4 Transcendental Functions

So far we have used only algebraic functions as examples when finding derivatives, that is, functions that can be built up by the usual algebraic operations of addition, subtraction, multiplication, division, and raising to constant powers. Both in theory and practice there are other functions, called transcendental, that are very useful. Most important among these are the trigonometric functions, the inverse trigonometric functions, exponential functions, and logarithms.

4.1 Trigonometric Functions

When you first encountered the trigonometric functions it was probably in the context of “triangle trigonometry,” defining, for example, the sine of an angle as the “side opposite the hypotenuse.” While this will still be useful in an informal way, we need to use a “triangle trigonometry,” defining, for example, the sine of an angle as the “side opposite the hypotenuse.” While this will still be useful in an informal way, we need to use a

4.1 Trigonometric Functions

...to any angles, as indicated in this figure:

The angle $x$ is subtended by the heavy arc, in the figure, that is, $x = 7\pi/6$. Both coordinates of point $A$ in this figure are negative, so the sine and cosine of $7\pi/6$ are both negative.

The remaining trigonometric functions can be most easily defined in terms of the sine and cosine, as usual:

$$\tan x = \frac{\sin x}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x}$$
$$\sec x = \frac{1}{\cos x}$$
$$\csc x = \frac{1}{\sin x}$$

and they also can be defined as the corresponding ratios of coordinates.

Although the trigonometric functions are defined in terms of the unit circle, the unit circle diagram is not what we normally consider the graph of a trigonometric function. (The unit circle is the graph of, well, the circle.) We can easily get a qualitatively correct idea of the graphs of the trigonometric functions from the unit circle diagram. Consider the sine function, $y = \sin x$. As $x$ increases from 0 in the unit circle diagram, the second coordinate of the point $A$ goes from 0 to a maximum of 1, then back to 0, then to a minimum of $-1$, then back to 0, and then it obviously repeats itself. So the graph of $y = \sin x$ must look something like this:

To define the radian measurement system, we consider the unit circle in the $xy$-plane:

An angle, $x$, at the center of the circle is associated with an arc of the circle which is said to subtend the angle. In the figure, this arc is the portion of the circle from point $(1, 0)$ to point $A$. The length of this arc is the radian measure of the angle $x$; the fact that the radian measure is an actual geometric length is largely responsible for the usefulness of radian measure. The circumference of the unit circle is $2\pi$, so the radian measure of the full circular angle (that is, of the 360 degree angle) is $2\pi$.

While an angle with a particular measure can appear anywhere around the circle, we need a fixed, conventional location so that we can use the coordinate system to define properties of the angle. The standard convention is to place the starting radius for the angle on the positive $x$-axis, and to measure positive angles counterclockwise around the circle. In the figure, $x$ is the standard location of the angle $\pi/6$, that is, the length of the arc from $(1, 0)$ to $A$ is $\pi/6$. The angle $y$ in the picture is $-x/6$, because the distance from $(1, 0)$ to $B$ along the circle is also $\pi/6$, but in a clockwise direction.

Now the fundamental trigonometric definitions are: the cosine of $x$ and the sine of $x$ are the first and second coordinates of the point $A$, as indicated in the figure. The angle $x$ shown can be viewed as an angle of a right triangle, meaning the usual triangle definitions of the sine and cosine also make sense. Since the hypotenuse of the triangle is 1, the “side opposite over hypotenuse” definition of the sine is the second coordinate of point $A$ over 1, which is just the second coordinate; in other words, both methods give the same value for the sine.

The simple triangle definitions work only for angles that can “fit” in a right triangle, namely, angles between 0 and $\pi/2$. The coordinate definitions, on the other hand, apply...
4.3 A hard limit

We want to compute this limit:

$$\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}$$

Equivalently, to make the notation a bit simpler, we can compute

$$\lim_{x \to 0} \frac{\sin x}{x}.$$  

In the original context we need to keep $x$ and $\Delta x$ separate, but here it doesn’t hurt to rename $\Delta x$ to something more convenient.

**THEOREM 4.3.1 Squeeze Theorem** Suppose that $g(x) \leq f(x) \leq h(x)$ for all $x$ close to $a$ but not equal to $a$. If $\lim_{x \to a} g(x) = L = \lim_{x \to a} h(x)$, then $\lim_{x \to a} f(x) = L$.

This theorem can be proved using the official definition of limit. We won’t prove it here, but point out that it is easy to understand and believe graphically. The condition says that $f(x)$ is trapped between $g(x)$ below and $h(x)$ above, and that at $x = a$, both $g$ and $h$ approach the same value. This means the situation looks something like figure 4.3.1.

The wiggly curve is $x^2 \sin(\pi/x)$, the upper and lower curves are $x^2$ and $\pi^2/4$. The sine function is always between $-1$ and $1$, so $-x^2 \leq x^2 \sin(\pi/x) \leq x^2$, and it is easy to see that $\lim_{x \to \infty} -x^2 = 0 = \lim_{x \to \infty} x^2$. It is not so easy to see directly, that is algebraically, that $\lim_{x \to \infty} x^2 \sin(\pi/x) = 0$, because the $\pi/x$ prevents us from simply plugging in $x = 0$.

To do the hard limit that we want, $\lim_{x \to 0} \sin(x)/x$, we will find two simpler functions $g$ and $h$ so that $g(x) \leq \sin(x)/x \leq h(x)$, and so that $\lim_{x \to 0} g(x) = \lim_{x \to 0} h(x)$. Not too surprisingly, this will require some trigonometry and geometry. Referring to figure 4.3.2, $x$ is the measure of the angle in radians. Since the circle has radius $1$, the coordinates of point $A$ are $(\cos x, \sin x)$, and the area of the small triangle is $(\cos x \sin x)/2$. This triangle is completely contained within the circular wedge-shaped region bordered by two lines and the circle from $(1,0)$ to point $A$. Comparing the areas of the triangle and the wedge we see $(\cos x \sin x)/2 \leq x/2$, since the area of a circular region with angle $\theta$ and radius $r$ is $\theta r^2/2$. With a little algebra this turns into $\sin(x)/x \leq 1/\cos x$; giving us the $h$ we seek.

Finally, the two limits $\lim_{x \to 0} \cos x$ and $\lim_{x \to 0} 1/\cos x$ are easy, because $\cos(0) = 1$. By the squeeze theorem, $\lim_{x \to 0} \sin(x)/x = 0$ as well.

Before we can complete the calculation of the derivative of the sine, we need one other limit:

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$$

This limit is just as hard as $\sin x/x$, but closely related to it, so that we don’t have to do a similar calculation; instead we can do a bit of tricky algebra.

$$\frac{\cos x - 1}{x} = \frac{\cos x - 1}{\cos x + 1} \cdot \frac{\cos x + 1}{x} = \frac{\cos^2 x - 1}{x(\cos x + 1)} = \frac{-\sin^2 x}{x(\cos x + 1)} = \frac{-\sin x}{x} \cdot \frac{\sin x}{\cos x + 1}.$$

To compute the desired limit it is sufficient to compute the limits of the two final fractions, as $x$ goes to 0. The first of these is the hard limit we’ve just done, namely 1. The second turns out to be simple, because the denominator presents no problem:

$$\lim_{x \to 0} \frac{\sin x}{\cos x + 1} = \frac{\sin 0}{\cos 0 + 1} = \frac{0}{2} = 0.$$

Thus,

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$$

**Exercises 4.3.**

1. Compute $\lim_{x \to 0} \frac{\sin(x)}{x}$
2. Compute $\lim_{x \to 0} \frac{\sin(2x)}{2x}$
3. Compute $\lim_{x \to 0} \frac{\cos(3x)}{3x}$
4. Compute $\lim_{x \to 0} \frac{\sin x}{x}$
5. Compute $\lim_{x \to 0} \frac{\sin x}{x - \cos x}$
6. For all $x \geq 0$, $4x - 3 \leq f(x) \leq 7x$. Find $\lim_{x \to 0} f(x).
7. For all $x \leq 0$, $x \leq g(x) \leq x^2 + 2$. Find $\lim_{x \to 0} g(x).
8. Use the Squeeze Theorem to show that $\lim_{x \to 0} x^2 \sin(2x) = 0$.  

4.4 The Derivative of $\sin x$, Continued

Now we can complete the calculation of the derivative of the sine:

$$\frac{d}{dx} \sin x = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x - \cos x \sin \Delta x}{\Delta x} = \cos x \lim_{\Delta x \to 0} \frac{\sin x}{\cos x} = \cos x.$$

The derivative of a function measures the slope or steepness of the function; if we examine the graphs of the sine and cosine side by side, it should be that the latter appears to accurately describe the slope of the former, and indeed this is true:

$$\frac{d}{dx} \cos x = -\sin x.$$

Notice that where the cosine is zero the sine does appear to have a horizontal tangent line, and that the sine appears to be steeper where the cosine takes on its extreme values of 1 and −1.

Of course, now that we know the derivative of the sine, we can compute derivatives of more complicated functions involving the sine.

**EXAMPLE 4.4.1** Compute the derivative of $\sin(x^2)$.

$$\frac{d}{dx} \sin(x^2) = \cos(x^2) \cdot 2x = 2x \cos(x^2).$$

**EXAMPLE 4.4.2** Compute the derivative of $\sin^2(x^2 - 5x)$.

$$\frac{d}{dx} \sin^2(x^2 - 5x) = 2\sin(x^2 - 5x) \cdot \cos(x^2 - 5) \cdot (2x - 5) = 2(\sin^2 - 5) \cdot \cos(x^2 - 5) \cdot \sin(x^2 - 5).$$
4.5 Derivatives of the Trigonometric Functions

All of the other trigonometric functions can be expressed in terms of the sine, and so their derivatives can be calculated using the rules we already have. For the cosine we need to use two identities, $\cos x = \sin(x + \frac{\pi}{2})$, $\sin x = -\cos(x + \frac{\pi}{2})$.

Now:

1. $\frac{d}{dx}(\cos x) = -\sin x$, $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\tan x) = \sec^2 x$
3. $\frac{d}{dx}(\sec x) = \sec x \tan x$
4. $\frac{d}{dx}(\cot x) = -\csc^2 x$
5. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

The derivatives of the cotangent and cosecant are similar and left as exercises.

4.6 Exponential and Logarithmic Functions

An exponential function has the form $a^x$, where $a$ is a constant; examples are $2^x$, $10^x$, $e^x$. The logarithmic functions are the inverses of the exponential functions, that is, functions that "undo" the exponential functions. Just as, for example, the cube root function "undoes" the cube function: $\sqrt[3]{x^3} = x$.

Let $f(x) = 2^x$. The inverse of this function is called the logarithm base 2, denoted $\log_2(x)$ or (especially in computer science circles) $\log(x)$. What does this really mean? The logarithm must undo the action of the exponential function, so for example it must be that $\log(2^x) = x$ — starting with 2, the exponential function produces $2^x$, and the logarithm of $2^x$ must return us to 2.

Exercises 4.6.1

1. $\log_2(8) = 3$ — starting with 2, the exponential function produces $2^3 = 8$, and the logarithm of 8 must return us to 3. A little thought shows that this is not a coincidence that $\log(2^3)$ simply gives the exponent — the logarithm is the inverse of the exponential function, so that we must get back to 2.

In other words, the logarithm is the exponent. Remember this catchphrase, and what it means, and you won't go wrong. (You do have to remember what it means. Like any good mnemonic, "the logarithm is the exponent" leaves out a lot of detail, like "Which exponent?" and "Exponent of what?")

EXAMPLE 4.6.1 What is the value of $\log_{10}(1000)$? The "100" tells us the appropriate number to use for the base of the exponential function. The logarithm is the exponent, so the question is, what exponent $E$ makes $10^E = 1000$? If we can find such an $E$, then $\log_{10}(1000) = \log_{10}(10^E) = E$, finding the appropriate exponent is the same as finding the logarithm. In this case, of course, it is easy: $E = 3$ so $\log_{10}(1000) = 3$.

Let's review some laws of exponents and logarithms; let $a$ be a positive number. Since $a^x a^y = a^{x+y}$, it's clear that $a^x a^{-y} = a^{x-y}$ and in general that $a^{x+y} = a^x a^y$. Since "the logarithm is the exponent," it's no surprise that this translates directly into a fact about the logarithmic function. Here are three facts

whenever $x = p/q$ if we were to graph this we'd see something like this:

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But this is a poor picture, because you can't see that the "curve" is really a whole lot of individual points, above the rational numbers on the $x$-axis. There are really a lot of "holes" in the curve, above $x = \pi$, for example. But (this is the hard part) it is possible to prove that the holes can be "filled in," and that the resulting function, called $\tan x$, really does have the properties we want, namely that $\tan(2\pi) = \tan(2\pi + \pi) = \tan(2\pi + 2\pi)$.
There are two interesting things to note here: As in the case of the sine function we are left with a limit that involves $\Delta x$ but not $x$, which means that whatever $\lim_{\Delta x \to 0} (e^{\Delta x} - 1)/\Delta x$ is, we know that it is a number, that is, a constant. This means that $e^x$ has a remarkable property: its derivative is a constant times itself.

We earlier remarked that the hardest limit we would compute is $\lim_{x \to 0} \sin x/x = 1$; we now have a limit that is just a bit too hard to include here. In fact the hard part is to see that $\lim_{\Delta x \to 0} (e^{\Delta x} - 1)/\Delta x$ even exists—does this fraction really get closer and closer to some fixed value? Yes it does, but we will not prove this fact.

We can look at some examples. Consider $(2^x - 1)/x$ for some small values of $x$: 1, 0.828427124, 0.756824860, 0.724861964, 0.708393501, 0.70070877 when $x$ is 1, 1/2, 1/4, 1/8, 1/16, 1/32, respectively. It looks like this is settling in around 0.7, which turns out to be true (but the limit is not exactly 0.7). Consider next $(x^3 - 1)/x$: 2, 1.64610163, 1.264296052, 1.177621520, 1.13720773, 1.11768854, at the same values of $x$. It turns out to be true that in the limit this is about 1.1. Two examples don’t establish a pattern, but if you do more examples you will find that the limit varies directly with the value of $a$: bigger $a$, bigger limit; smaller $a$, smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between $a = 2$ and $a = 3$ the limit will be exactly 1; the value at which this happens is called $e$, so that

$$\lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1.$$

As you might guess from our two examples, $e$ is closer to 3 than to 2, and in fact $e \approx 2.718$.

Now we see that the function $e^x$ is a truly remarkable property:

$$\frac{d}{dx} e^x = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = e^x.$$

That is, $e^x$ is its own derivative, or in other words the slope of $e^x$ is the same as its height, or the same as its second coordinate: The function $f(x) = e^x$ goes through the point $(z, e^z)$ and has slope $e^z$ there, no matter what $z$ is. It is sometimes convenient to express the function $e^x$ without an exponent, since complicated exponents can be hard to read. In such cases we use exp$(x)$, e.g., exp$(1 + x^2)$ instead of $e^{1+x^2}$.

### Derivatives of the exponential and logarithmic functions

We have discussed this from the point of view of the graphs, which is easy to understand but is not normally considered a rigorous proof—it is too easy to be led astray by pictures that seem reasonable but that miss some hard point. It is possible to do this derivation without resorting to pictures, and indeed we will see an alternate approach soon.

Note that $\ln x$ is defined only for $x > 0$. It is sometimes useful to consider the function $\ln|x|$, a function defined for $x \neq 0$. When $x < 0$, $\ln|x| = \ln(-x)$ and $d/dx \ln |x| = d/dx \ln(-x) = 1/(-x) = 1/x$.

Thus whether $x$ is positive or negative, the derivative is the same.

What about the functions $a^x$ and $\log_a x$? We know that the derivative of $a^x$ is some constant times $a^x$ itself, but what constant? Remember that “the logarithm is the exponent” and you will see that $a = e^{\ln a}$. Then

$$a^x = (e^{\ln a})^x = e^{\ln a \cdot x},$$

and we can compute the derivative using the chain rule:

$$\frac{d}{dx} a^x = \frac{d}{dx} (e^{\ln a \cdot x}) = \frac{d}{dx} (e^{x \cdot \ln a}) = (\ln a) e^{x \cdot \ln a} = (\ln a) a^x.$$

The constant is simply $\ln a$. Likewise we can compute the derivative of the logarithm function $\log_a x$. Since $x = e^{\ln x}$, we can take the logarithm base $a$ of both sides to get

$$\log_a x = \log_a(e^{\ln x}) = \ln x \log_a e.$$

Then

$$\frac{d}{dx} \log_a x = \frac{1}{x \log_a e}.$$  

This is a perfectly good answer, but we can improve it slightly. Since

$$\log_a e = \log_a(e^{\ln e}) = \ln a \log_a e$$

we have

$$\frac{1}{\ln a} = \log_a e.$$

4.7 Derivatives of the exponential and logarithmic functions

We can replace $\log_a e$ to get

$$\frac{d}{dx} \log_a x = \frac{1}{x \log_a e}.$$  

You may if you wish memorize the formulas

$$\frac{d}{dx} \log_a x = \frac{1}{x \log_a e}$$

because the "trick" $a = e^{\ln a}$ is often useful, and sometimes essential, it may be better to remember the trick, not the formula.

**EXAMPLE 4.7.1** Compute the derivative of $f(x) = x^2$.

$$\frac{d}{dx} x^2 = \frac{d}{dx} (e^{\ln x^2}) = \frac{d}{dx} e^{\ln 2 \cdot \ln x} = (\ln 2) x \ln 2.$$

**EXAMPLE 4.7.2** Compute the derivative of $f(x) = 2x^2$.

$$\frac{d}{dx} 2x^2 = \frac{d}{dx} 2 e^{\ln x^2} = 2(\ln 2) x^2 \ln 2.$$  

**EXAMPLE 4.7.3** Compute the derivative of $f(x) = x^x$. At first this appears to be a tedious, or poorly defined function, it is not a constant power of $x$, and it does not seem to be an exponential function, since the base is not constant. But in fact it is no harder than the previous example.

$$\frac{d}{dx} x^x = \frac{d}{dx} (e^{\ln x^x}) = \frac{d}{dx} e^{x \ln x} = (1 + \ln x) x^x.$$  

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What about the logarithm function? This too is hard, but as the cosine function was easier to do once the sine was done, so the logarithm is easier to do now that we know the derivative of the exponential function. Let’s start with $\log_a x$, which as you probably know is often abbreviated $\ln x$ and called the “natural logarithm” function.

Consider the relationship between the two functions, namely, that they are inverses, that one is the “undo” of the other. Graphically this means that they have the same graph except that one is flipped or reflected through the line $y = x$, as shown in figure 4.7.1.

![Figure 4.7.1 The exponential and logarithm functions.](image)

This means that the slopes of these two functions are closely related as well. For example, the slope of $e^x$ is $e$; at $x = 1$; at the corresponding point on the $\ln(x)$ curve, the slope must be $1/e$, because the “rise” and the “run” have been interchanged. Since the slope of $e^x$ is $e$ at the point $(1, e)$; the slope of $\ln(x)$ is $1/x$ at the point $(e, 1)$.

![Figure 4.7.2 Slope of the exponential and logarithm functions.](image)

More generally, we know that the slope of $e^x$ is $e^x$ at the point $(z, e^z)$, so the slope of $\ln(x)$ is $1/e^z$ at $(z', e')$, as indicated in figure 4.7.2. In other words, the slope of $\ln x$ is the reciprocal of the first coordinate at any point; this means that the slope of $\ln x$ at $(x, \ln x)$ is $1/x$. The upshot is:

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$  

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We can replace $\log_a x$ to get

$$\frac{d}{dx} \log_a x = \frac{1}{x \log_a e}.$$  

You may if you wish memorize the formulas

$$\frac{d}{dx} \log_a x = \frac{1}{x \log_a e}.$$  

Because the “trick” $a = e^{\ln a}$ is often useful, and sometimes essential, it may be better to remember the trick, not the formula.
EXAMPLE 4.7.4 Recall that we have not justified the power rule except when the exponent is a positive or negative integer. We can use the exponential function to take care of other exponents. 

\[ \frac{d}{dx}e^x = e^x \ln e = e^x \]

\[ \frac{d}{dx}e^{mx} = \left( \frac{d}{dx} e^{mx} \right) e^{mx} = \left( e^{mx} \right) \frac{d}{dx} e^{mx} = x e^{mx} \]

\[ \frac{d}{dx}e^{-1} = \frac{1}{e} \]

Exercises 4.7.
In 1–19, find the derivatives of the functions.

1. \( y = \ln x \Rightarrow \)
2. \( \ln x \Rightarrow \)
3. \( (\ln x)^2 \Rightarrow \)
4. \( \sin(\ln x) \Rightarrow \)
5. \( e^{\ln x} \Rightarrow \)
6. \( x^{e^x} \Rightarrow \)
7. \( e^{x^2} \Rightarrow \)
8. \( x - 2 \Rightarrow \)
9. \( (1/3)^x \Rightarrow \)
10. \( x^{1/3} \Rightarrow \)
11. \( \ln(x^2 + 3x) \Rightarrow \)
12. \( \ln( \cos(x) ) \Rightarrow \)
13. \( \ln(\ln(3x)) \Rightarrow \)
14. \( f(x) = (x^2 + 2) (1/2) \Rightarrow \)
15. \( \ln(x^{2/3}) \Rightarrow \)
16. \( x \ln x \Rightarrow \)
17. \( 2x \ln(x^4) \Rightarrow \)
18. \( 1 + \ln(3x) \Rightarrow \)
19. \( (x^2 - 3)^{3/2} \Rightarrow \)
20. \( 27^{-3/2} (4x - 9)^3 \Rightarrow \)
21. \( f(x) = \ln(x^2 + 2) \Rightarrow \)

4.8 Implicit Differentiation
As we have seen, there is a close relationship between the derivatives of \( e^x \) and \( \ln x \) because these functions are inverses. Rather than relying on pictures for our understanding, we would like to be able to exploit this relationship computationally. In fact this technique can help us find derivatives in many situations, not just when we seek the derivative of an inverse function.

4.8 Implicit Differentiation

Chain rule where \( y \) appears.

\[ \frac{d}{dx} \frac{x^2 + y^2}{27^{3/2} (4x - 9)^3} \]

\[ y' = \frac{2x + 2y y'}{2y} = \frac{2x}{y} \]

Now we have an expression for \( y' \), but it contains \( y \) as well as \( x \). This means that if we want to compute \( y' \) for some particular value of \( x \) we’ll have to know or compute \( y \) at that value of \( x \) as well. It is at this point that we will need to know whether \( y = U(x) \) or \( L(x) \). Occasionally it will turn out that we can avoid explicit use of \( U(x) \) or \( L(x) \), by the nature of the problem.

EXAMPLE 4.8.1 Find the slope of the circle \( 4 = x^2 + y^2 \) at the point \((1, -\sqrt{3})\). Since we know both the \( x \) and \( y \) coordinates of the point of interest, we do not need to explicitly recognize that this point is on \( L(x) \), and we do not need to use \( U(x) \) to compute \( y' \) but we could. Using the calculation of \( y' \) from above.

\[ y' = \frac{x}{y} - \frac{1}{\sqrt{3}} = \frac{1}{3} \]

It is instructive to compare this approach to others.

We might have recognized at the start that \((1, -\sqrt{3})\) is on the function \( y = L(x) = -\sqrt{x^2 - 4} \). We could then take the derivative of \( L(x) \), using the power rule and the chain rule, to get

\[ L'(x) = \frac{1}{2} \frac{1}{2} \left( 1 - x^2 \right) (-2x) = -\frac{x}{\sqrt{x^2 - 4}} \]

Then we could compute \( L'(1) = 1/\sqrt{7} \) by substituting \( x = 1 \).

Alternatively, we could realize that the point is on \( L(x) \), use the fact that \( y' = y'/y \). Since the point is on \( L(x) \) we can replace \( y \) by \( L(x) \) to get

\[ y' = \frac{x}{L(x)} = \frac{2}{\sqrt{3}} \]

without computing the derivative of \( L(x) \) explicitly. Then we substitute \( x = 1 \) and get the same answer as before.

In the case of the circle it is possible to find the functions \( U(x) \) and \( L(x) \) explicitly, but there are potential advantages to using implicit differentiation anyway. In some cases it is more difficult or impossible to find an explicit formula for \( y \) and implicit differentiation is the only way to find the derivative.

EXAMPLE 4.8.2 Find the derivative of any function defined implicitly by \( y^2 + e^y = x \).

We treat \( y \) as an unspecified function and use the chain rule:

\[ \frac{d}{dx} (y^2 + e^y) = \frac{d}{dx} x \]

\[(y^2 + e^y) y' = 1 \]

\[ y^2 + e^y = 1 - 2y \]

\[ y' = \frac{1 - 2y}{e^y} \frac{1}{1 + e^y} \]

You might think that the step in which we solve for \( y' \) could sometimes be difficult—after all, we’re using implicit differentiation here because we can’t solve the equation \( y^2 + e^y = x \) for \( y \) so maybe after taking the derivative we get something that is hard to solve for \( y' \). In fact, this never happens. All occurrences of \( y' \) come from applying the chain rule, and whenever the chain rule is used it deposits a single \( y' \) multiplied by some other expression. So it will always be possible to group the terms containing \( y' \) together and factor out the \( y' \), just as in the previous example. If you ever get anything more difficult you have made a mistake and should fix it before trying to continue.

It is sometimes the case that a situation leads naturally to an equation that defines a function implicitly.

EXAMPLE 4.8.3 Consider all the points \((x, y)\) that have the property that the distance from \((x, y)\) to \((x_1, y_1)\) plus the distance from \((x, y)\) to \((x_2, y_2)\) is \(2a\) (a is some constant). These points form an ellipse, which is like a circle is not a function but can viewed as two functions pasted together. Because we know how to draw the distance between two points, we can write down an implicit equation for the ellipse:

\[ \sqrt{(x - x_1)^2 + (y - y_1)^2} + \sqrt{(x - x_2)^2 + (y - y_2)^2} = 2a \]

Then we can use implicit differentiation to find the slope of the ellipse at any point, though the computation is rather messy.

EXAMPLE 4.8.4 We have already justified the power rule by using the exponential function, but we could also do it for rational exponents by using implicit differentiation. Suppose that \( y = x^{n/m} \), where \( m \) and \( n \) are positive integers. We can write this implicitly as \( y^m = x^n \), then because we justified the power rule for integers, we can take the derivative...
of each side:

\[ n y^{n-1} y' = mx^{m-1} \]

\[ y' = \frac{mx^{m-1}}{n y^{n-1}} \]

\[ y' = \frac{m}{n} \frac{y^{(m-n)/n}}{x^{(m-n)/n}} \]

\[ y' = \frac{m}{n} \frac{1}{(m-n)/(n+1)} \]

\[ y' = \frac{m}{n} \frac{1}{(m-n)(n+1)} \]

\[ y' = \frac{m}{n} \frac{1}{(m-n)(n+1)} \]

4.8 Implicit Differentiation

In exercises 1–8, find a formula for the derivative \( y' \) at the point \((x, y)\):

1. \( y^3 + x^4 = 7 \)
2. \( y^3 + x^2 + y^2 = 7 \)
3. \( 4 \cos x \sin y = 1 \)
4. \( \sqrt{x} + \sqrt{y} = 9 \)
5. \( \tan(y/x) = x + y \)
6. \( \sin(x + y) = xy \)
7. \( \frac{1}{x} + \frac{1}{y} = 7 \)

9. A hyperbola passing through \((8, 6)\) consists of all points whose distance from the origin is a constant more than its distance from the point \((5, 2)\). Find the slope of the tangent line to the hyperbola at \((8, 6)\).

10. Compute \( y' \) for the ellipse of example 4.8.3.

11. If \( y = \log x \), then \( x^y = x \). Use implicit differentiation to find \( y' \).

12. The graph of the equation \( x^2 - xy + y^2 = 9 \) is an ellipse. Find the lines tangent to this curve at the two points where it intersects the \(x\)-axis. Show that these lines are parallel.

13. Repeat the previous problem for the points at which the ellipse intersects the \(y\)-axis.

14. Find the points on the ellipse from the previous two problems where the slope is horizontal and where it is vertical.

15. Find an equation for the tangent line to \( x^2 - y^2 + x^2 = 2\sqrt{2}\). (This curve is the lemniscate of Eudoxus.)

16. Find an equation for the tangent line to \( x^2 + y^2 = 2\) at a point \((x_1, y_1)\) on the curve, with \( x_1 \neq 0 \) and \( y_1 \neq 0 \). (This curve is an astroid.)

17. Find an equation for the tangent line to \( x^2 + y^2 = 2\) at a point \((x_1, y_1)\) on the curve, when \( x_1 \neq 0 \). (This curve is a lemniscate.)

4.9 Inverse Trigonometric Functions

The functions and their inverses undo each other in order, for example, \((\arcsin x)' = x + y \)

18. Show that \( y^2 + x^2 = 4 \) is orthogonal to \( 4x^2 + 9y^2 = 72 \). (Hint: You need to find the intersection points of the two curves and then show that the product of the derivatives at each intersection point is \(-1\).)

19. Show that \( y^2 + x^2 = 4 \) is orthogonal to \( y = mx \). Conclude that the family of circles centered at the origin is an orthogonal trajectory of the family of lines that pass through the origin.

Note that there is a technical issue when \( m = 0 \). The circles fail to be differentiable when they cross the \(y\)-axis. However, the circles are orthogonal to the \(x\)-axis. Explain why.

Likewise, the vertical line through the origin requires a separate argument.

20. For \( k \neq 0 \) and \( c \neq 0 \) show that \( y^2 - x^2 = k \) is orthogonal to \( y = mx \). In the case where \( k \) and \( c \) are both zero, the curves intersect at the origin. Are the curves \( y^2 - x^2 = 0 \) and \( y = x \) orthogonal to each other?

21. Suppose that \( m \neq 0 \). Show that the family of curves \( y = mx + h \) \((k \neq 0)\) is orthogonal to the family of curves \( y = -(x/m) + c \) \((k \neq 0)\).

4.9 Inverse Trigonometric Functions

The trigonometric functions frequently arise in problems, and often it is necessary to invert the functions, for example, to find an angle with a specified sine. Of course, there are many angles with the same sine, so the sine function doesn’t actually have an inverse that reliably “undoes” the sine function. If you know that \( \sin x = 0.5 \), you can’t reverse this to discover \( x \), that is, you can’t solve for \( x \), as there are infinitely many angles with sine 0.5. Nevertheless, it is useful to have something like an inverse to the sine, however imperfect. The usual approach is to pick out some collection of angles that produce all possible values of the sine exactly once. If we “discard” all other angles, the resulting function does have a proper inverse.

The sine takes on all values between \(-1\) and \(1\) exactly once on the interval \([\pi/2, \pi/2]\). When we truncate the sine, keeping only the interval \([\pi/2, \pi/2]\), as shown in figure 4.9.1, then this truncated sine has an inverse function. We call this the inverse sine or the arcsine, and write \( y = \arcsin(x) \).

Recall that a function and its inverse undo each other in either order, for example, \((\arcsin x)' = x + y \)

Note that \( y = \arcsin(x) \) we know that \( \arcsin(x) \in [\pi/2, \pi/2] \). The computation of the derivative of the arcsine is left as an exercise.

Finally we look at the tangent; the other trigonometric functions also have “partial inverses” but the sine, cosine and tangent are enough for most purposes. The tangent, truncated tangent and inverse tangent are shown in figure 4.9.2; the derivative of the arctangent is left as an exercise.

Figure 4.9.1 The sine, the truncated sine, the inverse sine.

Figure 4.9.2 The truncated cosine, the inverse cosine.

Figure 4.9.3 The tangent, the truncated tangent, the inverse tangent.

Exercises 4.9.

1. Show that the derivative of \( \arcsin x \) is \( 1 / \sqrt{1 - x^2} \).

2. Show that the derivative of \( \arctan x \) is \( 1 / (1 + x^2) \).
3. The inverse of cot is usually defined so that the range of arcot is $(0, \pi)$. Sketch the graph of $y = \cot x$. In the process you will make it clear what the domain of arcot is. Find the derivative of the arcotangent $\Rightarrow$
4. Show that arcot $x + \arctan x = \pi/2$.
5. Find the derivative of arcsec$(x^2)$ $\Rightarrow$
6. Find the derivative of arccot$(x^2)$ $\Rightarrow$
7. Find the derivative of arcsin$(x^2)$ $\Rightarrow$
8. Find the derivative of $\ln(\arccos(x^2))$ $\Rightarrow$
9. Find the derivative of $\sin(\arccos x)$ $\Rightarrow$
10. Find the derivative of $\arcsin(x + \arccos x) $ $\Rightarrow$
11. Find the derivative of $\log(x\arccos(x^2))$ $\Rightarrow$

### 4.10 LIMITS REVISITED

We have defined and used the concept of limit, primarily in our development of the derivative. Recall that $\lim_{x\to a} f(x) = L$ if true in a precise sense, $f(x)$ gets closer and closer to $L$ as $x$ gets closer and closer to $a$. While some limits are easy to see, others take some ingenuity; in particular, the limits that define derivatives are always difficult on their face; since in

$$\lim_{x \to a} \frac{f(x + dx) - f(x)}{dx}$$

both the numerator and denominator approach zero. Typically this difficulty can be resolved when $f$ is a “nice” function and we are trying to compute a derivative. Occasionally such limits are interesting for other reasons, and the limit of a fraction in which both numerator and denominator approach zero can be difficult to analyze. Note that we have the derivative available, there is another technique that can sometimes be helpful in such circumstances.

Before we introduce the technique, we will expand our concept of limit, in two ways. When the limit of $f(x)$ as $x$ approaches $a$ does not exist, it may be useful to note in what way it does not exist. We have already talked about one such case: one-sided limits. Another case is when $f(x)$ goes to infinity. We will also occasionally want to know what happens to $f(x)$ as $x$ goes to infinity.

#### EXAMPLE 4.10.1

What happens to $1/x$ as $x$ goes to 0? From the right, $1/x$ gets bigger and bigger, or goes to infinity. From the left it goes to negative infinity.

#### EXAMPLE 4.10.2

What happens to the function $\cos(1/x)$ as $x$ goes to infinity? It seems clear that as $x$ gets larger and larger, $1/x$ gets closer to zero, so $\cos(1/x)$ should be getting closer and closer to $\cos(0) = 1$.

### 4.10 LIMITS REVISITED

First we use L'Hôpital's Rule: Since the numerator and denominator both approach zero,

$$\lim_{x \to a} \frac{x^2 - x^2}{\sin x} = \lim_{x \to a} \frac{2x}{\cos x},$$

provided the latter exists. But in fact this is an easy limit, since the denominator now approaches $-1$, so

$$\lim_{x \to a} \frac{x^2 - x^2}{\sin x} = 2a.$$

We don't really need L'Hôpital's Rule to do this limit. Rewrite it as

$$\lim_{x \to a} \frac{x}{\sin x} \cdot \lim_{x \to a} \frac{(x-a)}{1}$$

and note that

$$\lim_{x \to a} \frac{(x-a)}{1} = x - a.$$

since $x - a$ approaches zero as $x$ approaches $a$. Now

$$\lim_{x \to a} \frac{x}{\sin x} \cdot \lim_{x \to a} \frac{x - a}{1} = \lim_{x \to a} \frac{x}{\sin x} \cdot (x - a).$$

as before.

#### EXAMPLE 4.10.7

Compute $\lim_{x \to 0} \frac{2x^2 - 3x + 7}{\sin x}$ in two ways.

As $x$ goes to infinity both the numerator and denominator go to infinity, so we may apply L'Hôpital's Rule:

$$\lim_{x \to 0} \frac{2x^2 - 3x + 7}{\sin x} = \lim_{x \to 0} \frac{4x - 3}{\cos x}.$$

In the second quotient, it is still the case that the numerator and denominator both go to infinity, so we are allowed to use L'Hôpital's Rule again:

$$\lim_{x \to 0} \frac{2x^2 - 3x + 7}{\sin x} = \lim_{x \to 0} \frac{4x - 3}{\cos x} = \lim_{x \to 0} \frac{2}{\cos x} = 2.$$

So the original limit is 2 as well.

Again, we don't really need L'Hôpital's Rule, and in fact a more elementary approach is easier—we divide the numerator and denominator by $x^2$:

$$\lim_{x \to 0} \frac{2x^2 - 3x + 7}{\sin x} = \lim_{x \to 0} \frac{2 - \frac{3}{x} + \frac{7}{x^2}}{\cos x} \cdot \lim_{x \to 0} \frac{1}{2 \frac{1}{x^2} + \frac{7}{x^2}}.$$

Now as $x$ approaches infinity, all the quotients with some power of $x$ in the denominator approach zero, leaving 2 in the numerator and 1 in the denominator, so the limit again is 2.

### Exercises 4.10

1. Compute the limits
   1. $\lim_{x \to 0} \frac{\cos x - 1}{\sin x}$
   2. $\lim_{x \to 0} \frac{e^x}{x}$
   3. $\lim_{x \to 0} \sqrt{x^2 + x - \sqrt{x^2 + x}}$
   4. $\lim_{x \to 0} \frac{e^{|x|}}{x}$
   5. $\lim_{x \to 0} \frac{\ln x}{x^2}$
   6. $\lim_{x \to 0} \frac{\ln x}{1 - e^{-x}}$
   7. $\lim_{x \to 0} \frac{e^{x^2} - 3}{x}$
   8. $\lim_{x \to 0} \frac{(1/x) - 1}{x}$
   9. $\lim_{x \to \infty} \frac{2 - \sqrt{x^2 + 2}}{x}$
   10. $\lim_{x \to \infty} \frac{2 + 2x^{-1} - 1/x^2}{x}$
   11. $\lim_{x \to \infty} \frac{\sqrt{x^2 + 2} + x^2}{y}$
   12. $\lim_{x \to \infty} \frac{1}{x^2}$
This immediately gives two additional identities:

\[ 1 - \tanh^2 x = \text{sech}^2 x \quad \text{and} \quad \cosh^2 x - 1 = \text{csch}^2 x. \]

The identity of the theorem also helps to provide a geometric motivation. Recall that the graph of \( x^2 - y^2 = 1 \) is a hyperbola with asymptotes \( x = \pm y \) whose \( x \)-intercepts are ±1. If \((x, y)\) is a point on the right half of the hyperbola, and if we let \( x = \cosh t \), then

\[ y = \pm \sqrt{x^2 - 1} = \pm \cosh t \cdot \sinh t = \pm \sinh t. \]

So for some suitable \( t \), \( \cosh t \) and \( \sinh t \) are the coordinates of a typical point on the hyperbola. In fact, it turns out that \( t \) is twice the area shown in the first graph of figure 4.11.2. Even this is analogous to trigonometric; \( \csc t \) and \( \sin t \) are the coordinates of a typical point on the unit circle, and \( t \) is twice the area shown in the second graph of figure 4.11.2.

The function \( f(x) = \frac{x}{\sqrt{x^2 + 1}} \) has two horizontal asymptotes. Find them and give a rough sketch of \( f \) with its horizontal asymptotes.

### 4.11 Hyperbolic Functions

The hyperbolic functions appear with some frequency in applications, and are quite similar in many respects to the trigonometric functions. This is a bit surprising given our initial definitions.

**DEFINITION 4.11.10** The hyperbolic cosine is the function

\[ \cosh x = \frac{e^x + e^{-x}}{2}, \]

and the hyperbolic sine is the function

\[ \sinh x = \frac{e^x - e^{-x}}{2}. \]

Notice that \( \cosh \) is even (that is, \( \cosh(-x) = \cosh(x) \)) while \( \sinh \) is odd (\( \sinh(-x) = -\sinh(x) \)), and \( \cosh x + \sinh x = e^x \). Also, for all \( x \), \( \cosh x > 0 \), while \( \sinh x = 0 \) if and only if \( x = 0 \), which is true precisely when \( x = 0 \).

**LEMMA 4.11.2** The range of \( \cosh x \) is \([1, \infty)\).

**Proof.** Let \( y = \cosh x \). We solve for \( x \):

\[ y = \frac{e^x + e^{-x}}{2} \]

\[ 2y = e^x + e^{-x} \]

\[ 2y = e^x - 2ye^{-x} + 1 \]

\[ 0 = e^{-2x} - 2ye^{-x} + 1 \]

\[ e^{-2x} - 2ye^{-x} + 1 \]

\[ e^{-2x} - 2ye^{-x} + 1 \]

From the last equation, we see \( y^2 \leq 1 \), and since \( y \geq 0 \), it follows that \( y \geq 1 \).

Now suppose \( y \geq 1 \), so \( y + \sqrt{y^2 - 1} > 0 \). Then \( x = \ln(y + \sqrt{y^2 - 1}) \) is a real number, and \( y = \cosh x \), so \( y \) is in the range of \( \cosh(x) \).

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**4.11 Hyperbolic Functions**

**DEFINITION 4.11.1** The other hyperbolic functions are:

\[ \tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}, \quad \sech x = \frac{1}{\cosh x}, \quad \text{and} \quad \text{csch} x = \frac{1}{\sinh x}. \]

The domain of \( \coth \) and \( \text{csch} \) is \( x \neq 0 \) while the domain of the other hyperbolic functions is all real numbers. Graphs are shown in figure 4.11.1.

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**THEOREM 4.11.4** For all \( x \in \mathbb{R}, \cosh^2 x - \sinh^2 x = 1 \).

**Proof.** The proof is a straightforward computation.

\[ \cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} = \frac{2e^{2x} + 2 + 2e^{-2x} - 2e^{2x} - 2e^{-2x}}{4} = 1. \]

This immediately gives two additional identities:

\[ 1 - \tanh^2 x = \text{sech}^2 x \quad \text{and} \quad \cosh^2 x - 1 = \text{csch}^2 x. \]

The identity of the theorem also helps to provide a geometric motivation. Recall that the graph of \( x^2 - y^2 = 1 \) is a hyperbola with asymptotes \( x = \pm y \) whose \( x \)-intercepts are ±1. If \((x, y)\) is a point on the right half of the hyperbola, and if we let \( x = \cosh t \), then

\[ y = \pm \sqrt{x^2 - 1} = \pm \cosh t \cdot \sinh t = \pm \sinh t. \]

So for some suitable \( t \), \( \cosh t \) and \( \sinh t \) are the coordinates of a typical point on the hyperbola. In fact, it turns out that \( t \) is twice the area shown in the first graph of figure 4.11.2. Even this is analogous to trigonometric; \( \csc t \) and \( \sin t \) are the coordinates of a typical point on the unit circle, and \( t \) is twice the area shown in the second graph of figure 4.11.2. Given the definitions of the hyperbolic functions, finding their derivatives is straightforward. Here again we see similarities to the trigonometric functions.

**THEOREM 4.11.5**

\[ \frac{d}{dx} \cosh x = \sinh x \quad \text{and} \quad \frac{d}{dx} \sinh x = \cosh x. \]

**Proof.**

\[ \frac{d}{dx} \cosh x = \frac{d}{dx} \frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x, \quad \text{and} \quad \frac{d}{dx} \sinh x = \frac{d}{dx} \frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x. \]

Since \( \cosh x > 0 \), \( \sinh x \) is increasing and hence injective, so \( \sinh x \) is an inverse, \( \text{arcosh} x \). Also, \( \sinh x > 0 \) when \( x > 0 \), so \( \cosh x \) is injective on \([0, \infty)\) and has a (partial) inverse, \( \text{arccosh} x \). The other hyperbolic functions have inverses as well, though \( \text{arcsech} x \) is only a partial inverse. We may compute the derivatives of these functions as we have other inverse functions.

**THEOREM 4.11.6**

\[ \frac{d}{dx} \text{arcosh} x = \frac{1}{\sqrt{1 + x^2}}. \]

**Proof.** Let \( y = \text{arcosh} x \), so \( \sinh y = x \). Then

\[ \frac{d}{dx} \sinh y = \cosh(y) \cdot y' = 1, \quad \text{and so} \]

\[ y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}}. \]
4.11 Hyperbolic Functions

The other derivatives are left to the exercises.

Exercises 4.11.

1. Show that the range of sinh $x$ is all real numbers. (Hint: show that if $y = \sinh x$ then $x = \ln(y + \sqrt{y^2 + 1})$)

2. Compute the following limits:
   a. $\lim_{x \to \infty} \cosh x$
   b. $\lim_{x \to \infty} \sinh x$
   c. $\lim_{x \to \infty} \tanh x$
   d. $\lim_{x \to \infty} (\cosh x - \sinh x)$

3. Show that the range of tanh $x$ is $(-1, 1)$. What are the ranges of coth, sech, and csch? (Use the fact that they are reciprocal functions)

4. Prove that for every $x, y \in \mathbb{R}$, $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$. Obtain a similar identity for $\sinh(x - y)$.

5. Prove that for every $x, y \in \mathbb{R}$, $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$. Obtain a similar identity for $\cosh(x - y)$.

6. Use exercises 4 and 5 to show that $\sinh(2x) = 2 \sinh x \cosh x$ and $\cosh(2x) = \cosh^2 x + \sinh^2 x$ for every $x$. Conclude also that $(\cosh(2x) - 1)/2 = \sinh^2 x$ and $(\cosh(2x) + 1)/2 = \cosh^2 x$.

7. Show that $\frac{d}{dx} \tanh x = \text{sech}^2 x$. Compute the derivatives of the remaining hyperbolic functions as well.

8. What are the domains of the six inverse hyperbolic functions?

9. Sketch the graphs of all six inverse hyperbolic functions.