Transcendental Functions

So far we have used only algebraic functions as examples when finding derivatives, that is, functions that can be built up by the usual algebraic operations of addition, subtraction, multiplication, division, and raising to constant powers. Both in theory and practice there are other functions, called transcendental, that are very useful. Most important among these are the trigonometric functions, the inverse trigonometric functions, exponential functions, and logarithms.

4.1 Trigonometric Functions

When you first encountered the trigonometric functions it was probably in the context of “triangle trigonometry,” defining, for example, the sine of an angle as the “side opposite over the hypotenuse.” While this will still be useful in an informal way, we need to use a more expansive definition of the trigonometric functions. First an important note: while degree measure of angles is sometimes convenient because it is so familiar, it turns out to be ill-suited to mathematical calculation, so (almost) everything we do will be in terms of radian measure of angles.

An angle, $x$, at the center of the circle is associated with an arc of the circle which is said to subtend the angle. In the figure, this arc is the portion of the circle from point $(1,0)$ to point $A$. The length of this arc is the radian measure of the angle $x$; the fact that the radian measure is an actual geometric length is largely responsible for the usefulness of radian measure. The circumference of the unit circle is $2\pi$, so the radian measure of the full circular angle (that is, of the 360 degree angle) is $2\pi$.

While an angle with a particular measure can appear anywhere around the circle, we need a fixed, conventional location so that we can use the coordinate system to define properties of the angle. The standard convention is to place the starting radius for the angle on the positive $x$-axis, and to measure positive angles counterclockwise around the circle. In the figure, $x$ is the standard location of the angle $\pi/6$, that is, the length of the arc from $(1,0)$ to $A$ is $\pi/6$. Angle $y$ in the picture is $-\pi/6$, but is $\pi/6$ in a clockwise direction.

Now the fundamental trigonometric definitions are: the cosine of $x$ and the sine of $x$ are the first and second coordinates of the point $A$, as indicated in the figure. The angle $x$ shown can be viewed as an angle of a right triangle, meaning the usual triangle definitions of the sine and cosine also make sense. Since the hypotenuse of the triangle is 1, the “side opposite over hypotenuse” definition of the sine and cosine also make sense. Since the hypotenuse of the triangle is 1, the “side opposite over hypotenuse” definition of the sine and cosine also make sense.

The principal use of the trigonometric functions is to relate an angle to a point on a unit circle. In the unit circle, the sine of an angle is equal to the $y$-coordinate of the end of the subtended arc, and the cosine of an angle is equal to the $x$-coordinate of the end of the subtended arc.

The sine function is also useful in the study of periodic functions. The sine function is periodic with period $2\pi$, meaning that $\sin(x + 2\pi) = \sin(x)$ for all values of $x$. This property is essential in the study of oscillatory phenomena such as waves and vibrations.

Exercises 4.1.

Similarly, as angle $x$ increases from 0 in the unit circle diagram, the first coordinate of the point $A$ goes from 1 to 0 then to $-1$, back to 0 and back to 1, so the graph of $y = \cos x$ must look something like this:

![Graph of cos(x)](image)

Exercises 4.1.

Some useful trigonometric identities are in Appendix B.

1. Find all values of $\theta$ such that $\sin(\theta) = -1$. Give your answer in radians.

2. Find all values of $\theta$ such that $\cos(\theta) = 1/2$. Give your answer in radians.

3. Use an angle sum identity to compute $\cos(\pi/12)$.

4. Use an angle sum identity to compute $\tan(5\pi/12)$.

5. Verify the identity $\cos^2(\theta) + \sin^2(\theta) = 1$.

6. Verify the identity $2 \cos(2\theta) = \cos(\theta) \cos(\theta) - \sin(\theta) \sin(\theta)$

7. Verify the identity $\sin(\theta) = \sin(\theta)$

8. Sketch $y = 2 \sin(x)$

9. Sketch $y = \sin(3x)$

10. Sketch $y = \sin(x + \pi)$

11. Find all of the solutions of $2 \sin(x) - 1 = \sin(\pi/2) = 0$ in the interval $[0, 2\pi]$. 

4.2 The Derivative of $\sin x$

What about the derivative of the sine function? The rules for derivatives that we have are no help, since $\sin x$ is not an algebraic function. We need to return to the definition of the derivative, set up a limit, and try to compute it. Here’s the definition:

$$\frac{d}{dx} \sin x = \lim_{\Delta x \to 0} \frac{\sin x + \Delta x - \sin x}{\Delta x}$$

Using some trigonometric identities, we can make a little progress on the quotient:

$$\frac{\sin x + \Delta x - \sin x}{\Delta x} = \frac{\sin x \cos \Delta x + \sin \Delta x \cos x - \sin x}{\Delta x}$$

$$= \sin x \frac{\cos \Delta x - 1}{\Delta x} + \cos x \frac{\sin \Delta x}{\Delta x}$$
This isolates the difficult bits in the two limits
\[ \lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} \quad \text{and} \quad \lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}. \]

Here we get a lucky break: it turns out that once we know the second limit the first is quite easy. The second is quite tricky, however. Indeed, it is the hardest limit we will actually compute, and we devote a section to it.

### 4.3 A HARD LIMIT

We want to compute this limit:
\[ \lim_{x \to 0} \frac{\sin x}{x}, \]

Equivalently, to make the notation a bit simpler, we can compute
\[ \lim_{x \to 0} \frac{\sin x}{x}. \]

In the original context we need to keep \( x \) and \( \Delta x \) separate, but here it doesn't hurt to rename \( \Delta x \) to something more convenient.

To do this we need to be quite clever, and to employ some indirect reasoning. The indirect reasoning is embodied in a theorem, frequently called the squeeze theorem.

**THEOREM 4.3.1 Squeeze Theorem** Suppose that \( g(x) \leq f(x) \leq h(x) \) for all \( x \) close to \( a \) but not equal to \( a \). If \( \lim_{x \to a} g(x) = L = \lim_{x \to a} h(x) \), then \( \lim_{x \to a} f(x) = L \).

This theorem can be proved using the official definition of limit. We won't prove it here, but point out that it is easy to understand and believe graphically. The condition says that \( f(x) \) is trapped between \( g(x) \) and \( h(x) \). This means the situation looks something like figure 4.3.1. The wiggly curve is \( x^2 \sin(1/x) \), the upper and lower curves are \( x^2 \) and \(-x^2 \). Since the sine function is always between \(-1\) and \(1\), \(-x^2 \leq x^2 \sin(1/x) \leq x^2 \), and it is easy to see that \( \lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0 \). It is not so easy to see directly, that is algebraically, that \( \lim_{x \to 0} x^2 \sin(1/x) = 0 \), because the \( 1/x \) prevents us from simply plugging in \( x = 0 \). The squeeze theorem makes this “hard limit” as easy as the trivial limits involving \( x^2 \).

To do the hard limit that we want, \( \lim_{x \to 0} \sin(x)/x \), we will find two simpler functions \( g \) and \( h \) so that \( g(x) \leq \sin(x)/x \leq h(x) \), and so that \( \lim_{x \to 0} g(x) = \lim_{x \to 0} h(x) = 0 \). Not too surprisingly, this will require some trigonometry and geometry. Referring to figure 4.3.2, \( x \) is the measure of the angle in radians. Since the circle has radius 1, the coordinates of point \( A \) are \((\cos x, \sin x)\), and the area of the small triangle is \((\cos x \sin x)/2\). This triangle is completely contained within the circular wedge-shaped region bordered by two lines and the circle from \((1,0)\) to point \( B \). Comparing the areas of the triangle and the wedge we see \((\cos x \sin x)/2 \leq x/2\), since the area of a circular region with angle \( \theta \) and radius \( r \) is \( \theta r^2/2\). With a little algebra this turns into \((\sin x)/x \leq 1/\cos x\), giving us the \( h \) we seek.

\[ \cos x \leq \frac{\sin x}{x} \leq \frac{1}{\cos x}. \]

To find \( g \), we note that the circular wedge is completely contained inside the larger triangle. The height of the triangle, from \((1,0)\) to point \( B \), is \( \tan x \), so comparing areas we get \( x/2 \leq (\tan x)/2 = \sin x/\cos x \). With a little algebra this becomes \( \cos x \leq (\sin x)/x \). So now we have
\[ \cos x \leq \frac{\sin x}{x} \leq \frac{1}{\cos x}. \]

Finally, the two limits \( \lim_{\Delta x \to 0} \cos \Delta x \) and \( \lim_{\Delta x \to 0} 1/\cos \Delta x \) are easy, because \( \cos(0) = 1 \). By the squeeze theorem, \( \lim_{x \to 0} \sin(x)/x = 1 \) as well.

Before we can complete the calculation of the derivative of the sine, we need one other limit:
\[ \lim_{x \to 0} \frac{\cos x - 1}{x}. \]

This limit is just as hard as \( \sin x/x \), but closely related to it, so that we don’t have to do a similar calculation; instead we can do a bit of tricky algebra.

\[ \frac{\cos x - 1}{x} = \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} = \frac{\cos^2 x - 1}{x(\cos x + 1)} = \frac{-\sin^2 x}{x(\cos x + 1)} = \frac{-\sin x \cdot \sin x}{x(\cos x + 1)}. \]

To compute the desired limit it is sufficient to compute the limits of the two final fractions, as \( x \) goes to \( 0 \). The first of these is the hard limit we’ve just done, namely 1. The second turns out to be simple, because the denominator presents no problem:
\[ \lim_{x \to 0} \frac{\sin x}{x} = \frac{\sin 0}{0 + 1} = 0. \]

Thus,
\[ \lim_{x \to 0} \frac{\cos x - 1}{x} = 0. \]

**Exercises 4.3.**

1. Compute \( \lim_{x \to 0} \frac{\sin(x)}{x} \)
2. Compute \( \lim_{x \to 0} \frac{\sin(2x)}{2x} \)
3. Compute \( \lim_{x \to 0} \frac{\cos(x)}{x} \)
4. Compute \( \lim_{x \to 0} \frac{x}{\sin(x)} \)
5. Compute \( \lim_{x \to 0} \frac{\sin(x)}{x} = \cos(x) \)

6. For all \( x \geq 0 \), \( 4x - 9 \leq f(x) \leq 4x + 4 \). Find \( \lim_{x \to 0} f(x) \).
7. For all \( x \geq 0 \), \( 4x - 9 \leq g(x) \leq x^2 + 2 \). Find \( \lim_{x \to 0} g(x) \).
8. Use the Squeeze Theorem to show that \( \lim_{x \to 0} x^2 \cos(2x) = 0 \).

**4.4 The Derivative of \( \sin x \), Continued**

Now we can complete the calculation of the derivative of the sine:
\[ \frac{d}{dx} \sin x = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x}. \]

Expand and simplify:
\[ = \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}. \]

The derivative of a function measures the slope or steepness of the function; if we examine the graphs of the sine and cosine side by side, it should be that the latter appears to accurately describe the slope of the former, and indeed this is true:
\[ \cos x \leq \frac{\sin x}{x} \leq \frac{1}{\cos x}. \]

Notice that where the cosine is zero the sine does appear to have a horizontal tangent line, and that the sine appears to be steepest where the cosine takes on its extreme values of \( 1 \) and \(-1 \).

Of course, now that we know the derivative of the sine, we can compute derivatives of more complicated functions involving the sine.

**EXAMPLE 4.4.1** Compute the derivative of \( \sin(x^2) \).
\[ \frac{d}{dx} \sin(x^2) = \cos(x^2) \cdot 2x = 2x \cos(x^2). \]

**EXAMPLE 4.4.2** Compute the derivative of \( \sin^2(x^2 - 5x) \).
\[ \frac{d}{dx} \sin^2(x^2 - 5x) = 2\sin(x^2 - 5x) \cdot (x^2 - 5x) = 2(x^2 - 5) \cos(x^2 - 5x) \sin(x^2 - 5x). \]
Find the derivatives of the following functions.

1. \( \sin^2(x) \) 
2. \( \sqrt{x} \) 
3. \( x^3 \) 
4. \( \tan(x) \) 
5. \( 1 - \sin^2(x) \)

### 4.5 DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS

All of the other trigonometric functions can be expressed in terms of the sine, and so their derivatives can easily be calculated using the rules we already have. For the cosine we need to use two identities,

\[
\cos x = \sin(x + \frac{\pi}{2}) \quad \sin x = -\cos(x + \frac{\pi}{2})
\]

Now,

\[
\frac{d}{dx} \cos x = -\sin(x + \frac{\pi}{2}) = -\cos x \\
\frac{d}{dx} \sin x = \cos(x + \frac{\pi}{2}) = \sin x
\]

\[
\frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x} = -\frac{\sin x}{\cos^2 x} = -\sec x \tan x
\]

The derivatives of the cotangent and cosecant are similar and left as exercises.

### 4.6 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

**Example 4.6.1** What is the value of \( \log_{10}(1000) \)? The "10" tells us the appropriate number to use for the base of the exponential function. The logarithm in the exponent, so the quantity is, what exponent \( E \) makes \( 10^E = 1000 \)? If we can find such an \( E \), then \( \log_{10}(1000) = \log_{10}(10^E) = E \), finding the appropriate exponent is the same as finding the logarithm. In this case, of course, it is easy: \( E = 3 \) so \( \log_{10}(1000) = 3 \).

Let’s review some laws of exponents and logarithms; let \( a \) be a positive number.

- \( a^m \cdot a^n = a^{m+n} \)
- \( a^m / a^n = a^{m-n} \)
- \( (a^m)^n = a^{mn} \)
- \( \sqrt[n]{a^m} = a^{m/n} \)

the cube root function “undoes” the cube function. \( \sqrt[3]{x} = x^{1/3} \). Note that the original function also undoes the inverse function: \( (\sqrt[3]{x})^3 = x \).

**Exercises 4.6.6**

1. Expand \( \log_{10}(x + 45)(x - 2) \).
2. Expand \( \log_{10} \left( \frac{x^2}{x^3 + 4} \right) \).
3. Write \( \log_{10} x - 1 + \log_{10}(x - 2) - 2\log_{10}(x^2 - 4 + 1) \) as a single logarithm.
4. Solve \( \log_{10}(1 + \sqrt{5}) = \frac{7}{6} \).
5. Solve \( 2^x = 8 \).
6. Solve \( \log_{10}(\tan(x)) = -1 \).


4.7 Derivatives of the exponential and logarithmic functions

We have discussed this from the point of view of the graphs, which is easy to understand but is not normally considered a rigorous proof—it is too easy to be led astray by pictures that seem reasonable but that miss some hard point. It is possible to do this derivation without resorting to pictures, and indeed we will see an alternate approach soon.

Note that $\ln x$ is defined only for $x > 0$. It is sometimes necessary to consider the function $\ln |x|$, a function defined for $x \neq 0$. When $x < 0$, $\ln |x| = \ln (-x)$ and $\frac{d}{dx} \ln |x| = \frac{d}{dx} \ln (-x) = \frac{1}{x}(-1) = \frac{1}{x}$.

Thus whether $x$ is positive or negative, the derivative is the same.

What about the functions $a^x$ and $\log_a x$? We know that the derivative of $a^x$ is some constant times $a^x$ itself, but what constant? Remember that “the logarithm is the exponent” and you will see that $a = e^{\ln a}$. Then

$$a^x = (e^{\ln a})^x = e^{\ln a \cdot x},$$

and we can compute the derivative using the chain rule:

$$\frac{d}{dx} a^x = \frac{d}{dx} (e^{\ln a \cdot x}) = \frac{d}{dx} e^{x \ln a} = (\ln a) e^{x \ln a} = (\ln a) a^x.$$

The constant is simply $\ln a$. Likewise we can compute the derivative of the logarithm function $\log_a x$. Since

$$x = e^{\ln x},$$

we can take the logarithm base $a$ of both sides to get

$$\log_a x = \log_a (e^{\ln x}) = \ln x \log_a e.$$

Then

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a} \log_a e.$$

This is a perfectly good answer, but we can improve it slightly. Since

$$a^x = e^{\ln a \cdot x}$$

$$\log_a (a^x) = \log_a (e^{\ln a \cdot x}) = \ln x \log_a e$$

$$\frac{1}{\ln a} = \log_a e,$$

we can replace $\log_a e$ to get

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a} \text{ and } \frac{d}{dx} \log_a x = \frac{1}{x \ln a}.$$
EXAMPLE 4.7.4 Recall that we have not justified the power rule except when the exponent is a positive or negative integer. We can use the exponential function to take care of other exponents.

\[
\frac{d}{dx} e^x = \frac{d}{dx} e^{\ln x} = \left( \frac{d}{dx} \ln x \right) e^{\ln x} = \left( \frac{1}{x} \right) x^1 = x^{-1}
\]

Exercises 4.7.
In 1–19, find the derivatives of the functions.

1. \[y' \Rightarrow\]
2. \[\frac{\sin x}{x} \Rightarrow\]
3. \[\sin(x^2) \Rightarrow\]
4. \[\sin(y^2) \Rightarrow\]
5. \[\cos(x^2) \Rightarrow\]
6. \[\cos(y^2) \Rightarrow\]
7. \[\sqrt{x} \Rightarrow\]
8. \[\sqrt{y} \Rightarrow\]
9. \[\ln(\sqrt{x}) \Rightarrow\]
10. \[\ln(\sqrt{y}) \Rightarrow\]
11. \[\ln(x^2 + 3x) \Rightarrow\]
12. \[\ln(\tan(x)) \Rightarrow\]
13. \[\ln(\ln(3x)) \Rightarrow\]
14. \[\ln(e(x) + \tan(x)) \Rightarrow\]
15. \[\ln(x + 3) \Rightarrow\]
16. \[\ln(x + y^2) \Rightarrow\]
17. \[\ln(1 + \ln(x)) \Rightarrow\]
18. \[\frac{1}{x + \ln(x)} \Rightarrow\]
19. \[\frac{x^2 + 3y}{2x^2 + 4y} \Rightarrow\]
20. Find the value of \(a\) so that the tangent line to \(y = \ln(x)\) at \(x = a\) is a line through the origin.
Sketch the resulting situation. \(\Rightarrow\)
21. If \(f(x) = \ln(x + 2)\) compute \(f'(x/2)\).

4.8 Implicit Differentiation
As we have seen, there is a close relationship between the derivatives of \(e^x\) and \(\ln x\) because these functions are inverses. Rather than relying on pictures for our understanding, we would like to be able to exploit this relationship computationally. In fact this technique can help us find derivatives in many situations, not just when we seek the derivative of an inverse function.

\[\frac{d}{dx} x^2 = \frac{d}{dx} (x^2 + y^2)\]
\[0 = 2x + 2y \frac{dy}{dx}\]
\[\frac{dy}{dx} = \frac{-x}{y}\]

Now we have an expression for \(y'\), but it contains \(y\) as well as \(x\). This means that if we want to compute \(y'\) for some particular value of \(x\) we'll have to know or compute \(y\) at that value of \(x\) as well. It is at this point that we will need to know whether \(y\) is \(U(x)\) or \(L(x)\). Occasionally it will turn out that we can avoid explicit use of \(U(x)\) or \(L(x)\) by the nature of the problem.

EXAMPLE 4.8.1 Find the slope of the circle \(4 = x^2 + y^2\) at the point \((1, -\sqrt{3})\). Since we know both the \(x\) and \(y\) coordinates of the point of interest, we do not need to explicitly recognize that this point is on \(L(x)\), and we do not need to use \(L(x)\) to compute \(y'\) but we could. Using the calculation of \(y'\) from above.

\[y' = \frac{x}{y} = \frac{-1}{\sqrt{3}} = \frac{-x}{y}\]

It is instructive to compare this approach to others.

We might have recognized at the start that \((1, -\sqrt{3})\) is on the function \(y = L(x) = -\sqrt{4 - x^2}\). We could then take the derivative of \(L(x)\), using the power rule and the chain rule, to get

\[L'(x) = -\frac{1}{2}(4-x^2)^{-1/2}(2x) = \frac{2x}{\sqrt{4-x^2}}\]

Then we could compute \(L'(1) = 1/\sqrt{3}\) by substituting \(x = 1\).

Alternatively, we could realize that the point is on \(L(x)\), but use the fact that \(y' = -x/y\).

Since the point is on \(L(x)\) we can replace \(y\) by \(L(x)\) to get

\[y' = \frac{x}{L(x)} = \frac{x}{\sqrt{4-x^2}}\]

without computing the derivative of \(L(x)\) explicitly. Then we substitute \(x = 1\) and get the same answer as before.

In the case of the circle it is possible to find the functions \(U(x)\) and \(L(x)\) explicitly, but there are potential advantages to using implicit differentiation anyway. In some cases it is more difficult or impossible to find an explicit formula for \(y\) and implicit differentiation is the only way to find the derivative.

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EXAMPLE 4.8.2 Find the derivative of any function defined implicitly by \(y^2 + x^3 = x\). We treat \(y\) as an unspecified function and use the chain rule:

\[\frac{d}{dx} (y^2 + x^3) = \frac{d}{dx} x\]

\[(y' \cdot 2y + y')^2 + 3x^2 y' = 1\]

\[y'^2 + 3x^2 y' = 1 - 2xy\]

\[y'^2 + x(3x^2 + 6y) = 1 - 2xy\]

You might think that the step in which we solve for \(y'\) could sometimes be difficult—after all, we're using implicit differentiation here because we can't solve the equation \(y'^2 + x(3x^2 + 6y) = 1\) for \(y'\), so maybe after taking the derivative we get something that is hard to solve for \(y'\). In fact, this never happens. All occurrences \(y'\) come from applying the chain rule, and whenever the chain rule is used it deposits a single \(y'\) multiplied by some other expression. So it will always be possible to group the terms containing \(y'\) together and factor out the \(y'\) just as in the previous example. If you ever get anything more difficult you have made a mistake and should fix it before trying to continue.

It is sometimes the case that a situation leads naturally to an equation that defines a function implicitly.

EXAMPLE 4.8.3 Consider all the points \((x, y)\) that have the property that the distance from \((x, y)\) to \((x_1, y_1)\) plus the distance from \((x, y)\) to \((x_2, y_2)\) is \(2a\) (a is some constant). These points form an ellipse, which like a circle is not a function but can viewed as two functions as given implicitly by \(x^2 + y^2 = a^2\) and \(x^2 + y^2 = a^2 - 2ax\). But it's somewhat easier, and quite useful, to view both functions as given implicitly by \(x^2 + y^2 = 2a^2\), which is the ellipse we are interested in. Notice that we can compute \(x^2 + y^2\) as defining both \(U(x)\) and \(L(x)\).

Now we can take the derivative of both sides as before, remembering that \(y\) is not simply a variable but a function—in this case, \(y\) is either \(U(x)\) or \(L(x)\) but we're not yet specifying which one. When we take the derivative we just have to remember to apply the chain rule where \(y\) appears.

\[\frac{d}{dx} x^2 + \frac{d}{dx} y^2 = \frac{d}{dx} x\]

\[2x + 2y y' = 1\]

\[y' = \frac{1}{2y} - \frac{x}{y}\]

You might think that the step in which we solve for \(y'\) could sometimes be difficult—after all, we're using implicit differentiation here because we can't solve the equation \(2x + 2y y' = 1\) for \(y'\), so maybe after taking the derivative we get something that is hard to solve for \(y'\). In fact, this never happens. All occurrences \(y'\) come from applying the chain rule, and whenever the chain rule is used it deposits a single \(y'\) multiplied by some other expression. So it will always be possible to group the terms containing \(y'\) together and factor out the \(y'\) just as in the previous example. If you ever get anything more difficult you have made a mistake and should fix it before trying to continue.

It is sometimes the case that a situation leads naturally to an equation that defines a function implicitly.

EXAMPLE 4.8.4 We have already justified the power rule by using the exponential function, but we could also do it for rational exponents using implicit differentiation. Suppose that \(y = x^{m/n}\), where \(m\) and \(n\) are positive integers. We can write this implicitly as \(y^n = x^m\), then because we justified the power rule for integers, we can take the derivative
of each side:

\[ \begin{align*}
\sin y &= \sin x^2 + y^2 - 1 \\
\cos y &= \cos x^2 + y^2 - 1.
\end{align*} \]

Then we use implicit differentiation to find the derivative of \( y' \) at the point \((x, y)\):

1. \( y' = 1 + x^2 \) \( \Rightarrow \)
2. \( y' = 7 \)
3. \( y' = 2y \)
4. \( 4 \cos x \sin y = 1 \) \( \Rightarrow \)
5. \( \frac{\sqrt{\pi}}{\pi} + x = 9 \) \( \Rightarrow \)
6. \( \tan(x/y) = x + y \)
7. \( \sin(x + y) = xy \)
8. \( k = \frac{1}{2} \) \( \Rightarrow \)

9. A hyperbola passing through \((8, 3)\) consists of all points whose distance from the origin is a constant more than its distance from the point \((0, 2)\) and \((2, 3)\).

10. Compute \( y' \) for the ellipse of example 4.8.3.
11. If \( y = \log x \) then \( a^n = x \) \( \Rightarrow \), use implicit differentiation to find \( y' \).
12. The graph of the equation \( x^2 + xy + y^2 = 9 \) is an ellipse. Find the line tangent to this curve at the two points where it intersects the \( x \)-axis. Show that these lines are parallel.
13. Repeat the previous problem for the points at which the ellipse intersects the \( y \)-axis.
14. Find the points on the ellipse from the previous two problems where the slope is horizontal and where it is vertical.
15. Find an equation for the tangent line to \( x^2 + y^2 = 2 \) at \((2, \sqrt{2})\). (This curve is the lemniscate of Eudoxus.)
16. Find an equation for the tangent line to \( x^2 + y^2 = 2 \) at a point \((x_1, y_1)\) on the curve, with \( x_1 \neq 0 \) and \( y_1 \neq 0 \). (This curve is an astroid.)
17. Find an equation for the tangent line to \( x^2 + y^2 = 2 \) at a point \((x_1, y_1)\) on the curve, when \( x_1 \neq 0 \). (This curve is a lemniscate.)

4.9 Inverse Trigonometric Functions

The trigonometric functions frequently arise in problems, and often it is necessary to invert the functions, for example, to find an angle with a specified sine. Of course, there are many angles with the same sine, so the sine function doesn’t actually have an inverse that reliably “undoes” the sine function. If you know that \( \sin x = 0.5 \), you can’t reverse this to discover \( x \), that is, you can’t solve for \( x \) as there are infinitely many angles with sine 0.5. Nevertheless, it is useful to have something like an inverse to the sine, however imperfect. The usual approach is to pick out some collection of angles that produce all possible values of the sine exactly once. If we “discard” all other angles, the resulting function has a proper inverse.

The sine takes on all values between \(-1 \) and \(1 \) exactly once on the interval \([-\pi/2, \pi/2] \). If we truncate the sine, keeping only the interval \([-\pi/2, \pi/2] \), then arcsin(1/2) = \( \pi/6 \). This does not work with the sine and the “inverse sine” because the inverse sine is the inverse of the truncated sine function, not the real sine function. It is true that \( \sin(\arcsin(x)) = x \), that is, the sine undoes the arcsine. It is not true that the arcsine undoes the sine, for example, \( \sin(\pi/6) = 1/2 \) and \( \arcsin(1/2) = \pi/6 \), so doing first the sine then the arcsine does not get us back where we started. This is because \( \pi/6 \)
3. The inverse of cot is usually defined so that the range of arccot is \((0, \pi)\). Sketch the graph of \(y = \text{arccot} x\). In the process you will make it clear what the domain of arccot is. Find the derivative of the arccotangent. 

4. Show that \(\text{arccot} x + \arctan x = \pi/2\).

5. Find the derivative of \(\arccos(x^3)\).

6. Find the derivative of \(\arctan(x^7)\).

7. Find the derivative of \(\arcsin(x^2)\).

8. Find the derivative of \(\ln(\arcsin(x^2))\).

9. Find the derivative of \(\arcsin x + \arccos x\).

10. Find the derivative of \(\arcsin x + \arccos r\).

11. Find the derivative of \(\ln(\arctan(x^2))\).

4.10 Limits revisited

We have defined and used the concept of limit, primarily in our development of the derivative. Recall that \(\lim f(x) = L\) if in a precise sense, \(f(x)\) gets closer and closer to \(L\) as \(x\) gets closer and closer to \(a\). While some limits are easy to see, others take some ingenuity; in particular, the limits that define derivatives are always difficult on their face, since in

\[
\lim_{x \to a} \frac{f(x + 2x) - f(x)}{2x},
\]

both the numerator and denominator approach zero. Typically this difficulty can be resolved when \(f\) is a “nice” function and we are trying to compute a derivative. Occasionally such limits are interesting for other reasons, and the limit of a fraction in which both numerator and denominator approach zero can be difficult to analyze. Now that we have the derivative available, there is another technique that can sometimes be helpful in such circumstances.

Before we introduce the technique, we will also expand our concept of limit, in two ways. When the limit of \(f(x)\) as \(x\) approaches \(a\) does not exist, it may be useful to note in what way it does not exist. We have already talked about such one-side limits. Another case is when \(f(x)\) goes to infinity. We will also occasionally want to know what happens to \(f\) when \(x\) “goes to infinity”.

**EXAMPLE 4.10.1** What happens to \(1/x\) as \(x\) goes to 0? From the right, \(1/x\) gets bigger and bigger, or goes to infinity. From the left it goes to negative infinity.

**EXAMPLE 4.10.2** What happens to the function \(\cos(1/x)\) as \(x\) goes to infinity? It seems clear that as \(x\) gets larger and larger, \(1/x\) gets closer and closer to zero, so \(\cos(1/x)\) should be getting closer and closer to \(\cos(0) = 1\).

4.10 Limits revisited

First we use L'Hôpital's Rule: Since the numerator and denominator both approach zero,

\[
\lim_{x \to a} \frac{x^2 - x}{x^2 + 2x} = \lim_{x \to a} \frac{2x}{2x + 2},
\]

provided the latter exists. But in fact this is an easy limit, since the denominator now approaches \(-1\), so

\[
\lim_{x \to a} \frac{x^2 - x}{x^2 + 2x} = -2a.
\]

We don’t really need L'Hôpital’s Rule to do this limit. Rewrite it as

\[
\lim_{x \to a} \frac{x + \pi}{x} = \lim_{x \to a} \frac{\pi}{x}.
\]

and note that

\[
\lim_{x \to a} \frac{x + \pi}{x} = \lim_{x \to a} \frac{x}{x} = \frac{\pi}{x}.
\]

since \(x - \pi\) approaches zero as \(x\) approaches \(\pi\). Now

\[
\lim_{x \to a} \frac{x + \pi}{x - \pi} = \lim_{x \to a} \frac{x}{x - \pi} = \lim_{x \to a} \frac{x}{x - \pi} = 2x(-1) = -2\pi
\]

as before. 

**EXAMPLE 4.10.7** Compute \(\lim_{x \to a} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1}\) in two ways.

As \(x\) goes to infinity both the numerator and denominator go to infinity, so we may apply L'Hôpital’s Rule:

\[
\lim_{x \to a} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = \lim_{x \to a} \frac{4x - 3}{2x + 47}.
\]

The second quotient, it is still the case that the numerator and denominator both go to infinity, so we are allowed to use L'Hôpital’s Rule again:

\[
\lim_{x \to a} \frac{4x - 3}{2x + 47} = \lim_{x \to a} \frac{4}{2} = 2.
\]

So the original limit is 2 as well.

Again, we don’t really need L'Hôpital’s Rule, and in fact a more elementary approach is easier—we divide the numerator and denominator by \(x^2\):

\[
\lim_{x \to a} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = \lim_{x \to a} \frac{2x - 3 + \frac{7}{x^2}}{1 + \frac{47}{x} + \frac{1}{x^2}} = \lim_{x \to a} \frac{2 - \frac{3}{x} + \frac{7}{x^2}}{1 + \frac{47}{x} + \frac{1}{x^2}}.
\]

Now as \(x\) approaches infinity, all the quotients with power of \(x\) in the denominator approach zero, leaving 2 in the numerator and 1 in the denominator, so the limit again is 2.

**Exercises 4.10.**

Compute the limits.

1. \(\lim_{x \to a} \frac{\cos x - 1}{\sin x} \Rightarrow \)

2. \(\lim_{x \to a} \frac{x^2}{x^2} \Rightarrow \)

3. \(\lim_{x \to a} \sqrt{x^2 + x - \sqrt{x^2 + x} - x} \Rightarrow \)

4. \(\lim_{x \to a} \frac{\ln x}{x^3} \Rightarrow \)

5. \(\lim_{x \to a} \frac{x^2}{x^3} \Rightarrow \)

6. \(\lim_{x \to a} \frac{x^3}{x^2} \Rightarrow \)

7. \(\lim_{x \to a} \frac{x^2 - 3}{x} \Rightarrow \)

8. \(\lim_{x \to a} \frac{(1/x) - 1}{x^2 + 1} \Rightarrow \)

9. \(\lim_{x \to a} \frac{2 - \sqrt{x^2 + 7}}{x^2 - 1} \Rightarrow \)

10. \(\lim_{x \to a} \frac{x^3 + 12x - 10}{x^3 - 1} \Rightarrow \)

11. \(\lim_{x \to a} \frac{\sqrt{x^2 + x} + \sqrt{x} \Rightarrow \)

12. \(\lim_{x \to a} \frac{\sqrt{x} - 1}{\sqrt{x} - 1} \Rightarrow \)
13. \( \lim_{x \to 0} \left( \frac{1 - x}{x} \right)^{1/2} - 1 \Rightarrow \)
15. \( \lim_{x \to 1} \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^{1/2} \left( \frac{3}{2} \right) - 1 \Rightarrow \)
17. \( \lim_{x \to 1} \left( \frac{1}{x} \right)^{1/2} \left( \frac{3}{x} \right) - 3 \Rightarrow \)
19. \( \lim_{x \to 1} \left( \frac{1}{x} \right)^{1/2} \left( \frac{3}{x} \right) - 3 \Rightarrow \)
21. \( \lim_{x \to 1} \left( \frac{1}{x} \right)^{1/2} \left( \frac{3}{x} \right) - 3 \Rightarrow \)
23. \( \lim_{x \to 1} \left( \frac{1}{x} \right)^{1/2} \left( \frac{3}{x} \right) - 3 \Rightarrow \)
25. \( \lim_{x \to 1} \left( \frac{1}{x} \right)^{1/2} \left( \frac{3}{x} \right) - 3 \Rightarrow \)
27. \( \lim_{x \to 1} \left( \frac{1}{x} \right)^{1/2} \left( \frac{3}{x} \right) - 3 \Rightarrow \)
29. \( \lim_{x \to 1} \left( \frac{1}{x} \right)^{1/2} \left( \frac{3}{x} \right) - 3 \Rightarrow \)
31. \( \lim_{x \to 1} \left( \frac{1}{x} \right)^{1/2} \left( \frac{3}{x} \right) - 3 \Rightarrow \)
32. \( \lim_{x \to 1} \left( \frac{1}{x} \right)^{1/2} \left( \frac{3}{x} \right) - 3 \Rightarrow \)
34. \( \lim_{x \to 1} \left( \frac{1}{x} \right)^{1/2} \left( \frac{3}{x} \right) - 3 \Rightarrow \)
36. \( \lim_{x \to 1} \left( \frac{1}{x} \right)^{1/2} \left( \frac{3}{x} \right) - 3 \Rightarrow \)
38. \( \lim_{x \to 1} \left( \frac{1}{x} \right)^{1/2} \left( \frac{3}{x} \right) - 3 \Rightarrow \)
40. \( \lim_{x \to 1} \left( \frac{1}{x} \right)^{1/2} \left( \frac{3}{x} \right) - 3 \Rightarrow \)
42. \( \lim_{x \to 1} \left( \frac{1}{x} \right)^{1/2} \left( \frac{3}{x} \right) - 3 \Rightarrow \)
44. \( \lim_{x \to 1} \left( \frac{1}{x} \right)^{1/2} \left( \frac{3}{x} \right) - 3 \Rightarrow \)
46. \( \lim_{x \to 1} \left( \frac{1}{x} \right)^{1/2} \left( \frac{3}{x} \right) - 3 \Rightarrow \)
48. \( \lim_{x \to 1} \left( \frac{1}{x} \right)^{1/2} \left( \frac{3}{x} \right) - 3 \Rightarrow \)

4.11 Hyperbolic Functions

DEFINITION 4.11.3 The other hyperbolic functions are
\[
\begin{align*}
\tanh x &= \frac{\sinh x}{\cosh x} \\
\coth x &= \frac{\cosh x}{\sinh x} \\
\sech x &= \frac{1}{\cosh x} \\
\csch x &= \frac{1}{\sinh x}
\end{align*}
\]
The domain of coth and csch is \( x \neq 0 \) while the domain of the other hyperbolic functions is all real numbers. Graphs are shown in figure 4.11.1

Figure 4.11.1 The hyperbolic functions: cosh, sinh, tanh, sech, csch, coth.

Certainly the hyperbolic functions do not closely resemble the trigonometric functions graphically. But they do have analogous properties, beginning with the following identity:

THEOREM 4.11.4 For all \( x \) in \( R \), \( \cosh^2 x - \sinh^2 x = 1 \).

Proof. The proof is a straightforward computation:
\[
\cosh^2 x - \sinh^2 x = \left( \frac{e^x + e^{-x}}{2} \right)^2 - \left( \frac{e^x - e^{-x}}{2} \right)^2 = \frac{e^{2x} + 2e^x - e^{-2x} - e^{2x} - 2 - e^{-2x}}{4} = \frac{4}{4} = 1.
\]

This immediately gives two additional identities:
\[
1 - \tanh^2 x = \text{sech}^2 x \quad \text{and} \quad \coth^2 x - 1 = \text{csch}^2 x.
\]
The identity of the theorem also helps to provide a geometric motivation. Recall that the graph of \( x^2 - y^2 = 1 \) is a hyperbola with asymptotes \( y = \pm x \) whose \( x \)-intercepts are 

50. The function \( f(x) = \frac{1}{\sqrt{x^2 + 1}} \) has two horizontal asymptotics. Find them and give a rough sketch of \( f \) with its horizontal asymptotics.

4.11 Hyperbolic Functions

The hyperbolic functions appear with some frequency in applications, and are quite similar in many respects to the trigonometric functions. This is a bit surprising given our initial definitions.

DEFINITION 4.11.1 The hyperbolic cosine is the function
\[
\cosh x = \frac{e^x + e^{-x}}{2},
\]
and the hyperbolic sine is the function
\[
\sinh x = \frac{e^x - e^{-x}}{2}.
\]

Notice that cosh is even (that is, \( \cosh(-x) = \cosh(x) \)) while sinh is odd (\( \sinh(-x) = -\sinh(x) \)), and \( \cosh x + \sinh x = e^x \). Also, for all \( x \), \( \cosh x > 0 \), while \( \sinh x = 0 \) if and only if \( e^x - e^{-x} = 0 \), which is true precisely when \( x = 0 \).

LEMMA 4.11.2 The range of \( \cosh x \) is \([1, \infty)\).

Proof. Let \( y = \cosh x \). We solve for \( x \):
\[
y = \frac{e^x + e^{-x}}{2} \Rightarrow 2y = e^x + e^{-x} \Rightarrow 2y^2 - 1 = e^x - e^{-x} = 2y^2 - 1 \Rightarrow e^x = 2y \pm \sqrt{4y^2 - 1} \Rightarrow e^x = y \pm \sqrt{y^2 - 1}.
\]
From the last equation, we see \( y^2 \geq 1 \), and since \( y \geq 0 \), it follows that \( y \geq 1 \).

Now suppose \( y \geq 1 \), so \( y \pm \sqrt{y^2 - 1} > 0 \). Then \( x = \log(y \pm \sqrt{y^2 - 1}) \) is a real number, and \( y = \cosh x \), so \( y \) is in the range of \( \cosh(x) \).

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±\( 1 \). If \( (x, y) \) is a point on the right half of the hyperbola, and if we let \( x = \cosh t \), then \( y = y = \sqrt{x^2 - 1} = \pm \sqrt{\cosh^2 t - 1} = \pm \sinh t \). So for some suitable \( t \), \( \cosh t \) and \( \sinh t \) are the coordinates of a typical point on the hyperbola. In fact, it turns out that \( t \) is twice the area shown in the first graph of figure 4.11.2. Even this is analogous to trigonometry; \( \cos t \) and \( \sin t \) are the coordinates of a typical point on the unit circle, and \( t \) is twice the area shown in the second graph of figure 4.11.2

Figure 4.11.2 Geometric definitions of \( \sin, \cos, \sinh, \cosh \). \( t \) is twice the shaded area in each figure.

Given the definitions of the hyperbolic functions, finding their derivatives is straightforward. Here again we see similarities to the trigonometric functions.

THEOREM 4.11.5 \( \frac{d}{dx} \cosh x = \sinh x \) and \( \frac{d}{dx} \sinh x = \cosh x \).

Proof. \( \frac{d}{dx} \cosh x = \frac{d}{dx} \frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x \), and \( \frac{d}{dx} \sinh x = \frac{d}{dx} \frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x \).

Since \( \cosh x > 0 \), \( \sinh x \) is increasing and hence injective, so \( \sinh x \) has an inverse, \( \text{arsinh} x \). Also, \( \sinh x > 0 \) when \( x > 0 \), so \( \cosh x \) is injective on \([0, \infty)\) and has a (partial) inverse, \( \text{arcosh} x \). The other hyperbolic functions have inverses as well, though \( \text{arcosh} x \) is only a partial inverse. We may compute the derivatives of these functions as we have other inverse functions.

THEOREM 4.11.6 \( \frac{d}{dx} \text{arsinh} x = \frac{1}{\sqrt{1 + x^2}} \).

Proof. Let \( y = \text{arsinh} x \), so \( \sinh y = x \). Then \( \frac{d}{dx} \sinh y = \cosh(y) \cdot y' = 1 \), and so \( y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}} \).
4.11 Hyperbolic Functions

The other derivatives are left to the exercises.

**Exercises 4.11.**

1. Show that the range of \( \sinh x \) is all real numbers. (Hint: show that if \( y = \sinh x \) then \( x = \ln(y + \sqrt{y^2 + 1}) \).)

2. Compute the following limits:
   - a. \( \lim_{x \to \infty} \cosh x \)
   - b. \( \lim_{x \to \infty} \sinh x \)
   - c. \( \lim_{x \to \infty} \tanh x \)
   - d. \( \lim_{x \to \infty} (\cosh x - \sinh x) \)

3. Show that the range of \( \tanh x \) is \((-1, 1)\). What are the ranges of \( \coth \), \( \text{sech} \), and \( \text{csch} \)? (Use the fact that they are reciprocal functions.)

4. Prove that for every \( x, y \in \mathbb{R} \), \( \sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y \). Obtain a similar identity for \( \sinh(x - y) \).

5. Prove that for every \( x, y \in \mathbb{R} \), \( \cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y \). Obtain a similar identity for \( \cosh(x - y) \).

6. Use exercises 4 and 5 to show that \( \sinh(2x) = 2 \sinh x \cosh x \) and \( \cosh(2x) = \cosh^2 x + \sinh^2 x \) for every \( x \). Conclude also that \( (\cosh(2x) - 1)/2 = \sinh^2 x \).

7. Show that \( \frac{d}{dx} \tanh x = \text{sech}^2 x \). Compute the derivatives of the remaining hyperbolic functions as well.

8. What are the domains of the six inverse hyperbolic functions?

9. Sketch the graphs of all six inverse hyperbolic functions.