8

Applications of Integration

8.1 Area between curves

We have seen how integration can be used to find an area between a curve and the x-axis. With very little change we can find some areas between curves; indeed, the area between a curve and the x-axis may be interpreted as the area between the curve and a second "curve" with equation $y = 0$. In the simplest of cases, the idea is quite easy to understand.

EXAMPLE 8.1.1 Find the area below $f(x) = -x^2 + 4x + 3$ and above $g(x) = -x^3 + 7x^2 - 10x + 5$ over the interval $1 \le x \le 2$. In figure 8.1.1 we show the two curves together, with the desired area shaded, then f alone with the area under f shaded, and then g alone with the area under g shaded.

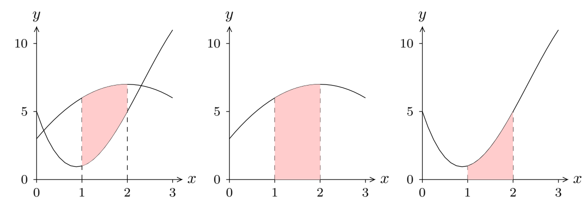

Figure 8.1.1 Area between curves as a difference of areas.

It is clear from the figure that the area we want is the area under f minus the area under g, which is to say

$$\int_1^2 f(x)\,dx - \int_1^2 g(x)\,dx = \int_1^2 f(x) - g(x)\,dx.$$

It doesn't matter whether we compute the two integrals on the left and then subtract or compute the single integral on the right. In this case, the latter is perhaps a bit easier:

$$\begin{aligned}
\int_1^2 f(x) - g(x)\,dx &= \int_1^2 -x^2 + 4x + 3 - (-x^3 + 7x^2 - 10x + 5)\,dx \\
&= \int_1^2 x^3 - 8x^2 + 14x - 2\,dx \\
&= \left.\frac{x^4}{4} - \frac{8x^3}{3} + 7x^2 - 2x\right|_1^2 \\
&= \frac{16}{4} - \frac{64}{3} + 28 - 4 - (\frac{1}{4} - \frac{8}{3} + 7 - 2) \\
&= 23 - \frac{56}{3} - \frac{1}{4} = \frac{49}{12}.
\end{aligned}$$

□

It is worth examining this problem a bit more. We have seen one way to look at it, by viewing the desired area as a big area minus a small area, which leads naturally to the difference between two integrals. But it is instructive to consider how we might find the desired area directly. We can approximate the area by dividing the area into thin sections and approximating the area of each section by a rectangle, as indicated in figure 8.1.2. The area of a typical rectangle is $\Delta x(f(x_i) - g(x_i))$, so the total area is approximately

$$\sum_{i=0}^{n-1} (f(x_i) - g(x_i))\Delta x.$$

This is exactly the sort of sum that turns into an integral in the limit, namely the integral

$$\int_1^2 f(x) - g(x)\,dx.$$

Of course, this is the integral we actually computed above, but we have now arrived at it directly rather than as a modification of the difference between two other integrals. In that example it really doesn't matter which approach we take, but in some cases this second approach is better.

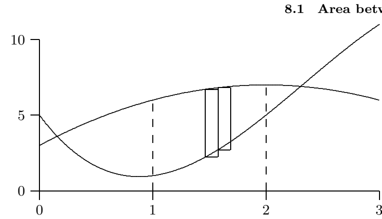

Figure 8.1.2 Approximating area between curves with rectangles.

EXAMPLE 8.1.2 Find the area below $f(x) = -x^2 + 4x + 1$ and above $g(x) = -x^3 + 7x^2 - 10x + 3$ over the interval $1 \le x \le 2$; these are the same curves as before but lowered by 2. In figure 8.1.3 we show the two curves together. Note that the lower curve now dips below the x-axis. This makes it somewhat tricky to view the desired area as a big area minus a smaller area, but it is just as easy as before to think of approximating the area by rectangles. The height of a typical rectangle will still be $f(x_i) - g(x_i)$, even if $g(x_i)$ is negative. Thus the area is

$$\int_1^2 -x^2 + 4x + 1 - (-x^3 + 7x^2 - 10x + 3)\,dx = \int_1^2 x^3 - 8x^2 + 14x - 2\,dx.$$

This is of course the same integral as before, because the region between the curves is identical to the former region—it has just been moved down by 2. □

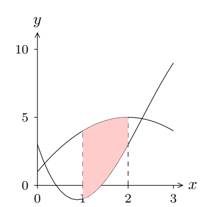

Figure 8.1.3 Area between curves.

EXAMPLE 8.1.3 Find the area between $f(x) = -x^2 + 4x$ and $g(x) = x^2 - 6x + 5$ over the interval $0 \le x \le 1$; the curves are shown in figure 8.1.4. Generally we should interpret

"area" in the usual sense, as a necessarily positive quantity. Since the two curves cross, we need to compute two areas and add them. First we find the intersection point of the curves:

$$\begin{aligned}
-x^2 + 4x &= x^2 - 6x + 5 \\
0 &= 2x^2 - 10x + 5 \\
x &= \frac{10 \pm \sqrt{100 - 40}}{4} = \frac{5 \pm \sqrt{15}}{2}.
\end{aligned}$$

The intersection point we want is $x = a = (5 - \sqrt{15})/2$. Then the total area is

$$\begin{aligned}
\int_0^a x^2 - 6x + 5 - (-x^2 + 4x)\,dx &+ \int_a^1 -x^2 + 4x - (x^2 - 6x + 5)\,dx \\
&= \int_0^a 2x^2 - 10x + 5\,dx + \int_a^1 -2x^2 + 10x - 5\,dx \\
&= \left.\frac{2x^3}{3} - 5x^2 + 5x\right|_0^a + \left.-\frac{2x^3}{3} + 5x^2 - 5x\right|_a^1 \\
&= -\frac{52}{3} + 5\sqrt{15},
\end{aligned}$$

after a bit of simplification. □

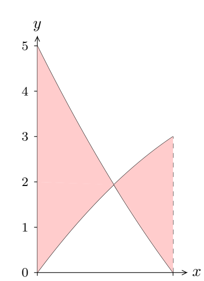

Figure 8.1.4 Area between curves that cross.

EXAMPLE 8.1.4 Find the area between $f(x) = -x^2 + 4x$ and $g(x) = x^2 - 6x + 5$; the curves are shown in figure 8.1.5. Here we are not given a specific interval, so it must be the case that there is a "natural" region involved. Since the curves are both parabolas, the only reasonable interpretation is the region between the two intersection points, which we found in the previous example:

$$\frac{5 \pm \sqrt{15}}{2}.$$

If we let $a = (5 - \sqrt{15})/2$ and $b = (5 + \sqrt{15})/2$, the total area is

$$\begin{aligned} \int_a^b -x^2 + 4x - (x^2 - 6x + 5)\,dx &= \int_a^b -2x^2 + 10x - 5\,dx \\ &= -\frac{2x^3}{3} + 5x^2 - 5x\Bigg|_a^b \\ &= 5\sqrt{15}. \end{aligned}$$

after a bit of simplification. □

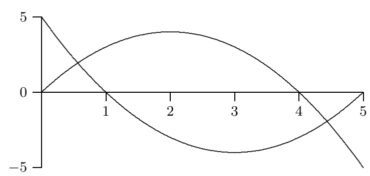

Figure 8.1.5 Area bounded by two curves.

Exercises 8.1.

Find the area bounded by the curves.

1. $y = x^4 - x^2$ and $y = x^2$ (the part to the right of the y-axis) ⇒
2. $x = y^3$ and $x = y^2$ ⇒
3. $x = 1 - y^2$ and $y = -x - 1$ ⇒
4. $x = 3y - y^2$ and $x + y = 3$ ⇒
5. $y = \cos(\pi x/2)$ and $y = 1 - x^2$ (in the first quadrant) ⇒
6. $y = \sin(\pi x/3)$ and $y = x$ (in the first quadrant) ⇒
7. $y = \sqrt{x}$ and $y = x^2$ ⇒
8. $y = \sqrt{x}$ and $y = \sqrt{x+1}$, $0 \le x \le 4$ ⇒
9. $x = 0$ and $x = 25 - y^2$ ⇒
10. $y = \sin x \cos x$ and $y = \sin x$, $0 \le x \le \pi$ ⇒

11. $y = x^{3/2}$ and $y = x^{2/3}$ ⇒
12. $y = x^2 - 2x$ and $y = x - 2$ ⇒

8.2 Distance, Velocity, Acceleration

We next recall a general principle that will later be applied to distance-velocity-acceleration problems, among other things. If $F(u)$ is an anti-derivative of $f(u)$, then $\int_a^b f(u)\,du = F(b) - F(a)$. Suppose that we want to let the upper limit of integration vary, i.e., we replace b by some variable x. We think of a as a fixed starting value x_0. In this new notation the last equation (after adding $F(a)$ to both sides) becomes:

$$F(x) = F(x_0) + \int_{x_0}^x f(u)\,du.$$

(Here u is the variable of integration, called a "dummy variable," since it is not the variable in the function $F(x)$. In general, it is not a good idea to use the same letter as a variable of integration and as a limit of integration. That is, $\int_{x_0}^x f(x)dx$ is bad notation, and can lead to errors and confusion.)

An important application of this principle occurs when we are interested in the position of an object at time t (say, on the x-axis) and we know its position at time t_0. Let $s(t)$ denote the position of the object at time t (its distance from a reference point, such as the origin on the x-axis). Then the net change in position between t_0 and t is $s(t) - s(t_0)$. Since $s(t)$ is an anti-derivative of the velocity function $v(t)$, we can write

$$s(t) = s(t_0) + \int_{t_0}^t v(u)du.$$

Similarly, since the velocity is an anti-derivative of the acceleration function $a(t)$, we have

$$v(t) = v(t_0) + \int_{t_0}^t a(u)du.$$

EXAMPLE 8.2.1 Suppose an object is acted upon by a constant force F. Find $v(t)$ and $s(t)$. By Newton's law $F = ma$, so the acceleration is F/m, where m is the mass of

the object. Then we first have

$$v(t) = v(t_0) + \int_{t_0}^t \frac{F}{m}\,du = v_0 + \frac{F}{m}u\Bigg|_{t_0}^t = v_0 + \frac{F}{m}(t - t_0),$$

using the usual convention $v_0 = v(t_0)$. Then

$$\begin{aligned} s(t) = s(t_0) + \int_{t_0}^t \left(v_0 + \frac{F}{m}(u - t_0)\right) du &= s_0 + (v_0 u + \frac{F}{2m}(u - t_0)^2)\Bigg|_{t_0}^t \\ &= s_0 + v_0(t - t_0) + \frac{F}{2m}(t - t_0)^2. \end{aligned}$$

For instance, when $F/m = -g$ is the constant of gravitational acceleration, then this is the falling body formula (if we neglect air resistance) familiar from elementary physics:

$$s_0 + v_0(t - t_0) - \frac{g}{2}(t - t_0)^2,$$

or in the common case that $t_0 = 0$,

$$s_0 + v_0 t - \frac{g}{2}t^2.$$

□

Recall that the integral of the velocity function gives the *net* distance traveled, that is, the displacement. If you want to know the *total* distance traveled, you must find out where the velocity function crosses the t-axis, integrate separately over the time intervals when $v(t)$ is positive and when $v(t)$ is negative, and add up the absolute values of the different integrals. For example, if an object is thrown straight upward at 19.6 m/sec, its velocity function is $v(t) = -9.8t + 19.6$, using $g = 9.8$ m/sec^2 for the force of gravity. This is a straight line which is positive for $t < 2$ and negative for $t > 2$. The net distance traveled in the first 4 seconds is thus

$$\int_0^4 (-9.8t + 19.6)dt = 0,$$

while the total distance traveled in the first 4 seconds is

$$\int_0^2 (-9.8t + 19.6)dt + \left|\int_2^4 (-9.8t + 19.6)dt\right| = 19.6 + |-19.6| = 39.2$$

meters, 19.6 meters up and 19.6 meters down.

EXAMPLE 8.2.2 The acceleration of an object is given by $a(t) = \cos(\pi t)$, and its velocity at time $t = 0$ is $1/(2\pi)$. Find both the net and the total distance traveled in the first 1.5 seconds.

We compute

$$v(t) = v(0) + \int_0^t \cos(\pi u)du = \frac{1}{2\pi} + \frac{1}{\pi}\sin(\pi u)\Bigg|_0^t = \frac{1}{\pi}\big(\frac{1}{2} + \sin(\pi t)\big).$$

The *net* distance traveled is then

$$\begin{aligned} s(3/2) - s(0) &= \int_0^{3/2} \frac{1}{\pi}\left(\frac{1}{2} + \sin(\pi t)\right) dt \\ &= \frac{1}{\pi}\left(\frac{t}{2} - \frac{1}{\pi}\cos(\pi t)\right)\Bigg|_0^{3/2} = \frac{3}{4\pi} + \frac{1}{\pi^2} \approx 0.340 \text{ meters}. \end{aligned}$$

To find the *total* distance traveled, we need to know when $(0.5 + \sin(\pi t))$ is positive and when it is negative. This function is 0 when $\sin(\pi t)$ is -0.5, i.e., when $\pi t = 7\pi/6$, $11\pi/6$, etc. The value $\pi t = 7\pi/6$, i.e., $t = 7/6$, is the only value in the range $0 \le t \le 1.5$. Since $v(t) > 0$ for $t < 7/6$ and $v(t) < 0$ for $t > 7/6$, the total distance traveled is

$$\begin{aligned} &\int_0^{7/6} \frac{1}{\pi}\left(\frac{1}{2} + \sin(\pi t)\right) dt + \left|\int_{7/6}^{3/2} \frac{1}{\pi}\left(\frac{1}{2} + \sin(\pi t)\right) dt\right| \\ &= \frac{1}{\pi}\left(\frac{7}{12} + \frac{1}{\pi}\cos(7\pi/6) + \frac{1}{\pi}\right) + \frac{1}{\pi}\left|\frac{3}{4} - \frac{7}{12} + \frac{1}{\pi}\cos(7\pi/6)\right| \\ &= \frac{1}{\pi}\left(\frac{7}{12} + \frac{1}{\pi}\frac{\sqrt{3}}{2} + \frac{1}{\pi}\right) + \frac{1}{\pi}\left|\frac{3}{4} - \frac{7}{12} + \frac{1}{\pi}\frac{\sqrt{3}}{2}.\right| \approx 0.409 \text{ meters}. \end{aligned}$$

□

Exercises 8.2.

For each velocity function find both the net distance and the total distance traveled during the indicated time interval (graph $v(t)$ to determine when it's positive and when it's negative):

1. $v = \cos(\pi t)$, $0 \le t \le 2.5$ ⇒
2. $v = -9.8t + 49$, $0 \le t \le 10$ ⇒
3. $v = 3(t - 3)(t - 1)$, $0 \le t \le 5$ ⇒
4. $v = \sin(\pi t/3) - t$, $0 \le t \le 1$ ⇒
5. An object is shot upwards from ground level with an initial velocity of 2 meters per second; it is subject only to the force of gravity (no air resistance). Find its maximum altitude and the time at which it hits the ground. ⇒
6. An object is shot upwards from ground level with an initial velocity of 3 meters per second; it is subject only to the force of gravity (no air resistance). Find its maximum altitude and the time at which it hits the ground. ⇒

7. An object is shot upwards from ground level with an initial velocity of 100 meters per second; it is subject only to the force of gravity (no air resistance). Find its maximum altitude and the time at which it hits the ground. ⇒

8. An object moves along a straight line with acceleration given by $a(t) = -\cos(t)$, and $s(0) = 1$ and $v(0) = 0$. Find the maximum distance the object travels from zero, and find its maximum speed. Describe the motion of the object. ⇒

9. An object moves along a straight line with acceleration given by $a(t) = \sin(\pi t)$. Assume that when $t = 0$, $s(t) = v(t) = 0$. Find $s(t)$, $v(t)$, and the maximum speed of the object. Describe the motion of the object. ⇒

10. An object moves along a straight line with acceleration given by $a(t) = 1 + \sin(\pi t)$. Assume that when $t = 0$, $s(t) = v(t) = 0$. Find $s(t)$ and $v(t)$. ⇒

11. An object moves along a straight line with acceleration given by $a(t) = 1 - \sin(\pi t)$. Assume that when $t = 0$, $s(t) = v(t) = 0$. Find $s(t)$ and $v(t)$. ⇒

8.3 Volume

We have seen how to compute certain areas by using integration; some volumes may also be computed by evaluating an integral. Generally, the volumes that we can compute this way have cross-sections that are easy to describe.

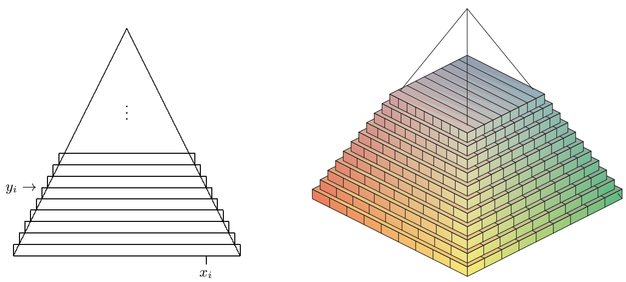

Figure 8.3.1 Volume of a pyramid approximated by rectangular prisms. (AP)

EXAMPLE 8.3.1 Find the volume of a pyramid with a square base that is 20 meters tall and 20 meters on a side at the base. As with most of our applications of integration, we begin by asking how we might approximate the volume. Since we can easily compute the volume of a rectangular prism (that is, a "box"), we will use some boxes to approximate

the volume of the pyramid, as shown in figure 8.3.1: on the left is a cross-sectional view, on the right is a 3D view of part of the pyramid with some of the boxes used to approximate the volume.

Each box has volume of the form $(2x_i)(2x_i)\Delta y$. Unfortunately, there are two variables here; fortunately, we can write x in terms of y: $x = 10 - y/2$ or $x_i = 10 - y_i/2$. Then the total volume is approximately

$$\sum_{i=0}^{n-1} 4(10 - y_i/2)^2 \Delta y$$

and in the limit we get the volume as the value of an integral:

$$\int_0^{20} 4(10 - y/2)^2\,dy = \int_0^{20} (20 - y)^2\,dy = -\left.\frac{(20-y)^3}{3}\right|_0^{20} = -\frac{0^3}{3} - -\frac{20^3}{3} = \frac{8000}{3}.$$

As you may know, the volume of a pyramid is (1/3)(height)(area of base) = (1/3)(20)(400), which agrees with our answer. □

EXAMPLE 8.3.2 The base of a solid is the region between $f(x) = x^2 - 1$ and $g(x) = -x^2 + 1$, and its cross-sections perpendicular to the x-axis are equilateral triangles, as indicated in figure 8.3.2. The solid has been truncated to show a triangular cross-section above $x = 1/2$. Find the volume of the solid.

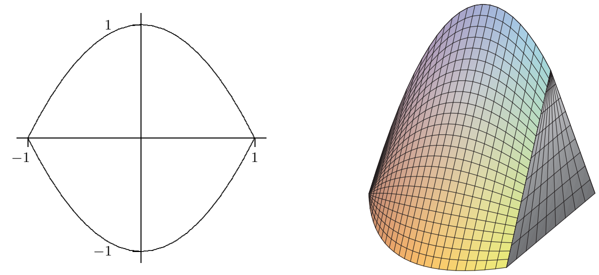

Figure 8.3.2 Solid with equilateral triangles as cross-sections. (AP)

A cross-section at a value x_i on the x-axis is a triangle with base $2(1 - x_i^2)$ and height $\sqrt{3}(1 - x_i^2)$, so the area of the cross-section is

$$\frac{1}{2}(\text{base})(\text{height}) = (1 - x_i^2)\sqrt{3}(1 - x_i^2),$$

and the volume of a thin "slab" is then

$$(1 - x_i^2)\sqrt{3}(1 - x_i^2)\Delta x.$$

Thus the total volume is

$$\int_{-1}^{1} \sqrt{3}(1 - x^2)^2\,dx = \frac{16}{15}\sqrt{3}.$$

□

One easy way to get "nice" cross-sections is by rotating a plane figure around a line. For example, in figure 8.3.3 we see a plane region under a curve and between two vertical lines; then the result of rotating this around the x-axis, and a typical circular cross-section.

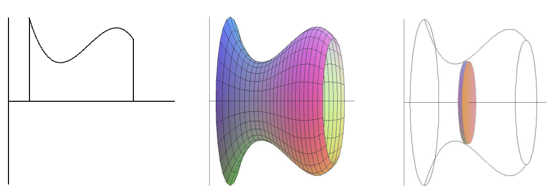

Figure 8.3.3 A solid of rotation. (AP)

Of course a real "slice" of this figure will not have straight sides, but we can approximate the volume of the slice by a cylinder or disk with circular top and bottom and straight sides; the volume of this disk will have the form $\pi r^2 \Delta x$. As long as we can write r in terms of x we can compute the volume by an integral.

EXAMPLE 8.3.3 Find the volume of a right circular cone with base radius 10 and height 20. (A right circular cone is one with a circular base and with the tip of the cone directly over the center of the base.) We can view this cone as produced by the rotation of the line $y = x/2$ rotated about the x-axis, as indicated in figure 8.3.4.

At a particular point on the x-axis, say x_i, the radius of the resulting cone is the y-coordinate of the corresponding point on the line, namely $y_i = x_i/2$. Thus the total volume is approximately

$$\sum_{i=0}^{n-1} \pi(x_i/2)^2\,dx$$

and the exact volume is

$$\int_0^{20} \pi\frac{x^2}{4}\,dx = \frac{\pi}{4}\frac{20^3}{3} = \frac{2000\pi}{3}.$$

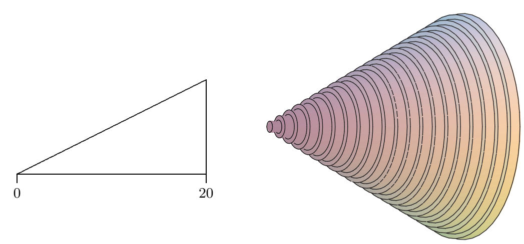

Figure 8.3.4 A region that generates a cone; approximating the volume by circular disks. (AP)

Note that we can instead do the calculation with a generic height and radius:

$$\int_0^h \pi\frac{r^2}{h^2}x^2\,dx = \frac{\pi r^2}{h^2}\frac{h^3}{3} = \frac{\pi r^2 h}{3},$$

giving us the usual formula for the volume of a cone. □

EXAMPLE 8.3.4 Find the volume of the object generated when the area between $y = x^2$ and $y = x$ is rotated around the x-axis. This solid has a "hole" in the middle; we can compute the volume by subtracting the volume of the hole from the volume enclosed by the outer surface of the solid. In figure 8.3.5 we show the region that is rotated, the resulting solid with the front half cut away, the cone that forms the outer surface, the horn-shaped hole, and a cross-section perpendicular to the x-axis.

We have already computed the volume of a cone; in this case it is $\pi/3$. At a particular value of x, say x_i, the cross-section of the horn is a circle with radius x_i^2, so the volume of the horn is

$$\int_0^1 \pi(x^2)^2\,dx = \int_0^1 \pi x^4\,dx = \pi\frac{1}{5},$$

so the desired volume is $\pi/3 - \pi/5 = 2\pi/15$.

As with the area between curves, there is an alternate approach that computes the desired volume "all at once" by approximating the volume of the actual solid. We can approximate the volume of a slice of the solid with a washer-shaped volume, as indicated in figure 8.3.5.

The volume of such a washer is the area of the face times the thickness. The thickness, as usual, is Δx, while the area of the face is the area of the outer circle minus the area of

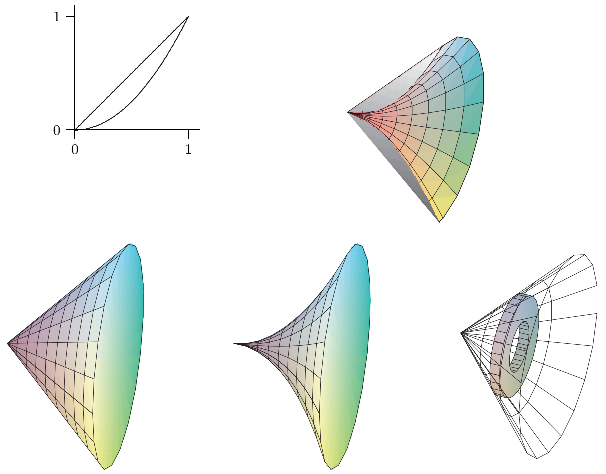

Figure 8.3.5 Solid with a hole, showing the outer cone and the shape to be removed to form the hole. (AP)

the inner circle, say $\pi R^2 - \pi r^2$. In the present example, at a particular x_i, the radius R is x_i and r is x_i^2. Hence, the whole volume is

$$\int_0^1 \pi x^2 - \pi x^4\,dx = \pi\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Bigg|_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}.$$

Of course, what we have done here is exactly the same calculation as before, except we have in effect recomputed the volume of the outer cone. □

Suppose the region between $f(x) = x + 1$ and $g(x) = (x-1)^2$ is rotated around the y-axis; see figure 8.3.6. It is possible, but inconvenient, to compute the volume of the resulting solid by the method we have used so far. The problem is that there are two "kinds" of typical rectangles: those that go from the line to the parabola and those that touch the parabola on both ends. To compute the volume using this approach, we need to

break the problem into two parts and compute two integrals:

$$\pi\int_0^1 (1+\sqrt{y})^2 - (1-\sqrt{y})^2\,dy + \pi\int_1^4 (1+\sqrt{y})^2 - (y-1)^2\,dy = \frac{8}{3}\pi + \frac{65}{6}\pi = \frac{27}{2}\pi.$$

If instead we consider a typical vertical rectangle, but still rotate around the y-axis, we get a thin "shell" instead of a thin "washer". If we add up the volume of such thin shells we will get an approximation to the true volume. What is the volume of such a shell? Consider the shell at x_i. Imagine that we cut the shell vertically in one place and "unroll" it into a thin, flat sheet. This sheet will be almost a rectangular prism that is Δx thick, $f(x_i) - g(x_i)$ tall, and $2\pi x_i$ wide (namely, the circumference of the shell before it was unrolled). The volume will then be approximately the volume of a rectangular prism with these dimensions: $2\pi x_i(f(x_i) - g(x_i))\Delta x$. If we add these up and take the limit as usual, we get the integral

$$\int_0^3 2\pi x(f(x) - g(x))\,dx = \int_0^3 2\pi x(x + 1 - (x-1)^2)\,dx = \frac{27}{2}\pi.$$

Not only does this accomplish the task with only one integral, the integral is somewhat easier than those in the previous calculation. Things are not always so neat, but it is often the case that one of the two methods will be simpler than the other, so it is worth considering both before starting to do calculations.

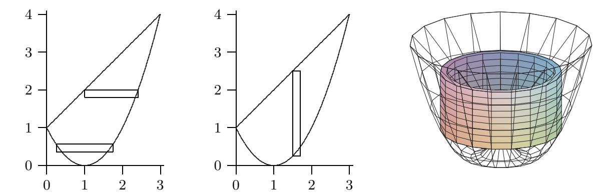

Figure 8.3.6 Computing volumes with "shells". (AP)

EXAMPLE 8.3.5 Suppose the area under $y = -x^2 + 1$ between $x = 0$ and $x = 1$ is rotated around the x-axis. Find the volume by both methods.

Disk method: $\displaystyle\int_0^1 \pi(1-x^2)^2\,dx = \frac{8}{15}\pi.$

Shell method: $\displaystyle\int_0^1 2\pi y\sqrt{1-y}\,dy = \frac{8}{15}\pi.$ □

Exercises 8.3.

1. Verify that $\displaystyle\pi\int_0^1 (1+\sqrt{y})^2 - (1-\sqrt{y})^2\,dy + \pi\int_1^4 (1+\sqrt{y})^2 - (y-1)^2 = \frac{8}{3}\pi + \frac{65}{6}\pi = \frac{27}{2}\pi.$

2. Verify that $\displaystyle\int_0^3 2\pi x(x + 1 - (x-1)^2)\,dx = \frac{27}{2}\pi.$

3. Verify that $\displaystyle\int_0^1 \pi(1-x^2)^2\,dx = \frac{8}{15}\pi.$

4. Verify that $\displaystyle\int_0^1 2\pi y\sqrt{1-y}\,dy = \frac{8}{15}\pi.$

5. Use integration to find the volume of the solid obtained by revolving the region bounded by $x + y = 2$ and the x and y axes around the x-axis. ⇒

6. Find the volume of the solid obtained by revolving the region bounded by $y = x - x^2$ and the x-axis around the x-axis. ⇒

7. Find the volume of the solid obtained by revolving the region bounded by $y = \sqrt{\sin x}$ between $x = 0$ and $x = \pi/2$, the y-axis, and the line $y = 1$ around the x-axis. ⇒

8. Let S be the region of the xy-plane bounded above by the curve $x^3 y = 64$, below by the line $y = 1$, on the left by the line $x = 2$, and on the right by the line $x = 4$. Find the volume of the solid obtained by rotating S around (a) the x-axis, (b) the line $y = 1$, (c) the y-axis, (d) the line $x = 2$. ⇒

9. The equation $x^2/9 + y^2/4 = 1$ describes an ellipse. Find the volume of the solid obtained by rotating the ellipse around the x-axis and also around the y-axis. These solids are called **ellipsoids**; one is vaguely rugby-ball shaped, one is sort of flying-saucer shaped, or perhaps squished-beach-ball-shaped. ⇒

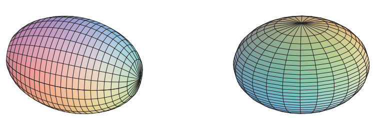

Figure 8.3.7 Ellipsoids.

10. Use integration to compute the volume of a sphere of radius r. You should of course get the well-known formula $4\pi r^3/3$.

11. A hemispheric bowl of radius r contains water to a depth h. Find the volume of water in the bowl. ⇒

12. The base of a tetrahedron (a triangular pyramid) of height h is an equilateral triangle of side s. Its cross-sections perpendicular to an altitude are equilateral triangles. Express its volume V as an integral, and find a formula for V in terms of h and s. Verify that your answer is (1/3)(area of base)(height).

13. The base of a solid is the region between $f(x) = \cos x$ and $g(x) = -\cos x$, $-\pi/2 \le x \le \pi/2$, and its cross-sections perpendicular to the x-axis are squares. Find the volume of the solid. ⇒

8.4 Average value of a function

The average of some finite set of values is a familiar concept. If, for example, the class scores on a quiz are 10, 9, 10, 8, 7, 5, 7, 6, 3, 2, 7, 8, then the average score is the sum of these numbers divided by the size of the class:

$$\text{average score} = \frac{10+9+10+8+7+5+7+6+3+2+7+8}{12} = \frac{82}{12} \approx 6.83.$$

Suppose that between $t = 0$ and $t = 1$ the speed of an object is $\sin(\pi t)$. What is the average speed of the object over that time? The question sounds as if it must make sense, yet we can't merely add up some number of speeds and divide, since the speed is changing continuously over the time interval.

To make sense of "average" in this context, we fall back on the idea of approximation. Consider the speed of the object at tenth of a second intervals: $\sin 0$, $\sin(0.1\pi)$, $\sin(0.2\pi)$, $\sin(0.3\pi)$,…, $\sin(0.9\pi)$. The average speed "should" be fairly close to the average of these ten speeds:

$$\frac{1}{10}\sum_{i=0}^{9}\sin(\pi i/10) \approx \frac{1}{10}6.3 = 0.63.$$

Of course, if we compute more speeds at more times, the average of these speeds should be closer to the "real" average. If we take the average of n speeds at evenly spaced times, we get:

$$\frac{1}{n}\sum_{i=0}^{n-1}\sin(\pi i/n).$$

Here the individual times are $t_i = i/n$, so rewriting slightly we have

$$\frac{1}{n}\sum_{i=0}^{n-1}\sin(\pi t_i).$$

This is almost the sort of sum that we know turns into an integral; what's apparently missing is Δt—but in fact, $\Delta t = 1/n$, the length of each subinterval. So rewriting again:

$$\sum_{i=0}^{n-1}\sin(\pi t_i)\frac{1}{n} = \sum_{i=0}^{n-1}\sin(\pi t_i)\Delta t.$$

Now this has exactly the right form, so that in the limit we get

$$\text{average speed} = \int_0^1 \sin(\pi t)\,dt = -\frac{\cos(\pi t)}{\pi}\Bigg|_0^1 = -\frac{\cos(\pi)}{\pi} + \frac{\cos(0)}{\pi} = \frac{2}{\pi} \approx 0.6366 \approx 0.64.$$

It's not entirely obvious from this one simple example how to compute such an average in general. Let's look at a somewhat more complicated case. Suppose that the velocity of an object is $16t^2+5$ feet per second. What is the average velocity between $t=1$ and $t=3$? Again we set up an approximation to the average:

$$\frac{1}{n}\sum_{i=0}^{n-1} 16t_i^2+5,$$

where the values t_i are evenly spaced times between 1 and 3. Once again we are "missing" Δt, and this time $1/n$ is not the correct value. What is Δt in general? It is the length of a subinterval; in this case we take the interval $[1,3]$ and divide it into n subintervals, so each has length $(3-1)/n = 2/n = \Delta t$. Now with the usual "multiply and divide by the same thing" trick we can rewrite the sum:

$$\frac{1}{n}\sum_{i=0}^{n-1} 16t_i^2+5 = \frac{1}{3-1}\sum_{i=0}^{n-1}(16t_i^2+5)\frac{3-1}{n} = \frac{1}{2}\sum_{i=0}^{n-1}(16t_i^2+5)\frac{2}{n} = \frac{1}{2}\sum_{i=0}^{n-1}(16t_i^2+5)\Delta t.$$

In the limit this becomes

$$\frac{1}{2}\int_1^3 16t^2+5\,dt = \frac{1}{2}\frac{446}{3} = \frac{223}{3}.$$

Does this seem reasonable? Let's picture it: in figure 8.4.1 is the velocity function together with the horizontal line $y = 223/3 \approx 74.3$. Certainly the height of the horizontal line looks at least plausible for the average height of the curve.

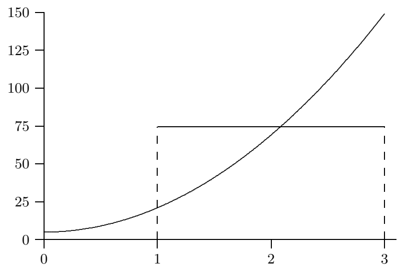

Figure 8.4.1 Average velocity.

Here's another way to interpret "average" that may make our computation appear even more reasonable. The object of our example goes a certain distance between $t=1$

and $t=3$. If instead the object were to travel at the average speed over the same time, it should go the same distance. At an average speed of 223/3 feet per second for two seconds the object would go 446/3 feet. How far does it actually go? We know how to compute this:

$$\int_1^3 v(t)\,dt = \int_1^3 16t^2+5\,dt = \frac{446}{3}.$$

So now we see that another interpretation of the calculation

$$\frac{1}{2}\int_1^3 16t^2+5\,dt = \frac{1}{2}\frac{446}{3} = \frac{223}{3}$$

is: total distance traveled divided by the time in transit, namely, the usual interpretation of average speed.

In the case of speed, or more properly velocity, we can always interpret "average" as total (net) distance divided by time. But in the case of a different sort of quantity this interpretation does not obviously apply, while the approximation approach always does. We might interpret the same problem geometrically: what is the average height of $16x^2+5$ on the interval $[1,3]$? We approximate this in exactly the same way, by adding up many sample heights and dividing by the number of samples. In the limit we get the same result:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} 16x_i^2+5 = \frac{1}{2}\int_1^3 16x^2+5\,dx = \frac{1}{2}\frac{446}{3} = \frac{223}{3}.$$

We can interpret this result in a slightly different way. The area under $y=16x^2+5$ above $[1,3]$ is

$$\int_1^3 16t^2+5\,dt = \frac{446}{3}.$$

The area under $y = 223/3$ over the same interval $[1,3]$ is simply the area of a rectangle that is 2 by 223/3 with area 446/3. So the average height of a function is the height of the horizontal line that produces the same area over the given interval.

Exercises 8.4.

1. Find the average height of $\cos x$ over the intervals $[0,\pi/2]$, $[-\pi/2,\pi/2]$, and $[0,2\pi]$. ⇒
2. Find the average height of x^2 over the interval $[-2,2]$. ⇒
3. Find the average height of $1/x^2$ over the interval $[1,A]$. ⇒
4. Find the average height of $\sqrt{1-x^2}$ over the interval $[-1,1]$. ⇒
5. An object moves with velocity $v(t) = -t^2+1$ feet per second between $t=0$ and $t=2$. Find the average velocity and the average speed of the object between $t=0$ and $t=2$. ⇒

6. The observation deck on the 102nd floor of the Empire State Building is 1,224 feet above the ground. If a steel ball is dropped from the observation deck its velocity at time t is approximately $v(t) = -32t$ feet per second. Find the average speed between the time it is dropped and the time it hits the ground, and find its speed when it hits the ground. ⇒

8.5 Work

A fundamental concept in classical physics is **work**: If an object is moved in a straight line against a force F for a distance s the work done is $W = Fs$.

EXAMPLE 8.5.1 How much work is done in lifting a 10 pound weight vertically a distance of 5 feet? The force due to gravity on a 10 pound weight is 10 pounds at the surface of the earth, and it does not change appreciably over 5 feet. The work done is $W = 10\cdot 5 = 50$ foot-pounds. □

In reality few situations are so simple. The force might not be constant over the range of motion, as in the next example.

EXAMPLE 8.5.2 How much work is done in lifting a 10 pound weight from the surface of the earth to an orbit 100 miles above the surface? Over 100 miles the force due to gravity does change significantly, so we need to take this into account. The force exerted on a 10 pound weight at a distance r from the center of the earth is $F = k/r^2$ and by definition it is 10 when r is the radius of the earth (we assume the earth is a sphere). How can we approximate the work done? We divide the path from the surface to orbit into n small subpaths. On each subpath the force due to gravity is roughly constant, with value k/r_i^2 at distance r_i. The work to raise the object from r_i to r_{i+1} is thus approximately $k/r_i^2\Delta r$ and the total work is approximately

$$\sum_{i=0}^{n-1}\frac{k}{r_i^2}\Delta r,$$

or in the limit

$$W = \int_{r_0}^{r_1}\frac{k}{r^2}\,dr,$$

where r_0 is the radius of the earth and r_1 is r_0 plus 100 miles. The work is

$$W = \int_{r_0}^{r_1}\frac{k}{r^2}\,dr = -\frac{k}{r}\Big|_{r_0}^{r_1} = -\frac{k}{r_1}+\frac{k}{r_0}.$$

Using $r_0 = 20925525$ feet we have $r_1 = 21453525$. The force on the 10 pound weight at the surface of the earth is 10 pounds, so $10 = k/20925525^2$, giving $k = 4378775965256250$.

Then

$$-\frac{k}{r_1}+\frac{k}{r_0} = \frac{491052320000}{95349} \approx 5150052 \quad \text{foot-pounds.}$$

Note that if we assume the force due to gravity is 10 pounds over the whole distance we would calculate the work as $10(r_1-r_0) = 10\cdot 100\cdot 5280 = 5280000$, somewhat higher since we don't account for the weakening of the gravitational force. □

EXAMPLE 8.5.3 How much work is done in lifting a 10 kilogram object from the surface of the earth to a distance D from the center of the earth? This is the same problem as before in different units, and we are not specifying a value for D. As before

$$W = \int_{r_0}^{D}\frac{k}{r^2}\,dr = -\frac{k}{r}\Big|_{r_0}^{D} = -\frac{k}{D}+\frac{k}{r_0}.$$

While "weight in pounds" is a measure of force, "weight in kilograms" is a measure of mass. To convert to force we need to use Newton's law $F = ma$. At the surface of the earth the acceleration due to gravity is approximately 9.8 meters per second squared, so the force is $F = 10\cdot 9.8 = 98$. The units here are "kilogram-meters per second squared" or "kg m/s^2", also known as a Newton (N), so $F = 98$ N. The radius of the earth is approximately 6378.1 kilometers or 6378100 meters. Now the problem proceeds as before. From $F = k/r^2$ we compute k: $98 = k/6378100^2$, $k = 3.986655642\cdot 10^{15}$. Then the work is:

$$W = -\frac{k}{D} + 6.250538000\cdot 10^{8} \quad \text{Newton-meters.}$$

As D increases W of course gets larger, since the quantity being subtracted, $-k/D$, gets smaller. But note that the work W will never exceed $6.250538000\cdot 10^8$, and in fact will approach this value as D gets larger. In short, with a finite amount of work, namely $6.250538000\cdot 10^8$ N-m, we can lift the 10 kilogram object as far as we wish from earth. □

Next is an example in which the force is constant, but there are many objects moving different distances.

EXAMPLE 8.5.4 Suppose that a water tank is shaped like a right circular cone with the tip at the bottom, and has height 10 meters and radius 2 meters at the top. If the tank is full, how much work is required to pump all the water out over the top? Here we have a large number of atoms of water that must be lifted different distances to get to the top of the tank. Fortunately, we don't really have to deal with individual atoms—we can consider all the atoms at a given depth together.

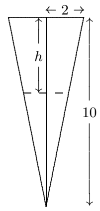

Figure 8.5.1 Cross-section of a conical water tank.

To approximate the work, we can divide the water in the tank into horizontal sections, approximate the volume of water in a section by a thin disk, and compute the amount of work required to lift each disk to the top of the tank. As usual, we take the limit as the sections get thinner and thinner to get the total work.

At depth h the circular cross-section through the tank has radius $r = (10-h)/5$, by similar triangles, and area $\pi(10-h)^2/25$. A section of the tank at depth h thus has volume approximately $\pi(10-h)^2/25\Delta h$ and so contains $\sigma\pi(10-h)^2/25\Delta h$ kilograms of water, where σ is the density of water in kilograms per cubic meter; $\sigma \approx 1000$. The force due to gravity on this much water is $9.8\sigma\pi(10-h)^2/25\Delta h$, and finally, this section of water must be lifted a distance h, which requires $h9.8\sigma\pi(10-h)^2/25\Delta h$ Newton-meters of work. The total work is therefore

$$W = \frac{9.8\sigma\pi}{25}\int_0^{10} h(10-h)^2\,dh = \frac{980000}{3}\pi \approx 1026254 \quad \text{Newton-meters.}$$

□

A spring has a "natural length," its length if nothing is stretching or compressing it. If the spring is either stretched or compressed the spring provides an opposing force; according to **Hooke's Law** the magnitude of this force is proportional to the distance the spring has been stretched or compressed: $F = kx$. The constant of proportionality, k, of course depends on the spring. Note that x here represents the *change* in length from the natural length.

EXAMPLE 8.5.5 Suppose $k = 5$ for a given spring that has a natural length of 0.1 meters. Suppose a force is applied that compresses the spring to length 0.08. What is the magnitude of the force? Assuming that the constant k has appropriate dimensions (namely, kg/s^2), the force is $5(0.1-0.08) = 5(0.02) = 0.1$ Newtons. □

EXAMPLE 8.5.6 How much work is done in compressing the spring in the previous example from its natural length to 0.08 meters? From 0.08 meters to 0.05 meters? How much work is done to stretch the spring from 0.1 meters to 0.15 meters? We can approximate the work by dividing the distance that the spring is compressed (or stretched) into small subintervals. Then the force exerted by the spring is approximately constant over the subinterval, so the work required to compress the spring from x_i to x_{i+1} is approximately $5(x_i - 0.1)\Delta x$. The total work is approximately

$$\sum_{i=0}^{n-1} 5(x_i - 0.1)\Delta x$$

and in the limit

$$W = \int_{0.1}^{0.08} 5(x-0.1)\,dx = \left.\frac{5(x-0.1)^2}{2}\right|_{0.1}^{0.08} = \frac{5(0.08-0.1)^2}{2} - \frac{5(0.1-0.1)^2}{2} = \frac{1}{1000}\ \text{N-m.}$$

The other values we seek simply use different limits. To compress the spring from 0.08 meters to 0.05 meters takes

$$W = \int_{0.08}^{0.05} 5(x-0.1)\,dx = \left.\frac{5(x-0.1)^2}{2}\right|_{0.08}^{0.05} = \frac{5(0.05-0.1)^2}{2} - \frac{5(0.08-0.1)^2}{2} = \frac{21}{4000}\quad \text{N-m}$$

and to stretch the spring from 0.1 meters to 0.15 meters requires

$$W = \int_{0.1}^{0.15} 5(x-0.1)\,dx = \left.\frac{5(x-0.1)^2}{2}\right|_{0.1}^{0.15} = \frac{5(0.15-0.1)^2}{2} - \frac{5(0.1-0.1)^2}{2} = \frac{1}{160}\quad \text{N-m.}$$

□

Exercises 8.5.

1. How much work is done in lifting a 100 kilogram weight from the surface of the earth to an orbit 35,786 kilometers above the surface of the earth? ⇒
2. How much work is done in lifting a 100 kilogram weight from an orbit 1000 kilometers above the surface of the earth to an orbit 35,786 kilometers above the surface of the earth? ⇒
3. A water tank has the shape of an upright cylinder with radius $r = 1$ meter and height 10 meters. If the depth of the water is 5 meters, how much work is required to pump all the water out the top of the tank? ⇒
4. Suppose the tank of the previous problem is lying on its side, so that the circular ends are vertical, and that it has the same amount of water as before. How much work is required to pump the water out the top of the tank (which is now 2 meters above the bottom of the tank)? ⇒

5. A water tank has the shape of the bottom half of a sphere with radius $r = 1$ meter. If the tank is full, how much work is required to pump all the water out the top of the tank? ⇒
6. A spring has constant $k = 10$ kg/s^2. How much work is done in compressing it 1/10 meter from its natural length? ⇒
7. A force of 2 Newtons will compress a spring from 1 meter (its natural length) to 0.8 meters. How much work is required to stretch the spring from 1.1 meters to 1.5 meters? ⇒
8. A 20 meter long steel cable has density 2 kilograms per meter, and is hanging straight down. How much work is required to lift the entire cable to the height of its top end? ⇒
9. The cable in the previous problem has a 100 kilogram bucket of concrete attached to its lower end. How much work is required to lift the entire cable and bucket to the height of its top end? ⇒
10. Consider again the cable and bucket of the previous problem. How much work is required to lift the bucket 10 meters by raising the cable 10 meters? (The top half of the cable ends up at the height of the top end of the cable, while the bottom half of the cable is lifted 10 meters.) ⇒