6.1 Optimization

Many important applied problems involve finding the best way to accomplish some task. Often this involves finding the maximum or minimum value of some function: the minimum time to make a certain journey, the minimum cost for doing a task, the maximum power that can be generated by a device, and so on. Many of these problems can be solved by finding the appropriate function and then using techniques of calculus to find the maximum or the minimum value required.

Generally such a problem will have the following mathematical form. Find the largest (or smallest) value of \( f(x) \) when \( a \leq x \leq b \). Sometimes \( a \) or \( b \) are infinite, but frequently the real world imposes some constraint on the values that \( x \) may have.

Such a problem differs in two ways from the local maximum and minimum problems we encountered when graphing functions. We are interested only in the function between \( a \) and \( b \), and we want to know the largest or smallest value that \( f(x) \) takes on, not merely values that are the largest or smallest in a small interval. That is, we seek not a local maximum or minimum but a global maximum or minimum, sometimes also called an absolute maximum or minimum.

Any global maximum or minimum must of course be a local maximum or minimum. If we find all possible local extrema, then the global maximum, if it exists, must be the largest of the local maxima and the global minimum, if it exists, must be the smallest of the local minima. We already know where local extrema can occur: only at those points at which \( f'(x) \) is zero or undefined. Actually, there are two additional points at which a maximum or minimum can occur if the endpoints \( a \) and \( b \) are not infinite, namely, at \( a \) and \( b \). We have not previously considered such points because we have not been interested in limiting a function to a small interval. An example should make this clear.

**EXAMPLE 6.1.1** Find the maximum and minimum values of \( f(x) = x^2 \) on the interval \([-2, 1]\), shown in figure 6.1.1. We compute \( f'(x) = 2x \), which is zero at \( x = 0 \) and is always defined.

Since \( f'(1) = 2 \) we would not normally flag \( x = 1 \) as a point of interest, but it is clear from the graph that when \( f(x) \) is restricted to \([-2, 1]\) there is a local maximum at \( x = 1 \). Likewise we would not normally pay attention to \( x = -2 \), but since we have truncated \( f \) at \(-2 \) we have introduced a new local maximum there as well.

In a technical sense nothing new is going on here: When we truncate \( f \) we actually create a new function, let’s call it \( g \), that is defined only on the interval \([-2, 1]\). If we try to compute the derivative of this new function we actually find that it does not have a derivative at \(-2 \) or 1. Why? Because to compute the derivative at 1 we must compute the limit

\[
\lim_{\Delta x \to 0} g(1 + \Delta x) - g(1)
\]

This limit does not exist because when \( \Delta x > 0 \), \( g(1 + \Delta x) \) is not defined. It is simpler, however, simply to remember that we must always check the endpoints.

So the function \( g \), that is, \( f \) restricted to \([-2, 1]\), has one critical value and two finite endpoints, any of which might be the global maximum or minimum. We could first determine which of these are local maximum or minimum points (or neither), then the largest local maximum must be the global maximum and the smallest local minimum must be the global minimum. It is usually easier, however, to compute the value of \( f \) at every point at which the global maximum or minimum might occur; the largest of these is the global maximum, the smallest the global minimum.

So we compute \( f(-2) = 4 \), \( f(0) = 0 \), \( f(1) = 1 \). The global maximum is 4 at \( x = -2 \) and the global minimum is 0 at \( x = 0 \).

**THEOREM 6.1.2 Extreme value theorem** If \( f \) is continuous on a closed interval \([a, b]\) always has both a global maximum and a global minimum, so examining the critical values and the endpoints is enough:

**EXAMPLE 6.1.2** Let \( f(x) = x^2 + 4x - 3 \). Find the maximum value of \( f(x) \) on the interval \([0, 4]\). First note that \( f'(x) = 2x + 4 = 0 \) when \( x = -2 \), and \( f(0) = -3 \) and \( f(4) = 15 \). The largest value of \( f(x) \) on the interval \([0, 4]\) is \( f(4) = 15 \).

**EXAMPLE 6.1.3** Let \( f(x) = -x^2 + 4x - 3 \). Find the maximum value of \( f(x) \) on the interval \([-1, 1]\). First note that \( f'(x) = -2x + 4 = 0 \) when \( x = 2 \). But \( x = 2 \) is not in the interval, so we don’t use it. Thus the only two points to be checked are the endpoints: \( f(-1) = -8 \) and \( f(1) = 0 \). So the largest value of \( f(x) \) on \([-1, 1]\) is \( f(1) = 0 \).

**EXAMPLE 6.1.4** Let \( f(x) = -x^2 + 4x - 3 \). Find the maximum value of \( f(x) \) on the interval \([-1, 1]\). First note that \( f'(x) = -2x + 4 = 0 \) when \( x = 2 \). But \( x = 2 \) is not in the interval, so we don’t use it. Thus the only two points to be checked are the endpoints: \( f(-1) = -8 \) and \( f(1) = 0 \). So the largest value of \( f(x) \) on \([-1, 1]\) is \( f(1) = 0 \).

**EXAMPLE 6.1.5** Find the maximum and minimum values of the function \( f(x) = 7 + |x - 2| \) for \( x \) between 1 and 4 inclusive. The derivative \( f'(x) \) is never zero, but \( f'(x) \) is undefined at \( x = 2 \), so we compute \( f(3) = 7 \). Checking the end points we get \( f(1) = 8 \) and \( f(4) = 9 \). The smallest of these numbers is \( f(3) = 7 \), which is, therefore, the minimum value of \( f(x) \) on the interval \( 1 \leq x \leq 4 \), and the maximum value of \( f(x) \) is \( f(4) = 9 \).

**EXAMPLE 6.1.6** Find all local maxima and minima for \( f(x) = x^3 - x \), and determine whether there is a global maximum or minimum on the open interval \((-2, 2)\). In example 5.1.2 we found a local maximum at \((-\sqrt{3}/3, 2\sqrt{3}/9)\) and a local minimum at \((\sqrt{3}/3, -2\sqrt{3}/9)\). Since the endpoints are not in the interval \((-2, 2)\) they cannot be considered. Is the lone local maximum a global maximum? Here we must look more closely at the graph.

We know that on the closed interval \([-\sqrt{3}/3, \sqrt{3}/3]\) there is a global maximum at \( x = -\sqrt{3}/3 \) and a global minimum at \( x = \sqrt{3}/3 \). So the question becomes: what happens between \(-2 \) and \(-\sqrt{3}/3 \), and between \(\sqrt{3}/3 \) and \(2 \)? Since there is a local minimum at \( x = \sqrt{3}/3 \), the graph must continue up to the right, since there are no more critical values. This means no value of \( f \) will be less than \(-2\sqrt{3}/3 \) between \(-\sqrt{3}/3 \) and \(2 \), but it says nothing about whether we might find a value larger than the local maximum \(2\sqrt{3}/3 \).

How can we tell? Since the function increases to the right of \(\sqrt{3}/3 \), we need to know which of the functions do “close to” \(2 \). Here the easiest test is to pick a number and do a computation to get some idea of what’s going on. Since \( f(1.9) = 4.959 > 2\sqrt{3}/3 \), there is no global maximum at \(-\sqrt{3}/3 \), and hence no global maximum at all. (How can we tell that \(4.959 > 2\sqrt{3}/3 \)? We can use a calculator to approximate the right hand side; if it is not even close to \(4.959 \) we can take this as decisive. Since \(2\sqrt{3}/3 \approx 0.3849 \), there’s really no question. Funny things can happen in the rounding done by computers and calculators, however, so we might be a little more careful, especially if the values come out quite close.

In this case we can convert the relation \(4.959 > 2\sqrt{3}/3 \) into \((2/25)^{2/3} > \sqrt{3} \) and ask whether this is true. Since the left side is clearly larger than \(4 \cdot 4 \) which is clearly larger than \(2\sqrt{3}/3 \), this settles the question.

A similar analysis shows that there is also no global minimum. The graph of \( f(x) \) on \((-2, 2) \) is shown in figure 6.1.2.

**EXAMPLE 6.1.7** Of all rectangles of area 100, which has the smallest perimeter? First we must translate this into a purely mathematical problem in which we want to find the minimum value of a function. If \( x \) denotes one of the sides of the rectangle, then the adjacent side must be 100/\( x \) in order that the area be 100. So the function we want to minimize is \( f(x) = 2x + \frac{200}{x} \). If we denote by \( f''(x) \) the second derivative of \( f \), we find

\[
f''(x) = 2 - \frac{200}{x^3}
\]

Since \( f''(10) = 2 - \frac{1}{50} > 0 \), there is a local minimum at \( x = 10 \). We must check the endpoints \( x = 0 \) and \( x = 100 \) to make sure they are not local minimums.

**Figure 6.1.1** The function \( f(x) = x^2 \) restricted to \([-2, 1]\).

**Figure 6.1.2** The function \( f(x) = x^3 - x \).
to minimize is
\[ f(x) = 2x + \frac{2100}{x} \]
since the perimeter is twice the length plus twice the width of the rectangle. Not all values of \( x \) make sense in this problem: lengths of sides of rectangles must be positive, so \( x > 0 \).

If \( x > 0 \) then so is \( 100/x \), so we need no second condition on \( x \).

We next find \( f'(x) \) and set it equal to zero: \( 0 = f'(x) = 2 - \frac{2100}{x^2} \). Solving \( f'(x) = 0 \) for \( x \) gives us \( x = \pm 10 \). We are interested only in \( x > 0 \), so only the value \( x = 10 \) is of interest. Since \( f''(x) \) is defined everywhere on the interval \((0, \infty)\), there are no more critical values, and there are no endpoints. Is there a local maximum, minimum, or neither at \( x = 10 \)? The second derivative is \( f''(x) = 400/x^3 \), and \( f''(10) > 0 \), so there is a local minimum. Since there is only one critical value, this is also the global minimum, so the rectangle with smallest perimeter is the \( 10 \times 10 \) square.

**EXAMPLE 6.1.8** You want to sell a certain number \( n \) of items in order to maximize your profit. Market research tells you that if you set the price at \$1.50, you will be able to sell 5000 items, and for every 10 cents you lower the price below \$1.50 you will be able to sell another 1000 items. Suppose that your fixed costs ("start-up costs") total \$2000, and the per item cost of production ("marginal cost") is \$0.50. Find the price to set per item and the number of items sold in order to maximize profit, and also determine the maximum profit you can get.

The first step is to convert the problem into a function maximization problem. Since we want to maximize profit by setting the price per item, we should look for a function \( P(x) \) representing the profit when the price per item is \( x \). Profit minus revenue minus costs gives us the profit function, and revenue is number of items times sales price, so we get \( P = nx - 2000 - 0.5nx \).

The number of items sold is itself a function of \( x \), namely, \( n = 5000 + 1000(1.5 - x)/10 \), because \((1.5 - x)/10 \) is the number of tens of 10 cents that the price is below \$1.50. Now we substitute for \( n \) in the profit function:
\[ P(x) = (5000 + 1000((1.5 - x)/10))x - 2000 - 0.5(5000 + 1000((1.5 - x)/10)) \]
\[ = -1000x^2 + 25000x - 12000 \]
We want to know the maximum value of this function when \( x \) is between 0 and 1.5. The derivative is \( P'(x) = -20000x + 25000 \), which is zero when \( x = 1.25 \). Since \( P''(x) = -20000 < 0 \), there must be a local maximum at \( x = 1.25 \), and since this is the only critical value it must be a global maximum as well. (Alternatively, we could compute \( P(0) = -12000 \), \( P(1.25) = 3625 \), and \( P(1.5) = 3000 \) and note that \( P(1.25) \) is the maximum of these.) Thus the maximum profit is \$3625, attained when we set the price at \$1.25 and sell 7500 items.

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**EXAMPLE 6.1.9** Find the largest rectangle (that is, the rectangle with largest area) that fits inside the graph of the parabola \( y = x^2 \) below the line \( y = x - a \) (an unspecified constant value), with the top side of the rectangle on the horizontal line \( y = a \); see figure 6.1.3. We want to find the maximum value of some function \( A(x) \) representing area. Perhaps the hardest part of this problem is deciding what \( x \) should represent. The lower right corner of the rectangle is at \((x, x^2)\), and once this is chosen the rectangle is completely determined. So we can let the \( x \) in \( A(x) \) be the \( x \) of the parabola \( f(x) = x^2 \). Then the area is \( A = (2x)(a-x^2) - 2x^2 + 2ax \). We want the maximum value of \( A(x) \) when \( x \) is in \([0, \sqrt{a}]\). (You might object to allowing \( x = 0 \) or \( x = \sqrt{a} \), since then the "rectangle" has either no width or no height, so is not "really" a rectangle. But the problem is somewhat easier if we simply allow such rectangles, which have zero area.)

Setting 0 = \( A'(x) = -6x^2 + 2ax \) we get \( x = \sqrt{a/3} \) as the only critical value. Testing this and the two endpoints, we have \( A(0) = A(\sqrt{a/3}) = 0 \) and \( A(\sqrt{a}) = (4/9)a^{3/2} \). The maximum area thus occurs when the rectangle has dimensions \( \sqrt{a/3} \times (2/3)a \).

**EXAMPLE 6.1.10** If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by "cone" we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.)

Let \( R \) be the radius of the sphere, and let \( r \) and \( h \) be the base radius and height of the cone inside the sphere. What we want to maximize is the volume of the cone: \( \pi r^2h/3 \). Here \( R \) is a fixed value, but \( r \) and \( h \) can vary. Namely, we could choose \( r \) to be as large as possible—equal to \( R \)—by taking the height equal to \( R \); or we could make the cone’s height \( h \) larger at the expense of making \( r \) a little less than \( R \). See the cross-section depicted in figure 6.1.4.

**EXAMPLE 6.1.11** You are making cylindrical containers to contain a given volume. Suppose that the top and bottom are made of a material that is \( N \) times as expensive (cost per unit area) as the material used for the lateral side of the cylinder. Find (in terms of \( N \)) the ratio of height to base radius of the cylinder that minimizes the cost of making the containers.

Let us first choose letters to represent various things: \( h \) for the height, \( r \) for the base radius, \( V \) for the volume of the cylinder, and \( c \) for the cost per unit area of the lateral side of the cylinder; \( V \) and \( c \) are constants, \( h \) and \( r \) are variables. Now we can write the cost of materials:
\[ c(2\pi rh + Nc(2\pi r^2)) \]
Again we have two variables; the relationship is provided by the fixed volume of the cylinder: \( V = \pi r^2h \). We use this relationship to eliminate \( h \) (we could eliminate \( r \), but it’s a little easier if we eliminate \( h \), which appears in only one place in the above formula for cost). The result is
\[ f''(r) = 2\pi r V \frac{V}{r^3} + 2Nc(2\pi r^2) = \frac{2V}{r^2} + 2Ncr^2 \]
We want to know the minimum value of this function when \( r \) is in \((0, \infty)\). We now set 0 = \( f'(r) = -2V/r^2 + 4Ncr \), giving \( r = \sqrt{V/(2Nc)} \). Since \( f''(r) = 4cV/r^3 + 4Ncr \) is positive when \( r \) is positive, there is a local minimum at the critical value, and hence a global minimum since there is only one critical value.

Finally, since \( h = V/(\pi r^2) \),
\[ \frac{h}{r} = \frac{V}{\pi r^3} = \frac{V}{(V/(2\pi c))^2} = 2N, \]
so the minimum cost occurs when the height \( h \) is \( 2N \) times the radius. If, for example, there is no difference in the cost of materials, the height is twice the radius (or the height is equal to the diameter).
EXAMPLE 6.1.12 Suppose you want to reach a point A that is located across the sand from a nearby road (see figure 6.1.5). Suppose that the road is straight, and B is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from B. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A?

Let x be the distance short of C where you turn off, i.e., the distance from B to C. We want to minimize the total travel time. Recall that when traveling at constant velocity, time is always divided by velocity.

You travel the distance $\sqrt{a^2 + x^2}$ at speed v, and then the distance $\sqrt{w^2 + x^2}$ at speed w. Since $\sqrt{a^2 + x^2} = a - x$ and, by the Pythagorean theorem, $\sqrt{w^2 + x^2} = \sqrt{x^2 + b^2}$, the total time for the trip is

$$f(x) = \frac{a - x}{v} + \frac{x}{w\sqrt{x^2 + b^2}}.$$

We want to find the minimum value of $f(x)$ when $x$ is between 0 and $a$. As usual we set $f'(x) = 0$ and solve for $x$:

$$0 - f''(x) = \frac{2w^2}{(x^2 + b^2)^{3/2}}.$$

Since this is always positive there is a local minimum at the critical point, and so it is a global minimum as well.

If the critical value is not in $[0,a]$ it is larger than $a$. In this case you must check at one of the endpoints. We can compute

$$f'(0) = \frac{a}{v} + \frac{1}{w},$$

$$f'(a) = \frac{\sqrt{a^2 + b^2}}{w}.$$

In example 6.1.12, what happens if $w = v$, i.e., your speed on sand is at least your speed on the road?

A container holding a fixed volume is being made in the shape of a cylinder with a hemispherical top. (The hemispherical top has the same radius as the cylinder.) Find the ratio of height to radius of the cylinder which minimizes the cost of the container if (a) the cost per unit area of the top is twice as great as the cost per unit area of the side; and the container is made with no bottom; (b) the same as in (a), except that the container is made with a circular bottom, for which the cost per unit area is 1.5 times the cost per unit area of the side.

A piece of cardboard is 1 meter by 1 meter. A square is to be cut from each corner and the sides folded up to make an open-top box. What are the dimensions of the box with maximum possible volume?
30. If you fit the cone with the largest possible surface area (lateral area plus area of base) into a sphere, what percent of the volume of the sphere is occupied by the cone? \( \Rightarrow \)

31. Two electrical charges, one a positive charge \( A \) of magnitude \( a \) and the other a negative charge \( B \) of magnitude \( b \), are located a distance \( c \) apart. A positively charged particle \( P \) is situated on the line between \( A \) and \( B \). Find where \( P \) should be put so that the pull away from \( A \) towards \( B \) is minimal. Here assume that the force from each charge is proportional to the strength of the source and inversely proportional to the square of the distance from the source. \( \Rightarrow \)

32. Find the fraction of the area of a triangle that is occupied by the largest rectangle that can be drawn in the triangle (with one of its sides along a side of the triangle). Show that this fraction does not depend on the dimensions of the given triangle. \( \Rightarrow \)

33. How are your answers to Problem 30 affected if the cost per item for the \( x \) items, instead of being simply \( 2x \), decreases below \( 2x \) in proportion to \( x \) (because of economy of scale and volume discounts) by 1 cent for each 25 items produced? \( \Rightarrow \)

34. You are standing near the side of a large wading pool of uniform depth when you see a child in trouble. You can run up at a speed \( v \) on \( c \) and at a slower speed \( v \) in the water. Your perpendicular distance from the side of the pool is \( a \), the child's perpendicular distance is \( b \), and the distance along the side of the pool between the closest point to you and the closest point to the child is \( c \) (see the figure below). Without stopping to do any calculations, you instinctively choose the quickest route (shown in the figure) and save the child. Our purpose is to derive a relation between the angle \( \theta \), your path makes with the perpendicular to the side of the pool when you've run on land, and the angle \( 
\begin{align*}
\frac{dx}{dt} &= 22 \quad \text{and} \quad \frac{dz}{dt} = 3, \\
\frac{dy}{dt} &= 2 \quad \text{and} \quad 6.3 = 36.
\end{align*}
\]

In many cases, particularly interesting ones, \( x \) and \( y \) will be related in some other way, for example \( x = f(y) \), or \( F(x, y) = k \), or perhaps \( F(x, y) = G(x, y) \), where \( F(x, y) \) and \( G(x, y) \) are expressions involving both variables. In all cases, you can solve the related rates problem by taking the derivative of both sides, plugging in all the known values (namely, \( x \), \( y \), and \( z \)), and then solving for \( z \).

To summarize, here are the steps in doing a related rates problem:

1. Decide what the two variables are.
2. Find an equation relating them.
3. Take \( \frac{dx}{dt} \) of both sides.
4. Plug in all known values at the instant in question.
5. Solve for the unknown rate.

**EXAMPLE 6.2.6.1** Suppose a boat is moving along a path described by \( y = x^2 \), that is, it is moving on a parabolic path. At a particular time, say \( t = 5 \), the \( x \) coordinate is 6 and we measure the speed at which the \( x \) coordinate of the object is changing and find that \( \frac{dx}{dt} = 3 \). At the same time, how fast is the \( y \) coordinate changing?

**EXAMPLE 6.2.6.2** A plane is flying directly away from you at 500 mph at an altitude of 3 miles. How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

To see what's going on, we first draw a schematic representation of the situation, as in figure 6.2.1.

Because the plane is in level flight directly away from you, the rate at which \( x \) changes is the speed of the plane, \( \frac{dx}{dt} = 500 \). The distance between you and the plane is \( y \), so it is \( \frac{dy}{dt} \) that we wish to know. By the Pythagorean Theorem we know that \( x^2 + y^2 = 49 \).
6.2 Related Rates

1. The point $P$ is known values and solve for the unknown rate.

2. With time, but in reality there are just two, as the others can be expressed in terms of just two. But sometimes there really are several variables that change with time; as long as you know the rates of change of all but one of them you can find the rate of change of the remaining one. As in the case when there were just two variables, take the derivative of both sides of the equation relating all of the variables, and then substitute all of the known values and solve for the unknown rate.

**EXAMPLE 6.2.6** A road running north to south crosses a road going east to west at the point $P$. Car A is driving north along the first road, and car B is driving east along the second road. At a particular time car A is 10 kilometers to the north of $P$ and traveling at 80 km/hr, while car B is 15 kilometers to the east of $P$ and traveling at 100 km/hr. How fast is the distance between the two cars changing?

3. Notice how this problem differs from example 6.2.2. In both cases we started with the Pythagorean Theorem and took derivatives on both sides. However, in example 6.2.2 one of the sides was a constant (the altitude of the plane), and so the derivative of the square of that side of the triangle was simply zero. In this example, on the other hand, all three sides of the right triangle are variables, even though we are interested in a specific value of each side of the triangle (namely, when the sides have lengths 10 and 15). Make sure that you understand at the start of the problem what are the variables and what are the constants.

**Exercises 6.2.**

1. A man 1.8 meters tall walks at the rate of 1 meter per second toward a streetlight that is 4 meters above the ground. At what rate is the tip of his shadow shortening? At what rate is his shadow lengthening?

2. A police helicopter is flying at 150 mph at a constant altitude of 0.5 mile above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 1 mile from the helicopter, and that this distance is decreasing at 190 mph. Find the speed of the car.

3. A police helicopter is flying at 200 kilometers per hour at a constant altitude of 1 km above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 2 kilometers from the helicopter, and that this distance is decreasing at 250 km/h. Find the speed of the car.

4. A baseball diamond is a square 90 ft on a side. A player runs from first base to second base at 15 ft/sec. At what rate is the player's distance from third base decreasing when she is half way from first to second base?

5. A woman 5 ft tall walks at the rate of 3.5 ft/sec away from a streetlight that is 12 ft above the ground. At what rate is the tip of her shadow moving? At what rate is her shadow lengthening?

6. A trough is full of water. At what rate (in ft/min) is the water level rising when the depth of water (at the deepest point) is 4 m? Note: the volume of any "conical" shape (including pyramids) is \( V = \frac{1}{3} \pi r^2 h \), where the radius is the top of the wall is being pulled up the wall at 1 meter per second. How fast is the foot of the ladder is pulled away from the wall at the rate of 0.6 ft/sec. How fast is the top sliding down the wall when the foot of the ladder is 5 ft from the wall?

7. Sand is poured onto a surface at 15 cm$^3$/sec, forming a conical pile whose base diameter is always equal to its altitude. How fast is the altitude of the pile increasing when the pile is 3 cm high?

8. A boat is pulled in to a dock by a rope with one end attached to the front of the boat and the other end passing through a ring attached to the dock at a point 5 ft higher than the front of the boat. The rope is being pulled through the ring at the rate of 0.6 ft/sec. How fast is the boat approaching the dock when 11 ft of rope are out?

9. A balloon is at a height of 50 meters, and is rising at the constant rate of 5 m/sec. A bicyclist passes beneath it, traveling in a straight line at the constant speed of 10 m/sec. How fast is the distance between the bicyclist and the balloon increasing 2 seconds later?

10. A right triangle has sides $a$, $b$, and $c$. Taking derivatives we get $a^2 + b^2 = c^2$. From this equation, we can solve for $c^2 = a^2 + b^2$. Differentiating with respect to time, we get $2a \frac{da}{dt} + 2b \frac{db}{dt} = 2c \frac{dc}{dt}$, so

\[
\frac{dc}{dt} = \frac{a \frac{da}{dt} + b \frac{db}{dt}}{c}.
\]

Substituting known values we get:

\[
\frac{dc}{dt} = \frac{80 \times 1.5 + 100}{\sqrt{80^2 + 100^2}} = \frac{660}{\sqrt{184000}} = \approx 127.6 \text{ km/hr}
\]

at the time of interest.

**Figure 6.2.5 Sunrise or sunset.**

11. **Figure 6.2.6 Trough.**

12. **Figure 6.2.7 Falling ball.**

13. The trough shown in figure 6.2.6 is constructed by fastening together three slabs of wood of dimensions 10 ft $\times$ 1 ft; and then attaching the construction to a wooden wall at each end. The angle $\theta$ was originally $30^{\circ}$, but because of poor construction the sides are collapsing. The trough is full of water. At what rate (in $\text{ft}^3/\text{sec}$) is the water spilling out over the top of the trough if the sides have each fallen to an angle of $45^\circ$, and are collapsing at the rate of $1^\circ$ per second?

14. Do example 6.2.6 assuming that the angle between the two roads is $120^\circ$ instead of $90^\circ$ (that is, the "north-south" road actually goes in a somewhat northeasterly direction from $P$). Recall the law of sines: $c = \frac{a \sin \theta}{\sin \alpha}$.

15. Do example 6.2.6 assuming that car $A$ is 300 meters north of $P$, car $B$ is 400 meters east of $P$, both cars are going at constant speed toward $P$, and the two cars will collide in 10 seconds.

16. Do example 6.2.6 assuming that 8 seconds ago car $A$ started from rest at $P$ and has been picking up speed at the steady rate of 5 m/sec$^2$, and 6 seconds after car $A$ started car $B$ passed $P$ moving east at constant speed 60 m/sec.

17. Referring again to example 6.2.6, suppose that instead of car $B$ an airplane is flying at speed 200 km/hr to the east of $P$ at an altitude of 2 km, as depicted in figure 6.2.8. How fast is the distance between car and airplane changing?

18. Referring again to example 6.2.6, suppose that instead of car $B$ an airplane is flying at speed 200 km/hr to the east of $P$ at an altitude of 2 km, that it is gaining altitude at 10 km/hr. How fast is the distance between car and airplane changing?

19. A light shines from the top of a pole 20 m high. An object is dropped from the same height from a point 10 m away, so that its height at time $t$ seconds is $h(t) = 20 - 9.8t^2/2$. How fast is the object's shadow moving on the ground one second later?
25. The two blades of a pair of scissors are fastened at the point A, as shown in Figure 6.2.9. Let \( x \) denote the distance from \( A \) to the tip of the blade (the point \( B \)). Let \( \beta \) denote the angle at the tip of the blade that is formed by the line \( AB \) and the bottom edge of the blade, line \( BC \), and let \( \theta \) denote the angle between \( AB \) and the horizontal. Suppose that a piece of paper is cut in such a way that the center of the scissors at \( A \) is fixed, and the paper is also fixed. As the blades are closed (i.e., the angle \( \theta \) is in the diagram is decreased), the distance \( x \) between \( A \) and \( C \) increases, cutting the paper.

a. Express \( x \) in terms of \( \theta \) and \( \beta \).

b. Express \( dx/d\theta \) in terms of \( a, \theta, \beta \), and \( db/d\theta \).

c. Suppose that the distance \( a \) is 20 cm, and the angle \( \beta \) is 5°. Further suppose that \( \theta \) is decreasing at 50 deg/sec. At the instant when \( \theta = 30^\circ \), find the rate (in cm/sec) at which the paper is being cut.

6.3 Newton’s Method

Suppose you have a function \( f(x) \), and you want to find as accurately as possible where it crosses the \( x \)-axis; in other words, you want to solve \( f(x) = 0 \). Suppose you know of no way to find an exact solution by any algebraic procedure, but you are able to use an approximation, provided it can be made quite close to the true value. Newton’s method is a way to find a solution to the equation to as many decimal places as you want. It is what is called an “iterative procedure,” meaning that it can be repeated again and again to get an answer of greater and greater accuracy. Iterative procedures like Newton’s method are well suited to programming for a computer. Newton’s method uses the fact that the tangent line to a curve is a good approximation to the curve near the point of tangency.

EXAMPLE 6.3.1 Approximate \( \sqrt{3} \). Since \( \sqrt{3} \) is a solution to \( x^2 - 3 = 0 \), we use \( f(x) = x^2 - 3 \). We start by guessing something reasonably close to the true value; this is usually easy to do; let’s use \( x_0 = 2 \). Now use the tangent line to the curve when \( x = 2 \) as an approximation to the curve, as shown in Figure 6.3.1. Since \( f'(x) = 2x \), the slope of this tangent line is 4 and its equation is \( y = 4x - 7 \). The tangent line is quite close to \( f(x) \), so it crosses the \( x \)-axis near the point at which \( f(x) \) crosses, that is, near \( \sqrt{3} \). It is easy to find where the tangent line crosses the \( x \)-axis; solve \(-4x + 7 = 0 \) to get \( x = 7/4 = 1.75 \). This is certainly a better approximation than 2, but let us say not close enough. We can improve it by doing the same thing again: find the tangent line at \( x = 1.75 \), find where this new tangent line crosses the \( x \)-axis, and use that as a better approximation. We can continue indefinitely, though it gets a bit tedious. Let’s see if we can shortcut the process. Suppose the host approximation to the intercept we have so far is \( x_i \). To find a better approximation we will always do the same thing: find the slope of the tangent line at \( x_i \), find the equation of the tangent line, and use the \( x \)-intercept. The slope is \( f'(x_i) \). The tangent line is \( y = (2x_i)(x - x_i) + (x_i^2 - 3) \), using the point-slope formula for a line. Finally, the intercept is found by solving \( 0 = (2x_i)(x - x_i) + (x_i^2 - 3) \). With a little algebra this turns into \( x = (x_i^2 + 3)/(2x_i) \), this is the next approximation, which we naturally call \( x_{i+1} \). Instead of doing the whole tangent line computation every time we can simply use this formula to get as many approximations as we want. Starting with \( x_0 = 2 \), we get \( x_1 = (2^2+3)/(2*2) = (2^2+3)/4 = 7/4 \) (the same approximation we got above, of course). \( x_2 = (7/4)^2 + 3)/(7/2) = 97/56 \equiv 1.73214 \), \( x_3 \approx 1.73205 \), and so on. This is still a bit tedious by hand, but with a calculator or, even better, a good computer program, it is quite easy to get many, many approximations. We might guess already that 1.73205 is accurate to two decimal places, and in fact it turns out that it is accurate to 5 places.

Let’s think about this process in more general terms. We want to approximate a solution to \( f(x) = 0 \). We start with a rough guess, which we call \( x_0 \). We use the tangent line to \( f(x) \) to get a new approximation that we hope will be closer to the true value. What is the equation of the tangent line when \( x = x_i \)? The slope is \( f'(x_i) \) and the line goes through \((x_i, f(x_i))\), so the equation of the line is \( y = f'(x_i)(x - x_i) + f(x_i) \). We now find where this crosses the \( x \)-axis by substituting \( y = 0 \) and solving for \( x \):

\[
x = x_i - \frac{f'(x_i)(x - x_i) + f(x_i)}{f'(x_i)} = x_i - \frac{f(x_i)}{f'(x_i)}
\]

We will typically want to compute more than one of these improved approximations, so we number them consecutively; from \( x_0 \) we have computed \( x_1 \)

\[
x_1 = x_i - \frac{f(x_i)}{f'(x_i)}\]

and in general from \( x_i \) we compute \( x_{i+1} \):

\[
x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}
\]

EXAMPLE 6.3.2 Returning to the previous example, \( f(x) = x^2 - 3 \), \( f'(x) = 2x \), and the formula becomes \( x_{i+1} = x_i - (x_i^2 - 3)/(2x_i) = (x_i^2 + 3)/(2x_i) \), as before.

In practice, which is to say, if you need to approximate a value in the course of designing a bridge or a building or an airplane, you will need to have some confidence that the approximation you settle on is accurate enough. As a rule of thumb, once a certain number of decimal places stop changing from one approximation to the next it is likely that those decimal places are correct. Still, this may not be enough assurance, in which case we can test the result for accuracy.

EXAMPLE 6.3.3 Find the \( x \)-coordinate of the intersection of the curves \( y = 2x \) and \( y = \tan x \), accurate to three decimal places. To put this in the context of Newton’s method,
6.4 Linear Approximations

Newton’s method is one example of the usefulness of the tangent line as an approximation to a curve. Here we explore another such application.

Recall that the tangent line to \( f(x) \) at a point \( x = a \) is given by \( L(x) = f(a)(x-a) + f(a) \). The tangent line in this context is also called the linear approximation to \( f \) at \( a \).

If \( f \) is differentiable at \( a \), then under a microscope \( f \) will look very much like a straight line. Figure 6.4.1 shows a tangent line to \( y = x^2 \) at three different magnifications.

If we want to approximate \( f(b) \), because computing it exactly is difficult, we can approximate the value using a linear approximation, provided that we can compute the tangent line at some close to \( b \).

![Figure 6.4.1 The linear approximation to \( y = x^2 \).](image)

**EXAMPLE 6.4.1** Let \( f(x) = \sqrt{x + 1} \). Then \( f'(x) = 1/(2\sqrt{x + 1}) \). The linear approximation to \( f \) at \( x = 5 \) is \( L(x) = 1/(2\sqrt{5 + 1}) \). As an immediate application we can approximate square roots of numbers near 9 by hand. To estimate \( \sqrt{10} \), we substitute \( 6 \) into the linear approximation instead of into \( f(x) \), so \( \sqrt{10} \approx 3 \). This rounds to 3.17 while the square root of 10 is actually 3.16 to two decimal places, so this estimate is only accurate to one decimal place. This is not too surprising, as 10 is really not very close to 9; on the other hand, for many calculations, 3.2 would be accurate enough.

With modern calculators and computer software it may not appear necessary to use linear approximations. But in fact they are quite useful. In cases requiring an explicit numerical approximation, they allow us to get a quick rough estimate which can be used as a "reality check" on a more complex calculation. In some complex calculations involving numerical approximation to \( f \), they allow us to get a quick rough estimate which can be used before you are convinced of the truth of this statement: Somewhere two endpoints have the same height and the slope of the line connecting the endpoints is zero. What can we say about the slope between the endpoints? It shouldn’t take much experimentation before you are convinced of the truth of this statement: Somewhere

**Exercises 6.4**

1. Let \( f(x) = x^2 \). If \( a = 1 \) and \( dx = \Delta x = 1/2 \), what are \( \Delta y \) and \( dy \)?
2. Let \( f(x) = \sqrt{x} \). If \( a = 1 \) and \( dx = \Delta x = 1/10 \), what are \( \Delta y \) and \( dy \)?
3. Let \( f(x) = \sin(2x) \). If \( \omega = \pi/4 \) and \( dx = \Delta x = \pi/100 \), what are \( \Delta y \) and \( dy \)?
4. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.02 cm thick to a sphere with diameter 48 meters. (Recall that the volume of a sphere of radius \( r \) is \( V = 4/3\pi r^3 \).) Notice that you are given that \( dr = 0.0002 \).\( r \)
5. Show in detail that the linear approximation of \( \sin(x) \) at \( x = 0 \) is \( L(x) = x \) and the linear approximation of \( \cos(x) \) at \( x = 0 \) is \( L(x) = 1 \).

6.5 The Mean Value Theorem

Here are two interesting questions involving derivatives:

1. Suppose two different functions have the same derivative; what can you say about the relationship between the two functions?
2. Suppose you drive a car from toll booth a on a toll road to another toll booth at an average speed of 70 miles per hour. What can be concluded about your actual speed during the trip? In particular, did you exceed the 65 mile per hour speed limit?

While those sound very different, it turns out that the two problems are very closely related. We know that “speed” is really the derivative by a different name; let’s start by translating the second question into something that may be easier to visualize. Suppose that if the function \( f(t) \) gives the position of your car on the toll road at time \( t \). Your change in position between one toll booth and the next is given by \( f(1) - f(0) \), assuming that at time \( t = 0 \) you were at the first booth and at time \( t = 1 \) you arrived at the second booth. Your average speed for the trip is \( (f(1) - f(0))/(t_1 - t_0) \). If we think about the graph of \( f(t) \), the average speed is the slope of the line that connects the two points \( (t_0, f(t_0)) \) and \( (t_1, f(t_1)) \). Your speed at any particular time \( t \) between \( t_0 \) and \( t_1 \) is \( f'(t) \), the slope of the curve. Now question 2 becomes a question about slope. In particular, if the slope between endpoints is \( 70 \), what can be said of the slopes at points between the endpoints?

As a general rule, faced with a new problem it is often a good idea to examine one or more simplified versions of the problem, in the hope that this will lead to an understanding of the original problem. In this case, the problem in its “slope” form is somewhat easier to simplify than the original, but equivalent, problem.

Here is a special instance of the problem. Suppose that \( f(t_0) = f(t_1) \). Then the two endpoints have the same height and the slope of the line connecting the endpoints is zero. What can we say about the slope between the endpoints? It shouldn’t take much experimentation before you are convinced of the truth of this statement: Somewhere

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Describe all functions that have derivative $5x - 3$. It’s easy to find one: $g(x) = (5/2)x^2 - 3x$ has $g'(x) = 5x - 3$. The only other functions with the same derivative are therefore of the form $f(x) = (5/2)x^2 - 3x + k$.

EXAMPLE 6.5.3

Describe all functions that have derivative $5x - 3$. It’s easy to find one: $g(x) = (5/2)x^2 - 3x$ has $g'(x) = 5x - 3$. The only other functions with the same derivative are therefore of the form $f(x) = (5/2)x^2 - 3x + k$.

EXAMPLE 6.5.5

Describe all functions with derivative $\sin(2x)$. One such function is $y = \sin(2x)$, since any function of the form $y = f(x) + C$, where $C$ is a constant, has derivative $\cos(2x)$. Another example is $y = \sin(2x) + 1$, which has the same derivative.

EXERCISES 6.5

1. Let $f(x) = x^3$. Find a value $c \in (-1, 2)$ so that $f'(c)$ equals the slope between the endpoints of $f(x)$ on $[-1, 2]$.

2. Verify that $f(x) = x^3 + 2$ satisfies the hypotheses of the Mean Value Theorem on the interval $[1, 4]$ and then find all of the values, $c$, that satisfy the conclusion of the theorem.

3. Verify that $f(x) = 3x^2 + 7$ satisfies the hypotheses of the Mean Value Theorem on the interval $[-2, 6]$ and then find all of the values, $c$, that satisfy the conclusion of the theorem.

4. Let $f(x) = \sin(x)$. Show that $f(x) = \sin(x)$ has no real roots.

5. Let $f(x) = \sin(2x)$. Show that there is no value $c \in (0, \pi)$ such that $f'(c) = \cos(2c)$.

6. Describe all functions with derivative $x^2 + 4x - 5$. \(\Rightarrow\)

7. Describe all functions with derivative $\sin(2x)$. \(\Rightarrow\)

8. Show that the equation $6x^2 - 7x + 1 = 0$ does not have more than two distinct real roots.

9. Let $f$ be differentiable on $R$. Suppose that $f'(x) \neq 0$ for every $x$. Prove that $f$ has at most one real root.

10. Prove that for all real $x$ and $y$, $\cos x - \cos y \leq |x - y|$. State and prove an analogous result involving sine.

11. Show that $\sqrt{1 + x} \leq 1 + (x/2)$ if $-1 < x < 1$. \(\Rightarrow\)