THE RANKING SYSTEMS OF INCOMPLETE TOURNAMENTS

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Introduction

As November approaches and the college football season nears an end, the annual debate begins: Who deserves to play for the national title? Who is number one? If each of the teams played each other, then it would be fairly easy to determine the best team; we could simply consider the team with the best record. However, the situation in Division I college football is slightly more complicated.

In many sports, each of the teams in a league has the opportunity to compete against each of the other teams (or at least a majority of the teams). In a situation such as this, the win-loss record is a reasonable method to rank the teams; the team with the highest winning percentage is said to be the best and the the team with the lowest winning percentage is said to be the worst. But what happens if this is not the case? What happens when each team only competes against a small fraction of the other teams? If that is the situation, how is the best team determined?

In baseball and basketball (and even professional football), some form of a playoff system is in place. Typically each division champion receives an automatic playoff berth, followed by wild-card or "at large" bids, which are given out to the teams with the best records. A playoff bracket is then set up and teams play each other until there is one undisputed champion. In the case of Division I college football, a playoff system has never been established.

For years, people have argued over how the national champion should be crowned. Prior to 1992, the national rankings were determined primarily by subjective methods such as the coach's poll. When the major conferences decided to establish the Bowl Championship Series (BCS) in 1997, they created a new mathematical formula to rank the teams. The ratings incorporated four different elements in the calculations: the subjective polls of coaches and writers, the teams' records, the average of three computer ratings (Jeff Sagarin of USA Today, Seattle

Times, and New York Times), and the team's strength of schedule (determined by the records of a team's opponents and their opponents' opponents). Since its creation, the BCS ranking system has continued to evolve by incorporating more computer ratings and polls into the calculations. Despite this evolution of the rating system, there remains controversy as to the accuracy of the rankings. [1]

The situation faced in college football is known as an incomplete tournament. An incomplete tournament can be defined as a tournament in which a team plays games against only a subset of the other teams in the league. There are 110 teams in Division I football, divided amongst eleven different conferences (and independents). Yet during the course of a season, each team plays only thirteen or fourteen games. If each team plays only a small fraction of other teams in the league, is the win-loss record still a valid way of comparing teams' performances? Would the teams with easier schedules not have an advantage over those with more difficult schedules? This is where the method of ranking the teams becomes much more complicated. It then becomes necessary to take into account the strength of each team's schedule. But what is the best way to calculate the strength of a team's schedule?

In this paper, we will examine a method of calculating schedule strength put forth by Charles Redmond in his paper "A Natural Generalization of the Win-Loss Rating System." We will then take the method a step further by looking at a case study of the Big 12 Conference during the 2007 season. [5]

An Illustration

To illustrate the method, we will use an example by Charles Redmond. Let us consider a small tournament consisting of four teams (A, B, C,and D) in which each team plays two games. The results of the games are as follows:

Game	**Results**
A $vs.$ B	$5 - 10$
A $vs.$ D	$57 - 45$
B $vs.$ C	$10 - 7$
C $vs.$ D	$3 - 10.$

The tournament results show that B has a record of 2-0, D has a record of 1-1, A has a record of 1-1, and C has a record of 0-2. We must now ask the question: "Do these records accurately represent the tournament?" Here, Redmond chooses to define the "dominance" of one team over another team as the point differential in the game played

TABLE 1. Average 1st-Generation Point-Difference Dominance

Team	**Average Point-Difference Dominance**
A	$7/3 = 2.33$
B	$8/3 = 2.67$
C	$-10/3 = -3.33$
D	$-5/3 = -1.67$

between the two teams. For example, in the game between A and B, B beats A by 5 points. Thus we can say that B's dominance over A is 5, while A's dominance over B is -5. Using the game results, we can calculate the average dominance for each team. In addition to the games actually played, we will include one imaginary game that each team plays against itself that has a dominance of zero.

Now we have developed statistics that are slightly more descriptive than the typical win-loss records. However, we still have not yet taken into account the strength of a team's schedule.

Here we can begin to look at how schedule strength can influence a ranking system. Consider team B. Team B does not play team D in the tournament, so we have do not have a direct way of comparing the two teams. However, both teams B and D played team A. Thus we can compare B and D based on how they did against team A. A beat D by 13 points and B beat A by 5 points. The basic idea is to consider an imaginary game in which B beats D by 13 + 5 = 18 points. In this case we can say that D is a second-generation opponent of B. In Figure 1 we see a tree of all nine possible second-generation games for B.

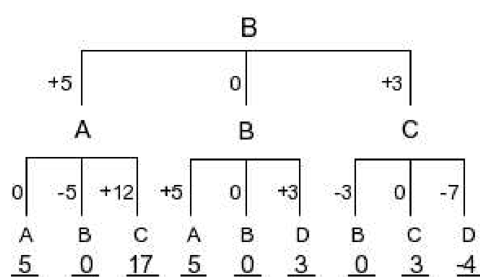

FIGURE 1. Second Generation Games for Team B (with B vs B game)

In the actual game between A and B, B beats A by 5 points. Note that this score is preserved in the second-generations games. Also note

Table 2. Average 2nd-Generation Point-Difference Dominance

Team	**Average 2nd Generation Dominance**
A	3.44
B	3.22
C	-4.11
D	-2.56

that the score in the B vs C game is preserved as well. Additionally, this generation of games includes the two possible "imaginary" games between between B and D (B $\rightarrow$ C $\rightarrow$ D and B $\rightarrow$ A $\rightarrow$ D). Since each of the games B vs A, B vs C, and B vs A appears twice in the second-generation bracket, and all B vs B game have a point difference of zero, the original relationships between the teams are preserved. Notice that if we had included the imaginary $BvsB$ game, we would have lost the information about the first generation games as seen in Figure 2.

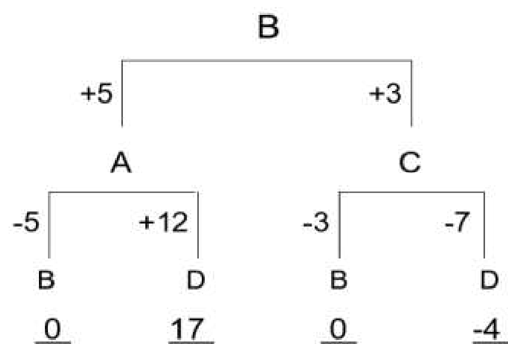

Figure 2. Second Generation Games for Team B (without B vs B game

To determine the second-generation dominance, we average the scores of the nine second-generation games. After updating our rankings to include second-generation dominance, in Figure 2 we see that by taking into account schedule strength, team A is now at the top.

The mathematical formulation for computing these second and third generation ratings becomes fairly interesting. Let us consider the diagram in Figure 3:

This diagram represents the tournament that has been played. Each line connecting the teams represents a game in which those two team

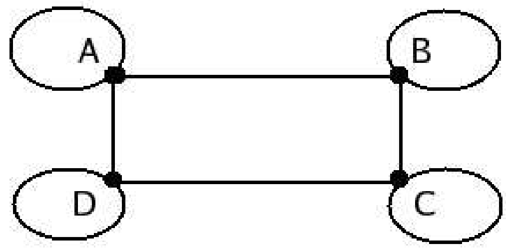

FIGURE 3. Connections Between Teams

competed. Note that no line connects B and D or A and C, and that there is a line that connects each team to itself. Now let's examine a 4 x 4 matrix M that represents the games played between each of the teams, where A corresponds to the first row and column, B corresponds to the second row and column, etc

$$M = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

In this matrix, a 1 represents a game played between the team of row i and column j and a 0 means that no game was played between these two teams. Now let's look at the matrix M^2,

$$M^2 = \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}.$$

It turns out that every entry in row i and column j is the number of distinct paths that can be drawn from team i to team j of length 2. Consider teams A and C. A and C never played each other directly, thus a zeros appeared in entries $(1, 3)$ and $(3, 1)$ of matrix M. However, we can say that C is a second generation of A. If we examine Figure 3, we can see that A can be linked to C in two distinct ways: (A $\rightarrow$ D $\rightarrow$ C and A $\rightarrow$ B $\rightarrow$ C). Looking in entries $(1, 3)$ and $(3, 1)$ of matrix M^2 we find the number 2, representing these two distinct paths. Other powers of the matrix M exhibit these same properties, where in the entry (i, j) of matrix M^n is the number of times that team j appears as a nth-generation opponent of team i. Next, let us define the vector

S, where the coordinates of S represent the net point difference of each team (the first entry represents A, the second entry represents B, etc.).

$$S = \begin{bmatrix} 7 \\ 8 \\ -10 \\ -5 \end{bmatrix}$$

We can calculate our first-generation ratings by multiplying S by the identity matrix M^0 and dividing by the total number of games played by each team (three games including the game played against themselves).

$$\frac{1}{3}M^0 \cdot S = \frac{1}{3}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 8 \\ -10 \\ -5 \end{bmatrix} = \begin{bmatrix} 2.33 \\ 2.67 \\ -3.33 \\ -1.67 \end{bmatrix}$$

These are our first-generation ratings. Now assume we want to calculate the second-generation ratings. Let's first look at our tree for the second-generation opponents of B. To find the second generation rating for B, we add up the net points for B for all nine second-generation games and divide by nine.

$$\frac{(5+0)+(5-5)+(5+12)+(0+5)+(0+0)+(0+3)+(3-3)+(3+0)+(3-7)}{9}$$

$$= \frac{31}{9} = 3.22$$

Let's take another look at the scores above. If we regroup the terms, we get:

$$\frac{5-0+3}{3} + \frac{(-5+5+12+0+0-3+3+0+3-7)}{9}$$

$$= 2.67 + 0.55 = 3.22$$

Notice that 2.67 is the first-generating rating for team B. If we also look at the second-generation scores for the other teams, we see that they can also be regrouped in the same fashion. It turns out that this second number (0.55) is the first coordinate in the vector resulting from the multiplication of M^1 and S.

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 8 \\ -10 \\ -5 \end{bmatrix} = 0.55$$

By repeating this process for each of the teams, we see that the ratings also come stem from the multiplication of M^1 and S. Thus we can use our first-generation matrix M^1 and the point difference vector S to calculate our second generation ratings. We can say that the second-generation ratings are equal to

$$\frac{1}{3}M^0 \cdot S + \frac{1}{3^2}M^1 \cdot S.$$

Similarly, we can write the nth-generation ratings as

$$\sum_{j=1}^{n} \frac{1}{3}\left(\frac{M}{3}\right)^{j-1}.$$

Now the question is, does this sum approach a limit? If we calculate the third-generation rating for B, we find that the third term ($\frac{1}{27}M^2 \cdot S$) has a value of .296. This is smaller than either of the first two terms (2.67 and .55), which leads us to believe that a limit could exist.

Using a proof of the Spectral Theorem for Symmetric Matrices (see Appendix) and a proof involving the eigenvalues of Markov matrices (see Appendix), we can show that a limit does indeed exist.

Our proof of the Spectral Theorem shows that any real symmetric matrix can be rewritten in terms of its eigenvalues. Since every tournament matrix is symmetric (we must look at the tournament matrix and convince ourselves of this point), we can find its eigenvalues and corresponding eigenvectors. Consider the first generation matrix M from our example.

$$\frac{M}{3} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Notice that M has been normalized (divided by the number of games played). This ensures that the eigenvalues will all be less than or equal to one (see Markov Matrices in the Appendix). Solving for the characteristic equation of $\frac{M}{3}$, we find that the eigenvalues are 1, 1/3, 1/3, and -1/3 (we can see that 1/3 has a multiplicity of 2). Because $\frac{M}{3}$ is symmetric, we can then use these eigenvalues to find corresponding eigenvectors that form an linearly independent orthonormal set. Doing this, we find the following set of orthonormal eigenvectors:

$$\overrightarrow{v_0} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \overrightarrow{v_1} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \overrightarrow{v_2} = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \text{ and } \overrightarrow{v_3} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

The vectors $\overrightarrow{v_0},\ \overrightarrow{v_1},\ \overrightarrow{v_2}$, and $\overrightarrow{v_3}$ correspond to the eigenvalues $1,\ 1/3,\ 1/3$, and $-1/3$, respectively. Because these eigenvectors are linearly independent, they form a basis. This allows us to express S as a linear combination of $\overrightarrow{v_0},\ \overrightarrow{v_1},\ \overrightarrow{v_2}$, and $\overrightarrow{v_3}$. To find the coefficients for the vectors in this linear combination, we can compute the scalar product $S \cdot \overrightarrow{s_i}$ for $i = 0, 1, 2, 3$. This will give us the coefficient for each of the vectors.

$$\begin{aligned} S = \begin{bmatrix} 7 \\ 8 \\ -10 \\ -5 \end{bmatrix} &= (S \cdot \overrightarrow{v_0})\overrightarrow{v_0} + (S \cdot \overrightarrow{v_1})\overrightarrow{v_1} + (S \cdot \overrightarrow{v_2})\overrightarrow{v_2} + (S \cdot \overrightarrow{v_3})\overrightarrow{v_3} \\ &= 0 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} - \frac{17}{\sqrt{2}} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} - \frac{13}{\sqrt{2}} \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} + 3 \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{17}{2} \\ 0 \\ -\frac{17}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{13}{2} \\ 0 \\ -\frac{13}{2} \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \\ \frac{3}{2} \end{bmatrix} \end{aligned}$$

Notice that these three vectors are still eigenvectors of $\frac{M}{3}$ corresponding to eigenvalues of 1/3, 1/3, and -1/3 respectively. We will name these new vectors $\overrightarrow{s_1},\ \overrightarrow{s_2}$, and $\overrightarrow{s_3}$. These new vectors also remain eigenvectors for $\left(\frac{M}{3}\right)^j$ corresponding to eigenvalues $\left(\frac{1}{3}\right)^j,\ \left(\frac{1}{3}\right)^j$, and $\left(-\frac{1}{3}\right)^j$ respectively, when j is a positive integer. We can then write the series $\sum_{j=1}^{n} \frac{1}{3}\ \left(\frac{M}{3}\right)^{j-1} \cdot S$ and use the properties of series to break it up into three different series.

$$\begin{aligned}
\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{3}\,\left(\frac{M}{3}\right)^{j-1}\cdot S &= \frac{1}{3}\lim_{n\to\infty}\sum_{j=1}^{n}\left(\frac{M}{3}\right)^{j-1}\cdot(\overrightarrow{s_1},\overrightarrow{s_2},\overrightarrow{s_3}) \\
&= \frac{1}{3}\lim_{n\to\infty}\sum_{j=0}^{n-1}\left[\left(\frac{1}{3}\right)^{j}\overrightarrow{s_1}+\left(\frac{1}{3}\right)^{j}\overrightarrow{s_2}+\left(-\frac{1}{3}\right)^{j}\overrightarrow{s_3}\right] \\
&= \frac{1}{3}\sum_{j=0}^{\infty}\left(\frac{1}{3}\right)^{j}\overrightarrow{s_1}+\frac{1}{3}\sum_{j=0}^{\infty}\left(\frac{1}{3}\right)^{j}\overrightarrow{s_2}+\frac{1}{3}\sum_{j=0}^{\infty}\left(-\frac{1}{3}\right)^{j}\overrightarrow{s_3}
\end{aligned}$$

Note that we now have three separate geometric series. Here we can use a property of geometric series to find the limit as n goes to infinity: $\lim_{n\to\infty}\sum_{j=0}^{n} ar^j = \frac{a}{1-r}$, whenever $|r| < 1$. Thus,

$$\begin{aligned}
\lim_{n\to\infty}\sum_{j=1}^{n}\frac{1}{3}\left(\frac{M}{3}\right)^{j-1}\cdot S &= \frac{1}{3}\left(\frac{1}{1-\frac{1}{3}}\right)\overrightarrow{s_1}+\frac{1}{3}\left(\frac{1}{1-\frac{1}{3}}\right)\overrightarrow{s_2}+\frac{1}{3}\left(\frac{1}{1+\frac{1}{3}}\right)\overrightarrow{s_3} \\
&= \frac{1}{2}\overrightarrow{s_1}+\frac{1}{2}\overrightarrow{s_2}+\frac{1}{4}\overrightarrow{s_3} \\
&= \frac{1}{2}\begin{bmatrix}\frac{17}{2}\\ 0\\ -\frac{17}{2}\\ 0\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0\\ \frac{13}{2}\\ 0\\ -\frac{13}{2}\end{bmatrix}+\frac{1}{4}\begin{bmatrix}-\frac{3}{2}\\ \frac{3}{2}\\ -\frac{3}{2}\\ \frac{3}{2}\end{bmatrix} \\
&= \begin{bmatrix}3.875\\ 3.625\\ -4.625\\ -2.875\end{bmatrix}
\end{aligned}$$

Thus we can see that our ratings do indeed approach a limit as all n-generation games are taken into consideration when n increases without bound.

Various Methods

Before we discuss an application of our system, we will first examine some of the different ways by which we can approach schedule strength. The three different methods that we will use to look at involving schedule strength are the Point-Difference (this method was used in the illustration), Point-Ratio, and Number of Wins. To explain these concepts, we will use the results of the 4-team tournament we have already discussed.

Wins. We begin by looking at by far the simplest method. This approach looks only at the number of victories of each team and does not take into account the point difference or point ratio of any game in the tournament. To determine the overall dominance of a team, we must adjust each teams number of wins such that the total number of wins is zero. To do this, we subtract one half of the total number of games each team plays. For example, if a team wins 7 of the 10 games they play, then their dominance is $7 - 5 = 2$. As another example, let us determine the dominance for each team in our 4-team illustration:

Teams	*# of Wins*	*Adjusted # of Wins*
A	1	0
B	2	1
C	0	-1
D	1	0

Notice that the relations between the teams remain the same and that the average number of wins has been shifted. Also note that this method only affects the ratings after schedule calculation is taken into account, which we will explore later.

Point-Difference. Next we will look at a more familiar method: the Point-Difference Method. This is the approach used in the 4-team tournament that we examined previously. When using the Point-Difference Method, we used the point-difference in a game to determine one team's dominance over the other. For example, if team B beats team A by a score of 10-5, then we can say that the dominance of team B over team A is 5. Similarly, we can say that the dominance of team A over team B is -5. The individual game dominances of each team are then added together to determine a team's overall dominance.

Like every method, the Point-Difference Method has its advantages and disadvantages. While this approach is a simple way to determine who has the best offensive/defensive statistics, it focuses entirely on the number of points a team scores. This system neglects to consider whether the points were scored against a strong team or a weak team, or whether a game resulted in a victory or a loss. Thus it gives an advantage to those teams who have a large margin of victory in one or two games, while giving those who win every game by a small margin a disadvantage. Due to these weaknesses in the system, a team can lose most of their games by one or two points and win one game by fifty points and still be considered a good team.

Point-Ratio. In an attempt to correct some of the weaknesses of the Point-Difference Method, the final method we will examine involves the ratio of two teams' scores. In the Point-Ratio Method, we look at the number of points a team scores with respect to the total number of points scored in a game. Consider the game in which team B beats team A by a score of 10-5. To determine the dominance of B over A, we take the number of points B scored (10 pts) and divide it by the total number of points scored in the game (15 pts). We then subtract 0.5 so that the sum of the dominances adds to zero.

$$\text{Team B: 10 pts} \qquad \text{Total: 15 pts} \qquad \Rightarrow \qquad \frac{10}{15} - 0.5 = 0.1667$$

Thus the dominance of team B over team A is 0.1667 and the dominance of team A over team B is -0.1667. Similarly to the Point-Difference Method, all of the dominances of each team are added together to determine a team's overall dominance.

While this method still does not take into account whether a team obtained a victory, it does address some of the weaknesses of the Point-Difference Method. Notice that this approach has a maximum dominance that can be achieved (0.5). This makes the games that have a large point differential much less influential in the final ratings than with the Point-Difference Method, but more influential in the final rankings than with the Win Method.

Another aspect of this method that must be considered is the higher importance that is placed on scoring in a low-scoring game. Since the dominance calculation depends on the total number of points scored in a game, a touchdown is more valuable in low-scoring games. This calculation translates logically to a real game situation. For example, it is apparently much more difficult to score a touchdown in a low-scoring defensive game than in a high-scoring offensive game. Thus we can logically conclude that a touchdown in a low-scoring game is much more valuable than a touchdown in a high-scoring game.

An Application

Having determined that schedule strength rankings do approach a limit, and having examined the different ways that we can rate the teams, we can now apply our system to a real life situation. In this section, we will apply the concepts that have been discussed to the Big 12 Conference results from the 2007 regular season.[2]

Consider Table 3. Each row shows the individual game results for the team corresponding to that row. In each of these entries, the score of the team corresponding to that row appears first and the score of their opponent second. Note that each team does not play each of the

Table 3. Big 12 Results

	Mis	Kan	Col	KanSt	Neb	IowaSt	Okl	Tex	OklSt	TexTech	TexA&M	Bay
Missouri	0	36-28	55-10	49-32	41-6	42-28	31-41	-	-	41-10	40-26	-
Kansas	28-36	0	19-14	30-24	76-39	45-7	-	-	43-28	-	19-11	58-10
Colorado	10-55	14-19	0	20-47	65-51	28-31	27-24	-	-	31-26	-	43-23
Kansas St	32-49	24-30	47-20	0	31-73	20-31	-	41-21	39-41	-	-	51-13
Nebraska	6-41	39-76	51-65	73-31	0	35-17	-	25-28	14-45	-	14-36	-
Iowa St	28-42	7-45	31-28	31-20	17-35	0	7-17	3-56	-	17-42	-	-
Oklahoma	41-31	-	24-27	-	-	17-7	0	28-21	49-17	27-34	42-14	52-21
Texas	-	-	-	21-41	28-25	56-3	21-28	0	38-35	59-43	30-38	31-10
Oklahoma St	-	28-43	-	41-39	45-14	-	17-49	35-38	0	49-45	23-24	45-14
Texas Tech	10-41	-	26-31	-	-	42-17	34-27	43-59	45-49	0	35-7	38-7
Texas A&M	26-40	11-19	-	-	36-14	-	14-42	38-30	24-23	7-35	0	34-10
Baylor	-	10-58	23-43	13-51	-	-	21-52	10-31	14-45	7-38	10-34	0

other teams, and that a score of zero appears in the game where the team plays itself.

Wins Application. First we will use the Wins method to calculate the schedule strength ratings for the Big 12 Conference. Let us examine Table 4.

TABLE 4. Schedule Strength: Wins

	Record	Wins-4	Ranking	SS Rating	SS Ranking
Missouri	7-1	3	1	3.4323	1
Kansas	7-1	3	1	2.7344	2
Oklahoma	6-2	2	3	2.0351	3
Texas	5-3	1	4	0.3202	4
Oklahoma St	4-4	0	5	-0.2018	8
Colorado	4-4	0	5	-0.0487	7
Texas Tech	4-4	0	5	0.0351	6
Texas A&M	4-4	0	5	0.3202	4
Kansas St	3-5	-1	9	-1.2656	9
Nebraska	2-6	-2	10	-1.7846	11
Iowa St	2-6	-2	10	-1.5677	10
Baylor	0-8	-4	12	-4.0089	12

In this table, we have ranked the teams based on their win-loss records; tiebreakers were not used for teams with equal records. Here, each team plays a total of eight games. Thus, in order to make the overall number of wins equal to zero, we must subtract $\frac{8}{2} = 4$ from each team's win total (see "Wins-4" in Table 4). Since the net number of wins now equals zero, we can use "Wins-4" as the point vector S from our 4-team example and calculate the schedule strength ratings for this method (see "SS Rating" in Table 4).

Notice that the schedule strength ratings have broken the ties between teams with equal records. Let us look at an example: Colorado has a record of 4-4, including notable victories over Texas Tech and Oklahoma. Oklahoma State also has a record of 4-4; however, of the four victories, only one was against a team with a record of 4-4 or better. We can see that the relative difficulties of these schedules are reflected in the final ratings; Colorado has a higher rating than Oklahoma State. Additionally, Colorado's 4-4 performance with a difficult schedule gives Colorado a schedule strength rating equal to that of Texas, who has a record of 5-3. In examining Table 4, we can see that the schedule strength calculation broke the ties between other teams with the same records as well. Thus the final ratings are determined not only by a team's record, but by a team's record with respect to records of its opponents (e.g. the strength of its schedule).

Point-Difference Application. Now we will apply the Point-Difference method to the Big 12 Conference. In Table 5, each entry represents the point-difference in the game played between the teams corresponding to that row and column. A positive entry signifies a victory for the team corresponding to that row, and a negative entry signifies a loss for that team.

Each of the entries in Table 5 represents one team's dominance over another team. For example, the first entry in the row corresponding to Nebraska has the number -35, implying that Nebraska lost to Missouri (the corresponding column) by 35 points. Thus Nebraska has a dominance of -35 over Missouri and Missouri has a dominance of 35 over Nebraska. By adding up the entries in each row, we can find the overall dominance for each team. These results appear in the "Point-Difference" column of Table 6.

Having calculated the overall dominance for each team, we can calculate the schedule strength ratings; these ratings appear in the "SS Rating" column of Table 6. For this case study, the schedule only changes the the rankings of Kansas State and Texas A & M.

First let us consider Kansas State and Texas A&M. While the initial dominances have Kansas State ahead by a 30-point margin, the schedule strength ratings place Texas A & M ahead by nearly a 3-point margin. This result implies that the teams performed at similar levels with respect to their schedules despite the initial point discrepancy. A similar result can be seen with the ratings for Texas and Texas Tech.

Next let us examine the dominances of Missouri and Kansas in order to gain a deeper understanding of why the ratings change in the manner that they do. From the initial ratings, the two teams appear to perform at similar levels. However, the schedule strength calculation puts Missouri far ahead of Kansas. Why is this?

First, consider the schedule of Missouri. Missouri had notable victories over Kansas (8pts), Texas Tech (31pts), and Kansas State (17pts). Furthermore, each Missouri victory (with the exception of the victory over Kansas) was by a margin of at least 14 points, including games with a margin of victory of over 30 points. On the other hand, Kansas played only one game against a top-5 team: an 8-point loss to Missouri. Although Kansas did win their remaining games, 123 of their 149 points in their overall dominance can be accounted for in the games

Table 5. Big 12 Game Results: Point-Difference

	Mis	Kan	Col	KanSt	Neb	IowaSt	Okl	Tex	OklSt	TexTech	TexA&M	Bay
Missouri	0	8	45	17	35	14	-10	-	-	31	14	-
Kansas	-8	0	5	6	37	38	-	-	15	-	8	48
Colorado	-45	-5	0	-27	14	-3	3	-	-	5	-	20
Kansas St	-17	-6	27	0	-38	-11	-	20	-2	-	-	38
Nebraska	-35	-37	-14	42	0	18	-	-3	-31	-	-22	-
Iowa St	-14	-38	3	11	-18	0	-10	-53	-	-25	-	-
Oklahoma	10	-	-3	-	-	10	0	7	32	-7	28	31
Texas	-	-	-	-20	3	53	-7	0	3	16	-8	21
Oklahoma St	-	-15	-	2	31	-	-32	-3	0	4	-1	31
Texas Tech	-31	-	-5	-	-	25	7	-16	-4	0	28	31
Texas A&M	-14	-8	-	-	22	-	-28	8	1	-28	0	24
Baylor	-	-48	-20	-38	-	-	-31	-21	-31	-31	-24	0

Table 6. Strength of Schedule: Point-Difference

	Point-Difference	Ranking	SOS Rating	SOS Ranking
Missouri	154	1	175.64	1
Kansas	149	2	130.86	2
Oklahoma	108	3	101.60	3
Texas	61	4	31.54	4
Texas Tech	35	5	28.60	5
Oklahoma St	17	6	15.23	6
Kansas St	7	7	-7.14	8
Texas A&M	-23	8	-4.46	7
Colorado	-38	9	-42.87	9
Nebraska	-80	10	-67.63	10
Iowa St	-144	11	-118.36	11
Baylor	-244	12	-243.02	12

played against the three worst teams in the league (see Table 5). Additionally, three of their four other victories were by margins of less than ten points.

Upon closer examination, we can see that although the performances of the two teams appear similar in the initial dominance ratings, the performances are actually quite different. The difference in performance between Kansas and Missouri, with respect to the strength of their schedules, is the reason that Missouri has a schedule strength rating of 175.64 and Kansas has a schedule strength rating of 130.86. This same reasoning explains why the schedule strength ratings appear as they do in Table 6.

Point-Ratio Application. Let us now apply the Point-Ratio Method to the Big 12 Conference. The primary reason behind using a point-ratio rather than a point-difference is to minimize the effect of "landslide" victories on the ratings. The term "landslide" describes games in which one team wins by a very large margin. For example, we can call the 58-10 victory of Kansas over Baylor a "landslide" victory. This method minimizes these effects by creating a maximum dominance that can be earned from a single game.

A secondary reason to use this method is to give increased value to points scored in a low-scoring game. Consider the game in which Kansas defeats Texas A&M 19-11. In this game, it was apparently much more difficult to score a touchdown than in the 65-51 Colorado victory over Nebraska; thus a touchdown should be perceived as more valuable.

Table 7 shows the individual game dominances using the Point-Ratio Method. Take a moment to compare some of the values of Table 7 with those of the Point-Difference Method in Table 5 and with the game results in Table 3. Note how some games have similar point-ratio dominances, while their corresponding point-difference dominances vary greatly. Also note how some games with similar point-ratio dominances have much different game scores.

Using these game results we can calculate the overall dominance for each team and the corresponding schedule strength rating. These values appear in Table 8.

Unlike the application of schedule strength to the Point-Difference Method, the application of schedule strength to the Point-Ratio Method changed the rankings of several teams. The initial sixth and seventh ranked teams (Kansas State and Oklahoma State respectively) switched rankings, the fourth and fifth ranked teams (Texas and Texas Tech respectively) switched rankings, and Oklahoma overtook Kansas for the number two ranking.

Let us first consider the change in rankings between Texas and Texas Tech. Recall that the schedule strength calculation for the Point-Difference Method placed these two teams within a few points of each other. With this method, the Point-Ratio ratings of the two teams are initially very close; however, the schedule strength calculation places Texas Tech ahead of Texas by a notable margin of .1557. An examination of the Point-Difference results for Texas in Table 5 shows that a majority of their point-difference dominance was achieved in their 53-point victory of Iowa State and their 16-pt victory in a high-scoring game against Texas Tech. The limit that the Point-Ratio Method places on the dominance a team can earn from a single game (0.5) hurt Texas in their game against Iowa State, while a point-ratio instead of a point-difference hurt their ratings in the game against Texas Tech. However, Texas Tech had two large victories in two relatively low-scoring games and lost two games by five point or less, which helped them surpass Texas in the schedule strength rankings.

We now look at the situation between Kansas and Oklahoma. Recall that Kansas had an initial point-difference dominance of 149 and that Oklahoma had an initial point-difference dominance of 108; the two teams were separated by 41 points. However, looking at the initial point-ratio dominances we can see that Kansas and Oklahoma are separated by a much smaller margin: Kansas (1.1867), Oklahoma (1.0996). This observation shows that Oklahoma is already benefiting from the use of the Point-Ratio Method. Now let us examine the

Table 7. Big 12 Game Results: Point-Ratio

	Mis	Kan	Col	KanSt	Neb	IowaSt	Okl	Tex	OklSt	TexTech	TexA&M	Bay
Missouri	0	.0625	.3462	.1049	.3723	.1000	.0694	-	-	.3039	.1061	-
Kansas	-.0625	0	.0758	.0556	.1609	.3654	-	-	.1056	-	.1333	.3529
Colorado	-.3462	-.0758	0	-.2015	.0603	-.0254	.0294	-	-	.0439	-	.1515
Kansas St	-.1049	-.0556	.2015	0	-.2019	-.1078	-	.1613	-.0125	-	-	.2969
Nebraska	-.3723	-.1609	-.0603	.2019	0	.1731	-	-.0283	-.2627	-	-.2200	-
Iowa St	-.1000	-.3654	.0254	.1078	-.1731	0	-.2083	-.4492	-	-.2119	-	-
Oklahoma	.0694	-	-.0294	-	-	.2083	0	.0714	.3750	-.0574	.2500	.2123
Texas	-	-	-	-.1613	.0283	.4492	-.0714	0	.0205	.0784	-.0588	.2561
Oklahoma St	-	-.1056	-	.0125	.2627	-	-.3750	-.0205	0	.0213	-.0106	.2627
Texas Tech	-.3039	-	-.0439	-	-	.2119	.0574	-.0784	-.0213	0	.3333	.3444
Texas A&M	-.1061	-.1333	-	-	.2200	-	-.25	.0588	.0106	-.3333	0	.2727
Baylor	-	-.3529	-.1515	-.2969	-	-	-.2132	-.2561	-.2627	-.3444	-.2727	0

TABLE 8. Strength of Schedule: Point Ratio

	Ratio Difference	Ranking	SOS Rating	SOS Ranking
Missouri	1.3265	1	1.5226	1
Kansas	1.1867	2	0.9834	3
Oklahoma	1.0996	3	1.062	2
Texas	0.541	4	0.3062	5
Texas Tech	0.4995	5	0.4619	4
Kansas St	0.1770	6	0.018	7
Oklahoma St	0.0475	7	0.048	6
Texas A&M	-0.2606	8	-0.0929	8
Colorado	-0.3638	9	-0.3928	9
Nebraska	-0.7292	10	-0.6631	10
Iowa St	-1.3747	11	-1.1343	11
Baylor	-2.1495	12	-2.119	12

schedule strength ratings. While the schedule strength calculation appears to hurt both teams' ratings, Kansas is hurt far worse. Despite similar performances by Kansas and Oklahoma against their respective opponents, the 'weak' schedule of Kansas allows Oklahoma to move to number two. Had Kansas played one or two of the other top-5 teams and performed slightly worse, it is likely that their schedule strength rating would be significantly higher. Fortunately for Oklahoma, their schedule included games against Missouri, Texas, and Texas Tech.

As noted in the beginning of the section, the ratings of teams with "landslide" victories were heavily influenced by the Point-Ratio Method. Additionally, teams that had a large margin of victory in a high-scoring games were affected. Lastly it is important to note that a team's schedule strength ranking is influenced by the performances of the teams it plays; these performances have a greater influence in the Point-Ratio Method.

CONCLUSION

While Redmond's method is a good, basic model for schedule strength, there are several factors that limit its application. The most prominent of these factors is the necessity that the tournament be connected. A connected tournament implies that each team can eventually be linked to every other team in the league when the *nth*-generation games are taken into account. If a tournament is not connected, there is no way to link and compare two non-connected teams, and our method does not work. However, most professional sports tend to be connected tournaments.

The second limiting factor is that each team must play the same number of games. If the teams do not all play the same number of games, it becomes extremely difficult to calculate the schedule strength ratings and our calculation method must be altered. Not only do these calculations become difficult, but they also become skewed as the ratings of the teams that play less games are focused on the performances of fewer teams. However, Redmond proposes a solution to this problem. Consider a tournament in which team A plays four games and team B plays five games. In order for both teams to play the same number of games, we will assume that team A plays itself twice instead of once when establishing the schedule strength matrix. Thus it is still possible to calculate the schedule strength ratings for a tournament in which teams play a varying number of games.

The last and possibly most prominent weakness that we will note is the model's inability to distinguish a team's performance against other specific teams. In this model, only a team's overall dominance and the other teams with which that team competes influence the final ratings. It is not important if team A defeats the best team in the league by 20 points and loses to the worst team in the league by 5 points; it only matters that team A scored a total of 15 more points than its opponents. In this sense, points that are scored against good teams are weighted equally with those scored against weak teams. Thus a team that does play the most competitive teams and yet performs well, such as Kansas in our application, suffers in the final schedule strength ratings.

While every sports fanatic and statistician nationwide has a distinct method of ranking college football teams, none of them seems simple. Some of them include complex formulas, others consider factors such as home-field advantage and when losses and victories occur. The method that we have examined in this paper focuses solely on a team's overall performance and the performances of the teams it played. Although Redmond's model for schedule strength chooses to disregard "extra" factors and is conceptually simple, its application provides a logical manner in which to rank the teams. This simplicity also provides those less knowledgeable in the field of sports the opportunity to grasp the idea of schedule strength. While our method for calculating schedule strength may only change one or two final rankings, when a national title is on the line, that change can make all of the difference.

Appendix

Definitions

In this section, we will define some terms necessary to prove the Spectral Decomposition Theorem For Symmetric Matrices (Theorem 8) and the eigenvalue proof regarding Markov matrices (Theorem 9). The following definitions are taken from Lay's book on linear algebra. [4]

Definition 1 (Probability Vector)**.** *A vector with nonnegative entries that add up to one is called a probability vector.*

$$\text{Example: } \begin{bmatrix} .1 \\ .3 \\ .6 \end{bmatrix}$$

Definition 2 (Unit Vector)**.** *A vector* $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ *such that* $v_1^2 + \cdots + v_n^2 = 1$.

Definition 3 (Dot Product)**.**

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} {}_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 u_1 + v_2 u_2 + \cdots + u_n v_n$$

Definition 4 (Real Matrix)**.** *A matrix with all real number entries.*

Definition 5 (Stochastic Matrix)**.** *A stochastic matrix is a square matrix whose columns are probability vectors.*

$$\text{Example: } \begin{bmatrix} .1 & .5 & .2 \\ .3 & .4 & .4 \\ .6 & .1 & .4 \end{bmatrix}$$

Definition 6 (Regular Stochastic Matrix)**.** *A stochastic matrix P is called regular if some power of P, P^k, contains only strictly positive entries.*

Definition 7 (Steady-state Vector)**.** *If P is a stochastic matrix, then a steady-state vector for P is a probability vector $\mathbf{q}$ such that*

$$P\mathbf{q} = \mathbf{q}.$$

Definition 8 (Markov Chain). *A Markov chain is a sequence of probability vectors* $\mathbf{x_0},\ \mathbf{x}_1,\ \mathbf{x}_2, ...$ *together with a stochastic matrix* P, *such that*

$$\mathbf{x}_1 = P\mathbf{x}_0,\ \mathbf{x}_2 = P\mathbf{x}_1,\ \mathbf{x}_3 = P\mathbf{x}_2, ...$$

Thus the Markov chain is described by the equation $\mathbf{x}_{k+1} = P\mathbf{x}_k$ *for* $k = 0,\ 1,\ 2, ...$ *When a Markov chain of vectors in* $\mathbb{R}^n$ *describes a system or sequence of experiments, the entries in* $\mathbf{x}_k$ *list the probabilities that the system is in each of* n *possible states, or the probabilities that the outcome of the experiment is one of the* n *possible outcomes. For this reason,* $\mathbf{x}_k$ *is often called a state vector.*

Definition 9 (Orthogonality). *Two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal to each other if* $\mathbf{u} \cdot \mathbf{v} = 0$.

Definition 10 (Orthogonal Complement). *If a vector* $\mathbf{z}$ *is orthogonal to every vector in a subspace* W *of* $\mathbb{R}^n$, *then* $\mathbf{z}$ *is said to be orthogonal to* W. *The set of all vectors* $\mathbf{z}$ *that are orthogonal to* W *is called the orthogonal complement of* W *and is denoted by* $W^{\perp}$.

Definition 11 (Span$\{\mathbf{v}_1, ...\mathbf{v}_p\}$). *If* $\mathbf{v}_1, ...\mathbf{v}_p$ *are in* $\mathbb{R}^n$, *then the set of all linear combinations of* $\mathbf{v_1}, ..., \mathbf{v}_p$ *is denoted by* $Span\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ *and is called the subset of* $\mathbb{R}^n$ *spanned by* $\mathbf{v}_1, ...\mathbf{v}_p$. *That is,* $Span\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ *is the collection of all vectors that can be written in the form*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_p\mathbf{v}_p$$

with $c_1, ..., c_p$ *scalars.*

Definition 12 (Orthogonal Set). *A set* S *is an orthogonal set if each vector in* S *is orthogonal to each of the other vectors in* S.

Definition 13 (Orthonormal Set). *A set* $\mathbf{u}_1, ..., \mathbf{u}_p$ *is an orthonormal set if it is an orthogonal set of unit vectors.*

Definition 14 (Orthonormal Basis). *If* W *is the subspace of* $\mathbb{R}^n$ *spanned by an orthonormal set* $\mathbf{u}_1, ..., \mathbf{u}_p$, *then* $\mathbf{u}_1, ..., \mathbf{u}_p$ *is also an orthonormal basis for* W *because the set is automatically linearly independent by the theorem for orthogonal bases.*

Definition 15 (Orthogonal Matrix). *An orthogonal matrix is a square invertible matrix* U *such that* $U^{-1} = U^T$.

Definition 16 (Symmetric Matrix). *A symmetric matrix is a matrix* A *such that* $A^T = A$. *Such a matrix is necessarily square.*

Example: $\quad$ Symmetric $\rightarrow \begin{bmatrix} 1 & -3 & 0 \\ -3 & 5 & 6 \\ 0 & 6 & 7 \end{bmatrix}$

Non symmetric $\rightarrow \begin{bmatrix} 7 & 9 & -6 \\ 1 & -2 & 1 \\ 2 & 4 & -5 \end{bmatrix}$

Theorems

In this section, we will identify some basic theorems necessary to prove the Spectral Decomposition Theorem For Symmetric Matrices (Theorem 8) and the eigenvalue proof regarding Markov matrices (Theorem 9). The following theorems are cited and proved in Lay's book on linear algebra. [4]

Theorem 1. *If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector $\mathbf{q}$. Further, if $\mathbf{x}_0$ is any initial state and $\mathbf{x}_{k+1} = P\mathbf{x}_k$ for $k = 1, 2, ...$, then the Markov chain $\mathbf{x}_k$ converges to $\mathbf{q}$ as $k \rightarrow \infty$.*

Theorem 2 (The Basis Theorem)**.** *Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is a basis for H. Also, any set of p elements of H that spans H is a basis for H.*

Theorem 3 (Orthogonal Basis)**.** *If $S = \{\mathbf{u}_1, ..., \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence a basis for the subspace spanned by S.*

Theorem 4 (Square Orthogonal Matrix)**.** *A square matrix with orthonormal columns is an orthogonal matrix. Such a matrix also has orthonormal rows.*

Theorem 5 (Orthogonal Decomposition Theorem)**.** *Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form*

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$. In fact, if $\mathbf{u}_1, ..., \mathbf{u}_p$ is any orthogonal basis of W, then

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + ... + \frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}\mathbf{u}_p$$

and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$. Here the vector $\hat{\mathbf{y}}$ is called the orthogonal projection of $\mathbf{y}$ onto W and is often denoted as $proj_W\mathbf{y}$.

Theorem 6. *If / $\mathbf{u}_1, ..., \mathbf{u}_p$ / is an orthonormal basis for a subspace W of $\mathbb{R}^n$, then*

$$proj_W \mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2)\mathbf{u}_2 + ... + (\mathbf{y} \cdot \mathbf{u}_p).$$

If $U = [\mathbf{u}_1 \ \mathbf{u}_2 \ ... \ \mathbf{u}_p]$, then $proj_W \mathbf{y} = UU^T \mathbf{y}$ for all $\mathbf{y}$ in $\mathbb{R}^n$.

Theorem 7. *If A is symmetric, then any two eigenvectors from different eigenspaces have a dot product equal to zero.*

The Spectral Decomposition Theorem for Symmetric Matrices

The following proof was written with the guidance of Patrick Keef. [3]

Theorem 8 (The Spectral Decomposition Theorem for Symmetric Matrices)**.** *Given a symmetric $n \times n$ matrix S with real coefficients, there exists an orthonormal matrix U such that $U^T SU$ is a diagonal matrix.*

We will proceed using the methods of induction. In Lemma 1 we will prove the base cases $n = 1$ and $n = 2$.

Lemma 1. *Any symmetric 1×1 or 2×2 matrix is orthogonally diagonalizable.*

Proof. We begin with the 1×1 case. Let $S = [x]$, and let U be the orthogonal matrix $U = [1]$. Then $U^T SU = [1][x][1] = [x]$. Since $[x]$ is a diagonal matrix, $U^T SU$ is also a diagonal matrix.

Next we consider the 2×2 case. Let S be a symmetric matrix $S = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$. To show that S is orthogonally diagonalizable, we begin by solving for the eigenvectors of S by subtracting λ times the identity matrix from the matrix S,

$$S - \lambda I = \begin{bmatrix} x - \lambda & y \\ y & x - \lambda \end{bmatrix}.$$

We find that the characteristic equation of S is $\lambda^2 - 2x\lambda + (x^2 - y^2) = 0$. Using the quadratic formula, we discover that the eigenvalues are $\lambda = x \pm y$. Using these eigenvalues, we find the corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for eigenvalues $\lambda = x + y$ and $\lambda = x - y$ respectively. We can now create an orthogonal matrix, U, by normalizing the eigenvectors. Then $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $U^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. We can now show that U^TSU is a diagonal matrix:

$$\begin{aligned} U^TSU &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x & y \\ y & x \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}}(x-y) & \frac{1}{\sqrt{2}}(y-x) \\ \frac{1}{\sqrt{2}}(x+y) & \frac{1}{\sqrt{2}}(x+y) \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}(2x-2y) & 0 \\ 0 & \frac{1}{2}(2x+2y) \end{bmatrix} \\ &= \begin{bmatrix} x-y & 0 \\ 0 & x+y \end{bmatrix} \end{aligned}$$

The resulting matrix is indeed a diagonal. Thus any 2×2 symmetric matrix S is orthogonally diagonalizable. ☐

We will now assume that $n > 2$. The following steps will use the knowledge that any 1×1 or 2×2 symmetric matrix can be orthogonally diagonalized to prove that any symmetric $n \times n$ matrix can indeed be orthogonally diagonalized as well.

Definition 1. *A function f is self-adjoint if for all column vectors $v, w \in \mathbf{R}^n$, $f(v) \cdot w = v \cdot f(w)$.*

Lemma 2. *If S is a symmetric matrix and x is a column vector, the function f(x)=Sx is self-adjoint.*

Proof. Let v and w be column vectors in $\mathbf{R}^n$. Then

$$\begin{aligned} f(v)\cdot w &= Sv \cdot w \\ &= (Sv)^T w \\ &= v^T S^T w \\ &= v^T S w \\ &= v \cdot Sw \\ &= v \cdot f(w) \end{aligned}$$

Thus we have shown that if $f(v) = Sv$ where S is symmetric, then f is "self-adjoint". ☐

Definition 2. *A subspace V is called f-invariant if for all $v \in V$, $f(v) \in V$.*

For Lemma 3, suppose $f(x) = Sx$ for some $n \times n$ symmetric matrix S.

Lemma 3. *If V is f-invariant, then $V^{\perp} = \{w \in \mathbf{R}^n : v \cdot w = 0$ for all $v \in V\}$, is an f-invariant subspace of $\mathbf{R}^n$.*

Proof. Suppose $w \in V^{\perp}$ and $v \in V$. Then $v \cdot f(w) = f(v) \cdot w = u \cdot w$ for some $u \in V$. Since $u \cdot w = 0$ for all $u \in V$, it follows that $v \cdot f(w) = 0$ and $f(w) \in V^{\perp}$. □

Using the same notation used in Lemma 3, suppose $v_1, v_2, ..., v_j$ is an orthonormal basis for V, and $w_1, w_2, ..., w_k$ is an orthonormal basis for $V^{\perp}$. Then $v_1, v_2, ..., v_j, w_1, w_2, ..., w_k$ is an orthonormal basis for $\mathbf{R}^n$. Without loss of generality, and because it is an orthonormal change of basis, we can assume that S looks like:

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix},$$

where S_1 and S_2 are the symmetric matrices corresponding to V and $V^{\perp}$ respectively.

By showing that S is orthogonally diagonalizable, we can reduce our original proposition (given a symmetric $n \times n$ matrix S, then there is an orthonormal matrix U such that $U^T SU$ is a diagonal) to the proposition that if $n > 2$, then there is an f-invariant subspace $V \subseteq \mathbf{R}^n$ such that $0 < dimV < n$.

Let us define polynomial $P(x)$ as

$$P(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0,$$

where x is an arbitrary variable. It then follows that

$$P(S) = a_m S^m + a_{m-1} S^{m-1} + \cdots + a_1 S + a_0 I,$$

where S is a symmetric matrix.

Lemma 4. *There exists a non-zero polynomial $P(S)$ with coefficients not all zero such that $P(S) = 0$.*

Proof. Since S is a symmetric $n \times n$ matrix, it follows that $P(S)$ is also an $n \times n$ matrix. Let us now define M_n as the set of all $n \times n$ matrices. Note that if $e_{i,j} = [\alpha_{r,s}]$ where $\alpha_{r,s} = \begin{cases} 1 \text{ if } r = i,\ s = j \\ 0 \text{ otherwise} \end{cases}$, then $\{e_{i,j}\}$ is a basis for M_n. For example, if $n = 4,\ i = 2$, and $j = 3$, then $e_{i,j} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Because there are n rows and n columns in each matrix, there are n^2 different elements $\alpha_{r,s}$, implying that the dimension of the basis of M_n is n^2. Given that the dimension of the vector space is n^2, the set $S^{n^2+1}, S^{n^2}, ..., S, I$ is not linearly independent. Thus we can find $a_0, a_1, ..., a_{n^2+1}$ not all zero such that

$$P(S) = a_{n^2+1}S^{n^2+1} + a_{n^2}S^{n^2} + \cdots + a_1 S + a_0 I = 0.$$

□

Lemma 5. *There exists a non-zero vector $v \in \mathbf{R}^n$ and an irreducible polynomial $r(x)$ such that $r(S)v = 0$.*

Proof. Now let $P(x) = r_1(x)\cdots r_l(x)$ be the factorization of the polynomial $P(x)$ from Lemma 4 into irreducible polynomials. Therefore, each $r_i(x)$ has a degree of at most 2. Previously, we showed that there does exist some polynomial $P(S) = a_m S^m + a_{m-1}S^{m-1} + \cdots + a_1 S + a_0 = 0$, where $a_m, a_{m-1}, ..., a_0$ are not all zero. Let w be an arbitrary non-zero vector in $\mathbf{R}^n$. Consider the sequence

$$\begin{aligned} &w, \\ &r_1(S)w, \\ &r_2(S)r_1(S)w, \\ &r_3(S)r_2(S)r_1(S)w, ... \end{aligned}$$

and let $v = r_i(S)r_{i-1}(S)\cdots r_1(S)w$ be the last vector that is nonzero. Since $P(S)w = 0w = 0$, this sequence will have a last nonzero factor. Then let $0 = r_{i+1}(S)\cdot = v$, and we have shown that there is a polynomial $r(x)$ such that $r(S)v = 0$. □

Lemma 6. *If the degree of $r(x)$ is 1, then $V = span\{v\}$ is f-invariant for $f(v) = Sv$.*

Proof. Suppose that $r(x) = a_1 x + a_0$. Knowing that v is a nonzero vector, that S is a symmetric matrix, and that $r(S)v = 0$, we can write

$$\begin{aligned} r(S)v = 0 &= (a_1 S + a_0 I)v \\ &= a_1 Sv + a_0 Iv \\ &= a_1 Sv + a_0 v \end{aligned}$$

Using this equation, we can solve for Sv in terms of v:

$$\begin{aligned} 0 &= a_1 Sv + a_0 v \\ a_1 Sv &= -a_0 v \\ Sv &= -\frac{a_0}{a_1} v. \end{aligned}$$

Let w, then $w = \rho v$ for some constant ρ. Then,

$$f(w) = Sw = \rho S v = \rho\left(-\frac{a_0}{a_1}\right) v = -\rho\left(\frac{a_0}{a_1}\right) v \in V.$$

Because $w \in V$ implies that $f(w) \in V$, we can conclude that $V = span\{v\}$ is f-invariant and that v is also an eigenvector for $\lambda = -\left(\frac{a_0}{a_1}\right)$. □

Now consider $r(x)$ of degree 2.

Lemma 7. *If the degree of $r(x)$ is 2, then $V = span\{v, sV\}$ is f-invariant for $f(v) = Sv$.*

Proof. If $r(x)$ is of degree 2, then $r(x) = a_2x^2 + a_1x + a_0$ and

$$\begin{aligned} 0 = r(S)v &= \left(a_2S^2 + a_1S + a_0\right) v \\ &= a_2S^2v + a_1Sv + a_0v. \end{aligned}$$

It follows that $S^2v = -\frac{a_1}{a_2}Sv - \frac{a_0}{a_2}v$. Now suppose that $w = \alpha v + \beta Sv \in span\{v, Sv\}$ where α and β are constants, then

$$\begin{aligned} Sw &= \alpha Sv + \beta S^2 v \\ &= \alpha Sv + \beta(-\frac{a_1}{a_2}Sv - \frac{a_0}{a_2}v) \\ &= (\alpha - \beta\frac{a_1}{a_2})Sv + (-\beta\frac{a_0}{a_2})v \end{aligned}$$

Since $\alpha - \beta\frac{a_1}{a_2}$ and $-\beta\frac{a_0}{a_2}$ are constants, $Sw \in span\{v, Sv\}$ and $V = span\{v, Sv\}$ is f-invariant. □

Here we have shown that if the degree of factor $r(S)$ is 1, then $V = span\{v\}$ is f-invariant. We also showed that if the degree of $r(S)$ is 2, then $V = span\{v, Sv\}$ is f-invariant. Since $r(S)$ has at most degree 2, the polynomial can be represented by a vector space of at most dimension 2. It follows that $P(S)$ must also have an f-invariant vector space V where $0 < dimV \leq 2$. Thus if $n > 2$, there exists an f-invariant subspace $V \in \mathbf{R}^n$ such that $0 < dimV < n$.

Proof. If $j, k < n$ for V and $V^\perp$, then by induction it is possible to orthogonally diagonalize symmetric matrices S_1 and S_2. Let U_1 denote the matrix of orthonormal vectors $v_1, v_2, ..., v_j$ and let U_1^T be its transpose. Also let U_2 denote the matrix of orthonormal vectors

$w_1, w_2, ..., w_j$ and let U_2^T be its transpose. Now we can write the matrices U and U^T in the same form as we did for S:

$$U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \rightarrow U^T = \begin{bmatrix} U_1^T & 0 \\ 0 & U_2^T \end{bmatrix}$$

We can now write $U^T SU$ as represented by these matrices and show that the resulting matrix is a diagonal, showing that symmetric matrix S is orthogonally diagonalizable.

$$\begin{aligned} U^T SU &= \begin{bmatrix} U_1^T & 0 \\ 0 & U_2^T \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \\ &= \begin{bmatrix} U_1^T S_1 & 0 \\ 0 & U_2^T S_2 \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \\ &= \begin{bmatrix} U_1^T S_1 U_1 & 0 \\ 0 & U_2^T S_2 U_2 \end{bmatrix} \end{aligned}$$

Because S_1 and S_2 are symmetric, we know that $U_1^T S_1 U_1 = D_1$ and $U_2^T S_2 U_2 = D_2$ where D_1 and D_2 are diagonals. Thus

$$\begin{bmatrix} U_1^T S_1 U_1 & 0 \\ 0 & U_2^T S_2 U_2 \end{bmatrix} = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} = D.$$

Since D is a diagonal, then S is orthogonally diagonalizable. □

Markov Matrices

For our calculations for schedule strength to be feasible, the tournament matrix S must have eigenvalues λ such that $|\lambda| < 1$. The following proof was written with the guidance of Patrick Keef. [3]

Theorem 9 (Markov Matrices)**.** *If S is a real symmetric stochastic matrix with all positive entries and eigenvalues λ, then $|\lambda| < 1$.*

Suppose that S is a real symmetric stochastic matrix. Then each entry α_{ij} is equal to entry α_{ji}. Additionally, the entries of each row and each column add to one. We will now define vector spaces P, P', and P^0 :

- $P := \{(x_1, x_2, ..., x_n)^T;\ x_1 + \cdots + x_n = 1$, where $x_1, x_2, ..., x_n \geq 0\}$
- $P' := \{x_1, x_2, ...x_n)^T \in P;\ x_i = 0$ for some $i\}$

- $P^0 := \{(x_1, x_2, ..., x_n)^T \in P;$ where $x_1, x_2, ..., x_n > 0\}$.

We look at P as the entire region, P' as the boundary of that region, and P^0 as everything inside that boundary. Let $f(v) = Sv$. In Lemma 1, we will show that $f(P) \subseteq P$, and that if all of the entries of S are positive, then $f(P) = f(P^0) \subseteq P$.

Lemma 1. *$f(P) \subseteq P$ and if all of the entries of S are positive, then $f(P) \subseteq f(P^0) \subseteq P^0$.*

Proof. Let $S = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{1n} & \cdots & \alpha_{nn} \end{bmatrix}$ and choose $p = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in P$. Then,

$$\begin{aligned} f(p) = Sp &= \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{1n} & \cdots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^n \alpha_{1i} x_i \\ \sum_{i=1}^n \alpha_{2i} x_i \\ \vdots \\ \sum_{i=1}^n \alpha_{ni} x_i \end{bmatrix} \end{aligned}$$

By our definition of P, $f(p) \in P$ if the components of $f(p)$ add to one. The summation of the components of $f(P)$is equal to:

$$\sum_{j=1}^n \sum_{i=1}^n \alpha_{ji} x_i = \sum_{i=1}^n \left[\sum_{j=1}^n \alpha_{ji} \right] x_i = \sum_{i=1}^n (1) x_i = 1.$$

Because S and P are defined as having all non-negative entries, then Sp must have all non-negative entries. Since all of the entries of Sp also add up to one, $f(p) = Sp \in P$. Thus $f(P) \subseteq P$. Further, if all entries of S are positive, then all entries of Sp will also be positive, and $f(P^0) \in P^0$. □

Next we will show that the column vector

$$c = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n} \right)^T$$

is an eigenvector corresponding to the eigenvalue 1 by showing that $f(c) = Sc = c$.

Lemma 2. *The column vector*

$$c = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n} \right)^T$$

is an eigenvector of S corresponding to the eigenvalue 1.

Proof. We can write $f(c)$ as follows:

$$\begin{aligned} f(c) = Sc &= \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{j=1}^{n} \alpha_{1j} \cdot \frac{1}{n} \\ \cdots \\ \sum_{j=1}^{n} \alpha_{nj} \cdot \frac{1}{n} \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot \frac{1}{n} \\ \vdots \\ 1 \cdot \frac{1}{n} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix} \end{aligned}$$

Because S is symmetric, we know that the sum of the entries of each row, $\sum_{j=1}^{n} \alpha_{ij}$, must equal one. Thus the multiplication of Sc gives us the vector c itself. This shows that c is indeed an eigenvector for the eigenvalue 1. □

Next we will define a set $Q := \{p - c : p \in P\}$.

Lemma 3. $Q \subseteq \{c\}^{\perp}$.

Proof. Let $q \in Q$. Since $Q := \{p - c\}$ for some $p \in P$,

$$q = p - c = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} - \begin{bmatrix} \frac{1}{n} \\ \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix} = \begin{bmatrix} x_1 - \frac{1}{n} \\ x_2 - \frac{1}{n} \\ \vdots \\ x_n - \frac{1}{n} \end{bmatrix}.$$

To show that $Q \subseteq \{c\}^{\perp}$, we show that the dot product of q and c is equal to zero:

$$\begin{aligned}
q \cdot c &= [p - c] \cdot [c] \\
&= \begin{bmatrix} x_1 - \frac{1}{n} \\ x_2 - \frac{1}{n} \\ \vdots \\ x_n - \frac{1}{n} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{n} \\ \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix} \\
&= \begin{bmatrix} x_1 - \frac{1}{n} & x_2 - \frac{1}{n} & \cdots & x_n - \frac{1}{n} \end{bmatrix} \begin{bmatrix} \frac{1}{n} \\ \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix} \\
&= \sum_{i=1}^{n} \frac{1}{n} \left(x_i - \frac{1}{n} \right) \\
&= \frac{1}{n} \sum_{i=1}^{n} \left(x_i - \frac{1}{n} \right) \\
&= \frac{1}{n} \left(\sum_{i=1}^{n} x_i - n \left(\frac{1}{n} \right) \right) \\
&= \frac{1}{n}(1 - 1) \\
&= 0.
\end{aligned}$$

Since the dot product is equal to zero, $Q \subseteq \{c\}^{\perp}$. □

Note : Notice that in the process of showing that $Q \subseteq \{c\}^{\perp}$, we also showed that the sum of all of the components of q, $\sum_{i=1}^{n} \left(x_i - \frac{1}{n} \right)$, is equal to zero. Additionally, we can show that each component of q is greater than or equal to $-\frac{1}{n}$. From our definition of p, we know that the components p add up to one and that each of these components is non-negative. Since each component of q is equal to $x_i - \frac{1}{n}$ and x_i is non-negative, the smallest value of $x_i - \frac{1}{n}$ is obtained when $x_i = 0$; this value is $-\frac{1}{n}$. Thus each component of q is greater than or equal to $-\frac{1}{n}$.

Now let us define $Q' := \{p' - c : p' \in P'\}$ and $Q^0 := \{p^0 - c : p^0 \in P^0\}$.

Lemma 4. *$f(Q) \subseteq Q$.*

Proof. Let $q \in Q$. Then $q = p - c$ for some p. We can then write $f(q)$ as

$$f(q) = f(p - c) = f(p) - f(c).$$

We previously determined that $f(c) = c$ and that $f(p) \in P$. Thus it follows that $f(q) = f(p) - f(c) = f(p) - c$ and $f(Q) \subseteq Q$. □

Similarly, we find that for $q^0 \in Q^0$, $q^0 = p^0 - c$ for some $p^0 \in P^0$ and

$$f(q^0) = f(p^0 - c) = f(p^0) - f(c) = f(p^0) - c.$$

Since $f(P^0) \subseteq P$, it follows that $f(q^0) \in Q^0$.

Lemma 5. *If v is a non-zero vector in $\{c\}^\perp$, then there exist positive numbers ρ and σ such that*

(1) $\alpha v \in Q$ *iff* $-\rho \leq \alpha \leq \sigma$;

(2) $\alpha v \in Q^0$ *iff* $-\rho < \alpha < \sigma$;

(3) $\alpha v \in Q'$ *iff* $\alpha = -\rho$ *or* $\alpha = \sigma$.

Let a be the smallest component in the vector v and let b be the largest component; note that $a < 0$ and $b > 0$. Additionally, let $\rho = \frac{1}{bn}$ and $\sigma = -\frac{1}{an}$. We will begin by proving case 1.

Case 1 :

Proof. Assume that $\alpha v \in Q$, thus each component of v must be greater than or equal to $-\frac{1}{n}$. Then

$-\frac{1}{n} \leq b\alpha$, which implies $-\frac{1}{nb} \leq \alpha$, and $-\frac{1}{n} \leq a\alpha$, which implies $-\frac{1}{an} \geq \alpha$.

In the second inequality, the inequality switches because we divide by a, which is a negative number. Putting these two inequalities together we arrive at

$$-\frac{1}{bn} \leq \alpha \leq -\frac{1}{an}$$

, implying that $-\rho \leq \alpha \leq \sigma$.. □

Case 1 Converse :

Proof. We will assume that $-\rho \leq \alpha \leq \sigma$, with ρ and σ as defined above. If $-\rho \leq \alpha \leq \sigma$, then $\alpha a \geq -\frac{1}{n}$ and $\alpha b \geq -\frac{1}{n}$. It follows that for any component of v_i between a and b, $\alpha v_i \geq -\frac{1}{n}$. Therefore, if $-\rho \leq \alpha \leq \sigma$, then $\alpha v \in Q$. □

We can use this same argument, but with strict inequalities, to prove case 2:

Case 2 :

Proof. If $\alpha v \in Q^0$, then

$-\frac{1}{n} < b\alpha$, which implies $-\frac{1}{nb} < \alpha$, and $-\frac{1}{n} < a\alpha$, implying that $-\frac{1}{an} > \alpha$.

Thus $-\rho < \alpha < \sigma$. □

Case 2 Converse :

Proof. We assume that $-\rho < \alpha < \sigma$. If $-\rho < \alpha < \sigma$, then $\alpha a > -\frac{1}{n}$ and $\alpha b > -\frac{1}{n}$. It follows that for any element v_i between a and b, $\alpha v_i > -\frac{1}{n}$. Thus, if $-\rho < \alpha < \sigma$, then $\alpha v \in Q'$. □

Having proved cases 1 and 2, it follows that case 3 must be true as well.

Lemma 6. *Suppose that v is an eigenvector for the matrix S corresponding to the eigenvalue λ. Using ρ and σ as defined previously, we can show that $\rho\lambda, \sigma\lambda \in [-\rho, \sigma]$.*

Proof. We previously determined that if $v \in Q$, then $f(v) = \lambda v \in Q$. In Lemma 5 we showed that $-\rho v \in Q$ and that $\sigma v \in Q$. It follows that $f(-\rho p v) = -\lambda\rho v$ and $f(\sigma v) = \lambda\sigma v$ are also in the set Q. Therefore, $-\rho \leq -\lambda\rho \leq \sigma$ and $-\rho \leq \lambda\sigma \leq \sigma$. □

We can also use this information to show that $|\lambda| \leq 1$. If $\lambda > 0$, then $0 \leq \lambda \leq 1$ because λv is still in Q, thus mapping the interval $[-\rho, \sigma]$ to itself. If $\lambda < 0$, then we need to look at two cases:

Case 1 : First we will consider the case where $\sigma \leq \rho$.

Proof. If we rewrite the interval $-\rho \leq -\lambda\rho \leq \sigma$ we arrive at $-1 \leq -\lambda \leq \frac{\sigma}{\rho}$. Since $\sigma \leq \rho$, it follows that $\frac{\sigma}{\rho} \leq 1$ and $-1 \leq -\lambda \leq 1$. □

Case 2 : The second case we must consider is the case where $\sigma \geq \rho$.

Proof. If we rewrite the interval $-\rho \leq \lambda\sigma \leq \sigma$, we arrive at $-1 \leq -\lambda \leq \frac{\rho}{\sigma}$. Since $\sigma \geq \rho$, it follows that $\frac{\rho}{\sigma} \leq 1$ and $-1 \leq -\lambda \leq 1$ □

Thus if v is an eigenvector of S, then the corresponding eigenvalue λ has an absolute value that is less than or equal to one. For the case where all of the coefficients of S are positive, we can use this same method to show that the absolute value of λ is strictly less than 1. To do that, would only need to make the inequalities strict inequalities. Thus when all of the entries of S have positive coefficients, $|\lambda| < 1$.

Now let us suppose that there exists some power of the matrix S, say S^k, such that each of its entries is positive. (Note that for our purposes, the significance that each entry is positive signifies that when the k^{th}-generation games are taken into account, every team in the tournament can be linked to every other team. We say the tournament is connected.) It follows from Lemma 6 that all of the eigenvalues of the matrix S^k have an an absolute value strictly less than 1. We can show that this also implies that all eigenvalues of the matrix S have an absolute value strictly less than 1 as well. If we let the eigenvalues for S be represented by λ, then the eigenvalues of S^k can be represented by λ^k.

Lemma 7. $|\lambda| < 1$ *if and only if* $|\lambda^k| < 1$.

Proof. First we will assume that $|\lambda^k| < 1$. If $|\lambda^k| < 1$, then

$$|\sqrt[k]{\lambda^k}| < \sqrt[k]{1} \text{ and } |\lambda| < 1.$$

Next we will assume that $|\lambda| < 1$. If $|\lambda| < 1$, then

$$|\lambda^k| < 1^k \text{ and } |\lambda^k| < 1.$$

Thus when the matrix S^k has all positive entries (implying that the tournament is connected), the matrix S has eigenvalues λ such that $|\lambda| < 1$.

□

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