5

Curve Sketching

Whether we are interested in a function as a purely mathematical object or in connection with some application to the real world, it is often useful to know what the graph of the function looks like. We can obtain a good picture of the graph using certain crucial information provided by derivatives of the function and certain limits.

5.1 Maxima and Minima

A local maximum point on a function is a point \((x, y)\) on the graph of the function whose \(y\) coordinate is larger than all other \(y\) coordinates on the graph at points “close to” \((x, y)\). More precisely, \((x, f(x))\) is a local maximum if there is an interval \((a, b)\) with \(a < x < b\) and \(f(x) \geq f(z)\) for every \(z\) in both \((a, b)\) and the domain of \(f\). Similarly, \((x, y)\) is a local minimum point if it has locally the smallest \(y\) coordinate. Again being more precise: \((x, f(x))\) is a local minimum if there is an interval \((a, b)\) with \(a < x < b\) and \(f(x) \leq f(z)\) for every \(z\) in both \((a, b)\) and the domain of \(f\). A local extremum is either a local minimum or a local maximum.

Local maximum and minimum points are quite distinctive on the graph of a function, and are therefore useful in understanding the shape of the graph. In many applied problems we want to find the largest or smallest value that a function achieves (for example, we might want to find the minimum cost at which some task can be performed) and so identifying maximum and minimum points will be useful for applied problems as well. Some examples of local maximum and minimum points are shown in figure 5.1.1.

Figure 5.1.1 Some local maximum points (A) and minimum points (B).

If \((x, f(x))\) is a point where \(f(x)\) reaches a local maximum or minimum, and if the derivative of \(f\) exists at \(x\), then the graph has a tangent line and the tangent line must be horizontal. This is important enough to state as a theorem, though we will not prove it.

THEOREM 5.1.1 Fermat’s Theorem

If \((x, f(x))\) has a local extremum at \(x = a\) and \(f\) is differentiable at \(a\), then \(f'(a) = 0\).

Thus, the only points at which a function can have a local maximum or minimum are points at which the derivative is zero, as in the left hand graph in figure 5.1.1, or the derivative is undefined, as in the right hand graph. Any value of \(x\) for which \(f(x)\) is zero or undefined is called a critical value for \(f\), and the point \((x, f(x))\) on the curve is called a critical point for \(f\). When looking for local maximum and minimum points, you are likely to make two sorts of mistakes. You may forget that a maximum or minimum can occur where the derivative does not exist, and so forget to check whether the derivative exists everywhere. You might also assume that any place that the derivative is zero is a local maximum or minimum point, but this is not true. A portion of the graph of \(f(x) = x^3\) is shown in figure 5.1.2. The derivative of \(f\) is \(f'(x) = 3x^2\), and \(f'(0) = 0\), but there is neither a maximum nor minimum at \((0, 0)\).

Since the derivative is zero or undefined at both local maximum and local minimum points, we need a way to determine which, if either, actually occurs. The most elementary approach, but one that is often tedious or difficult, is to test directly whether the \(y\) coordinates “near” the potential maximum or minimum are above or below the \(y\) coordinate of points on a horizontal. This is important enough to state as a theorem, though we will not prove it.

EXAMPLE 5.1.3

Find all local maximum and minimum points for the function \(f(x) = x^3 - x\). The derivative is \(f'(x) = 3x^2 - 1\). This is defined everywhere and is zero at \(x = \pm \sqrt[3]{1}/\sqrt[3]{3}\). Looking first at \(x = \sqrt[3]{3}/\sqrt[3]{4}\), we see that \(f(\sqrt[3]{3}/\sqrt[3]{4}) = 2\sqrt[3]{3}/\sqrt[3]{9}\). Now we test two points on either side of \(x = \sqrt[3]{3}/\sqrt[3]{3}\), making sure that neither is farther away than the nearest critical value; since \(\sqrt[3]{3} < 1 < \sqrt[3]{3}/\sqrt[3]{4}\) and \(\sqrt[3]{3}/\sqrt[3]{3} = \sqrt[3]{3}/\sqrt[3]{3}/\sqrt[3]{3})\) there must be a local minimum at \(x = \sqrt[3]{3}/\sqrt[3]{3}\). For \(x = \sqrt[3]{3}/\sqrt[3]{3}\), we see that \(f(\sqrt[3]{3}/\sqrt[3]{3}) = 2\sqrt[3]{3}/\sqrt[3]{9}\). This time we can use \(x = 0\) and \(x = -1\), and we find that \(f(-1) = f(0) = 0 < 2\sqrt[3]{3}/\sqrt[3]{9}\), so there must be a local maximum at \(x = -\sqrt[3]{3}/\sqrt[3]{3}\).

Of course this example is made very simple by our choice of points to test, namely \(x = -1, 0, 1\). We could have used other values, say \(-5/4, 1/3, 3/4, 3/4\), and this would have made the calculations considerably more tedious.

EXAMPLE 5.1.4

Find all local maximum and minimum points for \(f(x) = \sin x - \sin x\). The derivative is \(f'(x) = \cos x - \sin x\). This is always defined and is zero whenever \(\cos x = \sin x\). Recalling that \(\cos x \leq 1\) and \(\sin x \leq 1\), we have a local minimum when \(\cos x = \sin x\). Since both sine and cosine have a period of \(2\pi\), we need only determine the status of \(x = \pi/4\) and \(x = 5\pi/4\). We can use 0 and \(\pi/2\) to test the critical value \(x = \pi/4\). We find that \(f(\pi/4) = -\sqrt[3]{3}/\sqrt[3]{3}\). Hence \(f(x) = 1 < \sqrt[3]{3}/\sqrt[3]{3}\) and \(f(\pi/2) = 0\). We can summarize this more neatly by saying that there are local maxima at \(\pi/4 + 2\pi k\) for every integer \(k\).

We use \(\pi\) and \(2\pi\) to test the critical value \(x = 5\pi/4\). The relevant values are \(f(5\pi/4) = -\sqrt[3]{3}/\sqrt[3]{3}\). Hence \(f(x) = 1 < \sqrt[3]{3}/\sqrt[3]{3}\) and \(f(\pi/2) = 0\). We can summarize this more neatly by saying that there are local minima at \(5\pi/4 + 2\pi k\) for every integer \(k\).

Exercises 5.1

In problems 1–12, find all local maximum and minimum points \((x, y)\) by the method of this section.

1. \(y = x^2 - x \Rightarrow\)
2. \(y = 2x + 3x^3 \Rightarrow\)
3. \(y = 3x^2 - 4x^3 + 2x \Rightarrow\)
4. \(y = x^2 - 2x^3 + 3 \Rightarrow\)
5. \(y = x^2 - 1/x^2 \Rightarrow\)
6. \(y = \cos(x^2) - x \Rightarrow\)
7. \(y = 3x^2 - (1/x^2) \Rightarrow\)
8. \(z = x^3 - z^3 \Rightarrow\)
9. \(y = z^3 - 3x^3 \Rightarrow\)
10. \(z = 1/x^3 \Rightarrow\)
11. \(f(x) = x^2 - 8x + 4 \Rightarrow\)
12. \(f(x) = -2/x \Rightarrow\)
13. Find the local extrema of \( f(x) = |x| + |x-1| \).
14. For any real number \( x \) there is a unique integer \( n \) such that \( n \leq x < n+1 \), and the greatest integer function is defined as \( [x] = n \). Where are the critical values of the greatest integer function? Which are local maxima and which are local minima?
15. Explain why the function \( f(x) = \frac{1}{x} \) has no local maxima or minima.
16. How many critical points can a quadratic polynomial function have?
17. Show that a cubic polynomial can have at most two critical points. Give examples to show that a cubic polynomial can have zero, one, or two critical points.

5.2 The first derivative test

The method of the previous section for deciding whether there is a local maximum or minimum at a critical value is not always convenient. We can instead use information about the derivative \( f'(x) \) to decide; since we have already had to compute the derivative to find the critical values, there is often relatively little extra work involved in this method.

How can the derivative tell us whether there is a maximum, minimum, or neither at a point? Suppose that \( f'(x) = 0 \).

EXAMPLE 5.3.1 Find all local maximum and minimum points for \( f(x) = \sin x + \cos x \) using the first derivative test. The derivative is \( f'(x) = \cos x - \sin x \) and from example 5.1.3 the critical values we need to consider are \( \pi/4 \) and \( 5\pi/4 \).

The graphs of \( \sin x \) and \( \cos x \) are shown in figure 5.2.1. Just to the left of \( \pi/4 \) the cosine is larger than the sine, so \( f'(x) \) is positive; just to the right the cosine is smaller than the sine, so \( f'(x) \) is negative. This means there is a local maximum at \( \pi/4 \). Just to the left of \( 5\pi/4 \) the sine is larger than the cosine, so \( f'(x) \) is negative; just to the right the cosine is smaller than the sine, so \( f'(x) \) is positive. This means there is a local minimum at \( 5\pi/4 \).

5.3 The second derivative test

The basis of the first derivative test is that if the derivative changes from positive to negative at a point at which the derivative is zero then there is a local maximum at the point, and similarly for a local minimum. If \( f''(x) \) changes from positive to negative it is decreasing; this means that the derivative of \( f''(x) \) is negative; and if in fact \( f''(x) \) is negative then \( f''(x) \) is definitely decreasing, so there is a local maximum at the point in question. Note well that \( f''(x) \) might change from positive to negative while \( f'(x) \) is zero, in which case \( f''(x) \) gives us no information about the critical value. Similarly, if \( f''(x) \) changes from negative to positive there is a local minimum at the point, and \( f''(x) \) is increasing. If \( f''(x) > 0 \) at the point, this tells us that \( f''(x) \) is increasing, and so there is a local minimum.

EXAMPLE 5.3.1 Consider again \( f(x) = \sin x + \cos x \) with \( f'(x) = \cos x - \sin x \) and \( f''(x) = -\sin x - \cos x \). Since \( f''(\pi/4) = -\sqrt{2}/2 - -\sqrt{2}/2 = -\sqrt{2} < 0 \), we know there is a local maximum at \( \pi/4 \). Since \( f''(5\pi/4) = -\sqrt{2}/2 - +\sqrt{2}/2 = 0 \), there is a local minimum at \( 5\pi/4 \).

When it works, the second derivative test is often the easiest way to identify local maximum and minimum points. Sometimes the test fails, and sometimes the second derivative is quite difficult to evaluate; in such cases we must fall back on one of the previous tests.

EXAMPLE 5.3.2 Let \( f(x) = x^4 \). The derivatives are \( f'(x) = 4x^3 \) and \( f''(x) = 12x^2 \). Zero is the only critical value; but \( f''(0) = 0 \), so the second derivative test tells us nothing. However, \( f(x) \) is positive everywhere except at zero, so clearly \( f(x) \) has a local minimum at zero. On the other hand, \( f(x) = x^4 \) also has zero as its only critical value, and the second derivative is again zero, but \(-x^4 \) has a local maximum at zero. Finally, if \( f(x) = x^2 \), \( f'(x) = 2x \) and \( f''(x) = 2x \). Again, zero is the only critical value and \( f''(0) = 0 \), but \( x^2 \) has neither a maximum nor a minimum at zero.

Exercises 5.3.

Find all local maximum and minimum points by the second derivative test, when possible.

1. \( y = x^2 - x \)
2. \( y = 2 + 3x - x^2 \)
3. \( y = x^3 - 9x^2 + 24x \)
4. \( y = x^3 + 3x^2 - 2x + 1 \)
5. \( y = 3x^3 - 4x^3 \)
6. \( y = (x^2 - 1)/x \)
7. \( y = 3x^{1/2} - 1/x^2 \)
8. \( y = \cos(2x) - x \)
9. \( y = 4x + 1/x \)
10. \( y = (x + 1)/x^{1/2} \)
11. \( y = x^2 - 3x \)
12. \( y = 6x + 3x^2 \)
13. \( y = x + 1/x \)
14. \( y = x^2 + 1/x \)

5.4 Concavity and inflection points

We know that the sign of the derivative tells us whether a function is increasing or decreasing; for example, if \( f'(x) > 0 \), \( f(x) \) is increasing. The sign of the second derivative \( f''(x) \) tells us whether \( f''(x) \) is increasing or decreasing, we have seen that if \( f''(x) \) is zero and increasing at a point then there is a local minimum at the point, and if \( f''(x) \) is zero and decreasing at a point then there is a local maximum at the point. Thus, we extracted information about \( f(x) \) from information about \( f''(x) \).

We can get information from the sign of \( f''(x) \) even when \( f''(x) \) is zero. Suppose that \( f''(x) = 0 \). This means that \( f'(x) \) is increasing. If \( f'(x) \) is decreasing, this means that \( f(x) \) is decreasing and is getting steeper. The two situations are shown in figure 5.4.1. A curve that is shaped like this is called concave up.

Exercises 5.4.

Find all local maximum and minimum points by the second derivative test, when possible.

1. \( y = x^2 - x \)
2. \( y = 2 + 3x - x^2 \)
3. \( y = x^3 - 9x^2 + 24x \)
4. \( y = x^3 + 3x^2 - 2x + 1 \)
5. \( y = 3x^3 - 4x^3 \)
6. \( y = (x^2 - 1)/x \)
7. \( y = 3x^{1/2} - 1/x^2 \)
8. \( y = \cos(2x) - x \)
9. \( y = 4x + 1/x \)
10. \( y = (x + 1)/x^{1/2} \)
11. \( y = x^2 - 3x \)
12. \( y = 6x + 3x^2 \)
13. \( y = x + 1/x \)
14. \( y = x^2 + 1/x \)
Exercises 5.4.

Describe the concavity of the functions in 1–18.

1. \( y = x^2 - x \Rightarrow \)
2. \( y = 2 + 3x - x^2 \Rightarrow \)
3. \( y = x^2 - 9x^4 + 24x \Rightarrow \)
4. \( y = x^2 - 2x^3 + 3 \Rightarrow \)
5. \( y = 3x^3 - 4x^3 \Rightarrow \)
6. \( y = (x^2 - 1)/x \Rightarrow \)
7. \( y = 3x^2 - (1/x^4) \Rightarrow \)
8. \( y = \sin x + \cos x \Rightarrow \)
9. \( y = 4x + \sqrt{1-x} \Rightarrow \)
10. \( y = (x + 1)/\sqrt{x^4 + 30} \Rightarrow \)
11. \( y = x^2 - x \Rightarrow \)
12. \( y = 6x^2 + 3x \Rightarrow \)
13. \( y = x + 1/x \Rightarrow \)
14. \( y = x^2 + 1/x \Rightarrow \)
15. \( y = (x + 5)^{1/4} \Rightarrow \)
16. \( y = \tan x \Rightarrow \)
17. \( y = \cos x - \sin x \Rightarrow \)
18. \( y = \sin x \Rightarrow \)

Identify the intervals on which the graph of the function \( f(x) = x^3 - 4x^4 + 10 \) is of one of these four shapes: concave up and increasing; concave up and decreasing; concave down and increasing; concave down and decreasing. Θ

Describe the concavity of \( y = x^4 + 6x^2 + 5x + 4 \). You will need to consider different cases, depending on the values of the coefficients.

Let \( n \) be an integer greater than or equal to two, and suppose \( f \) is a polynomial of degree \( n \). How many inflection points can \( f \) have? Hint: Use the second derivative test and the Fundamental Theorem of Algebra (see exercise 18 in section 5.1).

Exercises 5.5.

Sketch the curves. Identify clearly any interesting features, including local maximum and minimum points, inflection points, asymptotes, and intercepts. You can use this Sage worksheet to check your answers or do some of the computations.

1. \( y = x^3 - 5x^4 + 5x^3 \)
2. \( y = x^3 - 3x^2 - 9x + 5 \)
3. \( y = (x - 1)/(x + 3)^{1/2} \)
4. \( y = x^3 - x^2 - 6x + 3 \), \( a > 0 \).
5. \( y = x^2 \)
6. \( y = (r + e^{1/2})/2 \)
7. \( y = x^3 \cos x \)
8. \( y = x^2 - \sin x \)
9. \( y = e^{x^2} \)
10. \( y = 4x + \sqrt{x^2 + 3} \)
11. \( y = (x + 1)/\sqrt{x^2 + 3} \)
12. \( y = x^2 - x \)
13. \( y = 6x + \sin 3x \)
14. \( y = x + 1/x \)
15. \( y = x^2 + 1/x \)
16. \( y = (x + 5)^{1/2} \)
17. \( y = \tan x \)
18. \( y = \cos x - \sin x \)
19. \( y = \sin x \)
20. \( y = x(x^2 + 1) \)
21. \( y = x^3 + 6x^2 + 9x \)
22. \( y = x(x^2 - 9) \)
23. \( y = x^3(x^3 + 9) \)
24. \( y = 2x^2 - x \)
25. \( y = 3\sin(x) - \sin^3(x), \text{ for } x \in [0, 2\pi] \)
26. \( y = (x - 1)(x^2) \)

For each of the following five functions, identify any vertical and horizontal asymptotes, and identify intervals on which the function is concave up and increasing; concave up and decreasing; concave down and increasing; concave down and decreasing.

27. \( f(x) = \sec(x) \)
28. \( f(x) = (1 + x^3) \)
29. \( f(x) = (x - 3)/(2x - 2) \)
30. \( f(x) = (x)/(1 - x - x^2) \)
31. \( f(x) = 1 + 1/(x^3) \)

32. Let \( f(x) = 1/(x^2 - a^2), \text{ where } a \geq 0 \). Find any vertical and horizontal asymptotes and the intervals upon which the given function is concave up and increasing; concave up and decreasing; concave down and increasing; concave down and decreasing. Discuss how the value of \( a \) affects these features.