It is tedious to compute a limit every time we need to know the derivative of a function. Fortunately, we can develop a small collection of examples and rules that allow us to compute the derivative of almost any function we are likely to encounter. Many functions involve quantities raised to a constant power, such as polynomials and more complicated combinations like \(y = (\sin x)^2\). So we start by examining powers of a single variable; this gives us a building block for more complicated examples.

### 3.1 The Power Rule

We start with the derivative of a power function, \(f(x) = x^n\). Here \(n\) is a number of any kind: integer, rational, positive, negative, even irrational, as in \(x^\pi\). We have already computed some simple examples, so the formula should not be a complete surprise:

\[
\frac{d}{dx}x^n = nx^{n-1}.
\]

It is not easy to show this is true for any \(n\). We will do some of the easier cases now, and discuss the rest later.

The easiest, and most common, is the case that \(n\) is a positive integer. To compute the derivative we need to compute the following limit:

\[
\frac{d}{dx}x^n = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}.
\]

For a specific, fairly small value of \(n\), we could do this by straightforward algebra.

#### Example 3.1.1

Find the derivative of \(f(x) = x^3\).

\[
\frac{d}{dx}x^3 = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x\Delta^2 x + \Delta^3 x^3 - x^3}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x\Delta^2 x + \Delta^3 x^3}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x\Delta^2 x + \Delta^3 x^3}{\Delta x} = 3x^2 + 3x\Delta x + \Delta^2 x^3 = 3x^2.
\]

The general case is really not much harder as long as we don’t try to do too much. The key is understanding what happens when \((x + \Delta x)^n\) is multiplied out.

\[
(x + \Delta x)^n = x^n + nx^{n-1}\Delta x + \text{terms with more than one factor of } \Delta x.
\]

We know that multiplying out will give a large number of terms all of the form \(x^m\Delta^n\), and in fact that \(i + j = n\) in every term. One way to see this is to understand that one method for multiplying out \((x + \Delta x)^n\) is the following: In every \((x + \Delta x)\) factor, pick either the \(x\) or the \(\Delta x\), then multiply the \(n\) choices together; do this in all possible ways. For example, for \((x + \Delta x)^3\), there are eight possible ways to do this:

\[
(x + \Delta x)(x + \Delta x)(x + \Delta x) = x^3 + 3x^2\Delta x + 3x\Delta^2 x + \Delta^3 x^3.
\]

No matter what \(n\) is, there are \(n\) ways to pick \(\Delta x\) in one factor and \(x\) in the remaining \(n - 1\) factors, this means one term is \(nx^{n-1}\Delta x^n\). The other coefficients are somewhat harder to understand, but we don’t really need them, so in the formula above they have simply been called \(a_2, a_3\), and so on. We know that every one of these terms contains \(\Delta x\) to at least the power 2. Now let’s look at the limit:

\[
\frac{d}{dx}x^n = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\Delta x + \text{terms with more than one factor of } \Delta x - x^n}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + \text{terms with more than one factor of } \Delta x}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + \text{terms with more than one factor of } \Delta x}{\Delta x} = nx^{n-1}.
\]

We will later see why the other cases of the power rule work, but from now on we will use the power rule whenever \(n\) is any real number. Let’s note here a simple case in which the power rule applies, or almost applies, but is not really needed. Suppose that \(f(x) = 1\); remember that this “1” is a function, not merely a number, and that \(f(x) = 1\) has a graph that is a horizontal line, with slope zero everywhere. So we know that \(f'(x) = 0\).

We might also write \(f(x) = x^0\), though there is some question about just what this means that \(x = 0\). If we apply the power rule, we get \(f'(x) = 0x^{-1} = 0\); \(x = 0\), again noting that there is a problem at \(x = 0\). So the power rule “works” in this case, but it’s really best to just remember that the derivative of any constant function is zero.

### Exercises 3.1.

Find the derivatives of the given functions.

1. \(x^{100}\) ⇒
2. \(x^{-100}\) ⇒
3. \(x^{-1/3}\) ⇒
4. \(x^0\) ⇒
5. \(x^{1/2}\) ⇒
6. \(x^{-1/7}\) ⇒

---

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3.2 Linearity of the Derivative

An operation is linear if it behaves “nicely” with respect to multiplication by a constant and addition. The name comes from the equation of a line through the origin, \(f(x) = mx\), and the following two properties of this equation. First, \(f(x + y) = f(x) + f(y)\), so the constant \(c\) can be “moved outside” or “moved through” the function \(f\). Second, \(f(x + y) = m(x + y) = mx + my = f(x) + f(y)\), so the addition symbol likewise can be moved through the function.

The corresponding properties for the derivative are:

\[
(f(x + y))' = \frac{df}{dx}(x + y) = \frac{df}{dx}(x) + \frac{df}{dx}(y) = f'(x) + f'(y).
\]

and

\[
(f(x) + g(x))' = \frac{d}{dx}(f(x) + g(x)) = \frac{df}{dx} + \frac{dg}{dx} = f'(x) + g'(x).
\]

It is easy to see, or at least to believe, that these are true by thinking of the distance/speed interpretation of derivatives. If one object is at position \(f(t)\) at time \(t\), we know its speed is given by \(f'(t)\). Suppose another object is at position \(f(t)\) at time \(t\), namely, that it is always 5 times as far along the route as the first object. Then it “must” be going 5 times as fast at all times.

The second rule is somewhat more complicated, but here is one way to picture it. Suppose a flatbed railroad car is at position \(f(t)\) at time \(t\), so the car is traveling at a speed of \(f'(t)\) (to be specific, let’s say that \(f(t)\) gives the position on the track of the rear end of the car). Suppose that an ant is crawling from the back of the car to the front so that its position on the car is \(g(t)\). Then in reality, at time \(t\), the ant is at position \(f(t) + g(t)\) along the track, and its speed is “obviously” \(f'(t) + g'(t)\).

We don’t want to rely on some more-or-less obvious physical interpretation to determine what is true mathematically, so let’s see how to verify these rules by computation.

---

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We'll do one and leave the other for the exercises.

\[ \frac{d}{dx}(f(x) + g(x)) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) + g(x + \Delta x) - f(x) - g(x)}{\Delta x} \]

This is sometimes called the sum rule for derivatives.

**EXAMPLE 3.2.1** Find the derivative of \( f(x) = x^2 + 5x^4 \). We have to invoke linearity twice here:

\[ f'(x) = \frac{d}{dx}(x^2 + 5x^4) = \frac{d}{dx}x^2 + \frac{d}{dx}5x^4 = 2x + 20x^3 \]

Because it is so easy with a little practice, we can usually combine all uses of linearity into a single step. The following example shows an acceptable detailed computation.

**EXAMPLE 3.2.2** Find the derivative of \( f(x) = x^3 - 2x^2 + 6x - 7 \).

\[ f'(x) = \frac{d}{dx}(x^3 - 2x^2 + 6x - 7) = 3x^2 - 4x + 6 \]

**Exercises 3.2.**

Find the derivatives of the functions in 1-6.

1. \( 5x^3 + 12x^2 - 15 \)
2. \( -2x^5 + 3x^4 - 2x^2 + 5x \)
3. \( 5x^3 - 3x^2 + 5x + 1 \)
4. \( f(x), \) where \( f(x) = x^2 - 3x + 2 \) and \( g(x) = 2x^3 - 5x \)
5. \( (x + 1)(x^2 + 2x - 3) \)
6. \( \sqrt{265 - x^3} + 3x^2 + 7 \)

Find an equation for the tangent line to \( f(x) = x^2/4 - 1/x \) at \( x = -2 \).

**3.3 The Product Rule**

\( g(x) \) must actually exist for this to make sense. We also replaced \( \lim_{\Delta x \to 0} f(x + \Delta x) \) with \( f(x) \) in this section to be careful.

What we really need to know here is that \( \lim_{\Delta x \to 0} f(x + \Delta x) = f(x) \), or in the language of section 2.5, that \( f \) is continuous at \( x \). We already know that \( f(x) \) exists (or the whole approach, writing the derivative of \( f \) in terms of \( f' \) and \( g' \), doesn’t make sense). This turns out to imply that \( f \) is continuous as well. Here’s why:

\[ \lim_{\Delta x \to 0} f(x + \Delta x) = \lim_{\Delta x \to 0} \left( f(x + \Delta x) - f(x) + f(x) \right) = \lim_{\Delta x \to 0} f(x + \Delta x) - f(x) + \lim_{\Delta x \to 0} f(x) = f'(x) + 0 + f(x) = f(x) \]

To summarize: the product rule says that

\[ \frac{d}{dx} (f(x)g(x)) = f'(x)g(x) + f(x)g'(x) \]

**EXAMPLE 3.3.1** Compute the derivative of \( f(x) = \sqrt{265 - x^3} \). We have already computed

\[ \frac{d}{dx} \sqrt{265 - x^3} = -\frac{x}{\sqrt{265 - x^3}} \]

Now

\[ f'(x) = \sqrt{265 - x^3} \cdot -\frac{x}{\sqrt{265 - x^3}} = \frac{d}{dx}(\sqrt{265 - x^3}) = \frac{-3x^2 + 1250x}{\sqrt{265 - x^3}} \]

**Exercises 3.3.**

In 1-4, find the derivatives of the functions using the product rule.

1. \( x^2(x^3 - 3x + 10) \)
2. \( (x^7 + 5x^3 - 3)(x^2 + 2x^4 - 3) \)
3. \( \sqrt{265 - x^3} \)
4. \( \sqrt{265 - x^3} \)

5. Use the product rule to compute the derivative of \( f(x) = (2x - 3)x^2 \). Sketch the function.

Find an equation of the tangent line to the curve at \( x = 2 \). Sketch the tangent line at \( x = 2 \).

**3.4 The Quotient Rule**

What is the derivative of \((x^2 + 1)/(x^2 - 3x)\)? More generally, we’ll like to have a formula to compute the derivative of \( f(x)/g(x) \) if we already know \( f'(x) \) and \( g'(x) \). Instead of attacking this problem head-on, let’s notice that we’ve already done part of the problem: \( f(x)/g(x) = f(x)(1/g(x)) \). Is this “really” a product, and can we compute the derivative if we know \( f(x) \) and \( 1/g(x) \)?
3.4 The Quotient Rule

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far we can get:

\[
\frac{d}{dx} g(x) = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{g(x) - g(x + \Delta x)}{-\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{g(x) - g(x + \Delta x)}{\Delta x} \cdot \frac{1}{-\Delta x}
\]

\[
= g'(x) \cdot \frac{1}{\Delta x} - \frac{1}{\Delta x}
\]

\[
= g'(x)
\]

Now we can put this together with the product rule:

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{[f(x)/g(x)] + f'(x) \cdot g(x) - g(x) \cdot f'(x)}{g(x)^2}
\]

\[
\text{EXAMPLE 3.4.1:} \quad \text{Compute the derivative of } (x^2 + 1)/(x^2 - 3x).
\]

\[
\frac{d}{dx} \left( \frac{x^2 + 1}{x^2 - 3x} \right) = \frac{2x(x^2 - 3x) - (x^2 + 1)(2x - 3)}{(x^2 - 3x)^2}
\]

It is often possible to calculate derivatives in more than one way, as we have already seen. Since every quotient can be written as a product, it is always possible to use the product rule to compute the derivative, though it is not always simpler.

\[
\text{EXAMPLE 3.4.2:} \quad \text{Find the derivative of } \sqrt{625 - x^2}/\sqrt{x} \text{ in two ways: using the quotient rule, and using the product rule.}
\]

Quotient rule:

\[
\frac{d}{dx} \left( \frac{\sqrt{625 - x^2}}{\sqrt{x}} \right) = \sqrt{625 - x^2} \cdot \frac{1}{2\sqrt{25 - x^2}} - \sqrt{x} \cdot \frac{-x}{2\sqrt{625 - x^2}}
\]

Product rule:

\[
\frac{d}{dx} \left( \sqrt{625 - x^2} \cdot \frac{1}{\sqrt{x}} \right) = \frac{1}{2\sqrt{625 - x^2}} \cdot \frac{-x}{\sqrt{x}} - \frac{1}{2\sqrt{625 - x^2}} \cdot \frac{-x}{\sqrt{x}}
\]

With a bit of algebra, both of these simplify to:

\[
-\frac{x^2 + 625}{2\sqrt{625 - x^2}^3}
\]

9. If \( f'(4) = 5 \), \( g'(4) = 12 \), \( f(g(4)) = f(12) \cdot 2 = 24 \), and \( g(4) = 6 \), compute \( f(4) \) and \( \frac{d}{dx} g \) at 4.

\[
3.5 \text{ The Chain Rule}
\]

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So far we have seen how to compute the derivative of a function built up from other functions by addition, subtraction, multiplication and division. There is another very important way that we combine simple functions to make more complicated functions: function composition, as discussed in section 2.3. For example, consider \( \sqrt{625 - x^2} \). This function has many simpler components, like 625 and \( x^2 \), and then there is that square root symbol, so the square root function \( \sqrt{.} \) is involved. The obvious question is: can we compute the derivative using the derivatives of the constituents \( 625 - x^2 \) and \( \sqrt{.} \)? We can indeed. In general, if \( f(x) \) and \( g(x) \) are functions, we can compute the derivatives of \( f(g(x)) \) and \( g(f(x)) \) in terms of \( f' \) and \( g' \).

\[
\text{EXAMPLE 3.5.1:} \quad \text{Form the two possible compositions of } f(x) = \sqrt{7} \text{ and } g(x) = 625 - x^2 \text{ and compute the derivatives. First, } f(g(x)) = \sqrt{625 - x^2}, \text{ and the derivative is } -x/\sqrt{625 - x^2} \text{ as we have seen. Second, } g(f(x)) = 625 - (\sqrt{7})^2 = 625 - x \text{ with derivative } -1. \text{ Of course, those calculations do not use anything new, and in particular the derivative of } f(g(x)) \text{ was somewhat tedious to compute from the definition.}
\]

Suppose we want the derivative of \( f(g(x)) \). Again, let’s set up the derivative and play some algebraic tricks:

\[
\frac{d}{dx} \left( f(g(x)) \right) = \lim_{\Delta x \to 0} \frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \cdot \frac{g(x + \Delta x) - g(x)}{\Delta x}
\]

Now we see immediately that the second fraction turns into \( g'(x) \) when we take the limit. The first fraction is more complicated, but it too looks something like a derivative. The denominator \( g(x + \Delta x) - g(x) \), is a change in the value of \( g \), so let’s abbreviate it as \( \Delta g = g(x + \Delta x) - g(x) \), which also means \( g(x + \Delta x) = g(x) + \Delta g \). This gives us

\[
\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

Now let \( \Delta x \to 0 \), it is also true that \( \Delta x \to 0 \), because \( g(x + \Delta x) \to g(x) \). So we can rewrite this limit as

\[
\lim_{\Delta x \to 0} \frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g}
\]

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Unfortunately, there is a small flaw in the argument. Recall that what we mean by \( \lim_{\Delta x \to 0} \) involves what happens when \( \Delta x \) is close to 0 but not equal to 0. The qualification is very important, since we must be able to divide by \( \Delta x \). But when \( \Delta x \) is close to 0 but not equal to 0, \( \Delta x = g(x + \Delta x) - g(x) \) is close to 0 and possibly equal to 0. This means it doesn’t really make sense to divide by \( \Delta x \). Fortunately, it is possible to rectify the argument to avoid this difficulty, but it is a bit tricky. We will not include the details, which can be found in many calculus books. Note that many functions do have the property that \( g(x + \Delta x) - g(x) \neq 0 \) when \( \Delta x \) is small, and for those functions the argument above is fine.

The chain rule has a particularly simple expression if we use the Leibniz notation for the derivative. The quantity \( f'(g(x)) \) is the derivative of \( f \) with \( x \) replaced by \( y \); this can be written \( df/\,dy \). As usual, \( g'(y) = dg/dx \). Then the chain rule becomes

\[
\frac{df}{dx} = \frac{df}{dy} \cdot \frac{dy}{dx}
\]

This looks like trivial arithmetic, but it is not: \( df/\,dy \) is not a fraction, that is, not literal division, but a single symbol that means \( g'(x) \). Nevertheless, it turns out that what looks like trivial arithmetic, and is therefore easy to remember, is really true.

It will take a bit of practice to make the use of the chain rule come naturally—it is more complicated than the earlier differentiation rules we have seen.

\[
\text{EXAMPLE 3.5.2:} \quad \text{Compute the derivative of } \frac{1}{\sqrt{625 - x^2}}. \text{ We already know that the answer is } -x/\sqrt{625 - x^2}, \text{ computed directly from the limit. In the context of the chain rule, we have } f(x) = \sqrt{x}, \text{ and } g(x) = 625 - x^2. \text{ We know that } f'(x) = (1/2)x^{-1/2}, \text{ so } f'(g(x)) = (1/2)(625 - x^2)^{-1/2}. \text{ Note that this is a two step computation: first compute } f'(x), \text{ then replace } x \text{ by } g(x). \text{ Since } g'(x) = -2x \text{ we have}
\]

\[
f'(g(x)) = \frac{1}{2\sqrt{625 - x^2}} \cdot (-2x) = \frac{-x}{\sqrt{625 - x^2}}
\]

\[
\text{EXAMPLE 3.5.3:} \quad \text{Compute the derivative of } 1/\sqrt{625 - x^2}. \text{ This is a quotient with a constant numerator, so we could use the quotient rule, but it is simpler to use the chain}
\]
We compute $f'(x) = -\frac{1}{2(2x+1)^{3/2}}$ using the power rule, and then

$$f'(g(x))g'(x) = \frac{-1}{2(2x+1)^{3/2}}(2x+1) = \frac{x}{(2x+1)^{3/2}}.$$  

In practice, of course, you will need to use more than one of the rules we have developed to compute the derivative of a complicated function.

**Example 3.5.4** Compute the derivative of

$$f(x) = \frac{x^2-1}{\sqrt{x^2+1}}.$$  

The “last” operation here is division, so to get started we need to use the quotient rule first. This gives

$$f'(x) = \frac{(x^2-1)\sqrt{x^2+1} - (x^2-1)(x\sqrt{x^2+1})}{(x^2+1)^{3/2}}$$

$$= \frac{2x\sqrt{x^2+1} - (x^2-1)(x+1)}{2(x^2+1)}.$$  

Now we need to compute the derivative of $\sqrt{x^2+1}$. This is a product, so we use the product rule:

$$\frac{d}{dx}\sqrt{x^2+1} = \frac{1}{\sqrt{x^2+1}}\frac{d}{dx}(x^2+1)$$

$$= \frac{x}{\sqrt{x^2+1}}.$$  

Finally, we use the chain rule:

$$\frac{d}{dx}\frac{x^2-1}{\sqrt{x^2+1}} = \frac{d}{dx}\left(\frac{x^2-1}{(x^2+1)^{1/2}}\right)$$

$$= \frac{2x\sqrt{x^2+1} - (x^2-1)(x+1)}{2(x^2+1)} \cdot \frac{x}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}}.$$  

And putting it all together:

$$f'(x) = \frac{2x\sqrt{x^2+1} - (x^2-1)(x+1)}{2(x^2+1)} \cdot \frac{x}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}}.$$  

This can be simplified of course, but we have done all the calculus, so that only algebra is left.

**Exercises 3.5.**

Find the derivative of the functions. For extra practice, and to check your answers, do some of these in more than one way if possible.

1. $x^4 - 3x^3 + (1/2)x^2 + 7x - \pi$  
2. $x^2 - 2x + 4\sqrt{x}$  
3. $(x^2 + 1)^3$  
4. $\sqrt[3]{x} - x$  
5. $(x^3 - 4x + 5)\sqrt{x+3} - x^2$  
6. $\sqrt{x} - x$, $x$ is a constant  
7. $\sqrt{x} + x^2$  
8. $\sqrt{\frac{x}{\sqrt{x}}}$  
9. $(1 + 3x)^2$  
10. $\frac{3x^2 + 2x + 1}{x}$  
11. $\frac{\sqrt{x^2 + 1}}{x}$  
12. $\frac{1}{\sqrt{x} - 2x}$  
13. $\sqrt{x^2 - (1/2)}$  
14. $\frac{1}{(100 - x^3)^{1/2}}$  
15. $x^2 + x^2$  
16. $\frac{x^2 + 1}{\sqrt{1 + (x^2 + 1)^2}}$  
17. $(x + 8)^5$  
18. $(4 - 2x)^3$  
19. $(x^4 + 5)^3$  
20. $(6 - 2x^5)^3$  
21. $(1 - 4x)^{1/3}$  
22. $5(5x + 1 - x/s)$  
23. $(2x^2 - x + 3)^{-3}$  
24. $\frac{1}{x^2 + 2}$  
25. $\frac{3}{4x^2 - 2x + 1}$  
26. $(x^2 + 1)(5 - 2x)/2$  
27. $(3x^2 + 5)(x - 4)/3$  
28. $\frac{x + 1}{x - 3}$  
29. $\frac{f - 1}{f + 1}$  
30. $\frac{1}{2x} - x$  
31. $\frac{2x - 3}{3x^2 - 4x - 2}$  
32. $3(x^2 + 1)/(2x^2 - 1)(2x + 3)$  
33. $\frac{1}{(2x + 1)(x - 3)}$  
34. $\frac{1}{(2x^2 - x + 1)^3}$  
35. $(2x + 1)(x^2 + 1)^3$  
36. Find an equation for the tangent line to $f(x) = (x - 2)^{1/3}/(x^3 + 4x - 1)^2$ at $x = 1$.  
37. Find an equation for the tangent line to $g = 9x^2$ at $(1, 1)$.  
38. Find an equation for the tangent line to $(x^3 - 4x + 5)/\sqrt{25 - x^2}$ at $(3, 8)$.  
39. Find an equation for the tangent line to $\frac{x^3 + x + 1}{x^3}$ at $(2, -7)$.  
40. Find an equation for the tangent line to $\sqrt{x^2 + 1} + \sqrt{x + (x^2 + 1)}$ at $(1, \sqrt{2} + \sqrt{3})$.  

**Example 3.5.5** Compute the derivative of $\sqrt{1 + \sqrt{1 + \sqrt{x}}}$ here. We have a more complicated chain of compositions, so we use the chain rule twice. At the outermost “layer” we have the function $g(x) = 1 + \sqrt{1 + \sqrt{x}}$ plugged into $f(x) = \sqrt{x}$, so applying the chain rule once gives

$$\frac{d}{dx}\sqrt{1 + \sqrt{1 + \sqrt{x}}} = \frac{1}{2} \left(1 + \sqrt{1 + \sqrt{x}}\right)^{-1/2} \frac{d}{dx}\left(1 + \sqrt{1 + \sqrt{x}}\right).$$  

Now we need the derivative of $\sqrt{1 + \sqrt{1 + \sqrt{x}}}$ Using the chain rule again:

$$\frac{d}{dx}\left(1 + \sqrt{1 + \sqrt{x}}\right) = \frac{1}{2}(1 + \sqrt{1 + \sqrt{x}})^{-1/2} \frac{1}{2} \left(1 + \sqrt{1 + \sqrt{x}}\right)^{-1/2} \frac{1}{2} \left(1 + \sqrt{1 + \sqrt{x}}\right)^{-1/2}.$$  

So the original derivative is

$$\frac{1}{8x}\sqrt{1 + \sqrt{1 + \sqrt{x}}}.$$  

Using the chain rule, the power rule, and the product rule, it is possible to avoid using the quotient rule entirely.

**Example 3.5.6** Compute the derivative of $f(x) = \frac{x^3}{x^2 + 1}$. Write $f(x) = x^3(x^2 + 1)^{-1}$, then

$$f'(x) = x^3 \frac{d}{dx}(x^2 + 1)^{-1} + \frac{d}{dx}x^3(x^2 + 1)^{-1}$$

$$= -2x^3(x^2 + 1)^{-1} + 3x^2(x^2 + 1)^{-1}$$

$$= -2x^4 + 3x^2 + 1(x^2 + 1)^{-1} - 2x^4 + 3x^2 + 1(x^2 + 1)^{-1}$$

$$= \frac{-2x^4 + 3x^2 + 1}{(x^2 + 1)^1}.$$  

Note that we already had the derivative on the second line, all the rest is simplification. It is easier to get to this answer by using the quotient rule, so there’s a trade off: more work for fewer memorized formulas.