4

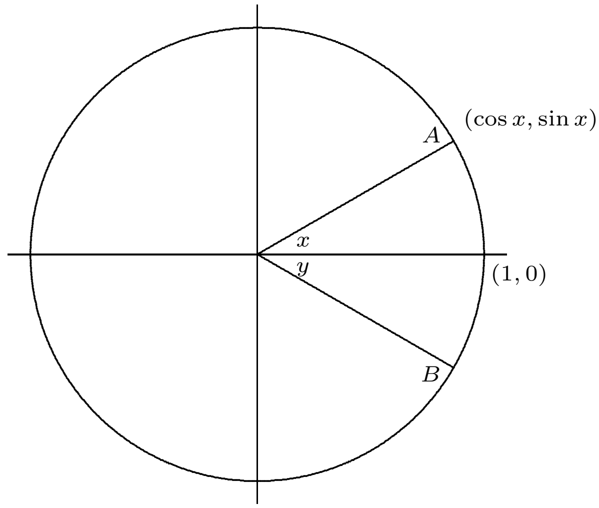
Trigonometric Functions

So far we have used only algebraic functions as examples when finding derivatives, that is, functions that can be built up by the usual algebraic operations of addition, subtraction, multiplication, division, and raising to constant powers. Both in theory and practice there are other functions, called transcendental, that are very useful. Most important among these are the trigonometric functions, the inverse trigonometric functions, exponential functions, and logarithms. In this chapter we investigate the trigonometric functions.

4.1 Trigonometric Functions

When you first encountered the trigonometric functions it was probably in the context of "triangle trigonometry," defining, for example, the sine of an angle as the "side opposite over the hypotenuse." While this will still be useful in an informal way, we need to use a more expansive definition of the trigonometric functions. First an important note: while degree measure of angles is sometimes convenient because it is so familiar, it turns out to be ill-suited to mathematical calculation, so (almost) everything we do will be in terms of **radian measure** of angles.

To define the radian measurement system, we consider the unit circle in the xy-plane:

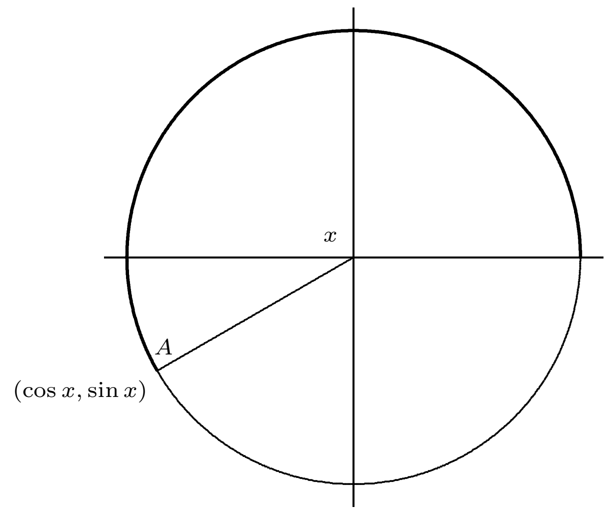


An angle, x, at the center of the circle is associated with an arc of the circle which is said to **subtend** the angle. In the figure, this arc is the portion of the circle from point $(1, 0)$ to point A. The length of this arc is the radian measure of the angle x; the fact that the radian measure is an actual geometric length is largely responsible for the usefulness of radian measure. The circumference of the unit circle is $2\pi r = 2\pi(1) = 2\pi$, so the radian measure of the full circular angle (that is, of the 360 degree angle) is 2π.

While an angle with a particular measure can appear anywhere around the circle, we need a fixed, conventional location so that we can use the coordinate system to define properties of the angle. The standard convention is to place the starting radius for the angle on the positive x-axis, and to measure positive angles counterclockwise around the circle. In the figure, x is the standard location of the angle $\pi/6$, that is, the length of the arc from $(1, 0)$ to A is $\pi/6$. The angle y in the picture is $-\pi/6$, because the distance from $(1, 0)$ to B along the circle is also $\pi/6$, but in a clockwise direction.

Now the fundamental trigonometric definitions are: the cosine of x and the sine of x are the first and second coordinates of the point A, as indicated in the figure. The angle x shown can be viewed as an angle of a right triangle, meaning the usual triangle definitions of the sine and cosine also make sense. Since the hypotenuse of the triangle is 1, the "side opposite over hypotenuse" definition of the sine is the second coordinate of point A over 1, which is just the second coordinate; in other words, both methods give the same value for the sine.

The simple triangle definitions work only for angles that can "fit" in a right triangle, namely, angles between 0 and $\pi/2$. The coordinate definitions, on the other hand, apply to any angles, as indicated in this figure:



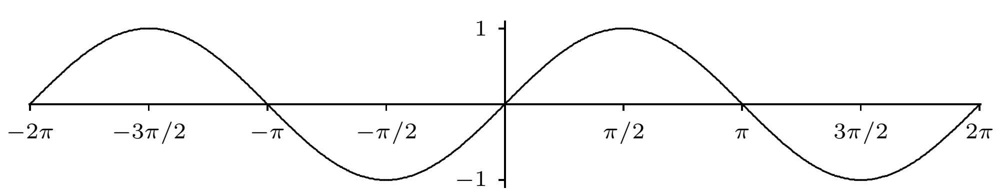
The angle x is subtended by the heavy arc in the figure, that is, $x = 7\pi/6$. Both coordinates of point A in this figure are negative, so the sine and cosine of $7\pi/6$ are both negative.

The remaining trigonometric functions can be most easily defined in terms of the sine and cosine, as usual:

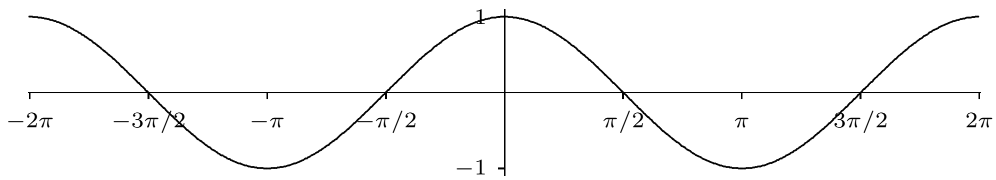
$$\tan x = \frac{\sin x}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x}$$
$$\sec x = \frac{1}{\cos x}$$
$$\csc x = \frac{1}{\sin x}$$

and they can also be defined as the corresponding ratios of coordinates.

Although the trigonometric functions are defined in terms of the unit circle, the unit circle diagram is not what we normally consider the graph of a trigonometric function. (The unit circle is the graph of, well, the circle.) We can easily get a qualitatively correct idea of the graphs of the trigonometric functions from the unit circle diagram. Consider the sine function, $y = \sin x$. As x increases from 0 in the unit circle diagram, the second coordinate of the point A goes from 0 to a maximum of 1, then back to 0, then to a minimum of -1, then back to 0, and then it obviously repeats itself. So the graph of $y = \sin x$ must look something like this:



Similarly, as angle x increases from 0 in the unit circle diagram, the first coordinate of the point A goes from 1 to 0 then to -1, back to 0 and back to 1, so the graph of $y = \cos x$ must look something like this:



Exercises 4.1.

Some useful trigonometric identities are in appendix B.

1. Find all values of θ such that $\sin(\theta) = -1$; give your answer in radians. ⇒

2. Find all values of θ such that $\cos(2\theta) = 1/2$; give your answer in radians. ⇒

3. Use an angle sum identity to compute $\cos(\pi/12)$. ⇒

4. Use an angle sum identity to compute $\tan(5\pi/12)$. ⇒

5. Verify the identity $\cos^2(t)/(1 - \sin(t)) = 1 + \sin(t)$.

6. Verify the identity $2\csc(2\theta) = \sec(\theta)\csc(\theta)$.

7. Verify the identity $\sin(3\theta) - \sin(\theta) = 2\cos(2\theta)\sin(\theta)$.

8. Sketch $y = 2\sin(x)$.

9. Sketch $y = \sin(3x)$.

10. Sketch $y = \sin(-x)$.

11. Find all of the solutions of $2\sin(t) - 1 - \sin^2(t) = 0$ in the interval $[0, 2\pi]$. ⇒

4.2 The Derivative of $\sin x$

What about the derivative of the sine function? The rules for derivatives that we have are no help, since $\sin x$ is not an algebraic function. We need to return to the definition of the derivative, set up a limit, and try to compute it. Here's the definition:

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}.$$

Using some trigonometric identities, we can make a little progress on the quotient:

$$\begin{aligned}\frac{\sin(x + \Delta x) - \sin x}{\Delta x} &= \frac{\sin x \cos \Delta x + \sin \Delta x \cos x - \sin x}{\Delta x} \\ &= \sin x \frac{\cos \Delta x - 1}{\Delta x} + \cos x \frac{\sin \Delta x}{\Delta x}.\end{aligned}$$

This isolates the difficult bits in the two limits

$$\lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}.$$

Here we get a little lucky: it turns out that once we know the second limit the first is quite easy. The second is quite tricky, however. Indeed, it is the hardest limit we will actually compute, and we devote a section to it.

4.3 A hard limit

We want to compute this limit:

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}.$$

Equivalently, to make the notation a bit simpler, we can compute

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

In the original context we need to keep x and Δx separate, but here it doesn't hurt to rename Δx to something more convenient.

To do this we need to be quite clever, and to employ some indirect reasoning. The indirect reasoning is embodied in a theorem, frequently called the **squeeze theorem**.

THEOREM 4.3.1 Squeeze Theorem Suppose that $g(x) \le f(x) \le h(x)$ for all x close to a but not equal to a. If $\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$, then $\lim_{x\to a} f(x) = L$. ■

This theorem can be proved using the official definition of limit. We won't prove it here, but point out that it is easy to understand and believe graphically. The condition says that $f(x)$ is trapped between $g(x)$ below and $h(x)$ above, and that at $x = a$, both g and h approach the same value. This means the situation looks something like figure 4.3.1. The wiggly curve is $x^2 \sin(\pi/x)$, the upper and lower curves are x^2 and $-x^2$. Since the sine function is always between -1 and 1, $-x^2 \le x^2 \sin(\pi/x) \le x^2$, and it is easy to see that $\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2$. It is not so easy to see directly, that is algebraically, that $\lim_{x\to 0} x^2 \sin(\pi/x) = 0$, because the π/x prevents us from simply plugging in $x = 0$. The squeeze theorem makes this "hard limit" as easy as the trivial limits involving x^2.

To do the hard limit that we want, $\lim_{x\to 0}(\sin x)/x$, we will find two simpler functions g and h so that $g(x) \le (\sin x)/x \le h(x)$, and so that $\lim_{x\to 0} g(x) = \lim_{x\to 0} h(x)$. Not too surprisingly, this will require some trigonometry and geometry. Referring to figure 4.3.2, x is the measure of the angle in radians. Since the circle has radius 1, the coordinates of

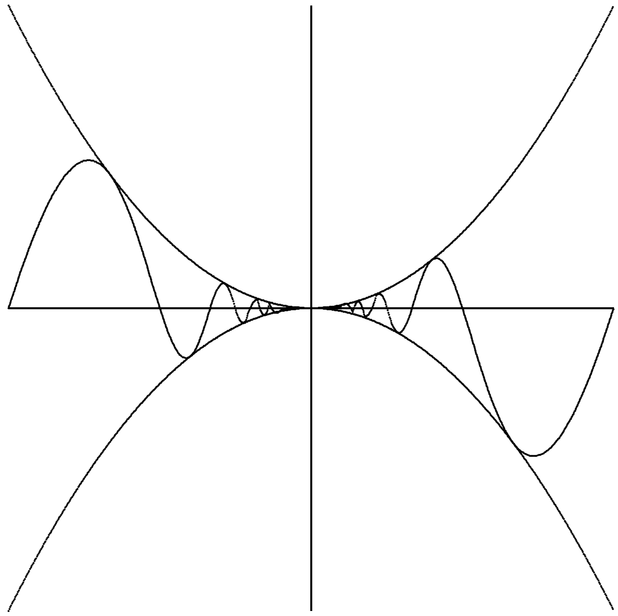


Figure 4.3.1 The squeeze theorem.

point A are $(\cos x, \sin x)$, and the area of the small triangle is $(\cos x \sin x)/2$. This triangle is completely contained within the circular wedge-shaped region bordered by two lines and the circle from $(1,0)$ to point A. Comparing the areas of the triangle and the wedge we see $(\cos x \sin x)/2 \le x/2$, since the area of a circular region with angle θ and radius r is $\theta r^2/2$. With a little algebra this turns into $(\sin x)/x \le 1/\cos x$, giving us the h we seek.

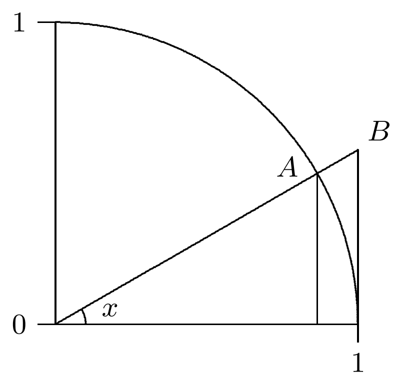


Figure 4.3.2 Visualizing $\sin x/x$.

To find g, we note that the circular wedge is completely contained inside the larger triangle. The height of the triangle, from $(1,0)$ to point B, is $\tan x$, so comparing areas we get $x/2 \le (\tan x)/2 = \sin x/(2\cos x)$. With a little algebra this becomes $\cos x \le (\sin x)/x$. So now we have

$$\cos x \le \frac{\sin x}{x} \le \frac{1}{\cos x}.$$

Finally, the two limits $\lim_{x\to 0}\cos x$ and $\lim_{x\to 0} 1/\cos x$ are easy, because $\cos(0) = 1$. By the squeeze theorem, $\lim_{x\to 0}(\sin x)/x = 1$ as well.

Before we can complete the calculation of the derivative of the sine, we need one other limit:

$$\lim_{x\to 0}\frac{\cos x - 1}{x}.$$

This limit is just as hard as $\sin x/x$, but closely related to it, so that we don't have to do a similar calculation; instead we can do a bit of tricky algebra.

$$\frac{\cos x - 1}{x} = \frac{\cos x - 1}{x}\frac{\cos x + 1}{\cos x + 1} = \frac{\cos^2 x - 1}{x(\cos x + 1)} = \frac{-\sin^2 x}{x(\cos x + 1)} = -\frac{\sin x}{x}\frac{\sin x}{\cos x + 1}.$$

To compute the desired limit it is sufficient to compute the limits of the two final fractions, as x goes to 0. The first of these is the hard limit we've just done, namely 1. The second turns out to be simple, because the denominator presents no problem:

$$\lim_{x\to 0}\frac{\sin x}{\cos x + 1} = \frac{\sin 0}{\cos 0 + 1} = \frac{0}{2} = 0.$$

Thus,

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$$

Exercises 4.3.

1. Compute $\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{x}$ ⇒

2. Compute $\displaystyle\lim_{x\to 0}\frac{\sin(7x)}{\sin(2x)}$ ⇒

3. Compute $\displaystyle\lim_{x\to 0}\frac{\cot(4x)}{\csc(3x)}$ ⇒

4. Compute $\displaystyle\lim_{x\to 0}\frac{\tan x}{x}$ ⇒

5. Compute $\displaystyle\lim_{x\to \pi/4}\frac{\sin x - \cos x}{\cos(2x)}$ ⇒

6. For all $x \ge 0$, $4x - 9 \le f(x) \le x^2 - 4x + 7$. Find $\displaystyle\lim_{x\to 4} f(x)$. ⇒

7. For all x, $2x \le g(x) \le x^4 - x^2 + 2$. Find $\displaystyle\lim_{x\to 1} g(x)$. ⇒

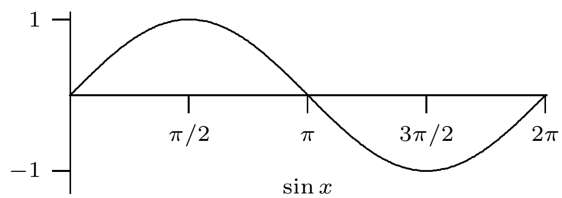
8. Use the Squeeze Theorem to show that $\displaystyle\lim_{x\to 0} x^4\cos(2/x) = 0$.

4.4 The Derivative of $\sin x$, continued

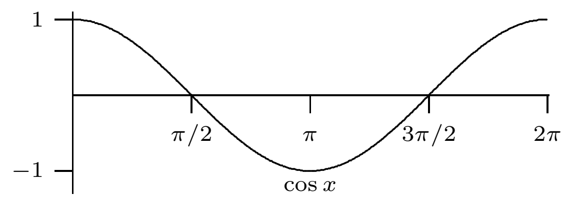
Now we can complete the calculation of the derivative of the sine:

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin x}{\Delta x} \\
&= \lim_{\Delta x\to 0}\sin x\frac{\cos\Delta x-1}{\Delta x}+\cos x\frac{\sin\Delta x}{\Delta x} \\
&= \sin x\cdot 0+\cos x\cdot 1=\cos x.
\end{aligned}$$

The derivative of a function measures the slope or steepness of the function; if we examine the graphs of the sine and cosine side by side, it should be that the latter appears to accurately describe the slope of the former, and indeed this is true:



$\sin x$



$\cos x$

Notice that where the cosine is zero the sine does appear to have a horizontal tangent line, and that the sine appears to be steepest where the cosine takes on its extreme values of 1 and −1.

Of course, now that we know the derivative of the sine, we can compute derivatives of more complicated functions involving the sine.

EXAMPLE 4.4.1 Compute the derivative of $\sin(x^2)$.

$$\frac{d}{dx}\sin(x^2)=\cos(x^2)\cdot 2x=2x\cos(x^2).$$

□

EXAMPLE 4.4.2 Compute the derivative of $\sin^2(x^3-5x)$.

$$\begin{aligned}
\frac{d}{dx}\sin^2(x^3-5x) &= \frac{d}{dx}(\sin(x^3-5x))^2 \\
&= 2(\sin(x^3-5x))^1\cos(x^3-5x)(3x^2-5) \\
&= 2(3x^2-5)\cos(x^3-5x)\sin(x^3-5x).
\end{aligned}$$

□

Exercises 4.4.

Find the derivatives of the following functions.

1. $\sin^2(\sqrt{x}) \Rightarrow$

2. $\sqrt{x}\sin x \Rightarrow$

3. $\dfrac{1}{\sin x} \Rightarrow$

4. $\dfrac{x^2+x}{\sin x} \Rightarrow$

5. $\sqrt{1-\sin^2 x} \Rightarrow$

4.5 Derivatives of the Trigonometric Functions

All of the other trigonometric functions can be expressed in terms of the sine, and so their derivatives can easily be calculated using the rules we already have. For the cosine we need to use two identities,

$$\cos x = \sin(x + \frac{\pi}{2}),$$
$$\sin x = -\cos(x + \frac{\pi}{2}).$$

Now:

$$\frac{d}{dx}\cos x = \frac{d}{dx}\sin(x+\frac{\pi}{2}) = \cos(x+\frac{\pi}{2})\cdot 1 = -\sin x$$
$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$
$$\frac{d}{dx}\sec x = \frac{d}{dx}(\cos x)^{-1} = -1(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

The derivatives of the cotangent and cosecant are similar and left as exercises.

Exercises 4.5.

Find the derivatives of the following functions.

1. $\sin x \cos x \Rightarrow$

2. $\sin(\cos x) \Rightarrow$

3. $\sqrt{x\tan x} \Rightarrow$

4. $\tan x/(1+\sin x) \Rightarrow$

5. $\cot x \Rightarrow$

6. $\csc x \Rightarrow$

7. $x^3\sin(23x^2) \Rightarrow$

8. $\sin^2 x + \cos^2 x \Rightarrow$

9. $\sin(\cos(6x)) \Rightarrow$

10. Compute $\dfrac{d}{d\theta}\dfrac{\sec\theta}{1+\sec\theta}$. $\Rightarrow$

11. Compute $\dfrac{d}{dt}t^5\cos(6t)$. $\Rightarrow$

12. Compute $\dfrac{d}{dt}\dfrac{t^3\sin(3t)}{\cos(2t)}$. $\Rightarrow$

13. Find all points on the graph of $f(x) = \sin^2(x)$ at which the tangent line is horizontal. $\Rightarrow$

14. Find all points on the graph of $f(x) = 2\sin(x) - \sin^2(x)$ at which the tangent line is horizontal. $\Rightarrow$

15. Find an equation for the tangent line to $\sin^2(x)$ at $x = \pi/3$. $\Rightarrow$

16. Find an equation for the tangent line to $\sec^2 x$ at $x = \pi/3$. $\Rightarrow$

17. Find an equation for the tangent line to $\cos^2 x - \sin^2(4x)$ at $x = \pi/6$. $\Rightarrow$

18. Find the points on the curve $y = x + 2\cos x$ that have a horizontal tangent line. $\Rightarrow$

19. Let C be a circle of radius r. Let A be an arc on C subtending a central angle θ. Let B be the chord of C whose endpoints are the endpoints of A. (Hence, B also subtends θ.) Let s be the length of A and let d be the length of B. Sketch a diagram of the situation and compute $\lim_{\theta\to 0^+} s/d$.

4.6 Implicit Differentiation

We have not yet verified the power rule, $\dfrac{d}{dx}x^a = ax^{a-1}$, for non-integer a. There is a close relationship between x^2 and $x^{1/2}$—these functions are inverses of each other, each "undoing" what the other has done. Not surprisingly, this means there is a relationship between their derivatives.

Let's rewrite $y = x^{1/2}$ as $y^2 = x$. We say that this equation defines the function $y = x^{1/2}$ implicitly because while it is not an explicit expression $y = \ldots$, it is true that if $x = y^2$ then y is in fact the square root function. Now, for the time being, pretend that all we know of y is that $x = y^2$; what can we say about derivatives? We can take the derivative of both sides of the equation:

$$\frac{d}{dx}x = \frac{d}{dx}y^2.$$

Then using the chain rule on the right hand side:

$$1 = 2y\left(\frac{d}{dx}y\right) = 2yy'.$$

Then we can solve for y':

$$y' = \frac{1}{2y} = \frac{1}{2x^{1/2}} = \frac{1}{2}x^{-1/2}.$$

This is the power rule for $x^{1/2}$.

There is one little difficulty here. To use the chain rule to compute $d/dx(y^2) = 2yy'$ we need to know that the function y *has* a derivative. All we have shown is that *if* it has a derivative then that derivative must be $x^{-1/2}/2$. When using this method we will always have to assume that the desired derivative exists, but fortunately this is a safe assumption for most such problems.

Here's another interesting feature of this calculation. The equation $x = y^2$ defines more than one function implicitly: $y = -\sqrt{x}$ also makes the equation true. Following exactly the calculation above we arrive at

$$y' = \frac{1}{2y} = \frac{1}{2(-x^{1/2})} = -\frac{1}{2}x^{-1/2}.$$

So the single calculation leading to $y' = 1/(2y)$ simultaneously computes the derivatives of both functions.

We can use the same technique to verify the product rule for any rational power. Suppose $y = x^{m/n}$. Write instead $x^m = y^n$ and take the derivative of each side to get $mx^{m-1} = ny^{n-1}y'$. Then

$$y' = \frac{mx^{m-1}}{ny^{n-1}} = \frac{mx^{m-1}}{n(x^{m/n})^{n-1}} = \frac{m}{n}x^{m-1}x^{-m(n-1)/n} = \frac{m}{n}x^{m/n-1}.$$

This example involves an inverse function defined implicitly, but other functions can be defined implicitly, and sometimes a single equation can be used to implicitly define more than one function. Here's a familiar example. The equation $r^2 = x^2 + y^2$ describes a circle of radius r. The circle is not a function $y = f(x)$ because for some values of x there are two corresponding values of y. If we want to work with a function, we can break the circle into two pieces, the upper and lower semicircles, each of which is a function. Let's call these $y = U(x)$ and $y = L(x)$; in fact this is a fairly simple example, and it's possible to give explicit expressions for these: $U(x) = \sqrt{r^2 - x^2}$ and $L(x) = -\sqrt{r^2 - x^2}$. But it's somewhat easier, and quite useful, to view both functions as given implicitly by $r^2 = x^2 + y^2$: both $r^2 = x^2 + U(x)^2$ and $r^2 = x^2 + L(x)^2$ are true, and we can think of $r^2 = x^2 + y^2$ as defining both $U(x)$ and $L(x)$.

Now we can take the derivative of both sides as before, remembering that y is not simply a variable but a function—in this case, y is either $U(x)$ or $L(x)$ but we're not yet specifying which one. When we take the derivative we just have to remember to apply the chain rule where y appears.

$$\begin{aligned}
\frac{d}{dx}r^2 &= \frac{d}{dx}(x^2 + y^2) \\
0 &= 2x + 2yy' \\
y' &= \frac{-2x}{2y} = -\frac{x}{y}
\end{aligned}$$

Now we have an expression for y', but it contains y as well as x. This means that if we want to compute y' for some particular value of x we'll have to know or compute y at that value of x as well. It is at this point that we will need to know whether y is $U(x)$ or $L(x)$.

Occasionally it will turn out that we can avoid explicit use of $U(x)$ or $L(x)$ by the nature of the problem

EXAMPLE 4.6.1 Find the slope of the circle $4 = x^2 + y^2$ at the point $(1, -\sqrt{3})$. Since we know both the x and y coordinates of the point of interest, we do not need to explicitly recognize that this point is on $L(x)$, and we do not need to use $L(x)$ to compute y—but we could. Using the calculation of y' from above,

$$y' = -\frac{x}{y} = -\frac{1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

It is instructive to compare this approach to others.

We might have recognized at the start that $(1, -\sqrt{3})$ is on the function $y = L(x) = -\sqrt{4 - x^2}$. We could then take the derivative of $L(x)$, using the power rule and the chain rule, to get

$$L'(x) = -\frac{1}{2}(4 - x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{4 - x^2}}.$$

Then we could compute $L'(1) = 1/\sqrt{3}$ by substituting $x = 1$.

Alternately, we could realize that the point is on $L(x)$, but use the fact that $y' = -x/y$. Since the point is on $L(x)$ we can replace y by $L(x)$ to get

$$y' = -\frac{x}{L(x)} = \frac{x}{\sqrt{4 - x^2}},$$

without computing the derivative of $L(x)$ explicitly. Then we substitute $x = 1$ and get the same answer as before. □

In the case of the circle it is possible to find the functions $U(x)$ and $L(x)$ explicitly, but there are potential advantages to using implicit differentiation anyway. In some cases it is more difficult or impossible to find an explicit formula for y and implicit differentiation is the only way to find the derivative.

EXAMPLE 4.6.2 Find the derivative of any function defined implicitly by $yx^2 + y^2 = x$. We treat y as an unspecified function and use the chain rule:

$$\begin{aligned}
\frac{d}{dx}(yx^2 + y^2) &= \frac{d}{dx}x \\
(y \cdot 2x + y' \cdot x^2) + 2yy' &= 1 \\
y' \cdot x^2 + 2yy' &= 1 - y \cdot 2x \\
y' &= \frac{1 - 2xy}{x^2 + 2y}
\end{aligned}$$

□

You might think that the step in which we solve for y' could sometimes be difficult—after all, we're using implicit differentiation here because we can't solve the equation $yx^2 + e^y = x$ for y, so maybe after taking the derivative we get something that is hard to solve for y'. In fact, *this never happens.* All occurrences y' come from applying the chain rule, and whenever the chain rule is used it deposits a single y' multiplied by some other expression. So it will always be possible to group the terms containing y' together and factor out the y', just as in the previous example. If you ever get anything more difficult you have made a mistake and should fix it before trying to continue.

It is sometimes the case that a situation leads naturally to an equation that defines a function implicitly.

EXAMPLE 4.6.3 Consider all the points (x, y) that have the property that the distance from (x, y) to (x_1, y_1) plus the distance from (x, y) to (x_2, y_2) is $2a$ (a is some constant). These points form an ellipse, which like a circle is not a function but can viewed as two functions pasted together. Because we know how to write down the distance between two points, we can write down an implicit equation for the ellipse:

$$\sqrt{(x-x_1)^2 + (y-y_1)^2} + \sqrt{(x-x_2)^2 + (y-y_2)^2} = 2a.$$

Then we can use implicit differentiation to find the slope of the ellipse at any point, though the computation is rather messy. □

Exercises 4.6.

In exercises 1–8, find a formula for the derivative y' at the point (x, y):

1. $y^2 = 1 + x^2$ ⇒

2. $x^2 + xy + y^2 = 7$ ⇒

3. $x^3 + xy^2 = y^3 + yx^2$ ⇒

4. $4\cos x \sin y = 1$ ⇒

5. $\sqrt{x} + \sqrt{y} = 9$ ⇒

6. $\tan(x/y) = x + y$ ⇒

7. $\sin(x + y) = xy$ ⇒

8. $\dfrac{1}{x} + \dfrac{1}{y} = 7$ ⇒

9. A hyperbola passing through $(8, 6)$ consists of all points whose distance from the origin is a constant more than its distance from the point (5,2). Find the slope of the tangent line to the hyperbola at $(8, 6)$. ⇒

10. Compute y' for the ellipse of example 4.6.3.

11. If $y = \log_a x$ then $a^y = x$. Use implicit differentiation to find y'.

12. The graph of the equation $x^2 - xy + y^2 = 9$ is an ellipse. Find the lines tangent to this curve at the two points where it intersects the x-axis. Show that these lines are parallel. ⇒

13. Repeat the previous problem for the points at which the ellipse intersects the y-axis. ⇒

14. Find the points on the ellipse from the previous two problems where the slope is horizontal and where it is vertical. ⇒

15. Find an equation for the tangent line to $x^4 = y^2 + x^2$ at $(2, \sqrt{12})$. (This curve is the **kampyle of Eudoxus**.) ⇒

16. Find an equation for the tangent line to $x^{2/3} + y^{2/3} = a^{2/3}$ at a point (x_1, y_1) on the curve, with $x_1 \neq 0$ and $y_1 \neq 0$. (This curve is an **astroid**.) ⇒

17. Find an equation for the tangent line to $(x^2+y^2)^2 = x^2 - y^2$ at a point (x_1, y_1) on the curve, when $y_1 \neq 0$. (This curve is a **lemniscate**.) ⇒

Definition. Two curves are **orthogonal** if at each point of intersection, the angle between their tangent lines is $\pi/2$. Two families of curves, $\mathcal{A}$ and $\mathcal{B}$, are **orthogonal trajectories** of each other if given any curve C in $\mathcal{A}$ and any curve D in $\mathcal{B}$ the curves C and D are orthogonal. For example, the family of horizontal lines in the plane is orthogonal to the family of vertical lines in the plane.

18. Show that $x^2 - y^2 = 5$ is orthogonal to $4x^2 + 9y^2 = 72$. (Hint: You need to find the intersection points of the two curves and then show that the product of the derivatives at each intersection point is -1.)

19. Show that $x^2+y^2 = r^2$ is orthogonal to $y = mx$. Conclude that the family of circles centered at the origin is an orthogonal trajectory of the family of lines that pass through the origin.

Note that there is a technical issue when $m = 0$. The circles fail to be differentiable when they cross the x-axis. However, the circles are orthogonal to the x-axis. Explain why. Likewise, the vertical line through the origin requires a separate argument.

20. For $k \neq 0$ and $c \neq 0$ show that $y^2 - x^2 = k$ is orthogonal to $yx = c$. In the case where k and c are both zero, the curves intersect at the origin. Are the curves $y^2 - x^2 = 0$ and $yx = 0$ orthogonal to each other?

21. Suppose that $m \neq 0$. Show that the family of curves $\{y = mx + b \mid b \in \mathbb{R}\}$ is orthogonal to the family of curves $\{y = -(x/m) + c \mid c \in \mathbb{R}\}$.

4.7 Limits revisited

We have defined and used the concept of limit, primarily in our development of the derivative. Recall that $\lim_{x\to a} f(x) = L$ is true if, in a precise sense, $f(x)$ gets closer and closer to L as x gets closer and closer to a. While some limits are easy to see, others take some ingenuity; in particular, the limits that define derivatives are always difficult on their face, since in

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

both the numerator and denominator approach zero. Typically this difficulty can be resolved when f is a "nice" function and we are trying to compute a derivative. Occasionally such limits are interesting for other reasons, and the limit of a fraction in which both numerator and denominator approach zero can be difficult to analyze. Now that we have the derivative available, there is another technique that can sometimes be helpful in such circumstances.

Before we introduce the technique, we will also expand our concept of limit, in two ways. When the limit of $f(x)$ as x approaches a does not exist, it may be useful to note in what way it does not exist. We have already talked about one such case: one-sided limits. Another case is when "f goes to infinity". We also will occasionally want to know what happens to f when x "goes to infinity".

EXAMPLE 4.7.1 What happens to $1/x$ as x goes to 0? From the right, $1/x$ gets bigger and bigger, or goes to infinity. From the left it goes to negative infinity. □

EXAMPLE 4.7.2 What happens to the function $\cos(1/x)$ as x goes to infinity? It seems clear that as x gets larger and larger, $1/x$ gets closer and closer to zero, so $\cos(1/x)$ should be getting closer and closer to $\cos(0) = 1$. □

As with ordinary limits, these concepts can be made precise. Roughly, we want $\lim_{x\to a} f(x) = \infty$ to mean that we can make $f(x)$ arbitrarily large by making x close enough to a, and $\lim_{x\to\infty} f(x) = L$ should mean we can make $f(x)$ as close as we want to L by making x large enough. Compare this definition to the definition of limit in section 2.3, definition 2.3.2.

DEFINITION 4.7.3 If f is a function, we say that $\lim_{x\to a} f(x) = \infty$ if for every $N > 0$ there is a $\delta > 0$ such that whenever $|x - a| < \delta$, $f(x) > N$. We can extend this in the obvious ways to define $\lim_{x\to a} f(x) = -\infty$, $\lim_{x\to a^-} f(x) = \pm\infty$, and $\lim_{x\to a^+} f(x) = \pm\infty$. □

DEFINITION 4.7.4 Limit at infinity If f is a function, we say that $\lim_{x\to\infty} f(x) = L$ if for every $\epsilon > 0$ there is an $N > 0$ so that whenever $x > N$, $|f(x) - L| < \epsilon$. We may similarly define $\lim_{x\to-\infty} f(x) = L$, and using the idea of the previous definition, we may define $\lim_{x\to\pm\infty} f(x) = \pm\infty$. □

We include these definitions for completeness, but we will not explore them in detail. Suffice it to say that such limits behave in much the same way that ordinary limits do; in particular there are some analogs of theorem 2.3.6.

Now consider this limit:

$$\lim_{x\to\pi} \frac{x^2 - \pi^2}{\sin x}.$$

As x approaches π, both the numerator and denominator approach zero, so it is not obvious what, if anything, the quotient approaches. We can often compute such limits by application of the following theorem.

THEOREM 4.7.5 L'Hôpital's Rule For "sufficiently nice" functions $f(x)$ and $g(x)$, if $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ or both $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, and if $\displaystyle\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$. This remains true if "$x \to a$" is replaced by "$x \to \infty$" or "$x \to -\infty$". ∎

This theorem is somewhat difficult to prove, in part because it incorporates so many different possibilities, so we will not prove it here. We also will not need to worry about the precise definition of "sufficiently nice", as the functions we encounter will be suitable.

EXAMPLE 4.7.6 Compute $\displaystyle\lim_{x\to\pi} \frac{x^2-\pi^2}{\sin x}$ in two ways.

First we use L'Hôpital's Rule: Since the numerator and denominator both approach zero,

$$\lim_{x\to\pi} \frac{x^2-\pi^2}{\sin x} = \lim_{x\to\pi} \frac{2x}{\cos x},$$

provided the latter exists. But in fact this is an easy limit, since the denominator now approaches -1, so

$$\lim_{x\to\pi} \frac{x^2-\pi^2}{\sin x} = \frac{2\pi}{-1} = -2\pi.$$

We don't really need L'Hôpital's Rule to do this limit. Rewrite it as

$$\lim_{x\to\pi} (x+\pi)\frac{x-\pi}{\sin x}$$

and note that

$$\lim_{x\to\pi} \frac{x-\pi}{\sin x} = \lim_{x\to\pi} \frac{x-\pi}{-\sin(x-\pi)} = \lim_{x\to 0} -\frac{x}{\sin x}$$

since $x-\pi$ approaches zero as x approaches π. Now

$$\lim_{x\to\pi} (x+\pi)\frac{x-\pi}{\sin x} = \lim_{x\to\pi} (x+\pi) \lim_{x\to 0} -\frac{x}{\sin x} = 2\pi(-1) = -2\pi$$

as before. □

EXAMPLE 4.7.7 Compute $\displaystyle\lim_{x\to\infty} \frac{2x^2-3x+7}{x^2+47x+1}$ in two ways.

As x goes to infinity both the numerator and denominator go to infinity, so we may apply L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = \lim_{x\to\infty} \frac{4x - 3}{2x + 47}.$$

In the second quotient, it is still the case that the numerator and denominator both go to infinity, so we are allowed to use L'Hôpital's Rule again:

$$\lim_{x\to\infty} \frac{4x - 3}{2x + 47} = \lim_{x\to\infty} \frac{4}{2} = 2.$$

So the original limit is 2 as well.

Again, we don't really need L'Hôpital's Rule, and in fact a more elementary approach is easier—we divide the numerator and denominator by x^2:

$$\lim_{x\to\infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = \lim_{x\to\infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2 - \frac{3}{x} + \frac{7}{x^2}}{1 + \frac{47}{x} + \frac{1}{x^2}}.$$

Now as x approaches infinity, all the quotients with some power of x in the denominator approach zero, leaving 2 in the numerator and 1 in the denominator, so the limit again is 2. □

EXAMPLE 4.7.8 Compute $\displaystyle\lim_{x\to 0} \frac{\sec x - 1}{\sin x}$.

Both the numerator and denominator approach zero, so applying L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\sec x - 1}{\sin x} = \lim_{x\to 0} \frac{\sec x \tan x}{\cos x} = \frac{1 \cdot 0}{1} = 0.$$

□

Exercises 4.7.

Compute the limits.

1. $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{\sin x}$ ⇒

2. $\displaystyle\lim_{x\to\infty} \sqrt{x^2 + x} - \sqrt{x^2 - x}$ ⇒

3. $\displaystyle\lim_{x\to 0} \frac{\sqrt{9 + x} - 3}{x}$ ⇒

4. $\displaystyle\lim_{t\to 1^+} \frac{(1/t) - 1}{t^2 - 2t + 1}$ ⇒

5. $\displaystyle\lim_{x\to 2} \frac{2 - \sqrt{x + 2}}{4 - x^2}$ ⇒

6. $\displaystyle\lim_{t\to\infty} \frac{t + 5 - 2/t - 1/t^3}{3t + 12 - 1/t^2}$ ⇒

7. $\displaystyle\lim_{y\to\infty} \frac{\sqrt{y + 1} + \sqrt{y - 1}}{y}$ ⇒

8. $\displaystyle\lim_{x\to 1} \frac{\sqrt{x} - 1}{\sqrt[3]{x} - 1}$ ⇒

9. $\displaystyle\lim_{x\to 0} \frac{(1 - x)^{1/4} - 1}{x}$ ⇒

10. $\displaystyle\lim_{t\to 0} \left(t + \frac{1}{t}\right)((4 - t)^{3/2} - 8)$ ⇒

11. $\lim_{t\to 0^+}\left(\frac{1}{t}+\frac{1}{\sqrt{t}}\right)(\sqrt{t+1}-1)\Rightarrow$

12. $\lim_{x\to 0}\frac{x^2}{\sqrt{2x+1}-1}\Rightarrow$

13. $\lim_{u\to 1}\frac{(u-1)^3}{(1/u)-u^2+3u-3}\Rightarrow$

14. $\lim_{x\to 0}\frac{2+(1/x)}{3-(2/x)}\Rightarrow$

15. $\lim_{x\to 0^+}\frac{1+5/\sqrt{x}}{2+1/\sqrt{x}}\Rightarrow$

16. $\lim_{x\to 0^+}\frac{3+x^{-1/2}+x^{-1}}{2+4x^{-1/2}}\Rightarrow$

17. $\lim_{x\to\infty}\frac{x+x^{1/2}+x^{1/3}}{x^{2/3}+x^{1/4}}\Rightarrow$

18. $\lim_{t\to\infty}\frac{1-\sqrt{\frac{t}{t+1}}}{2-\sqrt{\frac{4t+1}{t+2}}}\Rightarrow$

19. $\lim_{t\to\infty}\frac{1-\frac{t}{t-1}}{1-\sqrt{\frac{t}{t-1}}}\Rightarrow$

20. $\lim_{x\to-\infty}\frac{x+x^{-1}}{1+\sqrt{1-x}}\Rightarrow$

21. $\lim_{x\to\pi/2}\frac{\cos x}{(\pi/2)-x}\Rightarrow$

22. $\lim_{x\to 1}\frac{x^{1/4}-1}{x}\Rightarrow$

23. $\lim_{x\to 1^+}\frac{\sqrt{x}}{x-1}\Rightarrow$

24. $\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}\Rightarrow$

25. $\lim_{x\to\infty}\frac{x^{-1}+x^{-1/2}}{x+x^{-1/2}}\Rightarrow$

26. $\lim_{x\to\infty}\frac{x+x^{-2}}{2x+x^{-2}}\Rightarrow$

27. $\lim_{x\to\infty}\frac{5+x^{-1}}{1+2x^{-1}}\Rightarrow$

28. $\lim_{x\to\infty}\frac{4x}{\sqrt{2x^2+1}}\Rightarrow$

29. $\lim_{x\to 0}\frac{3x^2+x+2}{x-4}\Rightarrow$

30. $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{\sqrt{x+4}-2}\Rightarrow$

31. $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{\sqrt{x+2}-2}\Rightarrow$

32. $\lim_{x\to 0^+}\frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}\Rightarrow$

33. $\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{\sqrt{x+1}-1}\Rightarrow$

34. $\lim_{x\to\infty}(x+5)\left(\frac{1}{2x}+\frac{1}{x+2}\right)\Rightarrow$

35. $\lim_{x\to 0^+}(x+5)\left(\frac{1}{2x}+\frac{1}{x+2}\right)\Rightarrow$

36. $\lim_{x\to 1}(x+5)\left(\frac{1}{2x}+\frac{1}{x+2}\right)\Rightarrow$

37. $\lim_{x\to 2}\frac{x^3-6x-2}{x^3+4}\Rightarrow$

38. $\lim_{x\to 2}\frac{x^3-6x-2}{x^3-4x}\Rightarrow$

39. $\lim_{x\to 1+}\frac{x^3+4x+8}{2x^3-2}\Rightarrow$

40. The function $f(x)=\frac{x}{\sqrt{x^2+1}}$ has two horizontal asymptotes. Find them and give a rough sketch of f with its horizontal asymptotes. $\Rightarrow$