Oral Exam in Mathematics

The oral exam for mathematics majors is one hour in length, divided into roughly two thirty minute segments. One part consists of questions chosen from a common list of topics. This list, along with some explanation about the topics, is given later in this document. The other portion of the oral exam consists of topics from either Math 455 (Real Analysis I) or Math 475 (Abstract Algebra I). You must make your choice of which course to be tested on before the end of the fall semester so that we can determine the committees (two or three professors for each exam) for the oral exams and you can take some time over winter break to prepare for the exam. The exam is scheduled for the first Saturday following the beginning of the spring semester. Specific times and places will be determined the week prior to the exam.

The common portion of the oral exam is somewhat predictable since you already know the possible questions. We sometimes ask for further examples or clarification and make certain that you truly understand the topic. The analysis and algebra components of the exam vary depending on the person asking the questions. The focus, however, is on the basic concepts covered in the respective course. We typically ask you to discuss definitions of key concepts and statements of the more important theorems, including giving examples that illustrate these results and indicate an understanding of the hypotheses and conclusions. You should be able to sketch the proofs of some of these key theorems and be able to use the results to solve simple problems involving the concepts.

Some questions can be answered orally, but we often have students write answers to our questions on the board, using shorthand notation and abbreviations when that is helpful. If you get stuck on a problem, we typically offer some sort of hint. Usually, one person asks the questions but other professors may chime in at times. The number of questions that you answer does matter; if it takes you a long time to answer a simple question (for example, state the Mean Value Theorem and illustrate it graphically), that is an indication that you do not know the result very well. Consistent trouble answering questions is a basis for failing the oral exam. Overall, the atmosphere is low key, but we do want to see evidence that you can discuss the topics in a clear and comprehensible manner. A sage advisor (years ago) provided the following advice prior to an oral exam: "Don't worry, but study like crazy." These words may seem contradictory, but there is much truth in them.

If you are hoping to apply for honors in major study, then the bar for passing the oral exam is higher than for other students. We want to see evidence that you have a command of the material, not just an ability to regurgitate facts and examples.

If you have questions concerning any aspect of the oral exam, you can communicate with any member of the math faculty.

Some Things Every Math Major Should Know

1. Prove the Pythagorean Theorem.

   There are many approaches to proving this ancient theorem. The most common ones involve splitting up squares in clever ways or using similar triangles. It is important to note that the fact that the sum of the measures of the angles of a triangle is 180 degrees is needed in each of these proofs. You should know how to prove this property of triangles and realize that the parallel postulate is required. If you get a chance, reading Euclid's proof of the Pythagorean Theorem (Book I, Proposition 47 of *The Elements*) is quite interesting. In addition, be able to prove the converse of the Pythagorean Theorem.

2. Prove that the square root of 2 is an irrational number.

   This fact destroyed the Pythagorean philosophy that all was number and led the Greeks to abandon algebra and focus on geometry. It is hard today to appreciate this perspective but it certainly changed the face of Western civilization. The classic proof focuses on even and odd integers but you should also know how to prove that the square root of any positive integer that is not a perfect square is irrational. The simplest proof of this more general result involves the Fundamental Theorem of Arithmetic.

3. Prove that there are infinitely many primes.

   This again is a very ancient result and the proof appears in *The Elements* (see Book IX, Proposition 20). The proof there is very interesting as it illustrates the difficulty of the idea of representing a generic number of things with symbols. The typical proof given for this result is a classic example of proof by contradiction. You should be able to adapt this proof to show that there are an infinite number of primes of the form $3k + 2$.

4. Prove that the set $\mathbb{R}$ of real numbers is uncountable.

   There are several ways to proceed here. If you use decimal expansions, you should note that some numbers have more than one decimal expansion and this means that some care must be used in the proof. You should also know that a number has a terminating or repeating decimal expansion if and only if it is a rational number. For the record, you should be able to explain why the set of rational numbers is countably infinite. What do these two facts say about the size of the set of irrational numbers? You might want to consider knowing the basic definition of algebraic numbers and transcendental numbers.

5. Be able to do a proof by induction.

   This should be self-explanatory. However, it is important to know that there are two equivalent forms of mathematical induction and that the strong (or complete) form is only needed in some cases. We typically ask students to prove a result that is not too complicated, but we make certain that students truly understand the ideas behind this technique of proof.

6. Discuss the formula for the sum of a geometric series.

   You should certainly know the formula for the sum and be able to use it to find the sum of a geometric series. You should understand how to find a simple formula for the sequence of partial sums and then use this result to be able to explain why the series converges in some cases (and gives the resulting formula for the sum) and diverges in others. In general, you should know a little about infinite series and their properties. For instance, the divergence test is a rather simple but important result concerning infinite series.

7. Prove that the p-series diverges for $p = 1$ and converges for $p = 2$.

   It is possible to prove these results using the integral test, but it is preferable that you be able to prove them using other techniques. If you decide to use the integral test, you must be able to explain why that test works. Here is a brief description for ways to proceed without the integral test. Let s_n denote the nth partial sum of the harmonic series, verify that $s_{2n} - s_n > \frac{1}{2}$ for each n, and then explain why this shows that the series does not converge. For the case in which $p = 2$, use $1/(k^2 - k)$ as an overestimate of $1/k^2$ for all $k > 1$, then show that the partial sums of this larger series (with the sum starting at $k = 2$) are bounded above by 1. (This hint is intended to be a bit cryptic so that you need to figure part of this out yourself; partial fractions and telescoping sums play a role here.) In addition, you should be aware that it is possible to use a hand-waving argument (as discovered by Euler) to prove that the sum of this series is $\pi^2/6$ by writing $\sin x$ as a "polynomial" in two different ways.

8. Explain how to solve a general first order linear differential equation.

   You need to be able to use symbols to explain why an integrating factor might be helpful to solve these types of equations and show how one determines what the integrating factor should be in the general case ($y' + p(t)y = q(t)$). Be able to apply the theory to a special case, that is, be able to solve a particular problem (of our choosing) for this type of differential equation.

9. Prove Euler's formula $e^{ix} = \cos x + i \sin x$.

   This formula is quite amazing as it links exponential functions and trigonometric functions, something you would never expect given the nature of these functions when they are first introduced. You can derive the connection using power series or the theory of differential equations. Make certain you are aware of the "nerdy" formula $e^{i\pi} + 1 = 0$ and how it is truly astounding that five special numbers appear in one simple formula. Using Euler's formula, you should be able to find a value for i^i.

10. Know the definition of derivative and be able to explain its geometrical interpretation. Be prepared to use the definition to find the derivatives of x^n (where n is a positive integer) and $\sin x$.

    Be certain you can explain the ideas behind this definition and how the difference quotient appearing in the definition relates to the graph of the function. If derivatives are only about tangent lines, why have they become so important? For the x^n derivation, you have several options; the Binomial Theorem, a factoring formula, or mathematical induction assuming the product rule are common options. For $\sin x$, there are a couple of standard techniques, each one involves some trig identities. Be able to explain why the trig limits that appear are valid; typically a picture and an inequality are sufficient.

11. Derive Newton's method.

    Drawing a good picture for this situation (after explaining what the method is designed to do) and running through two or three iterations should reveal to the viewer how quickly Newton's method converges to the root in many cases. You should then carefully show the algebraic steps that give the iteration formula. Be prepared to give some graphical examples to illustrate that Newton's method can fail to find the root in some cases.

12. Know the three "value theorems" from calculus (Intermediate, Extreme, and Mean) and be able to explain them geometrically.

    You should be able to clearly state the hypotheses and conclusions of these theorems and explain why the hypotheses are necessary. You should also be able to give graphical representations for each of these results and illustrate how the conclusions are plausible.

13. Know the definition of the definite integral and be able to explain its geometrical interpretation. Be prepared to use the definition to integrate x^2 over the interval $[0, b]$, where b is a positive constant.

    As with the derivative definition, you need to link the symbols in the integral definition with the corresponding aspects of the associated figure. Do all integrals represent area? Why has integration become useful for applications other than area? The typical integral computation using the definition is much more difficult than the typical derivative computation from the definition. The simple examples usually given (including x^2) involve the sum formulas for powers of positive integers. What key result in the seventeenth century made evaluating many integrals much easier?

14. Show that $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

    This problem can be reduced to one involving a double integral in polar coordinates. There are some technical ideas that are necessary to show that the formal steps in the process are actually valid (why do integrating over a square and integrating over a circle give the same result?), but we typically do not stress these. You should be aware that e^{-x^2} does not have an antiderivative in the traditional sense; this is why some other approach is needed. You should also know that this integral is not just a curiosity as the integrand is rather important in the theory of probability.

The following two lists provide a very brief sample of the sorts of topics that may be part of your subject oral exam. This list is nowhere near an exhaustive set of topics and your oral exam may not include any of these questions. However, they do give you an idea of the sorts of things that you should know.

for algebra:

1. Give the definition of a group.

   When giving this definition, be certain that you are careful with quantifiers. Also, be certain to explain what is meant by an identity and what it means to be an inverse.

2. Be prepared to give examples of both Abelian and Non-Abelian groups, including being able to explain that your examples have the desired properties.

   It is best to know a variety of examples, some involving finite groups and some involving infinite groups.

3. Be able to state and illustrate Lagrange's Theorem.

   It is important that you not simply state this theorem from memory; you need to be able to explain what a subgroup is and what the order of a subgroup means. Again, you should be familiar with several examples that illustrate the theorem as well as being able to outline its proof.

4. Know the definitions for rings and fields, including giving simple examples of each concept.

   As with the definition of a group, be certain that you are careful with quantifiers and are able to explain the terms that appear in the definitions. Be able to give several examples of each concept.

for analysis:

1. Be able to give the definition of a convergent sequence and a Cauchy sequence.

   You should know that a sequence converges if and only if it is a Cauchy sequence. You should be able to quickly prove one part of this biconditional statement (and know which part is the easy part) and be able to sketch a proof of the more complicated part.

2. Be able to use the Completeness Axiom to prove that bounded increasing sequences are convergent.

   The Completeness Axiom is extremely important so be certain that you know exactly what it says. The result mentioned here is one of the simple consequences of the Completeness Axiom. A quick outline of the proof, perhaps with the aid of diagram, is all that is expected.

3. Know the ϵ-δ definition for $\lim\limits_{x \to c} f(x) = L$ and be able to use it to establish a simple limit.

   This is a very important definition in analysis so be certain that you understand it well and are able to solve simple problems using the concept.

4. Know the definition of uniform continuity and be able to prove that a continuous function on $[a, b]$ is uniformly continuous on $[a, b]$.

   It is important that you understand the distinction between a continuous function on an interval I and a uniformly continuous function on I. Be able to give examples of functions that are continuous but not uniformly continuous on the interval $(0, 1)$ (know a bounded example as well as an unbounded one) and corresponding examples on the interval $[0, \infty)$. There are several options for the proof; we will not tell you which approach to take.