8

Applications of Integration

8.1 Area between Curves

We have seen how integration can be used to find an area between a curve and the \( x \)-axis. With very little change we can find some areas between curves; indeed, the area between a curve and the \( x \)-axis may be interpreted as the area between the curve and a second “curve” with equation \( y = 0 \). In the simplest of cases, the idea is quite easy to understand.

**EXAMPLE 8.1.1** Find the area below \( f(x) = -x^2 + 4x + 3 \) and above \( g(x) = -x^2 + 7x^2 - 10x + 5 \) over the interval \( 1 \leq x \leq 2 \). In figure 8.1.1 we show the two curves together, with the desired area shaded, then \( f \) alone with the area under \( f \) shaded, and then \( g \) alone with the area under \( g \) shaded.

![Figure 8.1.1 Area between curves as a difference of areas.](image)

**EXAMPLE 8.1.2** Find the area below \( f(x) = -x^4 + 4x \) and above \( g(x) = -x^2 - 6x + 5 \) over the interval \( 1 \leq x \leq 2 \). These are the same curves as before but lowered by 2. In figure 8.1.2 we show the two curves together. Note that the lower curve now dips below the \( x \)-axis. This makes it somewhat tricky to view the desired area as a big area minus a smaller area, but it is just as easy as before to think of approximating the area by rectangles. The height of a typical rectangle will still be \( f(x) - g(x) \), even if \( g(x) \) is negative. Thus the area is

\[
\int_1^2 (-x^4 + 4x) \, dx - \int_1^2 (-x^2 - 6x + 5) \, dx = \int_1^2 x^4 - 8x^2 + 14x - 2 \, dx.
\]

This is of course the same integral as before, because the region between the curves is identical to the former region—it has just been moved down by 2.

![Figure 8.1.2 Approximating area between curves with rectangles.](image)

**EXAMPLE 8.1.3** Find the area between \( f(x) = -x^2 + 4x \) and \( g(x) = x^2 - 6x + 5 \) over the interval \( 0 \leq x \leq 1 \), the curves are shown in figure 8.1.3. Generally we should interpret “area” in the usual sense, as a necessarily positive quantity. Since the two curves cross, we need to compute two areas and add them. First we find the intersection point of the curves:

\[
\begin{align*}
-2x^2 + 4x &= x^2 - 6x + 5 \\
0 &= 2x^2 - 10x + 5 \\
x &= \frac{10 \pm \sqrt{100 - 40}}{4} = \frac{5 \pm \sqrt{15}}{2}.
\end{align*}
\]

The intersection point we want is \( x = a = \frac{5 - \sqrt{15}}{2} \). Then the total area is

\[
\int_0^a x^2 - 6x + 5 \, dx + \int_a^1 (-x^2 + 4x) \, dx = \int_0^1 x^2 - 4x + 5 \, dx - \int_0^1 x^2 - 4x + 5 \, dx
\]

after a bit of simplification.

![Figure 8.1.3 Area between curves.](image)

**EXAMPLE 8.1.4** Find the area between \( f(x) = -x^2 + 4x \) and \( g(x) = x^2 - 6x + 5 \), the curves are shown in figure 8.1.4. Here we are not given a specific interval, so it must

It is clear from the figure that the area we want is the area under \( f \) minus the area under \( g \), which is to say

\[
\int f(x) \, dx - \int g(x) \, dx = \int f(x) - g(x) \, dx.
\]

It doesn’t matter whether we compute the two integrals on the left and then subtract or compute the single integral on the right. In this case, the latter is perhaps a bit easier:

\[
\begin{align*}
\int f(x) - g(x) \, dx &= \int_1^2 x^2 - 4x + 3 - (-x^2 + 7x^2 - 10x + 5) \, dx \\
&= \int_1^2 x^2 - 8x^2 + 14x - 2 \, dx \\
&= \int_1^2 x^2 - 7x^2 + 14x - 2 \, dx \\
&= \frac{16}{4} - \frac{64}{3} \int_1^2 x \, dx - \frac{8}{3} + 7 - 2 \\
&= 25 \frac{56}{7} - \frac{49}{12}.
\end{align*}
\]

It is worth examining this problem a bit more. We have seen one way to look at it, by viewing the desired area as a big area minus a small area, which leads naturally to the difference between two integrals. But it is instructive to consider how we might find the desired area directly. We can approximate the area by dividing the area into thin sections and approximating the area of each section by a rectangle, as indicated in figure 8.1.2. The area of a typical rectangle is \( \Delta(x_i) \cdot (f(x_i) - g(x_i)) \), so the total area is approximately

\[
\sum_{i=0}^{n-1} (f(x_i) - g(x_i)) \Delta x.
\]

This is exactly the sort of sum that turns into an integral in the limit, namely the integral

\[
\int f(x) - g(x) \, dx.
\]

Of course, this is the integral we actually computed above, but we have now arrived at it directly rather than as a modification of the difference between two other integrals. In that example it really doesn’t matter which approach we take, but in some cases this second approach is better.

![Figure 8.1.4 Area between curves that cross.](image)
be the case that there is a “natural” region involved. Since the curves are both parabolas, the only reasonable interpretation is the region between the two intersection points, which we found in the previous example:

\[
\int_{a}^{b} \left( x^2 + 4x - (x^2 - 6x + 5) \right) \, dx = \int_{a}^{b} -2e^2 + 10x - 5 \, dx = \frac{2e^2}{7} \cdot 5x^2 - 5e \bigg|_{a}^{b} = 5\sqrt{5}.
\]

If we let \( a = (5 - \sqrt{5})/2 \) and \( b = (5 + \sqrt{5})/2 \), the total area is

\[
\int_{a}^{b} \left( x^2 + 4x - (x^2 - 6x + 5) \right) \, dx = \int_{a}^{b} -2e^2 + 10x - 5 \, dx = \frac{2e^2}{7} \cdot 5x^2 - 5e \bigg|_{a}^{b} = 5\sqrt{5}.
\]

Figure 8.1.5 Area bounded by two curves.

Exercises 8.1.

Find the area bounded by the curves.

1. \( y = x^3 - x^2 \) and \( y = x^2 \) (the part to the right of the \( y \)-axis)
2. \( x = y^3 \) and \( x = y^2 + 1 \)
3. \( x = 1 - y^2 \) and \( y = x - 1 \)
4. \( x = 3y - y^2 \) and \( x + y = 3 \)
5. \( y = \cos(\pi/2) \) and \( y = x^2 \) (in the first quadrant)
6. \( y = \sin(x/\pi) \) and \( y = x \) (in the first quadrant)
7. \( y = \sqrt{7} \) and \( y = x^2 \)
8. \( y = \sqrt{7} \) and \( y = \sqrt{x^2 + 1} \), \( 0 \leq x \leq 4 \)
9. \( x = 0 \) and \( x = 25 - y^2 \)
10. \( y = \sin x \) and \( y = \sin x \), \( 0 \leq x \leq \pi \)

8.2 Distance, Velocity, Acceleration

We next recall a general principle that will later be applied to distance-velocity-acceleration problems, among other things. If \( F(t) \) is an anti-derivative of \( f(t) \), then

\[
F(x) = F(a) + \int_{a}^{x} f(u) \, du.
\]

(Here \( a \) is the variable of integration, called a “dummy variable,” since it is not the variable in the function \( F(x) \). In general, it is not a good idea to use the same letter as a variable of integration and as a limit of integration. That is \( \int_{a}^{x} f(x) \, dx \) is bad notation, and can lead to errors and confusion.)

An important application of this principle occurs when we are interested in the position of an object at time \( t \) (say on the \( x \)-axis) and we know its position at time \( t_0 \). Let \( \alpha(t) \) denote the position of the object at time \( t \) (its distance from a reference point, such as the origin on the \( x \)-axis). Then the net change in position between \( t_0 \) and \( t \) is \( \alpha(t) - \alpha(t_0) \). Since \( \alpha(t) \) is an anti-derivative of the velocity function \( v(t) \), we can write

\[
v(t) = v(t_0) + \int_{t_0}^{t} v(u) \, du.
\]

Similarly, since the velocity is an anti-derivative of the acceleration function \( a(t) \), we have

\[
v(t) = v(t_0) + \int_{t_0}^{t} a(u) \, du.
\]

EXAMPLE 8.2.1 Suppose an object is acted upon by a constant force \( F \). Find \( v(t) \) and \( s(t) \). By Newton’s law \( F = ma \), so the acceleration is \( F/m \), where \( m \) is the mass of the object. Then we first have

\[
v(t) = v(t_0) + \int_{t_0}^{t} F/m \, du = v_0 + \int_{t_0}^{t} F/m \, (u - t_0) \, du,
\]

using the usual convention \( v_0 = v(t_0) \). Then

\[
a(t) = \alpha(t) = \alpha(t_0) + \int_{t_0}^{t} \left( v_0 + \frac{F}{m} (u - t_0) \right) \, du = \alpha(t_0) + \frac{F}{m} (t - t_0).
\]

For instance, when \( F/m = -g \), the constant of gravitational acceleration, then this is the falling body formula (if we neglect air resistance) familiar from elementary physics:

\[
a_0 = \alpha_0 (t - t_0) = \frac{g}{2} (t - t_0)^2,
\]

or in the common case that \( t_0 = 0 \)

\[
a_0 = \alpha_0 \cdot \frac{g}{2} t^2.
\]

Recall that the integral of the velocity function gives the net distance traveled, that is, the displacement. If you want to know the total distance traveled, you must find out where the velocity function crosses the \( x \)-axis, integrate separately over the time intervals when \( v(t) \) is positive and when \( v(t) \) is negative, and add up the absolute values of the different integrals. For example, if an object is thrown straight upward at 19.6 m/sec, its velocity function is \( v(t) = -9.8t + 19.6 \), using \( g = 9.8 \) m/sec\(^2\) for the force of gravity. This is a straight line which is positive for \( t < 2 \) and negative for \( t > 2 \). The net distance traveled in the first 4 seconds is thus

\[
\int_{0}^{4} (-9.8t + 19.6) \, dt = 0,
\]

while the total distance traveled in the first 4 seconds is

\[
\int_{0}^{2} (-9.8t + 19.6) \, dt + \int_{2}^{4} (-9.8t + 19.6) \, dt = 19.6 + \left| -19.6 \right| = 39.2
\]

19.6 meters up and 19.6 meters down.

EXAMPLE 8.2.2 The acceleration of an object is given by \( a(t) = \cos(\pi t) \), and its velocity at time \( t = 0 \) is \( v(0) = 1/2 \). Find both the net and the total distance traveled in the first 5 seconds.

We compute

\[
v(t) = v(0) + \int_{0}^{t} \cos(\pi u) \, du = \frac{1}{2} \cdot \frac{1}{\pi} \sin(\pi u) \bigg|_{0}^{t} = \frac{1}{2} \cdot \frac{1}{\pi} \sin(\pi t).
\]

The net distance traveled is then

\[
s(3/2) - s(0) = -\frac{3}{\pi} \cdot \frac{1}{2} \cdot \frac{1}{\pi} \sin(\pi t) \bigg|_{0}^{3/2} = -\frac{3}{\pi^2} \approx 0.3409 \text{ meters.}
\]

To find the total distance traveled, we need to know when \( (0.5 + \sin(x)) \) is positive and when it is negative. This function is \( 0 \) when \( \sin(x) = -0.5 \), i.e., when \( t = 7\pi/6 \) or \( 11\pi/6 \), etc. The value \( \pi/2 = 7\pi/6 \), i.e., \( t = 7\pi/6 \), is the only value in the range \( 0 \leq t \leq 1.5 \). Since \( v(t) > 0 \) for \( t < 7\pi/6 \) and \( v(t) < 0 \) for \( t > 7\pi/6 \), the total distance traveled is

\[
\int_{0}^{7\pi/6} \left( \frac{1}{2} + \sin(\pi t) \right) \, dt + \int_{7\pi/6}^{1.5} \left( \frac{1}{2} + \sin(\pi t) \right) \, dt
\]

\[
= \frac{1}{2} \cdot \frac{1}{\pi} \cos(\pi t/6) + \frac{1}{2} \cdot \frac{1}{\pi} \cos(7\pi/6) = \frac{1}{2} - \frac{1}{\pi} \approx 0.409 \text{ meters.}
\]

Exercises 8.2.

For each velocity function find both the net distance and the total distance traveled during the indicated time interval (graph \( v(t) \) to determine when it’s positive and when it’s negative):

1. \( v = \cos(\pi t), 0 \leq t \leq 2 \)
2. \( v = -9.8t + 40, 0 \leq t \leq 10 \)
3. \( v = (t - 3)(1 - t), 0 \leq t \leq 5 \)
4. \( v = \sin(\pi t) + t, 0 \leq t \leq 1 \)
5. An object is shot upwards from ground level with an initial velocity of 2 meters per second; it is subject only to the force of gravity (no air resistance). Find its maximum altitude and the time at which it hits the ground.
6. An object is shot upwards from ground level with an initial velocity of 3 meters per second; it is subject only to the force of gravity (no air resistance). Find its maximum altitude and the time at which it hits the ground.
8.3 Volume

We have seen how to compute certain areas by using integration; some volumes may also be computed by evaluating an integral. Generally, the volumes that we can compute this way have cross-sections that are easy to describe.

EXAMPLE 8.3.1

Find the volume of a right circular cone with base radius 10 and height 20. (A right circular cone is one with a circular base and with the tip of the cone directly over the center of the base.)

The cross-sections that are easy to describe.

EXAMPLE 8.3.2

The base of a solid is the region between \( f(x) = x^2 - 1 \) and \( g(x) = -x^2 + 1 \), and its cross-sections perpendicular to the \( x \)-axis are equilateral triangles, as indicated in figure 8.3.2. The solid has been truncated to show a triangular cross-section above \( x = 1/2 \). Find the volume of the solid.

8.3 Volume

The volume of the pyramid, as shown in figure 8.3.1, on the left is a cross-sectional view, on the right is a 3D view of part of the pyramid with some of the boxes used to approximate the volume.

Each box has volume of the form \( (2x_i)(2x_i)\Delta y \). Unfortunately, there are two variables here; fortunately, we can write \( x \) in terms of \( y \): \( x = 10 - y/2 \) or \( x = 10 - y/2 \). Then the total volume is approximately

\[
\sum_{i=0}^{n-1} (10 - y_i/2)^2 \Delta y
\]

and in the limit we get the volume as the value of an integral:

\[
\int_0^{20} 4(10 - y/2)^2 \, dy = \int_0^{20} (20 - y)^2 \, dy = \frac{(20 - y)^3}{3} \bigg|_0^{20} = \frac{0^3}{3} - \frac{20^3}{3} = \frac{8000}{3}
\]

As you may know, the volume of a pyramid is \( (1/3)(\text{height})(\text{area of base}) \) = \( (1/3)(20)(400) \), which agrees with our answer.

EXAMPLE 8.3.2

The base of a solid is the region between \( f(x) = x^2 - 1 \) and \( g(x) = -x^2 + 1 \), and its cross-sections perpendicular to the \( x \)-axis are equilateral triangles, as indicated in figure 8.3.2. The solid has been truncated to show a triangular cross-section above \( x = 1/2 \). Find the volume of the solid.

and the volume of a thin "slice" is then

\[
(1 - x^2)^2 \sqrt{1 - x^2} \, dx.
\]

Thus the total volume is

\[
\int_0^1 (1 - x^2)^2 \sqrt{1 - x^2} \, dx = \frac{16}{15} \sqrt{3}.
\]

One easy way to get "nice" cross-sections is by rotating a plane figure around a line. For example, in figure 8.3.3 we see a plane region under a curve and between two vertical lines; then the result of rotating this around the \( x \)-axis, and a typical circular cross-section.

8.3 Volume

Of course a real "slice" of this figure will not have straight sides, but we can approximate the volume of the slice by a cylinder or disk with circular top and bottom and straight sides; the volume of this disk will have the form \( \pi r^2 h \). As long as we can write \( r \) in terms of \( x \), we can compute the volume of a rectangular prism (that is, a "box"), we will use some boxes to approximate the problem. We have already computed the volume of a cone; in this case it is

\[
\frac{1}{3} \pi \left( \frac{5}{2} \right)^2 \cdot 20 = \frac{500}{3}.
\]

The integral gives the exact volume of the object.

EXAMPLE 8.3.3

Find the volume of a right circular cone with base radius 10 and height 20. A right circular cone is one with a circular base and with the tip of the cone directly over the center of the base. We can view this cone as produced by the rotation of the line \( y = x/2 \) rotated about the \( x \)-axis, as indicated in figure 8.3.4.

At a particular point on the \( x \)-axis, say \( x_i \), the radius of the resulting cone is the \( y \)-coordinate of the corresponding point on the line, namely \( y_i = x_i/2 \). Thus the total volume is approximately

\[
\sum_{i=0}^{n-1} \pi (x_i/2)^2 \Delta x
\]

and the exact volume is

\[
\int_0^{20} \frac{x^2}{4} \, dx = \frac{20^3}{12} - \frac{2000 \pi}{3}. \]

Note that we can instead do the calculation with a generic height and radius:

\[
\frac{1}{2} \text{base} \cdot \text{height} = (1 - x_i^2) \sqrt{1 - x_i^2}, \]

giving us the usual formula for the volume of a cone.

EXAMPLE 8.3.4

Find the volume of the object generated when the area between \( y = x^2 \) and \( y = -x^2 \) rotated around the \( x \)-axis. This solid has a "hole" in the middle; we can compute the volume by subtracting the volume of the hole from the volume enclosed by the outer surface of the solid. In figure 8.3.5 we show the region that is rotated, the resulting solid with the front half cut away, the cone that forms the outer surface, the horn-shaped hole, and a cross-section perpendicular to the \( x \)-axis.

We have already computed the volume of a cone; in this case it is \( \pi/3 \). At a particular value of \( x \), say \( x_i \), the cross-section of the horn is a circle with radius \( x_i \), so the volume of the horn is

\[
\int_0^{20} \pi (x_i^2) \, dx = \int_0^{20} \pi x^4 \, dx = \frac{20^5}{5} = \frac{320000}{5}
\]

so the desired volume is \( \pi/3 - \pi/5 = 2\pi/15 \).

As with the area between curves, there is an alternate approach that computes the desired volume "all at once" by approximating the volume of the actual solid. We can approximate the volume of a slice of the solid with a washer-shaped volume, as indicated in figure 8.3.5.

The volume of such a washer is the area of the face times the thickness. The thickness, as usual, is \( \Delta x \), while the area of the face is the area of the outer circle minus the area of the inner circle.
the inner circle, say $\pi R^2 - \pi r^2$. In the present example, at a particular $x_i$, the radius $R$ is $x_i$ and $r$ is $x_i^2$. Hence, the whole volume is

$$\int_0^1 \pi x^2 - \pi x^4 \, dx = \pi \left( \frac{x^3}{3} - \frac{x^5}{5} \right) \bigg|_0^1 = \pi \left( \frac{1}{3} - \frac{1}{5} \right) \cdot \frac{2\pi}{15}.$$

Of course, what we have done here is exactly the same calculation as before, except we have in effect recomputed the volume of the outer cone.

Suppose the region between $f(x) = x + 1$ and $g(x) = (x - 1)^2$ is rotated around the $y$-axis; see figure 8.3.6. It is possible, but inconvenient, to compute the volume of the resulting solid by the method we have used so far. The problem is that there are two "kinds" of typical rectangles: those that go from the line to the parabola and those that touch the parabola on both ends. To compute the volume using this approach, we need to break the problem into two parts and compute two integrals:

$$\pi \int_0^1 (1 + \sqrt{y})^2 - (1 - \sqrt{y})^2 \, dy + \pi \int_0^1 (1 + \sqrt{y})^2 - (y - 1)^2 \, dy = \frac{8}{3} \pi + \frac{65}{18} \pi - \frac{27}{2} \pi.$$

If instead we consider a typical vertical rectangle, but still rotate around the $y$-axis, we get a thin "shell" instead of a thin "washer". If we add up the volume of such thin shells we will get an approximation to the true volume. What is the volume of such a shell? Consider the shell at $x_i, \imath$. Imagine that we cut the shell vertically in one place and "unroll" it into a thin, flat sheet. This sheet will be almost a rectangular prism that is $\Delta x$ thick, $f(x_i) - g(x_i)$ tall, and $2\pi x$ wide (namely, the circumference of the shell before it was unrolled). The volume will then be approximately the volume of a rectangular prism with these dimensions: $2\pi x_i f(x_i) - g(x_i)\Delta x$. If we add these up and take the limit as usual, we get the integral

$$\int_0^1 2\pi x f(x) - g(x) \, dx = \int_0^1 2\pi x (x + 1 - (x - 1)^2) \, dx = \frac{27}{2} \pi\text{.}$$

Not only does this accomplish the task with only one integral, the integral is somewhat easier than those in the previous calculation. Things are not always so neat, but it is often the case that one of the two methods will be simpler than the other, so it is worth considering both before starting to do calculations.

8.3 Volume

Exercises 8.3.

1. Verify that $\pi \int_0^1 (1 + \sqrt{y})^2 - (1 - \sqrt{y})^2 \, dy + \pi \int_0^1 (1 + \sqrt{y})^2 - (y - 1)^2 \, dy = \frac{8}{3} \pi + \frac{65}{18} \pi - \frac{27}{2} \pi$.
2. Verify that $\int_0^1 2\pi x (x + 1 - (x - 1)^2) \, dx = \frac{27}{2} \pi$.
3. Verify that $\int_0^1 \pi (1 - x)^3 \, dx = \frac{8}{15} \pi$.
4. Verify that $\int_0^1 2\pi x \sqrt{1 - y} \, dy = \frac{8}{15} \pi$.
5. Use integration to find the volume of the solid obtained by revolving the region bounded by $x + y = 3$ and the $x$ and $y$ axes around the $x$-axis. ⇒
6. Find the volume of the solid obtained by revolving the region bounded by $y = x - x^2$ and the $x$-axis around the $y$-axis. ⇒
7. Find the volume of the solid obtained by revolving the region bounded by $y = \sqrt{2x}$ between $x = 0$ and $x = \frac{\pi}{2}$, the $y$-axis, and the line $y = 1$ around the $x$-axis. ⇒
8. Let $S$ be the region of the $xy$-plane above the curve $y = x^2$ and bounded by the line $y = 1$, on the left by the line $x = 2$, and on the right by the line $x = 4$. Find the volume of the solid obtained by rotating $S$ around (a) the $x$-axis, (b) the line $y = 1$, (c) the point $(1, 1)$, (d) the line $x = 2$. ⇒
9. The equation $x^2 + y^2 + z^2 = 1$ describes an ellipsoid. Find the volume of the solid obtained by rotating the ellipse around the $x$-axis and also around the $y$-axis. These solids are called ellipsoids: one is vaguely rugby-ball shaped, one is sort of flying-saucer shaped, or perhaps squashed-beach-ball-shaped. ⇒
10. Use integration to compute the volume of a sphere of radius $r$. You should of course get the well-known formula $\frac{4}{3} \pi r^3$. ⇒
11. A hemispheric bowl of radius $r$ contains water to a depth $h$. Find the volume of water in the bowl. ⇒
12. The base of a tetrahedron (a triangular pyramid) of height $h$ is an equilateral triangle of side $a$. Its cross-sections perpendicular to an altitude are equilateral triangles. Express its volume $V$ as an integral, and find a formula for $V$ in terms of $h$ and $a$. Verify that your answer is $(\sqrt{3}/2)(\text{area of base})(\text{height})$. ⇒
13. The base of a solid is the region between $f(x) = \cos x$ and $g(x) = -\cos x, -\pi/2 \leq x \leq \pi/2$, and its cross-sections perpendicular to the $x$-axis are squares. Find the volume of the solid.

8.4 Average Value of a Function

The average of some finite set of values is a familiar concept. If, for example, the class scores on a quiz are 10, 9, 10, 8, 7, 5, 7, 6, 2, 7, 8, then the average score is the sum of these numbers divided by the size of the class:

$$\text{average score} = \frac{10 + 9 + 10 + 8 + 7 + 5 + 7 + 6 + 3 + 7 + 8}{12} = 7\frac{7}{12} \approx 6.83.$$

Suppose that between $t = 0$ and $t = 1$ the speed of an object is $\sin(t)$. What is the average speed of the object over that time? The question sounds as if it must make sense, yet we can't merely add up some number of speeds and divide, since the speed is changing continuously over the time interval.

To make sense of "average" in this context, we fall back on the idea of approximation. Consider the speed of the object at tenth of a second interval: $\sin(0), \sin(0.1\pi), \sin(0.2\pi), \sin(0.3\pi), \ldots$, $\sin(0.9\pi)$. The average speed "should" be fairly close to the average of these ten speeds:

$$\frac{1}{10} \sum_{i=0}^{10} \sin(t_i/10) \approx 0.63\text{.}$$

Of course, if we compute more speeds at more times, the average of these speeds should be closer to the "real" average. If we take the average of $n$ speeds at evenly spaced times, we get:

$$\frac{1}{n} \sum_{i=0}^{n-1} \sin(t_i/\pi)\text{.}$$

Here the individual times are $t_i = i/\pi$, so rewriting slightly we have

$$\frac{1}{n} \sum_{i=0}^{n-1} \sin(t_i/\pi)\text{.}$$

This is almost the sort of sum that we know turns into an integral; what's apparently missing is $\Delta t$—but in fact, $\Delta t = 1/n$, the length of each subinterval. So rewriting again:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \sin(t_i/\pi)\Delta t = \int_0^\pi \sin(t) \, dt$$

Now this has exactly the right form, so that in the limit we get

$$\text{average speed} = \int_0^\pi \sin(t) \, dt = \frac{\cos(\pi)}{\pi} - \frac{\cos(0)}{\pi} = \frac{2}{\pi} \approx 0.6366 \approx 0.64.$$  

It’s not entirely obvious from this one simple example how to compute such an average in general. Let’s look at a somewhat more complicated case. Suppose that the velocity
of an object is $10^2 + 5$ feet per second. What is the average velocity between $t = 1$ and $t = 3$? Again we set up an approximation to the average:

$$\frac{1}{3-1} \sum_{i=0}^{n-1} (16r_i^2 + 5),$$

where the values $r_i$ are evenly spaced times between 1 and 3. Once again we are “missing” $\Delta t$, and this time $1/n$ is not the correct value. What is $\Delta t$ in general? It is the length of a subinterval; in this case we take the interval $[1, 3]$ and divide it into $n$ subintervals, so each has length $(3 - 1)/n = 2/n = \Delta t$. Now with the usual “multiply and divide by the same thing” trick we can rewrite the sum:

$$\frac{1}{2} \sum_{i=0}^{n-1} (16r_i^2 + 5)\Delta t = \frac{1}{2} \sum_{i=0}^{n-1} (16r_i^2 + 5)-(\Delta t)^2.$$

In the limit this becomes

$$\int_1^3 (16r^2 + 5)\,dt = \frac{446}{3}.$$ 

Does this seem reasonable? Let’s picture it: in figure 8.4.1 the velocity function together with the horizontal line $y = 223/3 \approx 74.3$. Certainly the height of the horizontal line looks at least plausible for the average height of the curve.

Example 8.5.2

How much work is done in lifting a 10 kilogram object from the surface of the earth, and it does not change appreciably over 5 feet. The work done is

$$\sum_{i=1}^{n} W_i = \sum_{i=1}^{n} \frac{98}{r_i} = \sum_{i=1}^{n} \frac{98}{r_i - 3},$$

or in the limit

$$W = \int_3^5 \frac{98}{r} \,dr.$$

where $r_0$ is the radius of the earth and $r_1$ is $r_0$ plus 10 miles. The work is

$$W = \int_3^5 \frac{98}{r} \,dr = \frac{98}{r_1} - \frac{98}{r_0} = \frac{98}{223} - \frac{98}{3} = \frac{446}{3}.$$ 

Using $r_0 = 20952525$ feet we have $r_1 = 21453525$. The force on the 10 pound weight at the surface of the earth is 10 pounds, so $10 = k/20952525$, giving $k = 4378775865256250$.

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Then

$$W = \int \frac{k}{r} \,dr = \int \frac{4378775865256250}{r} \,dr = 491552355000.$$

Note that if we assume the force due to gravity is 10 pounds over the whole distance we would calculate the work as $10r_1 - r_0 = 30 - 10 = 20$ foot-pounds, somewhat higher since we don’t account for the weakening of the gravitational force.

Example 8.5.3

How much work is done in lifting a 10 kilogram object from the surface of the earth to a distance $D$ from the center of the earth? This is the same problem as before in different units, and we are not specifying a value for $D$. As before

$$W = \int_3^D \frac{k}{r^2} \,dr - \int_3^5 \frac{k}{r^2} \,dr.$$

While “weight in pounds” is a measure of force, “weight in kilograms” is a measure of mass. To convert to force we need to use Newton’s law $F = ma$. At the surface of the earth the acceleration due to gravity is approximately 9.8 meters per second squared, so the force is $F = 10 \cdot 9.8 = 98$. The units here are “kilogram-meters per second squared” or “kg m/s^2”, also known as a Newton (N), so $F = 98$ N. The radium of the earth is approximately 6378 kilometers or 6378100 meters. Now the problem proceeds as before. From $F = k/r^2$ we compute $k = 98 \cdot 6378100 = 6250380000$ foot-pounds.

$$W = \int \frac{k}{r^2} \,dr - \int \frac{k}{r^2} \,dr = \frac{98}{3} - \frac{98}{3} = 0.$$ 

Next is an example in which the force is constant, but there are many objects moving different distances.

Example 8.5.4

Suppose that a water tank is shaped like a right circular cone with the top at the bottom, and has height 10 meters and radius 2 meters at the top. If the tank is full, how much work is required to pump all the water out over the top? Here we have a large number of atoms of water that must be lifted different distances to get to the top of the tank. Fortunately, we don’t really have to deal with individual atoms—we can consider all the atoms at a given depth together.
Example 8.5.5 Suppose \( k = 5 \) for a given spring that has a natural length of 1 meter. Suppose a force is applied that compresses the spring to length 0.8 meters. What is the magnitude of the force? Assuming that the constant \( k \) has appropriate dimensions (namely, \( \text{kg/m}^2 \)), the force is \( 5(0.1 - 0.08) = 5(0.02) = 0.1 \) Newtons.

Exercise 8.5.1

1. A water tank has the shape of the bottom half of a sphere with radius \( r = 1 \) meter. If the tank is full, how much work is required to pump all the water out the top of the tank?

2. A spring has constant \( k = 10 \text{ kg/m}^2 \). How much work is done in compressing it 1/10 meter from its natural length?

3. A force of 2 Newtons will compress a spring from 1 meter (its natural length) to 0.8 meters. How much work is required to stretch the spring from 0.8 meters to 1.5 meters?

4. A circular cross-section through the tank has radius \( r \). At depth \( h \) the circular cross-section has radius \( r - h \). If the tank is full, how much work is required to pump all the water out the top of the tank?

5. Suppose \( k = 5 \) for a given spring that has a natural length of 1 meter. Suppose a force is applied that compresses the spring to length 0.8 meters. Suppose a force is applied that compresses the spring to length 0.8 meters. How much work is done in stretching the spring from 0.8 meters to 1.5 meters?

8.5 Work

Example 8.5.6 How much work is done in compressing the spring in the previous example from its natural length to 0.08 meters? From 0.08 meters to 0.05 meters? How much work is done to stretch the spring from 0.05 meters to 0.15 meters? We can approximate the work by dividing the distance that the spring is compressed (or stretched) into small subintervals. Then the force exerted by the spring is approximately constant over the subinterval, so the work required to compress the spring from \( x_i \) to \( x_{i+1} \) is approximately \( 5(x - 0.1)|\Delta x| \). The total work is therefore

\[
W = \sum_{i=0}^{n-1} 5(x_i - 0.1)|\Delta x|
\]

and in the limit

\[
W = \int_{0.1}^{0.08} 5(x - 0.1)|dx| = 5(0.08 - 0.1)\int_{0.1}^{0.08} = 5(0.08 - 0.1)/2 = 1/1000 \text{ Nm.}
\]

The other values we seek simply use different limits. To compress the spring from 0.08 meters to 0.05 meters takes

\[
W = \int_{0.08}^{0.05} 5(x - 0.1)|dx| = 5(0.05 - 0.1)/2 = 5(0.08 - 0.1)/2 = 21/400 \text{ Nm.}
\]

and to stretch the spring from 0.05 meters to 0.15 meters requires

\[
W = \int_{0.05}^{0.15} 5(x - 0.1)|dx| = 5(0.15 - 0.1)/2 = 5(0.1 - 0.1)/2 = 1/100 \text{ Nm.}
\]