2 Instantaneous Rate of Change: The Derivative

2.1 The Slope of a Function

Suppose that \( y \) is a function of \( x \), say \( y = f(x) \). It is often necessary to know how sensitive the value of \( y \) is to small changes in \( x \).

**Example 2.1.1** Take, for example, \( y = f(x) = \sqrt{25 - x^2} \) (the upper semicircle of radius 5 centered at the origin). When \( x = 7 \), we find that \( y = \sqrt{25 - 49} = -24 \). Suppose we want to know how much \( y \) changes when \( x \) increases a little, say to 7.1 or 7.01.

In the case of a straight line \( y = ax + b \), the slope \( m = \Delta y/\Delta x \) measures the change in \( y \) per unit change in \( x \). This can be interpreted as a measure of “sensitivity”: for example, if \( y = 10x + 5 \), a small change in \( x \) corresponds to a change one hundred times as large in \( y \), so \( y \) is quite sensitive to changes in \( x \).

Let us look at the same ratio \( \Delta y/\Delta x \) for our function \( y = \sqrt{25 - x^2} \) when \( x \) changes from 7 to 7.1. Here \( \Delta x = 7.1 - 7 = 0.1 \) is the change in \( x \), and

\[
\Delta y = f(x + \Delta x) - f(x) = f(7.1) - f(7)
\]

\[
\approx \frac{\sqrt{25 - (7.1)^2}}{-24} - \frac{\sqrt{25 - 7^2}}{-24} = \frac{\sqrt{25 - 49} - \sqrt{25 - 49}}{0.1} = \frac{0}{0.1}.
\]

Thus, \( \Delta y/\Delta x \approx -0.0294/0.1 = -0.294 \). This means that \( y \) changes by less than one third the change in \( x \), so apparently \( y \) is not very sensitive to changes in \( x \) at \( x = 7 \). We say “apparently” here because we don’t really know what happens between 7 and 7.1. Perhaps \( y \) changes dramatically as \( x \) runs through the values from 7 to 7.1, but at 7.1 \( y \) just happens to be close to its value at 7. This is not in fact the case for this particular function, but we don’t yet know why.

2.1.1 The slope of a function

Instead of looking at more particular values of \( \Delta x \), let’s see what happens if we do some algebra with the difference quotient using just \( \Delta x \). The slope of a chord from \((7, 24)\) to a nearby point is given by

\[
\frac{\sqrt{25 - (7 + \Delta x)^2} - 24}{\Delta x} = \frac{\sqrt{25 - (7 + \Delta x)^2} - 24}{\Delta x} = \frac{\sqrt{25 - (7 + \Delta x)^2} + 24}{\Delta x} = \frac{625 - (7 + \Delta x)^2 + 24}{\Delta x} = \frac{625 - 49 - 49\Delta x - \Delta x^2 + 24}{\Delta x} = \frac{625 - (7 + \Delta x)^2 + 24}{\Delta x}.
\]

Now, can we tell by looking at this last formula what happens when \( \Delta x \) gets very close to zero? The numerator clearly gets very close to \(-14\) while the denominator gets very close to \(\sqrt{25 - 7^2} + 24 \approx 48\). If the fraction therefore very close to \(-14/48 = -7/24 \approx -0.29167\), it certainly seems reasonable, and in fact it is true: as \( \Delta x \) gets closer and closer to zero, the difference quotient does in fact get closer and closer to the tangent line, so we say that \( f \) is differentiable at \( 7 \) and the slope of the tangent line is \( -7/24 \).

What about the slope of the tangent line at \( x = 127 \)? Well, 127 can’t all be that different from 7, we just have to redo the calculation with 12 instead of 7. This won’t be hard, but it will be a bit tedious. What if we try to do all the algebra without using a specific value for \( x \)? Let’s copy from above, replacing 7 by \( x \). We’ll have to do a bit more than that—for example, the “24” in the calculation came from \( \sqrt{25 - x^2} \), so we’ll need to fix that too.

\[
\frac{\sqrt{25 - (x + \Delta x)^2} - 24}{\Delta x} = \frac{\sqrt{25 - (x + \Delta x)^2} - 24}{\Delta x} = \frac{\sqrt{25 - (x + \Delta x)^2} + 24}{\Delta x} = \frac{625 - (x + \Delta x)^2 + 24}{\Delta x} = \frac{625 - x^2 - 2x\Delta x - \Delta x^2 + 24}{\Delta x} = \frac{625 - x^2 - 2x\Delta x - \Delta x^2 + 24}{\Delta x} = \frac{625 - x^2 - 2x\Delta x - \Delta x^2 + 24}{\Delta x} = \frac{625 - (x + \Delta x)^2 + 24}{\Delta x}.
\]

Now what happens when \( \Delta x \) is very close to zero? Again it seems apparent that the quotient will be very close to

\[
\frac{2x - 2x\Delta x - \Delta x^2}{\Delta x}
\]

Replacing \( x \) by 7 gives \(-7/24\), as before, and now we can easily do the computation for 12 or any other value of \( x \) between -25 and 25.

So now we have a single, simple formula, \(-x/\sqrt{25 - x^2}\), that tells us the slope of the tangent line for any value of \( x \). This slope, in turn, tells us how sensitive the value of \( y \) is to changes in the value of \( x \).

What do we call such a formula? That is, a formula with one variable, so that substituting an “input” value for the variable produces a new “output” value? This is a function. Starting with one function, \( f(x) = \sqrt{25 - x^2} \), we have derived, by means of some slightly messy algebra, a new function, \(-x/\sqrt{25 - x^2}\), that gives us important information about the original function. This new function in fact is called the derivative of the original function. If the original is referred to as \( f \) or \( y \) then the derivative is often written \( f’ \) or \( y’ \) and pronounced “first prime” or “y prime”, so in this case we might write \( f’(x) = -x/\sqrt{25 - x^2} \).

At a particular point, say \( x = 7 \), we say that \( f’(7) = -7/24 \) or “the slope of the tangent line is \( -7/24 \)” or “the derivative of \( f \) at \( 7 \) is \(-7/24\)”.

To summarize, we compute the derivative of \( f(x) \) by forming the difference quotient

\[
\frac{f(x + \Delta x) - f(x)}{\Delta x},
\]

(2.1.1)

which is the slope of a line, then we figure out what happens when \( \Delta x \) gets very close to zero.
We should note that in the particular case of a circle, there’s a simple way to find the derivative. Since the tangent to a circle at a point is perpendicular to the radius drawn to the point of contact, its slope is the negative reciprocal of the slope of the radius. The radius joining (0, 0) to (7, 24) has slope $24/7$. Hence, the tangent line has slope $-7/24$. In general, a radius to the point $(x, \sqrt{25 - x^2})$ has slope $-x/\sqrt{25 - x^2}$, so the tangent line is $y = -x/\sqrt{25 - x^2}$ as before. It is NOT true that a tangent line is perpendicular to the origin—don’t use this shortcut in any other circumstance.

As above, and as you might expect, for different values of $x$ we generally get different values of the derivative $f'(x)$. Could it be that the derivative always has the same value? This would mean that the slope of $f$ or the slope of its tangent line is the same everywhere. One curve that always has the same slope is a line; it seems odd to talk about the tangent line positive? Negative? What do you notice about the graph at the point(s) where the sign of the slope changes from positive to negative and vice versa?

### 2.2 Exercises

1. Draw the graph of the function $y = f(x) = \sqrt{16 - x^2}$ between $x = 0$ and $x = 13$. Find the slope $\Delta y/\Delta x$ of the chord between the points of the circle lying over (a) $x = 2$ and $x = 13$, (b) $x = 12$ and $x = 12.1$, (c) $x = 12$ and $x = 12.01$, (d) $x = 12$ and $x = 12.001$. Now use the geometry of tangent lines on a circle to find (e) the exact value of the derivative $f'(12)$. Your answers to (a)–(d) should be getting closer and closer to your answer to (e).

2. Use geometry to find the derivative $f'(x)$ of the function $f(x) = \sqrt{25 - x^2}$ in the text for each of the following $x$: (a) 0, (b) 5, (c) 10, (d) 12.5. Draw a graph of the upper semicircle, and draw the tangent line at each of these four points.

3. Draw the graph of the function $y = f(x) = 1/x$ between $x = 1/2$ and $x = 4$. Find the slope of the chord between $(a, x) = 3$ and $(b, x) = 3.01$, $(c, x) = 3$ and $x = 3.001$. Now use algebra to find a simple formula for the slope of the chord between $(3, f(3))$ and $(3 + \Delta x, f(3 + \Delta x))$. Determine what happens when $\Delta x$ approaches 0. In your graph of $y = 1/x$, draw the straight line through the point $(3, 1/3)$ whose slope is this limiting value of the difference quotient as $\Delta x$ approaches 0.

4. Find an algebraic expression for the difference quotient $(f(1 + \Delta x) - f(1))/\Delta x$ when $f(x) = x^2 - 1$. Simplify the expression as much as possible. Then determine what happens as $\Delta x$ approaches 0. What is $f'(1)$?

5. Draw the graph of $y = f(x) = x^2$ between $x = 0$ and $x = 1.5$. Find the slope of the chord between $(a, x) = 1$ and $(b, x) = 1.1$, $(c, x) = 1$ and $x = 1.001$, $(d, x) = 1$ and $x = 1.00001$. Then use algebra to find a simple formula for the slope of the chord between $1$ and $1 + \Delta x$. (Use the expansion $(a + \Delta x)^2 = a^2 + 2a\Delta x + \Delta x^2$.) Determine what happens as $\Delta x$ approaches 0, and in your graph of $y = x^2$, draw the straight line through the point $(1, 1)$ whose slope is equal to the value you just found.

6. Find an algebraic expression for the difference quotient $(f(x + \Delta x) - f(x))/\Delta x$ when $f(x) = mx + b$. Simplify the expression as much as possible. Then determine what happens as $\Delta x$ approaches 0. What is $f'(x)$?

### 2.3 Chapter 2 Instantaneous Rate of Change: The Derivative

We sketch the unit circle. Discuss the behavior of the slope of the tangent line at various angles around the circle. Which trigonometric function gives the slope of the tangent line at an angle $\theta$? Hint: think in terms of ratios of sides of triangles.

7. Sketch the parabola $y = x^2$. For what values of $x$ on the parabola is the slope of the tangent line positive? Negative? What do you notice about the graph at the point(s) where the sign of the slope changes from positive to negative and vice versa?

8. We wanted to know what happens to this function as $\Delta x$ goes to zero. $\textbf{How}$ because we were able to simplify the fraction, it was easy to see the answer, but it was not quite as simple...
EXAMPLE 2.3.1
Does very close to zero.

\[ < \]

never even get as far as 1.4142, no matter how long we continue the sequence.

The quantity is meaningless, while the right hand one is

\[ = \]

19

and the variable of the limit was not

\( x \)

fixed value, meaning that the first quantity can be made "as close as we like" to the fixed

value. Consider again the quantities

\[
\frac{19.6 \Delta x - 4.9 \Delta x^2}{\Delta x}
\]

These two quantities are equal as long as \( \Delta x \) is not zero. If \( \Delta x \) is zero, the left hand quantity is meaningless, while the right hand one is

\( x \)

19.6, pick

\[ \delta = \]

19

4

as you have requested.

So if I pick any number \( \delta \) that is less than \( \epsilon/4 \), the algebra tells me that whenever \( \delta < \epsilon/4 \) and so \( -19.6 < 4.9 \Delta x \) is within \( \epsilon \) of \( -19.6 \). (This is exactly what I did in the example: I picked \( \delta = 0.000002 \) so \( 0.000002 \times 412 = 0.0000082 \) .) A similar calculation again works for negative \( \Delta x \). The important part is that this is now a completely general result; it shows that I can always win, no matter what move you make.

Now we can codify this by giving a precise definition to replace the fuzzy, "gets closer and closer" language we have used so far.

Henceforward, we will say something like "the limit of \((-19.6 \Delta x - 4.9 \Delta x^2) / \Delta x \) as \( \Delta x \) goes to zero is \( -19.6, \)" and abbreviate this mouthful

\[ \lim_{\Delta x \to 0} \frac{-19.6 \Delta x - 4.9 \Delta x^2}{\Delta x} = -19.6. \]

Here is the actual, official definition of "limit".

DEFINITION 2.3.2 Limit
Suppose \( f \) is a function. We say that \( \lim_{x \to a} f(x) = L \) if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) so that whenever \( 0 < |x - a| < \delta \), \( |f(x) - L| < \epsilon \).

The \( \epsilon \) and \( \delta \) here play exactly the role they did in the preceding discussion. The definition says, in a very precise way, that \( f(x) \) can be made as close as desired to \( L \) (that's the \( f(x) - L \) is less than \( \epsilon \)) by making \( x \) close enough to \( a \) (the \( 0 < |x - a| < \delta \) part). Note that we specifically make no mention of what must happen if \( x = a \), that is, if \( |x - a| = 0 \). This is because in the cases we are most interested in, substituting \( a \) for \( x \) doesn't even make sense.

Make sure you are not confused by the names of important quantities. The generic definition talks about \( f(x) \), but the function and the variable might have other names. In the discussion above, the function we analyzed was

\[
-19.6 \Delta x - 4.9 \Delta x^2
\]

and the variable of the limit was not \( x \) but \( \Delta x \). The \( x \) was the variable of the original function; when we were trying to compute a slope or a velocity, \( x \) was essentially a fixed quantity, telling us at what point we wanted the slope. (In the velocity problem, it was literally a fixed quantity, as we discussed on the time 2.) The quantity \( a \) or the definition in the all the examples was zero; we were always interested in what happened as \( \Delta x \) became very close to zero.

Armed with a precise definition, we can now prove that certain quantities behave in a particular way. The bad news is that even proofs for simple quantities can be quite tedious and complicated; the good news is that we rarely need to do such proofs, because most expressions act the way you would expect, and this can be proved once and for all.

EXAMPLE 2.3.3 Let's show carefully that \( \lim_{x \to 4} x = 4 \). This is not something we "need" to prove, since it is "obviously" true. But if we couldn't prove it using our official definition there would be something very wrong with the definition.

As is often the case in mathematical proofs, it helps to work backwards. We want to end up showing that under certain circumstances \( x = 4 \) is close to 6; precisely, we want to show that \( |x - 4| < \delta \), \( |x - 2| < \epsilon \). Under what circumstances? We want this to be true whenever \( 0 < |x - 2| < \delta \). So the question becomes: can we choose a value for \( \delta \) that guarantees that \( -19.6 < 4.9 \Delta x < 19.6 \).

So it turns out to be very easy to prove something "obvious," which is nice. It doesn't take long before things get trickier, however.

EXAMPLE 2.3.4 It seems clear that \( \lim_{x \to 2} x^2 = 4 \). Let's try to prove it. We will want to be able to show that \( |x^2 - 4| < \epsilon \) whenever \( 0 < |x - 2| < \delta \), by choosing \( \delta \) carefully. Is there any connection between \( |x - 2| \) and \( |2 - 4| = \epsilon \)? Yes, but it's not hard to spot, but it is not as simple as the previous example. We can write \( |x^2 - 4| = |(x + 2)(x - 2)| \) now when \( |x - 2| < 1 \), small part of \( |(x + 2)(x - 2)| < |x + 2| \) is small, namely |(x + 2)(x - 2)|. What about \( |x + 2| ? \) If \( x \) is close to 2, \( |x + 2| \) certainly can't be too big, but we need to somehow be precise about it.

Let's recall the "game" version of what is going on here. You get to pick an \( \epsilon \) and I have to pick a \( \delta \) that makes things work out. Presumably it is the truly tiny values of \( \epsilon \) one need to worry about, but I have to be prepared for anything, even an apparently "bad" move like \( \epsilon = 1000 \). I expect that \( x \) is going to be small, and that the \( \delta \) corresponding will be small, certainly less than 1. If \( \delta \) then \( |x - 2| < \delta \), I have to have some connection between \( |x - 2| \) and \( |x - 4| \). So now how can I pick \( \delta \) so that \( |x - 2| < \delta \) implies \( 5|x - 2| < \epsilon \)? This is easy: use \( \delta = \epsilon/5 \), so \( 5|2 - x| < 5(\epsilon/5) = \epsilon \). But if the \( \epsilon \) you choose is not small? If you choose \( \epsilon = 1000 \), should I pick \( \delta = 2000 \)? No, to keep things "sane" I will never pick a \( \delta \) bigger than 1. Here's the final "game strategy": When you pick a value for \( \epsilon \), I will pick \( \delta = \epsilon/5 \) or \( 1 \), whichever is smaller. Now when \( |x - 2| < \delta \), I know both that \( |x - 2| < \delta < \epsilon/5 \). Thus \( |(x + 2)(x - 2)| < 5(\epsilon/5) = \epsilon \).

This has been a long discussion, but most of it was explanation and scratch work. If this were written down as a proof, it would be quite short, like this:

Proof that \( \lim_{x \to 2} x^2 = 4 \). Given any \( \epsilon \), pick \( \delta = \epsilon/5 \) or \( 1 \), whichever is smaller. Then when \( |x - 2| < \delta \), \( |x + 2| < \delta \) and \( |x - 2| < \epsilon/5 \).

Hence \( \lim_{x \to 2} \{ x^2 - 4 \} = \lim_{x \to 2} \{ 2(x - 2) \} = 2 \epsilon/5 = \epsilon \).
we don't have to worry about it ever again. When we say that $x$ might be “complicated” we really mean that in practice it might be a function. Here is then what we want to know:

**THEOREM 2.3.5** Suppose \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \). Then
\[
\lim_{x \to a} f(x)g(x) = LM.
\]

**Proof.** We have to use the official definition of limit to make sense of this. So given any \( \varepsilon > 0 \) we need to find a \( \delta > 0 \) so that \( 0 < |x - a| < \delta \) implies \( |f(x)g(x) - LM| < \varepsilon \). What do we have to work with? We know that we can make \( f(x) \) close to \( L \) and \( g(x) \) close to \( M \), and we have to somehow connect these facts to make \( f(x)g(x) \) close to \( LM \).

We use, as is so often the case, a little algebraic trick:
\[
|f(x)g(x) - LM| = |f(x)(g(x) - M) + (f(x) - L)M| \leq |f(x)||g(x) - M| + |f(x) - L|M.
\]

This is all straightforward except perhaps for the ‘\( \varepsilon/2 \)’. That is an example of the triangle inequality, which says that if \( a \) and \( b \) are any real numbers then \( |a + b| \leq |a| + |b| \). If you look at a few examples, using positive and negative numbers in various combinations for \( a \) and \( b \) you should quickly understand why this is true; we will not prove it formally.

Since \( \lim_{x \to a} f(x) = L \), there is a \( \delta_1 \) so that \( 0 < |x - a| < \delta_1 \) implies \( |f(x) - L| < \varepsilon/2M \). This means that \( 0 < |x - a| < \delta_1 \) implies \( |f(x)| < L + \varepsilon/2M \). You can see where this is going: if we can make \( |f(x)| |g(x) - M| < \varepsilon/2 \) also, then we'll be done. We can make \( |g(x) - M| < \varepsilon/2M \) smaller than any fixed number by making \( x \) close enough to \( a \); unfortunately, \( \varepsilon/2M \) is not a fixed number, since \( x \) is a variable. Here we need another little trick: just like the one we used in analyzing \( x^2 \). We can find a \( \delta_2 \) so that \( 0 < |x - a| < \delta_2 \) implies \( |f(x)| < L + \varepsilon/2M \), meaning that \( -L < f(x) < L + \varepsilon/2M \). This means that \( f(x) < M \), where \( N \) is either \( L - 1 \) or \( L + 1 \), depending on whether \( L \) is positive or not.

The important point is that \( N \) doesn't depend on \( x \). Finally, we know that there is a \( \delta_3 \) so that \( 0 < |x - a| < \delta_3 \) implies \( |g(x) - M| < \varepsilon/2N \). Now we're ready to put everything together. Let \( \delta \) be the smallest of \( \delta_1 \), \( \delta_2 \), and \( \delta_3 \). Then \( 0 < |x - a| < \delta \) implies \( |f(x)||g(x) - M| < \varepsilon/2M \) and \( |g(x) - M| < \varepsilon/2N \). Therefore
\[
|f(x)g(x) - LM| \leq |f(x)||g(x) - M| + |f(x) - L|M < N \frac{\varepsilon}{2M} \frac{\varepsilon}{2N} = \varepsilon.
\]

This is just what we needed, so by the official definition, \( \lim_{x \to a} f(x)g(x) = LM \).

---

### 2.3 Limits

number. But 5 can, and should, be interpreted here as the function that has value 5 everywhere, \( f(x) = 5 \), with a graph a horizontal line. From this point of view it makes sense to ask what happens to the height of the function as \( x \) approaches 1.

Of course, as we’ve already seen, we’re primarily interested in limits that aren’t so easy, namely, limits in which a denominator approaches zero. There are a handful of algebraic tricks that work on many of these limits.

**EXAMPLE 2.3.8** Compute \( \lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1} \). We can’t simply plug in \( x = 1 \) because that makes the denominator zero. However:
\[
\lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 3)}{x - 1} = \lim_{x \to 1} (x + 3) = 4.
\]

While theorem 2.3.6 is very helpful, we need a lot more to work easily with limits. Since the theorem applies when some limits are already known, we need to know the behavior of some functions that cannot themselves be constructed from the simple arithmetic operations of the theorem, such as \( \sqrt{x} \). Also, there is one other extraordinarily useful way to put functions together: composition. If \( f(x) \) and \( g(x) \) are functions, we can form two functions by composition: \( f(g(x)) \) and \( g(f(x)) \). For example, if \( f(x) = \sqrt{x} \) and \( g(x) = x^2 + 5 \), then \( f(g(x)) = \sqrt{x^2 + 5} \) and \( g(f(x)) = (\sqrt{x})^2 = 5 + x + 5 \). Here is a companion to theorem 2.3.6 for composition:

**THEOREM 2.3.9** Suppose \( \lim_{x \to a} g(x) = L \) and \( \lim_{x \to L} f(x) = f(L) \). Then
\[
\lim_{x \to a} f(g(x)) = f(L).
\]

Note the special form of the condition on \( f \): it is not enough to know that \( \lim_{x \to a} f(x) = M \), though it is a bit tricky to see why. Many of the most familiar functions do have this property, and this theorem can therefore be applied. For example:

**EXAMPLE 2.3.10** Suppose that \( n \) is a positive integer. Then
\[
\lim_{x \to 0} \sqrt[n]{x} = \sqrt[n]{0},
\]
provided that \( n \) is positive if \( n \) is even.

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### 4.2 Chapter 2 Instantaneous Rate of Change: The Derivative

A handful of such theorems give us the tools to compute many limits without explicitly working with the definition of limit.

**THEOREM 2.3.6** Suppose that \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \) and \( k \) is some constant. Then
\[
\lim_{x \to a} k f(x) = k \lim_{x \to a} f(x), \quad \lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M, \quad \lim_{x \to a} f(x) - g(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M, \quad \lim_{x \to a} f(x)g(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x) = LM.
\]

Roughly speaking, these rules say that to compute the limit of an algebraic expression, it is enough to compute the limits of the “innermost bits” and then combine these limits. This often means that it is possible to simply plug in a value for the variable, since \( \lim_{x \to a} x = a \).

**EXAMPLE 2.3.7** Compute \( \lim_{x \to 1} \frac{x^2 - 3x + 5}{x - 2} \). If we apply the theorem in all its glory detail, we get
\[
\lim_{x \to 1} \frac{x^2 - 3x + 5}{x - 2} = \lim_{x \to 1} x^2 - 3x + 5 + \lim_{x \to 1} x - 2 = (1^2 - 3 \cdot 1 + 5) + (1 - 2) = -3.
\]

It is worth commenting on the trivial limit \( \lim_{x \to a} x \). From one point of view this might seem meaningless, as the number 5 can’t “approach” any value, since it is simply a fixed
lim \( x \to 1 \) \( 1 = 1 \). The limit of \( f(x) \) must be equal to both the left and right limits; since they are different, the limit \( \lim \frac{\Delta f}{\Delta x} \) does not exist.

### Exercises 2.3.

Compute the limits. If a limit does not exist, explain why.

1. \( \lim_{x \to 1} \frac{x^2 + x - 12}{x - 3} \)
2. \( \lim_{x \to 1} \frac{x^2 - 2}{x^2 + 1} \)
3. \( \lim_{x \to 3} \frac{4x - 5x^2}{x^2 - 1} \)
4. \( \lim_{x \to 1} \frac{x^2 - x - 12}{x^2 - 3} \)
5. \( \lim_{x \to 1} \sqrt{x + 3} - 3 \)
6. \( \lim_{x \to 1} \frac{2x + 1}{2x - 1} \)
7. \( \lim_{x \to 3} \frac{3x^2 - 5x}{x} \)
8. \( \lim_{x \to 1} \sqrt{3x^3} - 5x \)
9. \( \lim_{x \to 1} \frac{x^2 - 1}{x^2 - x + 1} \)
10. \( \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \)
11. Compute the limits. If a limit does not exist, explain why.
12. \( \lim_{x \to 1} \frac{x - 5}{x - 1} \), \( x \neq 1 \), \( x = 1 \).
13. \( \lim x \sin \left( \frac{x}{2} \right) \) (Hint: Use the fact that \( |\sin a| < 1 \) for any real number \( a \). You should probably use the definition of a limit here.)
14. \( \lim (x^2 + 4)^3 \)

### 2.4 The Derivative Function

We know that \( f' \) carries important information about the original function \( f \). In one example we saw that \( f'(x) \) tells us how steep the graph of \( f(x) \) is, and we can see that \( f'(x) \) tells us the velocity of an object if \( f(x) \) tells us the position of the object at time \( x \).

As we said earlier, this same mathematical idea is useful whenever \( f(x) \) represents some changing quantity and we want to know something about how it changes, or roughly, the “rate” at which it changes. Most functions encountered in practice are built up from a small collection of "primitive" functions in a few simple ways, for example, by adding or multiplying functions together to get new, more complicated functions. To make good use of the information provided by \( f'(x) \) we need to be able to compute it for a variety of such functions.

We will begin by using different notations for the derivative of a function. While initially confusing, each is often useful so it is worth maintaining multiple versions of the same thing.

Consider again the function \( f(x) = \sqrt{25 - x^2} \). We have computed the derivative \( f'(x) = -x/\sqrt{25 - x^2} \), and have already noted that if we use the alternate notation \( y = \sqrt{25 - x} \) then we might write \( y' = -x/\sqrt{25 - x} \). Another notation is quite different, and in time it will become clear why it is often a useful one. Recall that to compute the derivative of \( f \) we computed

\[
\lim_{\Delta x \to 0} \frac{\sqrt{25 - (1 + \Delta x)^2} - 24}{\Delta x}
\]

The denominator here measures a distance in the \( x \) direction, sometimes called the “run”, and the numerator measures a distance in the \( y \) direction, sometimes called the “rise,” and “rise over run” is the slope of a line. Recall that sometimes such a numerator is abbreviated \( \Delta y \), exchanging brevity for a more detailed expression. So in general, a derivative is given by

\[
y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}
\]

To recall the form of the limit, we sometimes say instead that

\[
\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}
\]

In other words, \( dy/dx \) is another notation for the derivative, and it reminds us that it is related to an actual slope between two points. This notation is called Leibniz notation, after Gottfried Leibniz, who developed the fundamentals of calculus independently, at about the same time that Isaac Newton did. Again, since we often use \( f \) and \( f(x) \) to mean the original function, we sometimes use \( df/dx \) and \( df(x)/dx \) to refer to the derivative. If

\[
\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

the function \( f(x) \) is written out in full we often write the last of these something like this

\[
f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

with the function written to the side, instead of trying to fit it into the numerator.

### Example 2.4.2

Find the derivative of \( y = f(t) = t^2 \).

We compute

\[
y' = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{(t + \Delta t)^2 - t^2}{\Delta t} = \lim_{\Delta t \to 0} \frac{t^2 + 2t \Delta t + \Delta t^2 - t^2}{\Delta t} = \lim_{\Delta t \to 0} \frac{2t \Delta t + \Delta t^2}{\Delta t} = \lim_{\Delta t \to 0} (2t + \Delta t) = 2t.
\]

Remember that \( \Delta t \) is a single quantity, not a “\( \Delta \)” times a “\( t \),” and so \( \Delta t^2 \) is \( (\Delta t)^2 \) not \( \Delta (t^2) \).

### Example 2.4.3

Find the derivative of \( y = f(x) = 1/x \).

The computation:

\[
y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta (1/x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{-1}{x(1 + \Delta x)} = \lim_{\Delta x \to 0} \frac{-x}{x(1 + \Delta x)} = \lim_{\Delta x \to 0} \frac{-1}{1 + \Delta x} = -\frac{1}{x^2}
\]

Note. If you happen to know some “derivative formulas” from an earlier course, for the time being you should pretend that you do not know them. In examples like the ones above and the exercises below, you are required to know how to find the derivative.
Some unusual behavior off-stage (i.e., outside the view of the graphs).

DEFINITION 2.5.1 Bounded
A function \( f \) is bounded if there is a number \( M \) such that \( |f(x)| < M \) for every \( x \) in the domain of \( f \).

For the function in (c), one such choice for \( M \) would be \( M = 1 \). In other cases, simply finding an \( M \) is enough to establish boundedness. No such \( M \) exists for the hyperbola in (d) and hence we say that it is unbounded.

Continuity. The graphs shown in (b) and (c) both represent continuous functions. Geometrically, this is because there are no jumps in the graphs. That is, if you pick a point on the graph and approach it from the left and right, the values of the function approach the value of the function at that point. For example, we can see that this is not true for function values near \( x = -1 \) on the graph in (a) which is not continuous at that location.
DEFINITION 2.5.2 Continuous at a Point A function $f$ is continuous at a point $a$ if \( \lim_{x \to a} f(x) = f(a) \).

DEFINITION 2.5.3 Continuous A function $f$ is continuous if it is continuous at every point in its domain.

Strangely, we also say that (d) is continuous even though there is a vertical asymptote. A careful reading of the definition of continuous reveals the phrase “at every point in its domain.” Because the location of the asymptote, $x = 0$, is not in the domain of the function, and because the rest of the function is continuous, we say that (d) is continuous.

Differentiability. If a function has a derivative at every point, we say the function is differentiable. We can see that the tangent line is well-defined at every point on the graph in (c). Therefore, we say that (c) is a differentiable function.

DEFINITION 2.5.4 Differentiable at a Point A function $f$ is differentiable at point $a$ if $f'(a)$ exists.

DEFINITION 2.5.5 Differentiable A function $f$ is differentiable if it is differentiable at every point (excluding endpoints and isolated points in the domain of $f$) in the domain of $f$.

Take note that, for technical reasons not discussed here, both of these definitions exclude endpoints and isolated points in the domain from consideration.

We close with a useful theorem about continuous functions:

THEOREM 2.5.6 Intermediate Value Theorem If $f$ is continuous on the interval $[a, b]$ and $d$ is between $f(a)$ and $f(b)$, then there is a number $c$ in $[a, b]$ such that $f(c) = d$.

This is most frequently used when $d = 0$.

EXAMPLE 2.5.7 Explain why the function $f = x^3 + 3x^2 + x - 2$ has a root between 0 and 1. By theorem 2.3.6, $f$ is continuous. Since $f(0) = -2$ and $f(1) = 3$, and 0 is between $-2$ and 3, there is a $c \in [0, 1]$ such that $f(c) = 0$.

This example also points the way to a simple method for approximating roots.

EXAMPLE 2.5.8 Approximate the root of the previous example to one decimal place.

If we compute $f(0.01)$, $f(0.02)$, and so on, we find that $f(0.05) < 0$ and $f(0.07) > 0$, so by the Intermediate Value Theorem, $f$ has a root between 0.05 and 0.07. Repeating the process with $f(0.061)$, $f(0.062)$, and so on, we find that $f(0.061) < 0$ and $f(0.062) > 0$, so $f$ has a root between 0.061 and 0.062, and the root is 0.6 rounded to one decimal place.

Exercises 2.5.

1. Along the lines of Figure 2.5.1, for each part below sketch the graph of a function that is:
   a. bounded, but not continuous.
   b. differentiable and unbounded.
   c. continuous at $x = 0$, not continuous at $x = 1$, and bounded.
   d. differentiable everywhere except at $x = -1$, continuous, and unbounded.
2. Is $f(x) = (x+1)^2$ a bounded function? If so, find the smallest $M$. ⇒
3. Is $g(x) = 1/(1 + x^2)$ a bounded function? If so, find the smallest $M$. ⇒
4. Is $h(x) = 2\ln|x|$ a bounded function? If so, find the smallest $M$. ⇒
5. Consider the function $h(x) = \begin{cases} 2x - 3, & \text{if } x < 1 \\ 0, & \text{if } x \geq 1. \end{cases}$ Show that it is continuous at the point $x = 0$. Is $h$ a continuous function? ⇒
6. Approximate a root of $f = x^3 - 4x^2 + 2x + 2$ to one decimal place. ⇒
7. Approximate a root of $f = x^3 + x^2 - 5x + 1$ to one decimal place. ⇒