over the interval \([-1, 1] \Rightarrow\).
5. An object moves with velocity \(v(t) = -t^3 + 3t + 1\) foot per second between \(t = 0\) and \(t = 2\). Find the average velocity and the average speed of the object between \(t = 0\) and \(t = 2\). \(\Rightarrow\).
6. The observation deck on the 102nd floor of the Empire State Building is 1,224 feet above the ground. If a steel ball is dropped from the observation deck its velocity at time \(t\) is approximately \(v(t) = -32t\) feet per second. Find the average speed between the time it is dropped and the time it hits the ground, and find its speed when it hits the ground.

Next is an example in which the force is constant, but there are many objects moving different distances.

EXAMPLE 8.5.4 Suppose that a water tank is shaped like a right circular cone with the tip at the bottom, and has height 10 meters and radius 2 meters at the top. If the tank is full, how much work is required to pump all the water out of the top? Here we have a large number of atoms of water that must be lifted different distances to get to the top of the tank. Fortunately, we don’t really have to deal with individual atoms—we can consider all the atoms at a given depth together.

To approximate the work, we can divide the water in the tank into horizontal sections, approximate the volume of water in a section by a thin disk, and compute the amount of work required to lift each disk to the top of the tank. As usual, we take the limit as the sections get thinner and thinner to get the total work.

At depth \(h\) the circular cross-section through the tank has radius \(r = (10 - h)/7\), by similar triangles, and area \(\pi(10-h)^2/25\). A section of the tank at depth \(h\) thus has volume approximately \(\pi(10-h)^2/25\Delta h\) and so contains \(\pi(10-h)^2/25\Delta h\) kilograms of water,
A spring has a “natural length,” its length if nothing is stretching or compressing it. If the spring is either stretched or compressed the spring provides an opposing force; according to Hooke’s Law the magnitude of this force is proportional to the distance the spring has been stretched or compressed: \( F = kx \). The constant of proportionality, \( k \), of course depends on the spring. Note that \( x \) here represents the change in length from the natural length.

**Example 8.5.5** Suppose \( k = 5 \) for a given spring that has a natural length of 0.1 meters. Suppose a force is applied that compresses the spring to length 0.08 meters. How much work is done in compressing it 0.08 meters to 0.05 meters? To compress the spring from 0.08 meters to 0.05 meters takes

\[
W = \int_{0.08}^{0.05} 5(x - 0.1) \, dx = 5(0.08 - 0.1)^2 = 0.01 N \cdot m.
\]

The other values we seek simply use different limits. To compress the spring from 0.1 meters to 0.05 meters takes

\[
W = \int_{0.1}^{0.05} 5(x - 0.1) \, dx = 5(0.1 - 0.1)^2 = 0 N \cdot m.
\]

And to stretch the spring from 0.1 meters to 0.15 meters requires

\[
W = \int_{0.1}^{0.15} 5(x - 0.1) \, dx = 5(0.15 - 0.1)^2 = 0.01 N \cdot m.
\]

**Exercises 8.5.**

1. How much work is done in lifting a 100 kilogram weight from the surface of the earth to an orbit 35,786 kilometers above the surface of the earth? ⇒
2. How much work is done in lifting a 100 kilogram weight from an orbit 1000 kilometers above the surface of the earth to an orbit 35,786 kilometers above the surface of the earth? ⇒
3. A water tank has the shape of an upright cylinder with radius \( r \) = 1 meter and height 10 meters. If the depth of the water is 5 meters, how much work is required to pump all the water out the top of the tank? ⇒
4. Suppose the tank of the previous problem is lying on its side, so that the circular ends are vertical, and that it has the same amount of water as before. How much work is required to pump the water out the top of the tank? (which is now 2 meters above the bottom of the tank) ⇒
5. A water tank has the shape of the bottom half of a sphere with radius \( r = 1 \) meter. If the tank is full, how much work is required to pump all the water out the top of the tank? ⇒
6. A spring has constant \( k = 10 \) kg/n. How much work is done in compressing it 1/10 meter from its natural length? ⇒
7. A force of 2 Newtons will compress a spring from 1 meter (its natural length) to 0.8 meters. How much work is required to stretch the spring from 1.1 meters to 1.5 meters? ⇒
8. A 20 meter long steel cable has density 2 kilograms per meter, and is hanging straight down. How much work is required to lift the entire cable to the height of its top end? ⇒

9. The cable in the previous problem has a 100 kilogram bucket of concrete attached to its lower end. How much work is required to lift the entire cable and bucket to the height of its top end? ⇒

10. Consider again the cable and bucket of the previous problem. How much work is required to lift the bucket 10 meters by raising the cable 10 meters? (The top half of the cable ends up at the height of the top end of the cable, while the bottom half of the cable is lifted 10 meters.) ⇒

\[ W = \frac{9.8 \pi}{25} \int_{0}^{10} h(10 - h) \, dh = \frac{980000}{3} \approx 1026254 \text{ Newton-meters.} \]