5

Curve Sketching

Whether we are interested in a function as a purely mathematical object or in connection with some application to the real world, it is often useful to know what the graph of the function looks like. We can obtain a good picture of the graph using certain crucial information provided by derivatives of the function and certain limits.

5.1 Maxima and Minima

A **local maximum point** on a function is a point (x, y) on the graph of the function whose y coordinate is larger than all other y coordinates on the graph at points "close to" (x, y). More precisely, $(x, f(x))$ is a local maximum if there is an interval (a, b) with $a < x < b$ and $f(x) \ge f(z)$ for every z in (a, b). Similarly, (x, y) is a **local minimum point** if it has locally the smallest y coordinate. Again being more precise: $(x, f(x))$ is a local minimum if there is an interval (a, b) with $a < x < b$ and $f(x) \le f(z)$ for every z in (a, b). A **local extremum** is either a local minimum or a local maximum.

Local maximum and minimum points are quite distinctive on the graph of a function, and are therefore useful in understanding the shape of the graph. In many applied problems we want to find the largest or smallest value that a function achieves (for example, we might want to find the minimum cost at which some task can be performed) and so identifying maximum and minimum points will be useful for applied problems as well. Some examples of local maximum and minimum points are shown in figure 5.1.1.

If $(x, f(x))$ is a point where $f(x)$ reaches a local maximum or minimum, and if the derivative of f exists at x, then the graph has a tangent line and the tangent line must be horizontal. This is important enough to state as a theorem, though we will not prove it.

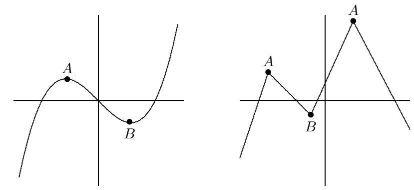

Figure 5.1.1 Some local maximum points (A) and minimum points (B).

THEOREM 5.1.1 Fermat's Theorem If $f(x)$ has a local extremum at $x = a$ and f is differentiable at a, then $f'(a) = 0$. ∎

Thus, the only points at which a function can have a local maximum or minimum are points at which the derivative is zero, as in the left hand graph in figure 5.1.1, or the derivative is undefined, as in the right hand graph. Any value of x for which $f'(x)$ is zero or undefined is called a **critical value** for f. When looking for local maximum and minimum points, you are likely to make two sorts of mistakes: You may forget that a maximum or minimum can occur where the derivative does not exist, and so forget to check whether the derivative exists everywhere. You might also assume that any place that the derivative is zero is a local maximum or minimum point, but this is not true. A portion of the graph of $f(x) = x^3$ is shown in figure 5.1.2. The derivative of f is $f'(x) = 3x^2$, and $f'(0) = 0$, but there is neither a maximum nor minimum at $(0, 0)$.

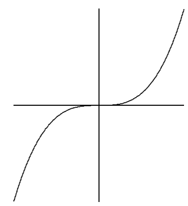

Figure 5.1.2 No maximum or minimum even though the derivative is zero.

Since the derivative is zero or undefined at both local maximum and local minimum points, we need a way to determine which, if either, actually occurs. The most elementary approach, but one that is often tedious or difficult, is to test directly whether the y coordinates "near" the potential maximum or minimum are above or below the y coordinate

at the point of interest. Of course, there are too many points "near" the point to test, but a little thought shows we need only test two provided we know that f is continuous (recall that this means that the graph of f has no jumps or gaps).

Suppose, for example, that we have identified three points at which f' is zero or nonexistent: (x_1, y_1), (x_2, y_2), (x_3, y_3), and $x_1 < x_2 < x_3$ (see figure 5.1.3). Suppose that we compute the value of $f(a)$ for $x_1 < a < x_2$, and that $f(a) < f(x_2)$. What can we say about the graph between a and x_2? Could there be a point $(b, f(b))$, $a < b < x_2$ with $f(b) > f(x_2)$? No: if there were, the graph would go up from $(a, f(a))$ to $(b, f(b))$ then down to $(x_2, f(x_2))$ and somewhere in between would have a local maximum point. (This is not obvious; it is a result of the Extreme Value Theorem, theorem 6.1.2.) But at that local maximum point the derivative of f would be zero or nonexistent, yet we already know that the derivative is zero or nonexistent only at x_1, x_2, and x_3. The upshot is that one computation tells us that $(x_2, f(x_2))$ has the largest y coordinate of any point on the graph near x_2 and to the left of x_2. We can perform the same test on the right. If we find that on both sides of x_2 the values are smaller, then there must be a local maximum at $(x_2, f(x_2))$; if we find that on both sides of x_2 the values are larger, then there must be a local minimum at $(x_2, f(x_2))$; if we find one of each, then there is neither a local maximum or minimum at x_2.

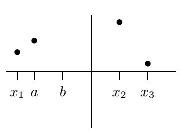

Figure 5.1.3 Testing for a maximum or minimum.

It is not always easy to compute the value of a function at a particular point. The task is made easier by the availability of calculators and computers, but they have their own drawbacks—they do not always allow us to distinguish between values that are very close together. Nevertheless, because this method is conceptually simple and sometimes easy to perform, you should always consider it.

EXAMPLE 5.1.2 Find all local maximum and minimum points for the function $f(x) = x^3 - x$. The derivative is $f'(x) = 3x^2 - 1$. This is defined everywhere and is zero at $x = \pm\sqrt{3}/3$. Looking first at $x = \sqrt{3}/3$, we see that $f(\sqrt{3}/3) = -2\sqrt{3}/9$. Now we test two points on either side of $x = \sqrt{3}/3$, making sure that neither is farther away than the nearest critical value; since $\sqrt{3} < 3$, $\sqrt{3}/3 < 1$ and we can use $x = 0$ and $x = 1$. Since $f(0) = 0 > -2\sqrt{3}/9$ and $f(1) = 0 > -2\sqrt{3}/9$, there must be a local minimum at

$x = \sqrt{3}/3$. For $x = -\sqrt{3}/3$, we see that $f(-\sqrt{3}/3) = 2\sqrt{3}/9$. This time we can use $x = 0$ and $x = -1$, and we find that $f(-1) = f(0) = 0 < 2\sqrt{3}/9$, so there must be a local maximum at $x = -\sqrt{3}/3$. □

Of course this example is made very simple by our choice of points to test, namely $x = -1, 0, 1$. We could have used other values, say $-5/4$, $1/3$, and $3/4$, but this would have made the calculations considerably more tedious.

EXAMPLE 5.1.3 Find all local maximum and minimum points for $f(x) = \sin x + \cos x$. The derivative is $f'(x) = \cos x - \sin x$. This is always defined and is zero whenever $\cos x = \sin x$. Recalling that the $\cos x$ and $\sin x$ are the x and y coordinates of points on a unit circle, we see that $\cos x = \sin x$ when x is $\pi/4$, $\pi/4 \pm \pi$, $\pi/4 \pm 2\pi$, $\pi/4 \pm 3\pi$, etc. Since both sine and cosine have a period of 2π, we need only determine the status of $x = \pi/4$ and $x = 5\pi/4$. We can use 0 and $\pi/2$ to test the critical value $x = \pi/4$. We find that $f(\pi/4) = \sqrt{2}$, $f(0) = 1 < \sqrt{2}$ and $f(\pi/2) = 1$, so there is a local maximum when $x = \pi/4$ and also when $x = \pi/4 \pm 2\pi$, $\pi/4 \pm 4\pi$, etc. We can summarize this more neatly by saying that there are local maxima at $\pi/4 \pm 2k\pi$ for every integer k.

We use π and 2π to test the critical value $x = 5\pi/4$. The relevant values are $f(5\pi/4) = -\sqrt{2}$, $f(\pi) = -1 > -\sqrt{2}$, $f(2\pi) = 1 > -\sqrt{2}$, so there is a local minimum at $x = 5\pi/4$, $5\pi/4 \pm 2\pi$, $5\pi/4 \pm 4\pi$, etc. More succinctly, there are local minima at $5\pi/4 \pm 2k\pi$ for every integer k. □

Exercises 5.1.

In problems 1–12, find all local maximum and minimum points (x, y) by the method of this section.

1. $y = x^2 - x \Rightarrow$

2. $y = 2 + 3x - x^3 \Rightarrow$

3. $y = x^3 - 9x^2 + 24x \Rightarrow$

4. $y = x^4 - 2x^2 + 3 \Rightarrow$

5. $y = 3x^4 - 4x^3 \Rightarrow$

6. $y = (x^2 - 1)/x \Rightarrow$

7. $y = 3x^2 - (1/x^2) \Rightarrow$

8. $y = \cos(2x) - x \Rightarrow$

9. $f(x) = \begin{cases} x - 1 & x < 2 \\ x^2 & x \ge 2 \end{cases} \Rightarrow$

10. $f(x) = \begin{cases} x - 3 & x < 3 \\ x^3 & 3 \le x \le 5 \\ 1/x & x > 5 \end{cases} \Rightarrow$

11. $f(x) = x^2 - 98x + 4 \Rightarrow$

12. $f(x) = \begin{cases} -2 & x = 0 \\ 1/x^2 & x \ne 0 \end{cases} \Rightarrow$

13. For any real number x there is a unique integer n such that $n \le x < n + 1$, and the greatest integer function is defined as $\lfloor x \rfloor = n$. Where are the critical values of the greatest integer function? Which are local maxima and which are local minima?

14. Explain why the function $f(x) = 1/x$ has no local maxima or minima.

15. How many critical points can a quadratic polynomial function have? $\Rightarrow$

16. Show that a cubic polynomial can have at most two critical points. Give examples to show that a cubic polynomial can have zero, one, or two critical points.

17. Explore the family of functions $f(x) = x^3 + cx + 1$ where c is a constant. How many and what types of local extremes are there? Your answer should depend on the value of c, that is, different values of c will give different answers.

18. We generalize the preceding two questions. Let n be a positive integer and let f be a polynomial of degree n. How many critical points can f have? (Hint: Recall the **Fundamental Theorem of Algebra**, which says that a polynomial of degree n has at most n roots.)

5.2 The first derivative test

The method of the previous section for deciding whether there is a local maximum or minimum at a critical value is not always convenient. We can instead use information about the derivative $f'(x)$ to decide; since we have already had to compute the derivative to find the critical values, there is often relatively little extra work involved in this method.

How can the derivative tell us whether there is a maximum, minimum, or neither at a point? Suppose that $f'(a) = 0$. If there is a local maximum when $x = a$, the function must be lower near $x = a$ than it is right at $x = a$. If the derivative exists near $x = a$, this means $f'(x) > 0$ when x is near a and $x < a$, because the function must "slope up" just to the left of a. Similarly, $f'(x) < 0$ when x is near a and $x > a$, because f slopes down from the local maximum as we move to the right. Using the same reasoning, if there is a local minimum at $x = a$, the derivative of f must be negative just to the left of a and positive just to the right. If the derivative exists near a but does not change from positive to negative or negative to positive, that is, it is positive on both sides or negative on both sides, then there is neither a maximum nor minimum when $x = a$. See the first graph in figure 5.1.1 and the graph in figure 5.1.2 for examples.

EXAMPLE 5.2.1 Find all local maximum and minimum points for $f(x) = \sin x + \cos x$ using the first derivative test. The derivative is $f'(x) = \cos x - \sin x$ and from example 5.1.3 the critical values we need to consider are $\pi/4$ and $5\pi/4$.

The graphs of $\sin x$ and $\cos x$ are shown in figure 5.2.1. Just to the left of $\pi/4$ the cosine is larger than the sine, so $f'(x)$ is positive; just to the right the cosine is smaller than the sine, so $f'(x)$ is negative. This means there is a local maximum at $\pi/4$. Just to the left of $5\pi/4$ the cosine is smaller than the sine, and to the right the cosine is larger than the sine. This means that the derivative $f'(x)$ is negative to the left and positive to the right, so f has a local minimum at $5\pi/4$. □

Exercises 5.2.

In 1–13, find all critical points and identify them as local maximum points, local minimum points, or neither.

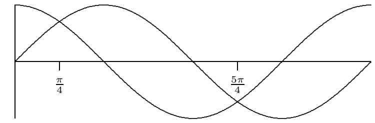

Figure 5.2.1 The sine and cosine.

1. $y = x^2 - x \Rightarrow$

2. $y = 2 + 3x - x^3 \Rightarrow$

3. $y = x^3 - 9x^2 + 24x \Rightarrow$

4. $y = x^4 - 2x^2 + 3 \Rightarrow$

5. $y = 3x^4 - 4x^3 \Rightarrow$

6. $y = (x^2 - 1)/x \Rightarrow$

7. $y = 3x^2 - (1/x^2) \Rightarrow$

8. $y = \cos(2x) - x \Rightarrow$

9. $f(x) = (5 - x)/(x + 2) \Rightarrow$

10. $f(x) = |x^2 - 121| \Rightarrow$

11. $f(x) = x^3/(x + 1) \Rightarrow$

12. $f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

13. $f(x) = \sin^2 x \Rightarrow$

14. Find the maxima and minima of $f(x) = \sec x$. ⇒

15. Let $f(\theta) = \cos^2(\theta) - 2\sin(\theta)$. Find the intervals where f is increasing and the intervals where f is decreasing in $[0, 2\pi]$. Use this information to classify the critical points of f as either local maximums, local minimums, or neither. ⇒

16. Let $r > 0$. Find the local maxima and minima of the function $f(x) = \sqrt{r^2 - x^2}$ on its domain $[-r, r]$.

17. Let $f(x) = ax^2 + bx + c$ with $a \neq 0$. Show that f has exactly one critical point. Give conditions on a and b which guarantee that the critical point will be a maximum. It is possible to see this without using calculus at all; explain.

5.3 The second derivative test

The basis of the first derivative test is that if the derivative changes from positive to negative at a point at which the derivative is zero then there is a local maximum at the point, and similarly for a local minimum. If f' changes from positive to negative it is decreasing; this means that the derivative of f', f'', might be negative, and if in fact f'' is negative then f' is definitely decreasing, so there is a local maximum at the point in question. Note well that f' might change from positive to negative while f'' is zero, in which case f'' gives us no information about the critical value. Similarly, if f' changes from negative to positive there is a local minimum at the point, and f' is increasing. If $f'' > 0$ at the point, this tells us that f' is increasing, and so there is a local minimum.

EXAMPLE 5.3.1 Consider again $f(x) = \sin x + \cos x$, with $f'(x) = \cos x - \sin x$ and $f''(x) = -\sin x - \cos x$. Since $f''(\pi/4) = -\sqrt{2}/2 - \sqrt{2}/2 = -\sqrt{2} < 0$, we know there is a local maximum at $\pi/4$. Since $f''(5\pi/4) = - -\sqrt{2}/2 - -\sqrt{2}/2 = \sqrt{2} > 0$, there is a local minimum at $5\pi/4$. □

When it works, the second derivative test is often the easiest way to identify local maximum and minimum points. Sometimes the test fails, and sometimes the second derivative is quite difficult to evaluate; in such cases we must fall back on one of the previous tests.

EXAMPLE 5.3.2 Let $f(x) = x^4$. The derivatives are $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Zero is the only critical value, but $f''(0) = 0$, so the second derivative test tells us nothing. However, $f(x)$ is positive everywhere except at zero, so clearly $f(x)$ has a local minimum at zero. On the other hand, $f(x) = -x^4$ also has zero as its only critical value, and the second derivative is again zero, but $-x^4$ has a local maximum at zero. □

Exercises 5.3.

Find all local maximum and minimum points by the second derivative test, when possible.

1. $y = x^2 - x \Rightarrow$

2. $y = 2 + 3x - x^3 \Rightarrow$

3. $y = x^3 - 9x^2 + 24x \Rightarrow$

4. $y = x^4 - 2x^2 + 3 \Rightarrow$

5. $y = 3x^4 - 4x^3 \Rightarrow$

6. $y = (x^2 - 1)/x \Rightarrow$

7. $y = 3x^2 - (1/x^2) \Rightarrow$

8. $y = \cos(2x) - x \Rightarrow$

9. $y = 4x + \sqrt{1 - x} \Rightarrow$

10. $y = (x + 1)/\sqrt{5x^2 + 35} \Rightarrow$

11. $y = x^5 - x \Rightarrow$

12. $y = 6x + \sin 3x \Rightarrow$

13. $y = x + 1/x \Rightarrow$

14. $y = x^2 + 1/x \Rightarrow$

15. $y = (x + 5)^{1/4} \Rightarrow$

16. $y = \tan^2 x \Rightarrow$

17. $y = \cos^2 x - \sin^2 x \Rightarrow$

18. $y = \sin^3 x \Rightarrow$

5.4 Concavity and inflection points

We know that the sign of the derivative tells us whether a function is increasing or decreasing; for example, when $f'(x) > 0$, $f(x)$ is increasing. The sign of the second derivative $f''(x)$ tells us whether f' is increasing or decreasing; we have seen that if f' is zero and increasing at a point then there is a local minimum at the point, and if f' is zero and decreasing at a point then there is a local maximum at the point. Thus, we extracted information about f from information about f''.

We can get information from the sign of f'' even when f' is not zero. Suppose that $f''(a) > 0$. This means that near $x = a$, f' is increasing. If $f'(a) > 0$, this means that f slopes up and is getting steeper; if $f'(a) < 0$, this means that f slopes down and is getting

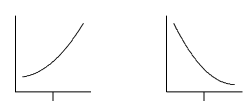

Figure 5.4.1 $f''(a) > 0$: $f'(a)$ positive and increasing, $f'(a)$ negative and increasing.

less steep. The two situations are shown in figure 5.4.1. A curve that is shaped like this is called **concave up**.

Now suppose that $f''(a) < 0$. This means that near $x = a$, f' is decreasing. If $f'(a) > 0$, this means that f slopes up and is getting less steep; if $f'(a) < 0$, this means that f slopes down and is getting steeper. The two situations are shown in figure 5.4.2. A curve that is shaped like this is called **concave down**.

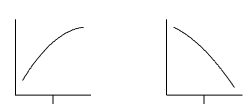

Figure 5.4.2 $f''(a) < 0$: $f'(a)$ positive and decreasing, $f'(a)$ negative and decreasing.

If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. Of particular interest are points at which the concavity changes from up to down or down to up; such points are called **inflection points**. If the concavity changes from up to down at $x = a$, f'' changes from positive to the left of a to negative to the right of a, and usually $f''(a) = 0$. We can identify such points by first finding where $f''(x)$ is zero and then checking to see whether $f''(x)$ does in fact go from positive to negative or negative to positive at these points. Note that it is possible that $f''(a) = 0$ but the concavity is the same on both sides; $f(x) = x^4$ at $x = 0$ is an example.

EXAMPLE 5.4.1 Describe the concavity of $f(x) = x^3 - x$. $f'(x) = 3x^2 - 1$, $f''(x) = 6x$. Since $f''(0) = 0$, there is potentially an inflection point at zero. Since $f''(x) > 0$ when $x > 0$ and $f''(x) < 0$ when $x < 0$ the concavity does change from down to up at zero, and the curve is concave down for all $x < 0$ and concave up for all $x > 0$. □

Note that we need to compute and analyze the second derivative to understand concavity, so we may as well try to use the second derivative test for maxima and minima. If for some reason this fails we can then try one of the other tests.

Exercises 5.4.

Describe the concavity of the functions in 1–18.

1. $y = x^2 - x \Rightarrow$ **2.** $y = 2 + 3x - x^3 \Rightarrow$

3. $y = x^3 - 9x^2 + 24x \Rightarrow$ **4.** $y = x^4 - 2x^2 + 3 \Rightarrow$

5. $y = 3x^4 - 4x^3 \Rightarrow$ **6.** $y = (x^2 - 1)/x \Rightarrow$

7. $y = 3x^2 - (1/x^2) \Rightarrow$ **8.** $y = \sin x + \cos x \Rightarrow$

9. $y = 4x + \sqrt{1 - x} \Rightarrow$ **10.** $y = (x + 1)/\sqrt{5x^2 + 35} \Rightarrow$

11. $y = x^5 - x \Rightarrow$ **12.** $y = 6x + \sin 3x \Rightarrow$

13. $y = x + 1/x \Rightarrow$ **14.** $y = x^2 + 1/x \Rightarrow$

15. $y = (x + 5)^{1/4} \Rightarrow$ **16.** $y = \tan^2 x \Rightarrow$

17. $y = \cos^2 x - \sin^2 x \Rightarrow$ **18.** $y = \sin^3 x \Rightarrow$

19. Identify the intervals on which the graph of the function $f(x) = x^4 - 4x^3 + 10$ is of one of these four shapes: concave up and increasing; concave up and decreasing; concave down and increasing; concave down and decreasing. $\Rightarrow$

20. Describe the concavity of $y = x^3 + bx^2 + cx + d$. You will need to consider different cases, depending on the values of the coefficients.

21. Let n be an integer greater than or equal to two, and suppose f is a polynomial of degree n. How many inflection points can f have? Hint: Use the second derivative test and the fundamental theorem of algebra.

5.5 Asymptotes and Other Things to Look For

A vertical asymptote is a place where the function becomes infinite, typically because the formula for the function has a denominator that becomes zero. For example, the reciprocal function $f(x) = 1/x$ has a vertical asymptote at $x = 0$, and the function $\tan x$ has a vertical asymptote at $x = \pi/2$ (and also at $x = -\pi/2$, $x = 3\pi/2$, etc.). Whenever the formula for a function contains a denominator it is worth looking for a vertical asymptote by checking to see if the denominator can ever be zero, and then checking the limit at such points. Note that there is not always a vertical asymptote where the denominator is zero: $f(x) = (\sin x)/x$ has a zero denominator at $x = 0$, but since $\lim\limits_{x\to 0}(\sin x)/x = 1$ there is no asymptote there.

A horizontal asymptote is a horizontal line to which $f(x)$ gets closer and closer as x approaches ∞ (or as x approaches $-\infty$). For example, the reciprocal function has the x-axis for a horizontal asymptote. Horizontal asymptotes can be identified by computing the limits $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$. Since $\lim\limits_{x\to\infty} 1/x = \lim\limits_{x\to-\infty} 1/x = 0$, the line $y = 0$ (that is, the x-axis) is a horizontal asymptote in both directions.

Some functions have asymptotes that are neither horizontal nor vertical, but some other line. Such asymptotes are somewhat more difficult to identify and we will ignore them.

If the domain of the function does not extend out to infinity, we should also ask what happens as x approaches the boundary of the domain. For example, the function $y = f(x) = 1/\sqrt{r^2 - x^2}$ has domain $-r < x < r$, and y becomes infinite as x approaches either r or $-r$. In this case we might also identify this behavior because when $x = \pm r$ the denominator of the function is zero.

If there are any points where the derivative fails to exist (a cusp or corner), then we should take special note of what the function does at such a point.

Finally, it is worthwhile to notice any symmetry. A function $f(x)$ that has the same value for $-x$ as for x, i.e., $f(-x) = f(x)$, is called an "even function." Its graph is symmetric with respect to the y-axis. Some examples of even functions are: x^n when n is an even number, $\cos x$, and $\sin^2 x$. On the other hand, a function that satisfies the property $f(-x) = -f(x)$ is called an "odd function." Its graph is symmetric with respect to the origin. Some examples of odd functions are: x^n when n is an odd number, $\sin x$, and $\tan x$. Of course, most functions are neither even nor odd, and do not have any particular symmetry.

Exercises 5.5.

Sketch the curves. Identify clearly any interesting features, including local maximum and minimum points, inflection points, asymptotes, and intercepts. You can use this Sage worksheet to check your answers. Note that you may need to adjust the interval over which the function is graphed to capture all the details.

1. $y = x^5 - 5x^4 + 5x^3$ **2.** $y = x^3 - 3x^2 - 9x + 5$

3. $y = (x - 1)^2(x + 3)^{2/3}$ **4.** $x^2 + x^2y^2 = a^2y^2$, $a > 0$.

5. $y = 4x + \sqrt{1 - x}$ **6.** $y = (x + 1)/\sqrt{5x^2 + 35}$

7. $y = x^5 - x$ **8.** $y = 6x + \sin 3x$

9. $y = x + 1/x$ **10.** $y = x^2 + 1/x$

11. $y = (x + 5)^{1/4}$ **12.** $y = \tan^2 x$

13. $y = \cos^2 x - \sin^2 x$ **14.** $y = \sin^3 x$

15. $y = x(x^2 + 1)$ **16.** $y = x^3 + 6x^2 + 9x$

17. $y = x/(x^2 - 9)$ **18.** $y = x^2/(x^2 + 9)$

19. $y = 2\sqrt{x} - x$ **20.** $y = 3\sin(x) - \sin^3(x)$, for $x \in [0, 2\pi]$

21. $y = (x - 1)/(x^2)$

For each of the following five functions, identify any vertical and horizontal asymptotes, and identify intervals on which the function is concave up and increasing; concave up and decreasing; concave down and increasing; concave down and decreasing.

22. $f(\theta) = \sec(\theta)$

23. $f(x) = 1/(1 + x^2)$

24. $f(x) = (x - 3)/(2x - 2)$

25. $f(x) = 1/(1 - x^2)$

26. $f(x) = 1 + 1/(x^2)$

27. Let $f(x) = 1/(x^2 - a^2)$, where $a \geq 0$. Find any vertical and horizontal asymptotes and the intervals upon which the given function is concave up and increasing; concave up and decreasing; concave down and increasing; concave down and decreasing. Discuss how the value of a affects these features.