8

Techniques of Integration

Over the next few sections we examine some techniques that are frequently successful when seeking antiderivatives of functions. Sometimes this is a simple problem, since it will be apparent that the function you wish to integrate is a derivative in some straightforward way. For example, faced with
\[ \int x^4 \, dx \]
we realize immediately that the derivative of \( x^5 \) will supply an \( x^5 \), \( (x^5)' = 11x^4 \). We don’t want the “11”, but constants are easy to alter, because differentiation “ignores” them in certain circumstances, so
\[ \frac{d}{dx} x^5 - \frac{1}{11} x^11 = x^5. \]

From our knowledge of derivatives, we can immediately write down a number of antiderivatives. Here is a list of those most often used:
\[
\begin{align*}
\int x^n \, dx &= \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \\
\int x^{-1} \, dx &= \ln |x| + C \\
\int e^x \, dx &= e^x + C \\
\int \sin x \, dx &= -\cos x + C
\end{align*}
\]

8.1 Substitution

of \( 1 - x^2 \), \( -2x \), multiplied on the outside. If we can find a function \( F(x) \) whose derivative is \(-(1/2)(2x - 4x)\sqrt{1-x^2}\) we’ll be done, since then
\[
\frac{d}{dx} F(1-x^2) = -2x F'(1-x^2) = -(2x) \left( \frac{1}{2} \right) (1 - (1-x^2)) \sqrt{1-x^2}
\]
\[= x \sqrt{1-x^2}. \]

But this isn’t hard:
\[
\int \frac{1}{2} (1-x) \sqrt{1-x^2} \, dx = \int \frac{1}{2} (x^{3/2} - x^{1/2}) \, dx
\]
\[= \frac{1}{2} \left( \frac{2}{5} x^{5/2} - \frac{2}{3} x^{3/2} \right) + C
\]
\[= \left( \frac{1}{5} - \frac{1}{3} \right) x^{3/2} + C. \]

So finally we have
\[
\int x \sqrt{1-x^2} \, dx = \left( \frac{1}{5} - \frac{1}{3} \right) (1-x^2)^{3/2} + C.
\]

So we succeeded, but it required a clever first step, rewriting the original function so that it looked like the result of using the chain rule. Fortunately, there is a technique that makes such problems simpler, without requiring cleverness to rewrite a function in just the right way. It sometimes does not work, or may require more than one attempt, but the idea is simple: guess at the most likely candidate for the “inside function”, then do some algebra to see what this requires the rest of the function to look like.

One frequently good guess is any complicated expression inside a square root, so we start by trying \( u = 1 - x^2 \), using a new variable, \( u \), for convenience in the manipulations that follow. Now we know that the chain rule will multiply by the derivative of this inner function:
\[
\frac{du}{dx} = -2x,
\]
so we need to rewrite the original function to include this:
\[
\int x \sqrt{1-x^2} = \int x \sqrt{2x} \frac{du}{dx} = \int \frac{x}{2} \sqrt{2x} \frac{du}{dx}.
\]

Recall that one benefit of the Leibniz notation is that it often turns out that what looks like ordinary arithmetic gives the correct answer, even if something more complicated is going on. For example, in Leibniz notation the chain rule is
\[
\frac{dy}{dx} = \frac{du}{dx} \frac{dy}{du},
\]
the same is true of our current expression:
\[
\int x^2 \sqrt{1-x^2} \, dx = \int x^2 \sqrt{1-u} \frac{du}{dx}.
\]

Now we’re almost there; since \( u = 1 - x^2 \), \( x^2 = 1 - u \) and the integral is
\[
\int \frac{x^2}{\sqrt{1-u}} \, du.
\]

It’s no coincidence that this is exactly the integral we computed in (8.1.1), we have simply renamed the variable \( u \) to make the calculations less confusing. Just as before:
\[
\int \frac{x^2}{\sqrt{1-u}} \, du = \left( \frac{u}{2} - \frac{1}{3} \right) (1-u)^{3/2} + C.
\]

Then since \( u = 1 - x^2 \):
\[
\int x^2 \sqrt{1-x^2} \, dx = \left( \frac{1}{2} - \frac{1}{3} \right) (1-x^2)^{3/2} + C.
\]

To summarize: if we suspect that a given function is the derivative of another via the chain rule, we let \( u \) denote a likely candidate for the inner function, then translate the given function so that it is written entirely in terms of \( u \), with no \( x \) remaining in the expression. If we can integrate this new function of \( u \), then the antiderivative of the original function is obtained by replacing \( u \) by the equivalent expression in \( x \).

Even in simple cases you may prefer to use this mechanical procedure, since it often helps to avoid silly mistakes. For example, consider again this simple problem:
\[
\int 2x \cos(x^2) \, dx.
\]

Let \( u = x^2 \), then \( du/dx = 2x \) or \( dx = du/2x \). Since we have exactly \( 2x \, dx \) in the original integral, we can replace it by \( du/2x \):
\[
\int 2x \cos(x^2) \, dx = \int \cos u \, du = \sin u + C = \sin(x^2) + C.
\]

This is not the only way to do the algebra, and typically there are many paths to the correct answer. Another possibility, for example, is: Since \( du/dx = 2x \), \( dx = du/2x \), and
\[
\int 2x \cos(x^2) \, dx = \int \cos x \, dx = \sin x + C.
\]

\[
\int \cos x \, dx = \sin x + C
\]
\[
\int \sec^2 x \, dx = \tan x + C
\]
\[
\int \sec x \tan x \, dx = \sec x + C
\]
\[
\int \frac{1}{1 + x^2} \, dx = \arctan x + C
\]
\[
\int \frac{1}{\sqrt{1 - x^2}} \, dx = \arcsin x + C
\]
then the integral becomes
\[ \int 2x \cos(x^2) \, dx = \int 2x \cos \left( \frac{dx}{2} \right) \cos \left( \frac{dx}{2} \right) \, du. \]

The important thing to remember is that you must eliminate all instances of the original variable \( x \).

**EXAMPLE 8.1.1** Evaluate \( \int (ax + b)^n \, dx \), assuming that \( a \) and \( b \) are constants, \( a \neq 0 \), and \( n \) is a positive integer. We let \( u = ax + b \) so \( du = a \, dx \) or \( dx = du/a \). Then
\[ \int (ax + b)^n \, dx = \int \frac{1}{a} u^n \, du = \frac{1}{a(n+1)} u^{n+1} + C = \frac{1}{a(n+1)} (ax + b)^{n+1} + C. \]

**EXAMPLE 8.1.2** Evaluate \( \int \sin(ax + b) \, dx \), assuming that \( a \) and \( b \) are constants and \( a \neq 0 \). Again we let \( u = ax + b \) so \( du = a \, dx \) or \( dx = du/a \). Then
\[ \int \sin(ax + b) \, dx = \int \frac{1}{a} \sin u \, du = -\frac{1}{a} \cos u + C = -\frac{1}{a} \cos(ax + b) + C. \]

**EXAMPLE 8.1.3** Evaluate \( \int \frac{1}{2} x \sin(x^2) \, dx \). First we compute the antiderivative, then evaluate the definite integral. Let \( u = x^2 \) so \( du = 2x \, dx \) or \( dx = du/2 \). Then
\[ \int \frac{1}{2} x \sin(x^2) \, dx = \int \frac{1}{2} \sin u \, du = -\frac{1}{2} \cos u + C = -\frac{1}{2} \cos(x^2) + C. \]

A somewhat neater alternative to this method is to change the original limits to match the variable \( u \). Since \( u = x^2 \), when \( x = 2 \), \( u = 4 \) and \( u = 0 \) when \( x = 0 \). So we can do this:
\[ \int_{0}^{4} \frac{1}{2} \sin u \, du = \left[ -\frac{1}{2} \cos u \right]_{0}^{4} = -\frac{1}{2} \cos(4) + \frac{1}{2} \cos(0). \]

An incorrect, and dangerous, alternative is something like this:
\[ \int \frac{1}{2} x \sin(x^2) \, dx = \int \frac{1}{2} \sin u \, du = -\frac{1}{2} \cos u + C = -\frac{1}{2} \cos(x^2) + \frac{1}{2} \cos(4). \]

This is incorrect because \( \int \frac{1}{2} x \sin u \, du \) means that \( u \) takes on values between \( 2 \) and \( 4 \), which is wrong. It is dangerous, because it is very easy to get to the point \( -\frac{1}{2} \cos(u) \) and forget to substitute \( x^2 \) back in for \( u \), thus getting the incorrect answer \( \frac{1}{2} \cos(4) + \frac{1}{2} \cos(2) \). A somewhat clumsy, but acceptable, alternative is something like this:
\[ \int 2 \sin u \, du = -\frac{1}{2} \cos u + C = -\frac{1}{2} \cos(2) + \frac{1}{2} \cos(4). \]

**EXAMPLE 8.1.4** Evaluate \( \int \frac{1}{\sqrt{1 - \sin^2(u)}} \, du \). Let \( u = \sin(\theta) \) so \( du = \cos(\theta) \, d\theta \) or \( d\theta = du/\cos(\theta) \). Then
\[ \int \frac{1}{\sqrt{1 - \sin^2(u)}} \, du = \int \frac{1}{\sqrt{1 - \sin^2(\theta)}} \, d\theta = \int \frac{1}{\sqrt{1 - \sin^2(\theta)}} \, d\theta = \frac{1}{\sqrt{\pi}} \int \frac{1}{\sqrt{1 - \sin^2(\theta)}} \, d\theta = \frac{1}{\sqrt{\pi}} \int \frac{1}{\sqrt{1 - x^2}} \, dx. \]

### Exercises 8.1.
Find the antiderivatives or evaluate the definite integral in each problem.

1. \( \int (1 - u^3) \, du \Rightarrow \)
2. \( \int (1 + 3u^2) \, du \Rightarrow \)
3. \( \int e^{x^2} + 1 \, dx \Rightarrow \)
4. \( \int \frac{1}{\sqrt{1 - u^2}} \, du \Rightarrow \)
5. \( \int \sin^2 x \cos x \, dx \Rightarrow \)
6. \( \int \sqrt{\csc^2 x - u^2} \, du \Rightarrow \)
7. \( \int \csc \, dx \Rightarrow \)
8. \( \int \cos(u) \sin(\sin(u)) \, du \Rightarrow \)
9. \( \int \cos(x) \cos(\sin(x)) \, dx \Rightarrow \)
10. \( \int \tan x \, dx \Rightarrow \)
11. \( \int \sec(3x) \cos(x) \, dx \Rightarrow \)
12. \( \int \sec^2 x \tan x \, dx \Rightarrow \)
13. \( \int \frac{1}{(x^2 - 1)^{1/2}} \, dx \Rightarrow \)
14. \( \int \frac{1}{2} \cot(\csc^2 x) \, dx \Rightarrow \)
15. \( \int \frac{1}{2} \, dx \Rightarrow \)
16. \( \int \sec^2(x^2 - 1) \, dx \Rightarrow \)
17. \( \int \frac{1}{2} \csc(2x^2) \, dx \Rightarrow \)
18. \( \int \frac{1}{2} \csc(2x^2) \, dx \Rightarrow \)
19. \( \int \csc \, dx \Rightarrow \)
20. \( \int f(x) f(x) \, dx \Rightarrow \)

### 8.2 Powers of sine and cosine
Functions consisting of products of the sine and cosine can be integrated by using substitution and trigonometric identities. These can sometimes be tedious, but the technique is straightforward. Some examples will suffice to explain the approach.

**EXAMPLE 8.2.1** Evaluate \( \int \sin^3 x \, dx \). We write the function:
\[ \int \sin^3 x \, dx = \int \sin x \sin^2 x \, dx = \int \sin x (1 - \cos^2 x) \, dx. \]

Now use \( u = \cos x \), \( du = -\sin x \, dx \):
\[ \sin x (1 - \cos^2 x) \, dx = -\int (1 - u^2) \, du \]
\[ = -\int (1 - u^2 + u^2) \, du \]
\[ = -u^2 + \frac{1}{3} u^3 + C \]
\[ = -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{3} \cos^2 x + C. \]

**EXAMPLE 8.2.2** Evaluate \( \int \sin^6 x \, dx \). Use \( \sin^2 x = (1 - \cos(2x))/2 \) to rewrite the function:
\[ \int \sin^6 x \, dx = \int (\sin^2 x)^3 \, dx = \int \left( \frac{1 - \cos(2x)}{2} \right)^3 \, dx \]
\[ = \frac{1}{8} \int 1 - 3 \cos 2x + 3 \cos^2 2x - \cos^3 2x \, dx. \]

Now we have four integrals to evaluate:
\[ \int 1 \, dx = x \]
and
\[ \int -3 \cos 2x \, dx = -\frac{3}{2} \sin 2x \]

**EXAMPLE 8.2.3** Evaluate \( \int \cos^3 x \, dx \). Use the formulas \( \sin x = (1 - \cos(2x))/2 \) and \( \cos^3 x = (1 + \cos(2x))/2 \) to get:
\[ \int \cos^3 x \, dx = \int \frac{1 + \cos(2x)}{2} \, dx = \frac{1}{2} \int (1 + \cos(2x)) \, dx = \frac{1}{2} \left( x + \frac{\sin 2x}{2} \right). \]

So at last we get
\[ \int \sin^6 x \, dx = \frac{3}{8} \int \sin 2x = \frac{1}{16} \int \sin 2x - \frac{3}{8} \int \sin^2 2x + \frac{1}{8} \int \sin 4x + C. \]

**EXAMPLE 8.2.4** Evaluate \( \int \csc^2 x \cos x \, dx \). Use the formulas \( \sin x = (1 - \cos(2x))/2 \) and \( \cos x = (1 + \cos(2x))/2 \) to get:
\[ \int \sin x \cos x \, dx = \frac{1}{2} \int \sin(2x) + \int \cos(2x) \, dx = \frac{1}{2} \int \sin(2x) + \int \cos(2x) \, dx = \frac{1}{2} \left( x + \frac{\sin 2x}{2} \right). \]

The remainder is left as an exercise.

### Exercises 8.2.
Find the antiderivatives.

1. \( \int \sin^2 x \, dx \Rightarrow \)
2. \( \int \sin^4 x \, dx \Rightarrow \)
3. \( \int \sin^6 x \, dx \Rightarrow \)
4. \( \int \cos^2 x \, dx \Rightarrow \)
5. \( \int \cos^4 x \, dx \Rightarrow \)
6. \( \int \cos^6 x \, dx \Rightarrow \)
7. \( \int \cos x \sin x \, dx \Rightarrow \)
8. \( \int \sin x \cos x \, dx \Rightarrow \)
9. \( \int \sec^2 x \cot x \, dx \Rightarrow \)
10. \( \int \tan^2 x \sec x \, dx \Rightarrow \)
### 8.3 Trigonometric Substitutions

So far we have seen that it sometimes helps to replace a subexpression of a function by a single variable. Occasionally it can help to replace the original variable by something more complicated. This seems like a “reverse” substitution, but it is really no different in principle than ordinary substitution.

**EXAMPLE 8.3.1** Evaluate \( \int \sqrt{1 - x^2} \, dx \). Let \( x = \sin u \) so \( dx = \cos u \, du \). Then

\[
\int \sqrt{1 - x^2} \, dx = \int \sqrt{1 - \sin^2 u} \, \cos u \, du = \int \cos^2 u \, du.
\]

We would like to replace \( \sqrt{\cos^2 u} \) by \( \cos u \), but this is valid only if \( \cos u \) is positive, since \( \sqrt{\cos^2 u} \) is positive. Consider again the substitution \( x = \sin u \). We could just as well think of this as \( u = \arcsin x \). If we do, then by the definition of the arcsec, \( -\pi/2 \leq u \leq \pi/2 \), \( \cos u \geq 0 \). Then we continue:

\[
\int \sqrt{\sin^2 u} \, \cos u \, du = \int \cos^2 u \, du = \int \frac{1 + \cos 2u}{2} \, du = \frac{u}{2} + \frac{\sin 2u}{4} + C.
\]

This is a perfectly good answer, though the term \( \sin(\arcsin x) \) is a bit unpleasant. It is possible to simplify this. Using the identity \( \sin 2u = 2 \sin u \cos u \), we can write \( \sin 2u = 2 \sin(\arcsin x) \sqrt{1 - \sin^2(\arcsin x)} = 2 \sqrt{1 - x^2} \). Then the full antiderivative is

\[
\frac{1}{2} \arcsin x \sqrt{1 - x^2} - \frac{1}{2} \arcsin x + \frac{1}{2} \sqrt{1 - x^2} + C.
\]

This type of substitution is usually indicated when the function you wish to integrate contains a polynomial expression that might allow you to use the fundamental identity \( \sin^2 x + \cos^2 x = 1 \) in one of three forms:

\[
\cos^2 x = 1 - \sin^2 x \quad \sec^2 x = 1 + \tan^2 x \quad \tan^2 x = \sec^2 x - 1.
\]

If your function contains \( 1 - x^2 \), as in the example above, try \( x = \sin u \); if it contains \( 1 + x^2 \) try \( x = \tan u \); and if it contains \( x^2 - 1 \), try \( x = \sec u \). Sometimes you will need to try something a bit different to handle constants other than one.

**8.3 Trigonometric Substitutions**

First we do \( \int \sec u \, du \), which we will need to compute \( \int \sec^3 u \, du \):

\[
\int \sec u \, du = \int \frac{\sec u + \tan u}{\sec u + \tan u} \, du = \int 1 \, du = \ln |\sec u + \tan u| + C.
\]

Now let \( w = \sec u + \tan u \), \( dw = \sec u \tan u + \sec^2 u \, du \), exactly the numerator of the function we are integrating. Thus

\[
\int \sec u \, du = \int \frac{\sec u + \tan u}{\sec u + \tan u} \, dw = \int 1 \, dw = \ln |w| + C = \ln |\sec u + \tan u| + C.
\]

Now for \( \int \sec^3 u \, du \):

\[
\int \sec^3 u \, du = \frac{\sec u}{2} + \frac{\sec u \tan u}{2} + \frac{\ln |\sec u + \tan u|}{2} + C.
\]

We already know how to integrate \( \sec u \), so we just need the first quotient. This is “simply” a matter of recognizing the product rule in action:

\[
\int \sec^3 u \, du = \sec u \tan u + \ln |\sec u + \tan u| + C.
\]

So putting those together we get

\[
\int \sec u \, du = \sec u \tan u + \ln |\sec u + \tan u| + C,
\]

and reverting to the original variable \( x \):

\[
\int \sqrt{1 + x^2} \, dx = \frac{\sec(\arctan x) \tan(\arctan x)}{2} + \ln |\sec(\arctan x) + \tan(\arctan x)| + C
\]

\[
= \frac{x \sqrt{1 + x^2}}{2} + \ln |x \sqrt{1 + x^2} + 1| + C,
\]

using \( \tan(\arctan x) = x \) and \( \sec(\arctan x) = \sqrt{1 + \tan^2(\arctan x)} = \sqrt{1 + x^2} \).

**EXAMPLE 8.3.2** Evaluate \( \int \sqrt{1 - x^2} \, dx \). We start by rewriting this so that it looks more like the previous example:

\[
\int \sqrt{1 - x^2} \, dx = \int \sqrt{1 - (3x/2)^2} \, dx = \int 2 \sqrt{1 - (3x/2)^2} \, dx.
\]

Now let \( 3x/2 = \sin u \) so \( (3x/2)^2 \cos u \, du = dx = (2/3) \cos u \, du \). Then

\[
\int 2 \sqrt{1 - (3x/2)^2} \, dx = \int 2 \sqrt{1 - \sin^2 u} (2/3) \cos u \, du = \int \frac{4 \sin u}{3} \, du + C
\]

\[
= \frac{2}{3} \arcsin(3x/2) + 2 \sin u \cos u + C.
\]

**EXAMPLE 8.3.3** Evaluate \( \int \sqrt{1 + x^2} \, dx \). Let \( x = \tan u \), \( dx = \sec^2 u \, du \), so

\[
\int \sqrt{1 + x^2} \, dx = \int \sqrt{1 + \tan^2 u} \, \sec^2 u \, du = \int \sec^3 u \, du.
\]

Since \( u = \arctan x \), \( -\pi/2 \leq u \leq \pi/2 \) and \( \sec x \geq 0 \), so \( \sqrt{\sec^2 u} = \sec u \). Then

\[
\int \sec^3 u \, du = \int \sec^3 u \, du.
\]

In problems of this type, two integrals come up frequently: \( \int \sec u \, du \) and \( \int \sec^3 u \, du \). Both have relatively nice expressions but they are a bit tricky to discover.

**8.4 Integration by Parts**

We have already seen that recognizing the product rule can be useful, when we noticed that

\[
\int \sec^3 u \, du = \sec u \tan u + \ln |\sec u + \tan u| + C.
\]

As with substitution, we do not have to rely on insight or cleverness to discover such antiderivatives; there is a technique that will often help to uncover the product rule.

Start with the product rule:

\[
\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x).
\]

We can rewrite this as

\[
f(x)g(x) = \int f'(x)g(x) \, dx + \int f(x)g'(x) \, dx,
\]

and then

\[
\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx.
\]
This may not seem particularly useful at first glance, but it turns out that in many cases we have an integral of the form

\[ \int f(x)g'(x)\,dx \]

but that

\[ \int f'(x)g(x)\,dx \]

easier. This technique for turning one integral into another is called integration by parts, and is usually written in more compact form. If we let \( u = f(x) \) and \( v = g(x) \) then \( du = f'(x)\,dx \) and \( dv = g'(x)\,dx \) and

\[ \int u\,dv = uv - \int v\,du. \]

To use this technique we need to identify likely candidates for \( u = f(x) \) and \( dv = g'(x)\,dx \).

EXAMPLE 8.4.1 Evaluate \( \int x\ln x\,dx \). Let \( u = \ln x \) so \( du = \frac{1}{x}\,dx \). Then we must let \( dv = x\,dx \) so \( v = \frac{x^2}{2} \) and

\[ \int x\ln x\,dx = \frac{x^2\ln x}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} \,dx = \frac{x^2\ln x}{2} - \frac{x^2}{4} + C. \]

EXAMPLE 8.4.2 Evaluate \( \int \sin x\,dx \). Let \( u = -\cos x \) so \( -\sin x \,dx \). Then we must let \( dv = \sin x\,dx \) so \( v = -\cos x \) and

\[ \int \sin x\,dx = -x\cos x - \int -\cos x \,dx = -x\cos x + \sin x + C. \]

EXAMPLE 8.4.3 Evaluate \( \int \sec^3 x\,dx \). Of course we already knew the answer to this, but we needed to be clever to discover it. Here we’ll use the new technique to discover the antiderivative. Let \( u = \sec x \) and \( dv = \sec^2 x\,dx \). Then \( du = \sec x\tan x\,dx \) and \( v = \tan x \)

\[ \int \sec^3 x\,dx = \sec x \tan x - \int \sec x\tan^2 x\,dx \]

\[ = \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx \]

\[ = \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx. \]

At first this looks useless—we’re right back to \( \int \sec^3 x\,dx \). But looking more closely:

\[ \int \sec^3 x\,dx + \int \sec x\,dx = \sec x \tan x - \frac{1}{2} \int \sec x\,dx \]

\[ = \sec x \tan x + \int \sec x\,dx \]

\[ = \sec x \tan x + \ln |\sec x + \tan x| + C. \]

EXAMPLE 8.4.4 Evaluate \( \int x^3 \sin x\,dx \). Let \( u = x^3 \), \( dv = \sin x\,dx \) then \( du = 3x^2\,dx \) and \( v = -\cos x \). Now \( \int x^3 \sin x\,dx = -x^3 \cos x + \int 3x^2 \cos x\,dx \). This is better than the original integral, but we need to do integration by parts again. Let \( u = 2x \), \( dv = \cos x\,dx \) then \( du = 2\,dx \) and \( v = \sin x \), and

\[ \int x^3 \sin x\,dx = -x^3 \cos x + \int 2x^2 \cos x\,dx \]

\[ = -x^3 \cos x + 2\sin x - \int 2\sin x\,dx \]

\[ = -x^3 \cos x + 2\sin x - 2\cos x + C. \]

Such repeated use of integration by parts is fairly common, but it can be a bit tedious to accomplish, and it is easy to make errors, especially sign errors involving the subtraction in the formula. There is a nice tabular method to accomplish the calculation that minimizes the chance for error and speeds up the whole process. We illustrate with the previous example. Here is the table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( u )</th>
<th>( dv )</th>
<th>( u )</th>
<th>( dv )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( x^2 \sin x )</td>
<td>-2x</td>
<td>( \sin x )</td>
<td>( \cos x )</td>
</tr>
<tr>
<td>1</td>
<td>( 2x \cos x )</td>
<td>-sin x</td>
<td>0</td>
<td>( \cos x )</td>
</tr>
<tr>
<td>2</td>
<td>( \cos x )</td>
<td>( \cos x )</td>
<td>( \cos x )</td>
<td>( \cos x )</td>
</tr>
</tbody>
</table>

To form the first table, we start with \( u \) at the top of the second column and repeatedly compute the derivative, starting with \( \frac{dx}{dx} \) at the top of the third column, we repeatedly compute the antiderivative. In the first column, we place a “−” in every second row. To form the second table we combine the first and second columns by ignoring the boundary; if you do this by hand, you may simply start with two columns and add a “−” to every second row.

To compute with this second table we begin at the top. Multiply the first entry in column \( u \) by the second entry in column \( dv \) to get \(-x^2 \cos x\), and add this to the integral of the product of the second entry in column \( u \) and second entry in column \( dv \). This gives:

\[ -x^2 \cos x + \int 2x \cos x\,dx \]

or exactly the result of the first application of integration by parts. Since this integral is not yet easy, we return to the table. Now we multiply twice on the diagonal, \((x^3)(\cos x)\) and \((-2x)(-\sin x)\) and then once straight across, \((2)(-\sin x)\), and combine these as

\[ -x^2 \cos x + 2x \sin x + \int 2 \sin x\,dx, \]

giving the same result as the second application of integration by parts. While this integral is easy, we may return yet once more to the table. Now multiply three times on the diagonal to get \((x^2)(-\cos x), (2)(-\sin x), \) and \((2)(\cos x)\), and once straight across, \((0)(\cos x)\). We combine these as before to get

\[ -x^2 \cos x + 2x \sin x + 2 \cos x + \int 0\,dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C. \]

Typically we would fill in the table one line at a time, until the “straight across” multiplication gives an easy integral. If we can see that the \( u \) column will eventually become zero, we can instead fill in the whole table; computing the products as indicated will then give the entire integral, including the “-1" term, as above.

8.4 Integration by Parts

Exercises 8.4.

Find the antiderivatives.

1. \( \int \cos x\,dx \) =>
2. \( \int \sec x\,dx \) =>
3. \( \int \cos^2 x\,dx \) =>
4. \( \int \cos^4 x\,dx \) =>
5. \( \int \sin^2 x\,dx \) =>
6. \( \int \tan x\,dx \) =>
7. \( \int \tan x\,dx \) =>
8. \( \int \sin x\,dx \) =>
9. \( \int \cos x\,dx \) =>
10. \( \int \sin x\,dx \) =>
11. \( \int \arctan x\,dx \) =>
12. \( \int \arcsin x\,dx \) =>
13. \( \int e^{x^2}\,dx \) =>
14. \( \int \sec^2 x\,dx \) =>

8.5 Rational Functions

A rational function is a fraction with polynomials in the numerator and denominator. For example,

\[ \frac{x^3}{x^2 + x - 6} \]

\[ \frac{1}{(x - 3)^2} \]

\[ \frac{x^2 + 1}{x^2 - 1} \]

are all rational functions of \( x \). There is a general technique called “partial fractions" that, in principle, allows us to integrate any rational function. The algebraic steps in the technique are rather cumbersome if the polynomial in the denominator has degree more than 2, and the technique requires that we factor the denominator, something that is not always possible. However, in practice one does not often run across rational functions with high degree polynomials in the denominator for which one has to find the antiderivative function. So we shall explain how to find the antiderivative of a rational function only when the denominator is a quadratic polynomial \( ax^2 + bx + c \).

We should mention a special type of rational function that we already know how to integrate: If the denominator has the form \((ax + b)^n\), the substitution \( u = ax + b \) will always work. The numerator becomes \( u^n \), and each \( x \) in the numerator is replaced by \((u - b)/a\), and \( dx = du/a\). While it may be tedious to complete the integration if the numerator has high degree, it is merely a matter of algebra.
EXAMPLE 8.5.1 Find \( \int \frac{x^3}{(3 - 2x)^3} \, dx \). Using the substitution \( u = 3 - 2x \) we get

\[
\int \frac{x^3}{(3 - 2x)^3} \, dx = -\frac{1}{16} \int \frac{u^3}{u^3} \, du = -\frac{1}{16} \int \frac{u^3 - 9u^2 + 27u - 27}{u^3} \, du = -\frac{1}{16} \left[ \frac{u^2}{2} - \frac{9u}{3} + \frac{27}{3} - 
\frac{27}{4} + C \right] = -\frac{1}{16} \left[ \frac{(3 - 2x)^2}{2} - 9(3 - 2x) + 27(3 - 2x)^{-1} - \frac{27}{4} + C \right] = \frac{1}{16} (3 - 2x)^2 - \frac{9}{2} (3 - 2x) - \frac{27}{4} + C.
\]

We now proceed to the case in which the denominator is a quadratic polynomial. We determine whether \( \frac{u}{u^2 + 1} \) factors, and factor it if possible. The quadratic formula tells us that \( \frac{u}{u^2 + 1} \) is equal to \( \frac{u}{u^2 + 1} = \frac{u}{u^2 + 1} \), so we have that \( \frac{u}{u^2 + 1} \) doesn’t factor. Again we can use long division to ensure that \( \frac{u}{u^2 + 1} \) doesn’t factor. Again we can use long division to ensure that the numerator has degree less than 2, then we complete the square.

\[
\int x^2 + x + 1 \, dx = \frac{1 + \sqrt{5}}{2} x + \frac{1 - \sqrt{5}}{2}.
\]

Therefore, \( x^2 + x = \left( \frac{1 + \sqrt{5}}{2} \right) x + \left( \frac{1 - \sqrt{5}}{2} \right). \]

For the second integral we complete the square: \( \frac{1}{8} x^2 + \frac{1}{4} x + \frac{1}{8} = \frac{1}{8} \left( \frac{2}{3} \right)^2 + 1 \), making the integral

\[
\int \frac{1}{8} \left( \frac{2}{3} \right)^2 + 1 \, dx.
\]

Using \( u = \frac{x + 2}{2} \) we get

\[
\int \frac{1}{8} \left( \frac{2}{3} \right)^2 + 1 \, dx = \frac{1}{8} \int \frac{1}{u^2 + 1} \, du = \frac{1}{2} \arctan \left( \frac{x + 2}{2} \right).
\]

The final answer is now

\[
\int \frac{1}{8} \left( \frac{2}{3} \right)^2 + 1 \, dx = \frac{1}{2} \arctan \left( \frac{x + 2}{2} \right) + C.
\]

8.6 Numerical Integration

We have now seen some of the most generally useful methods for discovering antiderivatives, and there are others. Unfortunately, some functions have no simple antiderivatives; in such cases if the value of a definite integral is needed it will have to be approximated. We will see two methods that work reasonably well and yet are fairly simple; in some cases more sophisticated techniques will be needed.

Of course, we already know one way to approximate an integral: if we think of the integral as computing an area, we can add up the areas of some rectangles. While this is quite simple, it is usually the case that a large number of rectangles is needed to get acceptable accuracy. A similar approach is much better: we approximate the area under a curve over a small interval as the area of a trapezoid. In figure 8.6.1 we see an area under a curve approximated by rectangles and by trapezoids; it is apparent that the trapezoids are more accurate than the rectangles. In some cases if the value of a definite integral is needed it will have to be approximated. We will see two methods that work reasonably well and yet are fairly simple; in some cases more sophisticated techniques will be needed.

As with rectangles, we divide the interval into \( n \) equal subintervals of length \( \Delta x \). A typical trapezoid is pictured in figure 8.6.2; it has area \( \frac{f(x_i) + f(x_{i+1})}{2} \Delta x \). If we add up
by using something other than a straight line? The obvious candidate is a parabola: if we can approximate a short piece of the curve with a parabola with equation \( y = ax^2 + bx + c \), we can easily compute the area under the parabola.

There are an infinite number of parabolas through any two given points, but only one through three given points. If we find a parabola through three consecutive points \((x_i, f(x_i)), (x_{i+1}, f(x_{i+1})), (x_{i+2}, f(x_{i+2}))\) on the curve, it should be quite close to the curve over the whole interval \([x_i, x_{i+2}]\), as is figure 8.6.3. If we divide the interval \([a, b]\) into an even number of subintervals, we can then approximate the curve by a sequence of parabolas, each covering two of the subintervals. For this to be practical, we would like a simple formula for the area under one parabola, namely, the parabola through \((x_i, f(x_i)), (x_{i+1}, f(x_{i+1})), (x_{i+2}, f(x_{i+2}))\). That is, we should attempt to write down the parabola \( y = ax^2 + bx + c \) through these points and then integrate it, and hope that the result is fairly simple. Although the algebra involved is messy, this turns out to be possible. The algebra is well within the capability of a good computer algebra system like Sage, so we will present the result without all of the algebra; you can see how to do it in this Sage worksheet.

To find the parabola, we solve these three equations for \( a, b, \) and \( c \):

\[
\begin{align*}
  f(x_i) &= a(x_{i+1}-x_i)^2 + b(x_{i+1}-x_i) + c \\
  f(x_{i+1}) &= a(x_{i+2}-x_{i+1})^2 + b(x_{i+2}-x_{i+1}) + c \\
  f(x_{i+2}) &= a(x_{i+1}+\Delta x)^2 + b(x_{i+1}+\Delta x) + c
\end{align*}
\]

Not surprisingly, the solutions turn out to be quite messy. Nevertheless, Sage can easily compute and simplify the integral to get

\[
\int_{x_{i+1}}^{x_{i+2}} ax^2 + bx + c\,dx = \frac{\Delta x}{3} (f(x_i) + 4f(x_{i+1}) + f(x_{i+2})).
\]

Now the sum of the areas under all parabolas is

\[
\frac{\Delta x}{3} \left( f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right) = \frac{\Delta x}{3} \left( f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right).
\]

This is just slightly more complicated than the formula for trapezoids; we need to remember the alternating 2 and 4 coefficients; note that \( n \) must be even for this to make sense. This approximation technique is referred to as Simpson’s Rule.

As with the trapezoid method, this is useful only with an error estimate:

\[
E(\Delta x) = \frac{b^3}{180} M(\Delta x)^3 = \frac{(b-a)^3}{180n^3} M.
\]

Let’s see how we can use this.

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the areas of all trapezoids we get

\[
\frac{f(x_0) + f(x_1)}{2} \Delta x + \frac{f(x_1) + f(x_2)}{2} \Delta x + \cdots + \frac{f(x_{n-1}) + f(x_n)}{2} \Delta x = \left( \frac{f(x_0)}{2} + f(x_1) + \frac{f(x_2)}{2} + \cdots + \frac{f(x_{n-1})}{2} + f(x_n) \right) \Delta x.
\]

This is usually known as the Trapezoid Rule. For a modest number of subintervals this is not too difficult to do with a calculator; a computer can easily do many subintervals.

THEOREM 8.6.1 Suppose \( f \) has a second derivative \( f'' \) everywhere on the interval \([a, b]\), and \(|f''(x)| \leq M \) for all \( x \) in the interval. With \( \Delta x = (b-a)/n \), an error estimate for the trapezoidal approximation is

\[
E(\Delta x) = \frac{b-a}{12} M(\Delta x)^3 = \frac{(b-a)^3}{12n^3} M.
\]

EXAMPLE 8.6.2 Approximate \( \int_0^1 e^{-x^2} \,dx \) to ten decimal places. The second derivative of \( f = e^{-x^2} \) is \((4x^2-2)e^{-x^2}\), and it is not hard to see that on \([0, 1], |(4x^2-2)e^{-x^2}| \leq 2 \). We begin by estimating the number of subintervals we are likely to need. To get two decimal places of accuracy, we will certainly need \( E(\Delta x) < 0.005 \) or

\[
\frac{1}{12} (2)^7 < 0.005\quad \frac{1}{12} (200)^2 < n^2
\]

\[
5.77 < \frac{\sqrt{12}}{6} < n
\]

With \( n = 6 \), the error estimate is thus \( 1/6^3 < 0.0047 \). We compute the trapezoidal approximation for six intervals:

\[
\frac{1}{2}(0) + f(1/6) + f(2/6) + \cdots + f(5/6) + f(1/2) \text{ for } n = 6 \approx 0.74512.
\]

So the true value of the integral is between 0.74512 – 0.0047 = 0.74042 and 0.74512 + 0.0047 = 0.74982. Unfortunately, the first round to 0.74 and the second rounds to 0.75, so we can’t be sure of the correct value in the second decimal place; we need to pick a larger \( n \). As it turns out, we need to go to \( n = 12 \) to get two bounds that both round to the same value, which turns out to be 0.75. For comparison, using 12 rectangles to approximate the area gives 0.7727, which is considerably less accurate than the approximation using six trapezoids.

In practice it generally pays to start by requiring better than the maximum possible error; for example, we might have initially required \( E(\Delta x) < 0.001 \), or

\[
\frac{1}{12} (2)^7 < 0.001\quad \frac{1}{12} (1000)^2 < n^2
\]

\[
12.91 < \frac{\sqrt{12}}{6} < n
\]

Had we immediately tried \( n = 13 \) this would have given us the desired answer.

The trapezoidal approximation works well, especially compared to rectangles, because the tops of the trapezoids form a reasonably good approximation to the curve when \( \Delta x \) gets small. Fortunately, for many functions, there is such an error estimate associated with the trapezoidal approximation.

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EXAMPLE 8.6.4 Let us again approximate \( \int_0^1 e^{-x^2} \,dx \) to two decimal places. The second derivative of \( f = e^{-x^2} \) is \((16x^4 - 48x^2 + 12)e^{-x^2}\); on \([0, 1]\) this is at most 12 in absolute value. We begin by estimating the number of subintervals we are likely to need. To get two decimal places of accuracy, we will certainly need \( E(\Delta x) < 0.005 \), but taking a cue from our earlier example, let’s require \( E(\Delta x) < 0.004 \):

\[
\frac{1}{12} (12)^7 < 0.004\quad \frac{1}{12} (200)^2 < n^2
\]

\[
2.86 < \frac{\sqrt{12}}{6} < n
\]

So we try \( n = 4 \), since we need an even number of subintervals. Then the error estimate is \( 12/160/4^4 < 0.003 \) and the approximation is

\[
\int_0^1 e^{-x^2} \,dx \approx 0.746855.
\]

So the true value of the integral is between 0.746855 – 0.000 3 = 0.746555 and 0.746855 + 0.0003 = 0.747155, both of which round to 0.75.

\[
\int_0^1 e^{-x^2} \,dx
\]
Exercises 8.6.

In the following problems, compute the trapezoid and Simpson approximations using 4 subintervals, and compute the error estimate for each. (Finding the maximum values of the second and fourth derivatives can be challenging for some of these; you may use a graphing calculator or computer software to estimate the maximum values.) If you have access to Sage or similar software, approximate each integral to two decimal places. You can use this Sage worksheet to get started.

1. \( \int_0^1 x \, dx \) ⇒
2. \( \int_0^1 x^2 \, dx \) ⇒
3. \( \int_0^1 x^3 \, dx \) ⇒
4. \( \int_1^2 1 \, dx \) ⇒
5. \( \int_0^1 \frac{1}{x + 2} \, dx \) ⇒
6. \( \int_0^1 \sqrt{x + 1} \, dx \) ⇒
7. \( \int_0^1 \frac{x}{x^2 + 1} \, dx \) ⇒
8. \( \int_0^1 \sqrt{1 + x^2} \, dx \) ⇒
9. \( \int_0^1 e^{x^2} \, dx \) ⇒
10. \( \int_0^1 \sqrt{1 + x^2} \, dx \) ⇒
11. Using Simpson’s rule on a parabola \( f(x) \), even with just two subintervals, gives the exact value of the integral, because the parabolas used to approximate \( f \) will be \( f \) itself. Remarkably, Simpson’s rule also computes the integral of a cubic function \( f(x) = ax^3 + bx^2 + cx + d \) exactly. Show this is true by showing that

\[
\int_{x_0}^{x_2} f(x) \, dx = \frac{x_2 - x_0}{12} (f(x_0) + 4f((x_0 + x_2)/2) + f(x_2)).
\]

Note that the right hand side of this equation is exactly the Simpson approximation for the cubic. This does require a bit of messy algebra, so you may prefer to use Sage.

8.7 Additional exercises

These problems require the techniques of this chapter, and are in no particular order. Some problems may be done in more than one way.

1. \( \int (t + 4) \, dt \) ⇒
2. \( \int (t^2 - 9)^{3/2} \, dt \) ⇒
3. \( \int e^{x^2} + 16 \, dx \) ⇒
4. \( \int \sin t \cos 2t \, dt \) ⇒
5. \( \int \tan t \sec^3 t \, dt \) ⇒
6. \( \int \frac{2t + 1}{t^3 + 3} \, dt \) ⇒
7. \( \int \frac{1}{\tan^2 t - 4} \, dt \) ⇒
8. \( \int \frac{1}{2 \sin^2 t} \, dt \) ⇒
9. \( \int \frac{\cot^2 t}{\csc^3 t} \, dt \) ⇒
10. \( \int \sec^2 t \, dt \) ⇒
11. \( \int \frac{e^t}{\sqrt{e^t + 1}} \, dt \) ⇒
12. \( \int \cos^3 t \, dt \) ⇒

13. \( \int \frac{1}{t^2 + 3} \, dt \) ⇒
14. \( \int \frac{1}{t^4 + t} \, dt \) ⇒
15. \( \int \sqrt{t^3 + 1} \, dt \) ⇒
16. \( \int e^{\sqrt{t} - 1} \, dt \) ⇒
17. \( \int \sqrt{t^2 + 1} \, dt \) ⇒
18. \( \int (\frac{\sqrt{t} + 4}{t}) \sqrt{t} \, dt \) ⇒
19. \( \int \frac{1}{(2-t^3)^{3/2}} \, dt \) ⇒
20. \( \int \frac{1}{(5 + 4t^3)^{1/3}} \, dt \) ⇒
21. \( \int \sqrt{t} \arctan \sqrt{t} \, dt \) ⇒
22. \( \int \frac{3}{t^2 - 2t + 5} \, dt \) ⇒
23. \( \int \frac{1}{t - \sqrt{2} + 3} \, dt \) ⇒
24. \( \int \frac{1}{t + \sqrt{5} - 1} \, dt \) ⇒
25. \( \int \frac{1}{\ln (1 + t^2)} \, dt \) ⇒
26. \( \int (\ln t)^3 \, dt \) ⇒
27. \( \int e^{3t} \, dt \) ⇒
28. \( \int \frac{1}{t - \sqrt{1 + 1}} \, dt \) ⇒