THE SATURN-JANUS-EPIMETHEUS SYSTEM

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ABSTRACT. The Saturn-Janus-Epimetheus system is a prime example of a circular restricted three body problem. In this paper we examine the difficulties behind the three-body problem, the assumptions that must be made in order to solve it, the limitations behind these assumptions, and possible methods of solving the three-body problem. Finally, we use the unique properties of the Saturn-Janus-Epimetheus system to show that Epimetheus' orbit traces out a horseshoe pattern in a rotating frame in which Saturn and Janus are fixed. The system is then numerically integrated and animated in MATLAB.

1. INTRODUCTION

The study of gravity has led to many revolutions in science, from Newtonian Dynamics to General Relativity to Quantum Field Theory. Despite these advances, however, gravity continues to puzzle scientists. In the 17th century, Sir Isaac Newton was able to successfully model the gravitational force on a macroscopic scale, but in the succeeding 400 years physicists have made little progress in understanding the precise nature of this elusive force.[1] Determining the gravitational force exerted by an object is fairly trivial; however, trying to model gravitational interactions between multiple objects quickly becomes nearly impossible. The simplest case, the two-body problem, is the most complicated gravitational interaction that can be fully understood without some basic and unrealistic assumptions. We examine the two-body problem now in order to compare the results to the three-body problem. Consider two masses, m_1 and m_2, and let the distance from the origin to m_1 and m_2 be r_1 and r_2, respectively. The force on m_1 due to m_2 is simply the gravitational force:

$$\mathbf{F}_1 = \frac{Gm_1m_2}{r^3}\mathbf{r}$$

where r is the distance between the two masses, $\mathbf{r}$ is the vector pointing from m_2 to m_1, and $G = 6.6726 \times 10^{-11}$ N$\cdot$ m^2 kg^{-2} is the gravitational constant. Newton's second law relates force to acceleration: $\mathbf{F} = m\mathbf{a} = m\ddot{r}$, where the dot notation implies differentiation with respect to time. This enables us to write the two-body equations of motion as

$$\mathbf{F}_1 = \frac{Gm_1m_2}{r^3}\mathbf{r} = m_1\ddot{\mathbf{r}}_1$$

[1]Current theories of gravity are extremely varied: Newtonian theories predict that gravity is an attractive force; General Relativity predicts that gravity is the curvature of space-time due to the presence of mass; and Quantum Field Theory says that gravity is due to the exchange of elementary particles called gravitons. Each of these theories has its own problems. The most likely candidate, General Relativity, cannot yet be reconciled with Quantum Mechanics.

and

$$\mathbf{F}_2 = -\frac{Gm_1m_2}{r^3}\mathbf{r} = m_2\ddot{\mathbf{r}}.$$

Because the two forces are of equal magnitude in opposite directions,

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = 0.$$

We can integrate this result directly to obtain equations for the motion of the center of mass of the system. Using the conservation of angular momentum, linear momentum, and energy, we can also show that the path of one body relative to the other and the paths of the two bodies relative to the center of mass are conic sections. At the end of a rather lengthy derivation, we can produce a definitive result for the position of the bodies at a given time.

The key to the two-body problem is that we know multiple relationships between variables that enable us to reduce the number of unknowns in order to obtain an exact solution. These relationships include the conservation of quantities such as linear momentum, angular momentum, and energy. When we move to the three-body problem, however, we add in six more variables (three space variables and three velocity variables) with the extra body; moreover, we lose some of our important quantities, such as the conservation of energy, as is shown in Section 3. For more than two bodies we end up with more unknowns than equations, which makes the problem analytically unsolvable. However, there are some cases in which the three-body problem can be simplified. One case occurs when the three-body problem is *restricted*, meaning we assume that the third body has such little mass compared to the other two bodies that it can be treated as a massless particle. Another case is to consider the orbits to be *circular*. When we combine the two, we have a circular restricted three-body problem.

The circular restricted three-body problem is a fairly good approximation to the Saturn-Janus-Epimetheus system. Janus and Epimetheus are moons of Saturn, located at orbital radii of $151,472$ km and $151,422$ km respectively. Janus has a mass of around 1.98×10^{18} kg and a diameter of approximately 175 km, while Epimetheus' mass and diameter are about 5.5×10^{17} kg and 105 km [6]. Because the orbits of Janus and Epimetheus are only 50 km apart, smaller than the radii of the moons, a naïve analysis would suggest that the two moons would eventually collide. However, we show that the gravitational interactions in the circular restricted three body problem prevent Janus and Epimetheus from colliding with each other. We then use numerical integration to determine and animate the trajectories of the moon in time.

In Section 2, we derive the equations of motion for the three-body problem, both in inertial and rotating frames of reference. Section 3 concerns the quantities that are conserved in the three-body problem. We then locate the equilibrium points of the three-body system in Section 4 and determine their stability in Section 5. In Section 6 we determine adequate initial conditions for the Saturn-Janus-Epimetheus system. We then discuss the methods, purposes, and accuracy of numerical integration Section 7. Finally, in Section 8 we animate the trajectories of the three-bodies in the Saturn-Janus-Epimetheus system in the rotating and inertial frames of reference. In the Appendices we include several snippets of MATLAB code that can be used to reproduce our animations of the system.

2. The Equations of Motion

In this section we derive the equations of motion for the circular restricted three-body problem. First we consider two masses m_1 and m_2 interacting via the gravitational force. A particle P orbits around the two masses at distances of r_1 and r_2 from m_1 and m_2, respectively. We assume that P "feels" the gravitational force of m_1 and m_2 but does not exert a gravitational force of its own. Our goal is to find an equation that gives the position of the particle at any time t.

To begin, we set up a coordinate system in the inertial reference frame.[2] Let the origin of the coordinate system be located at the center of mass of the system. Let ξ be the axis pointed along the line joining m_1 and m_2 at $t = 0$, let η be the axis perpendicular to the ξ axis and in the orbital plane, and let ζ be the axis perpendicular to the $\xi\eta$-plane (see Figure 1). In the circular three-body problem each of the masses will orbit with the same angular velocity about their center of mass so that they have a constant separation. We define the unit mass $\mu = G(m_1 + m_2) = \mu_1 + \mu_2 = 1$, with $\mu_1 > \mu_2$. It then follows that $\mu_1 = Gm_1$ and $\mu_2 = Gm_2$. We now use a similar approach as in the two-body problem to determine the three dimensional equations of motion of the particle P:

$$\ddot{\xi} = \mu_1 \frac{\xi_1 - \xi}{r_1^3} + \mu_2 \frac{\xi_2 - \xi}{r_2^3}, \tag{1}$$

$$\ddot{\eta} = \mu_1 \frac{\eta_1 - \eta}{r_1^3} + \mu_2 \frac{\eta_2 - \eta}{r_2^3}, \tag{2}$$

and

$$\ddot{\zeta} = \mu_1 \frac{\zeta_1 - \zeta}{r_1^3} + \mu_2 \frac{\zeta_2 - \zeta}{r_2^3} \tag{3}$$

where (ξ_1, η_1, ζ_1) is the position of μ_1, (ξ_2, η_2, ζ_2) is the position of μ_2, and r_1 and r_2, the distances from m_1 and m_2 to the particle P, are determined by the distance formula:

$$r_1^2 = (\xi_1 - \xi)^2 + (\eta_1 - \eta)^2 + (\zeta_1 - \zeta)^2$$

and

$$r_2^2 = (\xi_2 - \xi)^2 + (\eta_2 - \eta)^2 + (\zeta_2 - \zeta)^2.$$

[2]According to Taylor [8], "an inertial frame is any reference frame (that is, a system of coordinates x, y, z, and time t) in which all laws of physics hold in their usual form." It is called an inertial frame because Newton's First Law, the Law of Inertia, holds in these frames. Though it is hard to find a true inertial frame when everything in the universe is moving, for our purposes we can define the inertial frame as the frame that is fixed with respect to distant background stars.

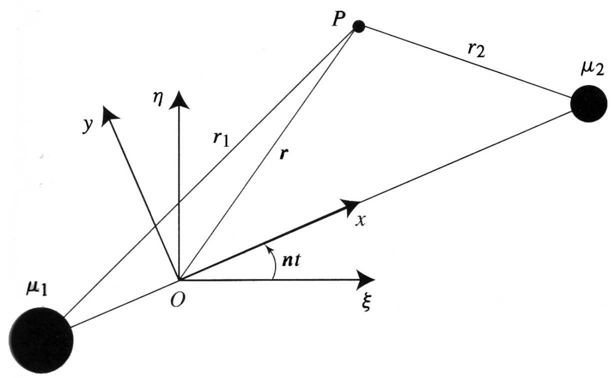


FIGURE 1. The three-body problem: two masses and a particle. Note that the z axis and the ζ axis are out of the page (from Murray & Dermott [6]).

In this reference frame all three bodies move in time. The motion of the particle P is dependent on the motion of the two other bodies, which makes solving the equations particularly difficult and computationally intensive. Suppose we consider a frame of reference that rotates with the masses so that the locations of m_1 and m_2 are fixed. Also suppose that the x-axis lies along the line joining the two masses, the y-axis lies perpendicular to the x-axis in the orbital plane, and the z-axis lies perpendicular to the xy-plane (Figure 1). We define the coordinates of m_1 as $(x_1, y_1, z_1) = (-\mu_2, 0, 0)$ and the coordinates of m_2 as $(x_2, y_2, z_2) = (\mu_1, 0, 0)$. If the particle is located at (x, y, z), then the distance formula shows that the squares of the distances between the particle and the masses are

$$r_1^2 = (x + \mu_2)^2 + y^2 + z^2 \tag{4}$$

and

$$r_2^2 = (x - \mu_1)^2 + y^2 + z^2. \tag{5}$$

To obtain our equations of motion in the rotating frame, we rotate from the inertial frame to the rotating frame with a rotation matrix below. If n is the average angular velocity, then the relationship is found by a rotation in the xy plane:

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} \cos nt & -\sin nt & 0 \\ \sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Differentiating with respect to t, we see that

$$\begin{aligned} \xi &= x\cos nt - y\sin nt \\ \eta &= x\sin nt + y\cos nt \\ \zeta &= z \end{aligned}$$

yields

$$\begin{aligned}\dot{\xi} &= \dot{x}\cos nt - nx\sin nt - \dot{y}\sin nt - ny\cos nt\\ \dot{\eta} &= \dot{x}\sin nt + nx\cos nt + \dot{y}\cos nt - ny\sin nt\\ \dot{\zeta} &= \dot{z}.\end{aligned}$$

We write this result as

$$\begin{bmatrix}\dot{\xi}\\ \dot{\eta}\\ \dot{\zeta}\end{bmatrix} = \begin{bmatrix}\cos nt & -\sin nt & 0\\ \sin nt & \cos nt & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\dot{x}-ny\\ \dot{y}+nx\\ \dot{z}\end{bmatrix}.$$

Differentiating with respect to t once again, we obtain

$$\ddot{\xi} = \ddot{x}\cos nt - 2n\dot{x}\sin nt - n^2x\cos nt - \ddot{y}\sin nt - 2n\dot{y}\cos nt + n^2y\sin nt,$$

$$\ddot{\eta} = \ddot{x}\sin nt + 2n\dot{x}\cos nt - n^2x\sin nt + \ddot{y}\cos nt - 2n\dot{y}\sin nt - n^2y\sin nt,$$

and

$$\ddot{\zeta} = \ddot{z},$$

which tells us that

$$\begin{bmatrix}\ddot{\xi}\\ \ddot{\eta}\\ \ddot{\zeta}\end{bmatrix} = \begin{bmatrix}\cos nt & -\sin nt & 0\\ \sin nt & \cos nt & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\ddot{x}-2n\dot{y}-n^2x\\ \ddot{y}+2n\dot{x}-n^2y\\ \ddot{z}\end{bmatrix}.$$

The switch to a rotating reference frame forces us to include the Coriolis acceleration (the terms $2n\dot{x}$ and $-2n\dot{y}$) and the centrifugal acceleration (the terms n^2x and n^2y).[3] Since we now know ξ, η, ζ, $\ddot{\xi}$, $\ddot{\eta}$, and $\ddot{\zeta}$ in terms of x, y, z, $\ddot{x}$, $\ddot{y}$, and $\ddot{z}$, we substitute these values into Equations (1), (2), and (3):

[3]The Coriolis acceleration occurs when an object appears to be accelerated in a curved path in a rotating frame even though it is moving in a straight line in the inertial frame. The centrifugal acceleration is the force that seems to push a rotating object outward from the center.

$$(\ddot{x} - 2n\dot{y} - n^2x)\cos nt - (\ddot{y} + 2n\dot{x} - n^2y)\sin nt = \left[\mu_1\frac{x_1 - x}{r_1^3} + \mu_2\frac{x_2 - x}{r_2^3}\right]\cos nt + \left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right] y \sin nt,$$

$$(\ddot{x} - 2n\dot{y} - n^2x)\sin nt + (\ddot{y} + 2n\dot{x} - n^2y)\cos nt = \left[\mu_1\frac{x_1 - x}{r_1^3} + \mu_2\frac{x_2 - x}{r_2^3}\right]\sin nt - \left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right] y \cos nt,$$

and

$$\ddot{z} = -\left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right] z.$$

Next we multiply the first equation by $\cos nt$ and the second equation by $\sin nt$:

$$(\ddot{x} - 2n\dot{y} - n^2x)\cos^2 nt - (\ddot{y} + 2n\dot{x} - n^2y)\sin nt\cos nt = \left[\mu_1\frac{x_1 - x}{r_1^3} + \mu_2\frac{x_2 - x}{r_2^3}\right]\cos^2 nt + \left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right] y \sin nt\cos nt$$

and

$$(\ddot{x} - 2n\dot{y} - n^2x)\sin^2 nt + (\ddot{y} + 2n\dot{x} - n^2y)\cos nt\sin nt = \left[\mu_1\frac{x_1 - x}{r_1^3} + \mu_2\frac{x_2 - x}{r_2^3}\right]\sin^2 nt - \left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right] y \cos nt\sin nt.$$

Their sum is

$$(\ddot{x} - 2n\dot{y} - n^2x) = \mu_1\frac{x_1 - x}{r_1^3} + \mu_2\frac{x_2 - x}{r_2^3}.$$

Continuing this process and noting the values of x_1, x_2, etc., we derive the following equations of motion:

$$\ddot{x} - 2n\dot{y} - n^2x = -\left[\mu_1\frac{x + \mu_2}{r_1^3} + \mu_2\frac{x - \mu_1}{r_2^3}\right], \tag{6}$$

$$\ddot{y} + 2n\dot{x} - n^2y = -\left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right] y, \tag{7}$$

and

$$\ddot{z} = -\left[\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right] z. \tag{8}$$

These equations can be written as the gradient of a scalar function, which we shall call U:

$$\ddot{x} - 2n\dot{y} = \frac{\partial U}{\partial x}, \tag{9}$$

$$\ddot{y} + 2n\dot{x} = \frac{\partial U}{\partial y}, \tag{10}$$

and

$$\ddot{z} = \frac{\partial U}{\partial z} \tag{11}$$

where

$$U = \frac{n^2}{2}(x^2 + y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}$$

and r_1 and r_2 are defined in Equations (4) and (5).

Notice that these second-order equations of motion are nonlinear and coupled. For these reasons, they are essentially unsolvable. In order to determine the motion of the particle P at a time t, we must find ways to study the system without determining exact solutions of the equations of motion.

3. The Jacobi Constant

In this section we find a constant of motion for the circular restricted three-body problem. Suppose we multiply Equation (9) by $\dot{x}$, Equation (10) by $\dot{y}$, and Equation (11) by $\dot{z}$ and then add the three expressions together:

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \dot{x}\frac{\partial U}{\partial x} + \dot{y}\frac{\partial U}{\partial y} + \dot{z}\frac{\partial U}{\partial z} = \frac{dU}{dt}.$$

We can integrate this expression:

$$\int (\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z})dt = \int dU$$

$$\frac{\dot{x}^2}{2} + \frac{\dot{y}^2}{2} + \frac{\dot{z}^2}{2} = U + c$$

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2U + 2c$$

where c is the constant of integration. We define $C_J = -2c$ to be the Jacobi constant.[4] Since $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2$, where v is the velocity,

$$v^2 = 2U - C_J.$$

Putting everything in terms of x, y, and z, we see that

$$C_J = n^2(x^2 + y^2) + 2\left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right) - \dot{x}^2 - \dot{y}^2 - \dot{z}^2. \tag{12}$$

[4]The Jacobi constant is also sometimes called the Jacobi integral.

The Jacobi constant is a constant of motion. A simple guess might assume that the Jacobi constant has something to do with the conservation of energy or momentum, since those quantities have to be conserved. As we shall soon see, however, our simple model fails to account for all the interactions that occur, meaning that in our model energy and momentum are *not* conserved—thus, our model is not completely accurate. However, the simplifications we have made are necessary in order to obtain a solution, and do not prevent a very basic analysis of the system.

Suppose we wish to determine C_J in the inertial frame of reference. Earlier, we derived the equations of motion in the rotating frame by using rotation matrices to transform the equations between reference frames. We use the same process now, but in the opposite direction. Recall that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos nt & \sin nt & 0 \\ -\sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

and

$$\begin{pmatrix} \dot{x} - ny \\ \dot{y} + nx \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \cos nt & \sin nt & 0 \\ -\sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{pmatrix}.$$

Separating the velocity vector into two parts, we write

$$\begin{pmatrix} \dot{x} - ny \\ \dot{y} + nx \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} + n \begin{pmatrix} \sin nt & -\cos nt & 0 \\ \cos nt & \sin nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}.$$

Combining these results, we see that

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \cos nt & \sin nt & 0 \\ -\sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{pmatrix} - n \begin{pmatrix} \sin nt & -\cos nt & 0 \\ \cos nt & \sin nt & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}.$$

In order to make our equations more readable, we now define the matrices $\mathbf{A}$ and $\mathbf{B}$ as

$$\mathbf{A} \equiv \begin{pmatrix} \cos nt & \sin nt & 0 \\ -\sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} \equiv \begin{pmatrix} \sin nt & -\cos nt & 0 \\ \cos nt & \sin nt & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then we see that

$$\mathbf{A}^T \cdot \mathbf{A} = \begin{pmatrix} \cos nt & -\sin nt & 0 \\ \sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos nt & \sin nt & 0 \\ -\sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{B}^T \cdot \mathbf{B} = \begin{pmatrix} \sin nt & \cos nt & 0 \\ -\cos nt & \sin nt & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sin nt & -\cos nt & 0 \\ \cos nt & \sin nt & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{A}^T \cdot \mathbf{B} = \begin{pmatrix} \cos nt & -\sin nt & 0 \\ \sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin nt & -\cos nt & 0 \\ \cos nt & \sin nt & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\mathbf{B}^T \cdot \mathbf{A} = \begin{pmatrix} \sin nt & \cos nt & 0 \\ -\cos nt & \sin nt & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos nt & \sin nt & 0 \\ -\sin nt & \cos nt & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we take the transpose of the velocity vector and dot it with the velocity vector, we get

$$(\dot{x} \;\; \dot{y} \;\; \dot{z}) \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = (\dot{\xi} \;\; \dot{\eta} \;\; \dot{\zeta})\mathbf{A}^T \cdot \mathbf{A} \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{pmatrix} - n(\dot{\xi} \;\; \dot{\eta} \;\; \dot{\zeta})\mathbf{A}^T \cdot \mathbf{B} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

$$-n(\xi \;\; \eta \;\; \zeta)\mathbf{B}^T \cdot \mathbf{A} \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{pmatrix} + n^2(\xi \;\; \eta \;\; \zeta)\mathbf{B}^T \cdot \mathbf{B} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}.$$

Multiplying through and simplifying, the equation becomes

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 + n^2(\xi^2 + \eta^2) + 2n(\dot{\xi}\eta - \dot{\eta}\xi).$$

Since distances are unchanged by rotations (in non-relativistic frames of reference), the quantity $x^2 + y^2 + z^2 = \xi^2 + \eta^2 + \zeta^2$. Therefore, combining our expressions for the Jacobi constant in terms of x, y, and z (Equation (12)) with our expressions for $\dot{x}$, $\dot{y}$, and $\dot{z}$ in terms of ξ, η, and ζ, we determine the Jacobi constant in terms of ξ, η, and ζ:

$$C_J = 2\left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right) + 2n(\xi\dot{\eta} - \eta\dot{\xi}) - \dot{\xi}^2 - \dot{\eta}^2 - \dot{\zeta}^2. \tag{13}$$

Rewriting Equation (13), it becomes

$$\frac{1}{2}\left(\dot{\xi}^2+\dot{\eta}^2+\dot{\zeta}^2\right)-\left(\frac{\mu_1}{r_1}+\frac{\mu_2}{r_2}\right)=\mathbf{h}\cdot\mathbf{n}-\frac{1}{2}C_J.$$

Here, $\mathbf{n}$ is a normal vector pointing out of the orbital plane. The vector $\mathbf{h}$ is a vector that roughly describes the angular momentum per unit mass.[5] We see here that $\mathbf{h}\cdot\mathbf{n}$ is not a constant in time—this is why energy is not conserved in the restricted three-body problem.

How does the Jacobi constant help us in our analysis? By applying physical constraints on the system, we can put limits on the Jacobi constant, thus constraining the position of the "massless" particle. In the rotating inertial frame, recall that $v^2 = 2U - C_J$. Velocity is a "real" quantity—it cannot be imaginary. Therefore, $2U - C_J$ must always be positive or zero, so $2U \geq C_J$. A special case occurs when $v = 0$, or $2U = C_J$. For particular values of C_J these surfaces defined by $2U - C_J$, known as zero-velocity surfaces, describe boundaries for allowed regions. If we project these surfaces into the xy-plane, we then have zero-velocity curves (see Figure 2). The particle is forbidden from being outside of these curves.

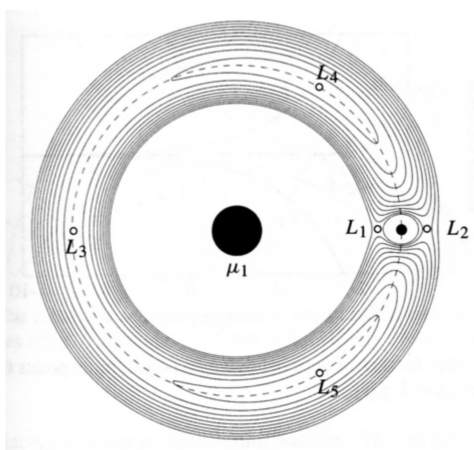


FIGURE 2. Several zero-velocity curves for $\mu_2 = 0.01$. The labeled points are the Lagrange points, which we discuss in Section 4. Note that μ_1 and μ_2 are held fixed in the rotating frame of reference (image from De Pater & Lissauer [3]).

The coupled, nonlinear nature of the equations of motion of the three-body problem prohibits us from determining exact solutions. Through the use of the Jacobi

[5]The vector $\mathbf{h}$ is a constant vector that satisfies the relation $\mathbf{r}\times\dot{\mathbf{r}}$ in the two-body problem. The quantity $h = |\mathbf{h}|$ is approximately equal to the angular momentum per unit mass in the center of mass reference frame for systems in which $m_2 \ll m_1$, but this is not true for systems in the inertial frame. For more information see Murray & Dermott [6].

constant, we can determine specific areas where the particle is not allowed. This means that a particle orbiting within one of the zero-velocity curves in Figure 2 can never escape from the system. These zero-velocity curves can tell us some very basic information about the system without requiring much computation.

4. The Lagrange Points

The equations of motion can be drastically simplified if we consider locations where the position of the particle P is fixed in the rotating frame of reference. At these locations the position and velocity of the particle do not change with time, meaning that $\dot{x} = \dot{y} = \dot{z} = \ddot{x} = \ddot{y} = \ddot{z} = 0$. From this we see that the Coriolis acceleration terms $-2n\dot{y}$ and $2n\dot{x}$ are zero. We further simplify the equations of motion if we assume that all motion takes place in the xy-plane, making $z = 0$. These points are particularly important for our analysis, since observations have determined that Epimetheus oscillates about two of the Lagrange points in the Saturn-Janus system.

To find the locations of the Lagrange points we first manipulate various known quantities into other, more useful forms. First we examine the equations of motion, Equations (9), (10), and (11). We rewrite the equations for r_1^2 and r_2^2 in the following way:

$$r_1^2 = (x + \mu_2)^2 + y^2 + z^2 = x^2 + 2\mu_2 x + \mu_2^2 + y^2 + z^2 \tag{14}$$

and

$$r_2^2 = (x - \mu_1)^2 + y^2 + z^2 = x^2 + 2\mu_1 x + \mu_1^2 + y^2 + z^2. \tag{15}$$

Multiplying Equation (14) by μ_1 and Equation (15) by μ_2, the equations become

$$\mu_1 r_1^2 = \mu_1 x^2 + 2\mu_1\mu_2 x + \mu_1\mu_2^2 + \mu_1 y^2 + \mu_1 z^2$$

and

$$\mu_2 r_2^2 = \mu_2 x^2 - 2\mu_1\mu_2 x + \mu_1^2\mu_2 + \mu_2 y^2 + \mu_2 z^2.$$

Adding the two equations together, factoring out $\mu_1 + \mu_2$, and noting that in our system of units $\mu_2 + \mu_1 = 1$, we obtain

$$\begin{aligned}\mu_1 r_1^2 + \mu_2 r_2^2 &= \mu_1 x^2 + \mu_2 x^2 + \mu_1\mu_2^2 + \mu_1^2\mu_2 + \mu_1 y^2 + \mu_2 y^2 + \mu_1 z^2 + \mu_2 z^2 \\ &= (\mu_1 + \mu_2)x^2 + (\mu_2 + \mu_1)\mu_1\mu_2 + (\mu_1 + \mu_2)y^2 + (\mu_1 + \mu_2)z^2 \\ &= x^2 + y^2 + z^2 + \mu_1\mu_2.\end{aligned}$$

If all motion is in the xy-plane, then

$$\mu_1 r_1^2 + \mu_2 r_2^2 = x^2 + y^2 + \mu_1\mu_2. \tag{16}$$

Now we examine the scalar function U. Using Equation (16) and noting that $n = 1$ in our system of units, we can rewrite:

$$\begin{aligned} U &= \frac{n^2}{2}(x^2+y^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} \\ &= \frac{1}{2}(\mu_1 r_1^2 + \mu_2 r_2^2 - \mu_1\mu_2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} \\ &= \left(\frac{\mu_1 r_1^2}{2} + \frac{\mu_1}{r_1}\right) + \left(\frac{\mu_2 r_2^2}{2} + \frac{\mu_2}{r_2}\right) - \frac{\mu_1\mu_2}{2} \end{aligned}$$

so that

$$U = \mu_1\left(\frac{1}{r_1} + \frac{r_1^2}{2}\right) + \mu_2\left(\frac{1}{r_2} + \frac{r_2^2}{2}\right) - \frac{\mu_1\mu_2}{2}. \tag{17}$$

With all the motion constrained to the xy-plane, Equations (9) and (10) show that

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r_1}\frac{\partial r_1}{\partial x} + \frac{\partial U}{\partial r_2}\frac{\partial r_2}{\partial x} = 0$$

and

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial r_1}\frac{\partial r_1}{\partial y} + \frac{\partial U}{\partial r_2}\frac{\partial r_2}{\partial y} = 0.$$

Therefore, the Lagrange points are located at positions where $\partial U/\partial x = \partial U/\partial y = 0$. To find these positions, we differentiate Equation (17) with respect to x and y:

$$\mu_1\left(-\frac{1}{r_1^2} + r_1\right)\frac{x+\mu_2}{r_1} + \mu_2\left(-\frac{1}{r_2^2} + r_2\right)\frac{x-\mu_1}{r_2} = 0 \tag{18}$$

and

$$\mu_1\left(-\frac{1}{r_1^2} + r_1\right)\frac{y}{r_1} + \mu_2\left(-\frac{1}{r_2^2} + r_2\right)\frac{y}{r_2} = 0. \tag{19}$$

The trivial solutions to Equations (18) and (19) are the simplest possible solutions; they occur when the coefficients of $(x - \mu_2)$, $(x + \mu_2)$ and y are zero, or

$$\mu_1\left(-\frac{1}{r_1^2}+r_1\right)=0 \quad\text{and}\quad \mu_2\left(-\frac{1}{\mathrm{r}_2^2}+\mathrm{r}_2\right)=0.$$

Therefore, $r_1 = r_2 = 1$, so $(x+\mu_2)^2 + y^2 = 1$ and $(x-\mu_1)^2 + y^2 = 1$. Solving for x, we get $x = \pm\sqrt{1-y^2} - \mu_2$ from the first equation and $x = \pm\sqrt{1-y^2} + \mu_1$ from the second. Setting these equations equal to each other the two expressions for x must be of opposite sign. Then, solving for y:

$$\begin{aligned}\sqrt{1-y^2}-\mu_2 &= -\sqrt{1-y^2}+\mu_1\\ \sqrt{1-y^2} &= -\sqrt{1-y^2}+\mu_1+\mu_2\\ &= -\sqrt{1-y^2}+1\\ 2\sqrt{1-y^2} &= 1\end{aligned}$$

which means that $y = \pm\sqrt{3}/2$. Using this value of y, we determine that the coordinates of these particular points are

$$x = \sqrt{1-3/4}-\mu_2 = \frac{1}{2}-\mu_2 \tag{20}$$

and

$$y = \pm\frac{\sqrt{3}}{2}. \tag{21}$$

We define these two points as the L_4 and L_5 Lagrange points, named after Joseph-Louis Lagrange. The L_4 Lagrange point, the leading point, is located at $(1/2-\mu_2, \sqrt{3}/2, 0)$, while the L_5 point, the trailing point, is located at $(1/2-\mu_2, -\sqrt{3}/2, 0)$. These points are the *triangular* equilibrium points.[6]

Equation (19) is also satisfied when $y = 0$. When $y = 0$, we have $r_1^2 = (x+\mu_2)^2$ and $r_2^2 = (x-\mu_1)^2$, which means that $r_1 = \pm(x+\mu_2)$ and $r_2 = \pm(x-\mu_1)$. On the x axis, which runs through the the two masses, there are three Lagrange points: L_1, L_2, and L_3, the *collinear* equilibrium points. The L_1 point is located between μ_1 and μ_2, the L_2 point is located on the opposite side of μ_2 from μ_1, and the L_3 point is located on the opposite side of μ_1 from μ_2 (see Figure 3).

For each of the three collinear Lagrange points where $y = 0$, our goal is to determine the relationships between r_1, r_2, and x. We use Equation (18) to determine the ratio μ_2/μ_1 in terms of x. Using Maple, we determine the series expansion of this ratio and then following Murray and Dermott's procedure [6], we use Lagrange's inversion formula to determine r_2 as a function of μ_1 and μ_2. We then use the relationships between r_1, r_2, and x to determine r_1. Then, for given masses μ_1 and μ_2

[6]These points are well named. If a particle is stationary at a point in its orbit, then the sum of the gravitational forces due to m_1 and m_2 must exactly balance the outward centrifugal acceleration from the origin. Simple vector calculations using Newton's second law show that $r_1 = r_2 = r_{\mathrm{cm}}$, where r_{cm} is the distance of the particle from the center of mass of the system. Therefore, a particle at an L_4 or L_5 Lagrange point must be located at the apex of an equilateral triangle connecting the particle and the two masses.

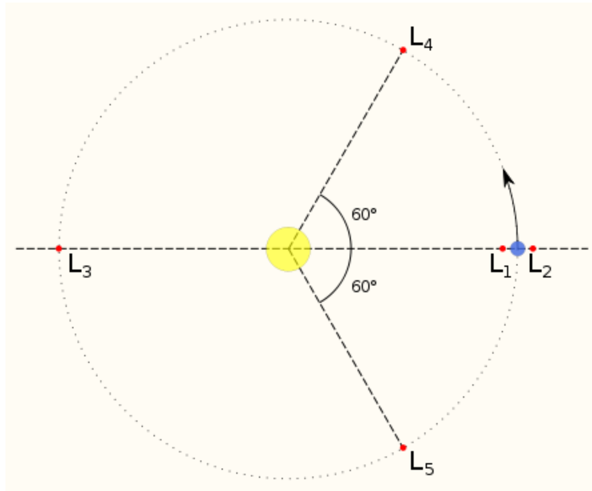


FIGURE 3. The locations of the Lagrange points (image from Wikipedia [10]).

we can calculate the precise positions of each of the three collinear Lagrange points.

At the L_1 point, we see that $r_1 + r_2 = 1$, $r_1 = x + \mu_2$, and $r_2 = -x + \mu_1$. We wish to determine r_1, r_2, and x in terms of one common quantity. Returning to Equation (18),

$$\mu_1 \left(-\frac{1}{(1-r^2)^2} + 1 - r_2 \right) - \mu_2 \left(-\frac{1}{r_2^2} + r_2 \right) = 0$$

which reduces to

$$\mu_1 \left(\frac{-1 + (1-r_2)^3}{(1-r_2)^3} \right) - \mu_2 \left(\frac{-1 + r_2^3}{r_2^3} \right) = 0.$$

Rearranging terms,

$$\frac{\mu_2}{\mu_1} = \frac{\left[-1 + (1-r_2)^3\right] r_2^3}{(1-r_2)^3(-1+r_2^3)}$$

$$= -\frac{\left[-1 + (1-r_2)(1-2r_2+r_2^2)\right] r_2^2}{(1-r_2)^3(r_2^2+r_2+1)}$$

$$= -\frac{(-3r_2 + 3r_2^2 - r_2^3)\, r_2^2}{(1-r_2)^3(r_2^2+r_2+1)}$$

which reduces to

$$\frac{\mu_2}{\mu_1} = 3r_2^3 \, \frac{1 - r_2 + r_2^2/3}{(1 - r_2)^3(r_2^2 + r_2 + 1)}. \tag{22}$$

We now define the quantity

$$\alpha = \left(\frac{\mu_2}{3\mu_1}\right)^{1/3}.$$

Using a series expansion, we then write Equation (22) as

$$\alpha = r_2 + \frac{1}{3}r_2^2 + \frac{1}{3}r_2^3 + \frac{53}{81}r_2^4 + \mathcal{O}(r_2^5)$$

where $\mathcal{O}(r_2^5)$ denotes the terms of order 5 and higher. Since $\mu_1 > \mu_2$, r_2 and α are pretty small; therefore, terms involving α^5 should also be pretty small. Next we follow the procedure used in *Solar System Dynamics* [6] to invert the series:

$$r_2 = \alpha - \frac{1}{3}\alpha^2 - \frac{1}{9}\alpha^3 - \frac{23}{81}\alpha^4 + \mathcal{O}(\alpha^5). \tag{23}$$

We can then solve for r_1:

$$r_1 = 1 - \alpha - \frac{1}{3}\alpha^2 - \frac{1}{9}\alpha^3 - \frac{23}{81}\alpha^4 + \mathcal{O}(\alpha^5) \tag{24}$$

and x:

$$x = \mu_1 - \alpha - \frac{1}{3}\alpha^2 - \frac{1}{9}\alpha^3 - \frac{23}{81}\alpha^4 + \mathcal{O}(\alpha^5). \tag{25}$$

Note that α only depends on μ_1 and μ_2, physically observable properties of the system. Therefore, we can determine r_1, r_2, and x given particular values of μ_1 and μ_2.

At the L_2 point, μ_2 is between the particle and μ_1. There, $r_1 - r_2 = 1$, $r_1 = x + \mu_2$, and $r_2 = x - \mu_1$. We can follow a similar procedure as with L_1 to obtain

$$r_2 = \alpha + \frac{1}{3}\alpha^2 - \frac{1}{9}\alpha^3 - \frac{31}{81}\alpha^4 + \mathcal{O}(\alpha^5). \tag{26}$$

Once again, we can solve for r_1:

$$r_1 = 1 + \alpha + \frac{1}{3}\alpha^2 - \frac{1}{9}\alpha^3 - \frac{31}{81}\alpha^4 + \mathcal{O}(\alpha^5) \tag{27}$$

and x:

$$x = \mu_1 + \alpha + \frac{1}{3}\alpha^2 - \frac{1}{9}\alpha^3 - \frac{31}{81}\alpha^4 + \mathcal{O}(\alpha^5). \tag{28}$$

The L_3 point is located on the opposite side of μ_1 from μ_2, where $r_2 - r_1 = 1$, $r_1 = -x - \mu_2$, and $r_2 = -x + \mu_1$. Solving for μ_2/μ_1,

$$\frac{\mu_2}{\mu_1} = \frac{(1-r_1^3)(1+r_1)^2}{r_1^3\,(r_1^2+3r_1+3)}. \tag{29}$$

We define $r_1 = 1+\beta$ and $r_2 = 2+\beta$ and simplify Equation (29) to get

$$\frac{\mu_2}{\mu_1} = -\frac{12}{7}\beta + \frac{144}{49}\beta^2 - \frac{1567}{343}\beta^3 + \mathcal{O}(\beta^4)$$

so that

$$\beta = -\frac{7}{12}\left(\frac{\mu_2}{\mu_1}\right) + \frac{7}{12}\left(\frac{\mu_2}{\mu_1}\right)^2 - \frac{13223}{20736}\left(\frac{\mu_2}{\mu_1}\right)^3 + \mathcal{O}\left(\frac{\mu_2}{\mu_1}\right)^4.$$

Now we can find r_1:

$$r_1 = 1 - \frac{7}{12}\left(\frac{\mu_2}{\mu_1}\right) + \frac{7}{12}\left(\frac{\mu_2}{\mu_1}\right)^2 - \frac{13223}{20736}\left(\frac{\mu_2}{\mu_1}\right)^3 + \mathcal{O}\left(\frac{\mu_2}{\mu_1}\right)^4, \tag{30}$$

r_2:

$$r_2 = 2 - \frac{7}{12}\left(\frac{\mu_2}{\mu_1}\right) + \frac{7}{12}\left(\frac{\mu_2}{\mu_1}\right)^2 - \frac{13223}{20736}\left(\frac{\mu_2}{\mu_1}\right)^3 + \mathcal{O}\left(\frac{\mu_2}{\mu_1}\right)^4, \tag{31}$$

and x:

$$x = \mu_1 + \frac{7}{12}\left(\frac{\mu_2}{\mu_1}\right) + \frac{7}{12}\left(\frac{\mu_2}{\mu_1}\right)^2 - \frac{13223}{20736}\left(\frac{\mu_2}{\mu_1}\right)^3 + \mathcal{O}\left(\frac{\mu_2}{\mu_1}\right)^4. \tag{32}$$

Once again, given μ_1 and μ_2 we can determine the precise location the L_3 point.

Now we know the locations of the five Lagrange equilibrium points. At these equilibrium points, a particle should remain at rest in the rotating frame of reference. The points are more than just mathematical curiosities—objects are frequently located at Lagrange points in real systems. For example, the Solar and Heliospheric Observatory (SOHO) is located at the L_1 point in the Sun-Earth system, while the Wilkinson Microwave Anisotropy Probe (WMAP) is located at the L_2 point. At the L_4 and L_5 points in the Sun-Jupiter system are the Trojan asteroids, a bunch of rocks clustered together and orbiting the Sun with Jupiter. Finally, Epimetheus oscillates about the L_4 and L_5 points in the Saturn-Janus system, as determined by observations.

5. The Stability of the Lagrange Points

If we consider the moon Epimetheus to be massless, and "drop" it into orbit around Saturn and Janus, the moon's trajectory will behave differently depending on the initial position. According to our analysis of the L_4 and L_5 Lagrange points, if we set Epimetheus at the point $(1/2-\mu_2, \sqrt{3}/2, 0)$, then the moon will not move—the definition of an equilibrium point is that a particle sitting at such a point will have zero velocity and zero acceleration. In the real world, however, any object sitting at an equilibrium point will eventually be perturbed, most likely due to interactions with other orbiting bodies. When such a perturbation occurs, what happens to the particle at the Lagrange point? There are two options:

(1) The particle could be displaced and continue to travel away from the Lagrange point. If this is the case the Lagrange point is considered to be *unstable*.

(2) The particle could start to move away, but then head back toward its equilibrium point. If this is the case the Lagrange point is said to be *stable*.

Unfortunately, our equations of motion are second order, nonlinear, differential equations; therefore, a stability analysis will not be easy. Because we cannot solve our equations analytically, we must attempt a graphical analysis. First, however, we must linearize the equations of motion, or convert the system into a linear system. First we develop the necessary skills to complete a stability analysis.

5.1. **The Stability of Linear Differential Equations.** Consider the system

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

where $\mathbf{x}$ is a 2×1 vector and $\mathbf{A}$ is a 2×2 constant matrix.[7] This system is a linear system with constant coefficients. Ideally we would like to find the eigenvalues λ_i and eigenvectors v_i so that we may obtain a solution of the form $\mathbf{x} = \mathbf{v}e^{\lambda t}$. The solution can be regarded as a parametric representation of a curve in the x_1x_2-plane. A particle located at a particular position with a particular velocity at time t will move through space as time goes on. We call the path of this particle the *trajectory*. For our analysis, we would like to understand the behavior of the particle around the critical points of the system.

Definition 1. *A* critical point*, or an* equilibrium point*, is a point where $\dot{\boldsymbol{x}} = 0$. If $\boldsymbol{A}$ is nonsingular, then the origin is the only critical point of the system.*

To understand how the system evolves, we need to examine the particle's trajectory as time progresses. If the equilibrium point is unstable then the trajectory should head away from the critical point as $t \to \infty$. To begin, let us define what we mean by a stable critical point.

Definition 2. *A critical point $\boldsymbol{x}_0$ of $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x})$ is* stable *if for every $\epsilon > 0$ there exists a $\delta > 0$ such that the solution $\boldsymbol{x} = \boldsymbol{\phi}(t)$:*

(1) *exists for all positive t,*
(2) *satisfies $||\boldsymbol{\phi}(0) - \boldsymbol{x}_0|| < \delta$ at $t = 0$ (in other words, the solution starts "sufficiently close" to $\boldsymbol{x}_0$), and*

[7]The following results generalize to nth order systems as well.

(3) *satisfies* $||\boldsymbol{\phi}(t) - \boldsymbol{x}_0|| < \epsilon$ *for all* $t \geq 0$ *(the solution stays "close" to* $\boldsymbol{x}_0$*).*

If a particle has an initial position close enough to the equilibrium position, then it will stay close to that point. Next we define a much more specific kind of stability.

Definition 3. *A critical point is said to be* asymptotically stable *if it is stable and if there is some* $\delta_0 > 0$ *such that if a solution* $x = \boldsymbol{\phi}(t)$ *satisfies*

$$||\boldsymbol{\phi}(0) - \boldsymbol{x}_0|| < \delta_0,$$

then

$$\lim_{t \to \infty} \boldsymbol{\phi}(t) = x_0.$$

Since the behavior of the system depends on the nature of the eigenvalues, we need to consider the different types of eigenvalues a system could have. There are five separate cases we need to consider.

Case 1—Real, unequal eigenvalues of the same sign: We know the general solution for a system with these types of eigenvalues is

$$\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}.$$

First, let us consider the case where $\lambda_1 < \lambda_2 < 0$. Then we see that $\mathbf{x} \to 0$ as $t \to \infty$—all solutions approach the critical point at the origin as $t \to \infty$. Moreover, if we rewrite the general solution as

$$\mathbf{x} = e^{\lambda_2 t}\left[c_1\mathbf{v}_1 e^{(\lambda_1 - \lambda_2)t} + c_2\mathbf{v}_2\right]$$

then we see that if $c_2 \neq 0$ (in other words, if we are not on the line that describes $\mathbf{v}_1$), then the trajectory will approach the line that describes $\mathbf{v}_2$. We note that λ_1 is "more negative" than λ_2; therefore, the term involving c_1 decays slightly faster than the term involving c_2. The trajectory will approach the line that goes through $\mathbf{v}_2$ as it heads toward the critical point. This can be seen in Figure 4. In this case, the critical point at the origin is called a *node*, or a *nodal sink*. Since

$$\lim_{t \to \infty} \mathbf{x} = \mathbf{0},$$

where $\mathbf{0}$ is the critical point, this system is asymptotically stable.

Next let us consider the situation where $\lambda_1 > \lambda_2 > 0$. In this case, the exponential terms cause $\mathbf{x} \to \infty$ as $t \to \infty$. Since $\lambda_1 > \lambda_2$, the term involving c_1 gets larger faster than the term involving c_2 does. The trajectory is slower to move away from the line through $\mathbf{v}_2$ than the line through $\mathbf{v}_1$. Therefore the trajectory looks very similar to the one in Figure 4, except that the directions of the arrows are reversed. Because the trajectory heads away from the critical point, the system is unstable.

Case 2—Real eigenvalues of opposite sign: Now suppose that $\lambda_1 > 0$ and $\lambda_2 < 0$. The general solution will then have a positive exponential and a negative exponential. The positive exponential will be the dominant term in this solution.

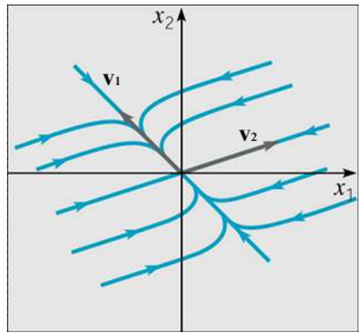


FIGURE 4. An example of Case 1, where $\lambda_1 < \lambda_2 < 0$ (image adapted from Boyce and DiPrima [2]).

Though solutions may be slow to move away from the line through $\mathbf{v}_2$, they will asymptotically approach the line through $\mathbf{v}_1$ as they head away from the critical point. In this configuration, the origin is called a *saddle point* and the system is unstable (see Figure 5).

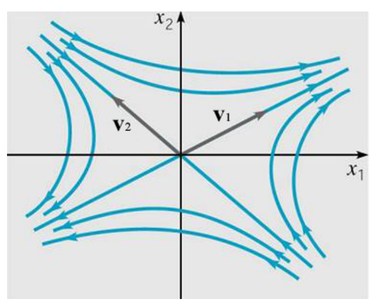
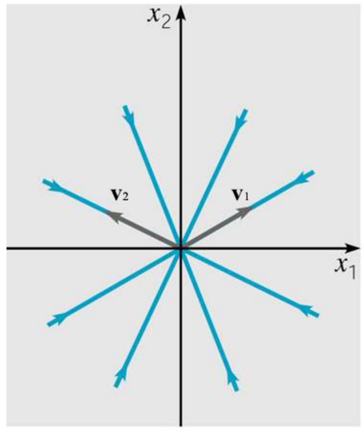


FIGURE 5. An example of Case 2, where $\lambda_1 > 0$ and $\lambda_2 < 0$ (image adapted from Boyce and DiPrima [2]).

Case 3—Equal eigenvalues: First consider the case where the system has two independent eigenvectors, but only one eigenvalue. The general solution can then be written as

$$\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda t} + c_2\mathbf{v}_2 e^{\lambda t}.$$

The ratio of x_2/x_1 does not depend on time, only on c_1, c_2, $\mathbf{v}_1$, and $\mathbf{v}_2$, which are constant in time. This means that all trajectories will follow a straight line. If the eigenvalue is negative, then $\mathbf{x} \to 0$ as $t \to \infty$, meaning that the system is asymptotically stable (Figure 6). In this case, the critical point is a *proper node*, or *star point*. If the eigenvalue is positive, then the system is unstable.



(a)

FIGURE 6. An example of Case 3, where the system has two independent eigenvectors (image adapted from Boyce and DiPrima [2]).

Now consider the case where the system has one independent eigenvector. We then write the general solution as

$$\mathbf{x} = c_1 \mathbf{v} e^{\lambda t} + c_2 \left(\mathbf{v} t e^{\lambda t} + \mathbf{w} e^{\lambda t} \right)$$

where $\mathbf{w}$ is the generalized eigenvector corresponding to λ. It solves the equation

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{w} = \mathbf{v}.$$

As $t \to \infty$, the dominant term is $c_2 \mathbf{v} t e^{\lambda t}$. The trajectory as $t \to \infty$ is therefore asymptotic to the line through $\mathbf{v}$. As $t \to -\infty$, $c_2 \mathbf{v} t e^{\lambda t}$ is once again the dominant term, meaning that the trajectory is once again asymptotic to the line through $\mathbf{v}$. A negative eigenvalue means that each trajectory will go to the critical point at the origin as $t \to \infty$. The case where $\lambda < 0$ is therefore asymptotically stable. If $\lambda > 0$, then the system is unstable. The critical point in these configurations is called an *improper* or *degenerate node*. The relative orientation of the trajectory depends on the relative positions of $\mathbf{v}$ and $\mathbf{w}$, as seen in Figure 7.

Case 4—Complex Eigenvalues: Now assume that the eigenvalues have the form

$$\lambda = j \pm ik$$

where j, k are real, $j \neq 0$, and $k > 0$. In general, systems with complex eigenvalues can be written as

$$\dot{\mathbf{x}} = \begin{pmatrix} j & k \\ -k & j \end{pmatrix} \mathbf{x}$$

or, in scalar form, $\dot{x}_1 = jx_1 + kx_2$ and $\dot{x}_2 = -jx_1 + kx_2$, where x_1 and x_2 are the elements of the vector $\mathbf{x}$. For simplicity we switch to polar coordinates. Recall that

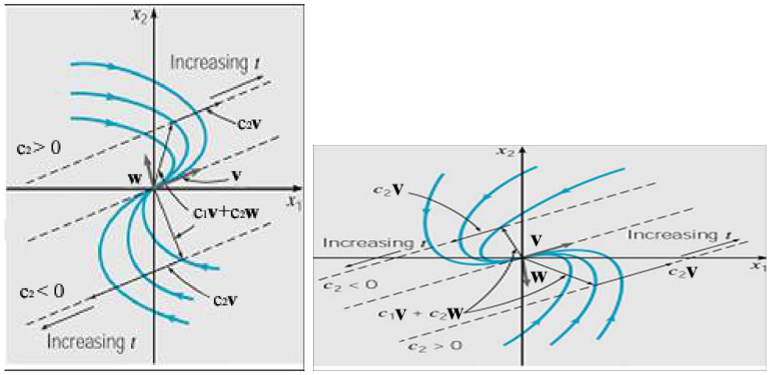


FIGURE 7. Two examples of Case 3, where the systems have one independent eigenvector each (image adapted from Boyce and DiPrima [2]).

$$r^2 = x_1^2 + x_2^2$$

and

$$\tan\theta = \frac{x_2}{x_1}$$

where r is the distance from the origin to a point located at (x_1, x_2) and θ is the angle between the x-axis and the vector from the origin to the point. Implicitly differentiating these results with respect to time, we obtain

$$r\dot{r} = x_1\dot{x}_1 + x_2\dot{x}_2$$

and

$$\sec^2\theta\,\dot{\theta} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2}.$$

Rearranging terms, using our definitions for $\dot{x}_1$ and $\dot{x}_2$ from above, and using geometrical arguments, [8]

$$\begin{aligned} r\dot{r} &= x_1(jx_1 + kx_2) + x_2(-kx_1 + jx_2) \\ &= jx_1^2 + kx_1x_2 - kx_1x_2 + jx_2^2 \\ \dot{r} &= \frac{k(x_1^2 + x_2^2)}{r} \\ &= jr \end{aligned}$$

[8]Note that for a right triangle where r is the hypotenuse and x_1 is the length of the base along the x-axis, $\cos\theta = x_1/r$. This implies that $\sec^2\theta = r^2/x_1^2$.

and

$$\begin{aligned}\sec^2\theta\,\dot{\theta} &= \frac{x_1(-kx_1+jx_2)-x_2(jx_1+kx_2)}{x_1^2}\\ &= \frac{-kx_1^2+jx_1x_2-jx_1x_2-kx_2^2}{x_1^2}\\ &= -k\frac{x_1^2+x_2^2}{x_1^2},\end{aligned}$$

which means that

$$\begin{aligned}\dot{\theta} &= -k\frac{x_1^2+x_2^2}{x_1^2}\cdot\frac{x_1^2}{r^2}\\ &= -k.\end{aligned}$$

Now we know dr/dt and $d\theta/dt$. Since $\dot{r} = jr$, the solution for r is $r = ce^{jt}$, where c is a constant. Since $\dot{\theta} = -k$, the solution for θ is $\theta = -kt+\theta_0$, where θ_0 represents the initial value for θ. A negative value of k implies that θ decreases as t increases; therefore the direction of motion is clockwise. If j is negative, then r goes to zero as $t \to \infty$, meaning the trajectory asymptotically approaches the critical point. If j is positive, then $r \to \infty$ as $t \to \infty$, meaning the system is unstable. Such trajectories are spirals; the critical point is either a *spiral sink* or a *spiral source* depending on the value of j (see Figure 8).

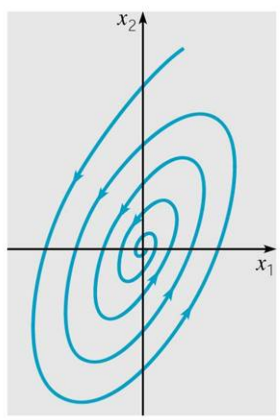


FIGURE 8. An example of Case 4, where the system is asymptotically stable (image adapted from Boyce and DiPrima [2]).

Case 5—Pure Imaginary Eigenvalues: If the eigenvalues are purely imaginary, then $j = 0$ and $\lambda = \pm ik$. We already determined that $r = ce^{jt}$; if $j = 0$, then $r = c$, a constant. The equation for θ is the same as given above. With a constant value of r, the trajectory is a circle around the origin with the direction of the trajectory determined by the value of k.

We can generalize this result to show that any system with purely imaginary eigenvalues must have elliptical trajectories.[9] First off, we consider a system where

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

To find out under what conditions this system will have purely imaginary eigenvalues, we compute the determinant of $\mathbf{A} - \lambda\mathbf{I}$, where λ is an eigenvalue of $\mathbf{A}$, and set it equal to zero:

$$\begin{aligned} \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} &= (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} \\ &= \lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) \\ &= 0. \end{aligned}$$

Using the quadratic formula, we see that

$$\lambda = \frac{a_{11}+a_{22} \pm \sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22}-a_{12}a_{21})}}{2}.$$

We want the eigenvalues to be purely imaginary. This means the real part of the eigenvalue must be zero, so $a_{11} + a_{22} = 0$. The part under the radical must be negative in order for the eigenvalue to be imaginary. Therefore,

$$(a_{11}+a_{22})^2 < 4(a_{11}a_{22} - a_{12}a_{21}).$$

The first quantity is zero, so we see that

$$a_{11}a_{22} - a_{12}a_{21} > 0$$

is the necessary condition for a system to have purely imaginary eigenvalues.

Now we return to our original system of equations. In scalar form, $\dot{x} = dx/dt = a_{11}x + a_{12}y$ and $\dot{y} = dy/dt = a_{21}x + a_{22}y$. We want to express these equations in the form

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a_{21}x + a_{22}y}{a_{11}x + a_{12}y}.$$

If we rearrange this equation into the form

$$(a_{11}x + a_{12}y)dy - (a_{21}x + a_{22}y)dx = 0$$

and define $N = a_{11}x + a_{12}y$ and $M = -a_{21}x - a_{22}y$, we see that

[9]This is Problem 9.1.19 in Boyce and DiPrima [2]

$$\frac{\partial M}{\partial y} = -a_{22} = \frac{\partial N}{\partial x} = a_{11}$$

which means this differential equation is exact. We can obtain a solution $\psi(x, y)$ by defining $\psi_x(x, y) = M$ and $\psi_y(x, y) = N$. Then we integrate each equation and compare them to obtain

$$\psi(x, y) = -\frac{a_{21}}{2}x^2 - a_{21}xy + \frac{a_{12}}{2}y^2 = k$$

such that

$$a_{21}x^2 + 2a_{22}xy - a_{12}y^2 = k \tag{33}$$

An ellipse is a curve defined by an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{34}$$

where the coefficients are real and where $B^2 < 4AC$. In Equation (33), we see that $A = a_{21}$, $B = 2a_{22}$, $C = -a_{12}$, and $F = -k$, where all of these values are real. To check that this system *does* have purely imaginary eigenvalues, we see that

$$\begin{aligned} B^2 - 4AC &= 4a_{22}^2 + 4a_{12}a_{21} \\ &= -4(a_{11}a_{22} - a_{12}a_{21}) < 0 \end{aligned}$$

which satisfies

$$a_{11}a_{22} - a_{12}a_{21} > 0.$$

Therefore, any system with pure imaginary eigenvalues can be expressed in the form of Equation (34) and will therefore have an elliptical trajectory (see Figure 9). In this case, the critical point is called a *center*.

Now we have considered the five possible scenarios for different eigenvalues. See Table 1 for a summary of these results.

For the Saturn-Janus-Epimetheus system, we only want to consider stable solutions that oscillate about the L_4 and L_5 points, as determined by observations. Therefore, any eigenvalues that give unstable solutions are not allowed. The Saturn-Janus-Epimetheus system is not as simple as the linear systems we have investigated in this section. Because we are dealing with second order equations, we have four differential equations even if we are dealing in only two dimensions—in fact, if we consider all three dimensions, we have six differential equations. A system of four differential equations means we are dealing with a 4×4 matrix and have four eigenvalues to find instead of two. The stability analysis should be similar, however, as we need to ensure that each term in our solution gives us a stable solution. Thus,

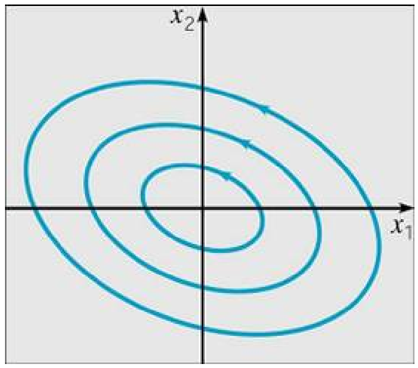


FIGURE 9. An example of Case 5, where the eigenvalues are purely imaginary (image adapted from Boyce and DiPrima [2]).

Eigenvalues	Type of Critical Point	Stability
$\lambda_1 > \lambda_2 > 0$	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically Stable
$\lambda_2 < 0 < \lambda_1$	Saddle Point	Unstable
$\lambda_1 = \lambda_2 > 0$	Proper or improper node	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper of improper node	Asymptotically stable
$\lambda_{1,2} = j \pm ik$	Spiral point	-
$\lambda > 0$	-	Unstable
$\lambda < 0$	-	Asymptotically stable
$\lambda_{1,2} = \pm ik$	Center	Stable

TABLE 1. Stability Properties of Linear Systems

we cannot have exponential terms heading off to infinity as $t \to \infty$. To determine the stability of the system, we must linearize the system and find the eigenvalues.

5.2. **Linearizing the Equations of Motion.** To analyze the eigenvalues of our equations of motion, we need to linearize the system, or put it into a form like the systems in Section 5.1. To do so, we assume that Epimetheus has been slightly displaced from its equilibrium position (x_0, y_0) so that its new position is $(x_0 + X, y_0 + Y)$, where X and Y are variables. We now investigate motion close to the equilibrium point, which should be approximately linear. Substituting our new positions into Equations (9) and (10), we obtain

$$\ddot{X} - 2n\dot{Y} = \frac{\partial U}{\partial x} \tag{35}$$

and

$$\ddot{Y} + 2n\dot{X} = \frac{\partial U}{\partial y} \tag{36}$$

where

$$U = \frac{n^2}{2}\left[(x_0 + X)^2 + (y_0 + Y)^2\right] + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2},$$

$$r_1 = \sqrt{(x_0 + X + \mu_2)^2 + (y_0 + Y)^2},$$

and

$$r_2 = \sqrt{(x_0 + X - \mu_1)^2 + (y_0 + Y)^2}.$$

Now we will expand the right hand sides of the Equations (35) and (36) in a Taylor series about the equilibrium point (the subscript 0 means that we evaluate the quantity at the critical points):

$$\begin{aligned}\ddot{X} - 2n\dot{Y} &\approx \left(\frac{\partial U}{\partial x}\right)_0 + X\left(\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)\right)_0 + Y\left(\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right)\right)_0 \\ &= X\left(\frac{\partial^2 U}{dx^2}\right)_0 + Y\left(\frac{\partial^2 U}{\partial x \partial y}\right)_0\end{aligned}$$

$$\begin{aligned}\ddot{Y} + 2n\dot{X} &\approx \left(\frac{\partial U}{\partial y}\right)_0 + X\left(\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right)\right)_0 + Y\left(\frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}\right)\right)_0 \\ &= X\left(\frac{\partial^2 U}{\partial x \partial y}\right)_0 + Y\left(\frac{\partial^2 U}{\partial y^2}\right)_0.\end{aligned}$$

Equations (35) and (36) show that $(\partial U/\partial x)_0 = (\partial U/\partial y)_0 = 0$ at the equilibrium points since the acceleration and velocity of X and Y ($\dot{X}$, $\dot{Y}$, $\ddot{X}$, and $\ddot{Y}$) are equal to zero. Because we are only considering small displacements from the equilibrium point, i.e. small values of X and Y, the higher order terms of the Taylor expansion are irrelevant. Now we have two linear equations:

$$\ddot{X} - 2\dot{Y} = XU_{xx} + YU_{xy}$$

and

$$\ddot{Y} + 2\dot{X} = XU_{xy} + YU_{yy}$$

where the constants U_{xx}, U_{xy}, and U_{yy} are the partial derivatives evaluated at the critical points. We can rewrite this system as four first-order differential equations in matrix form:

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \ddot{X} \\ \ddot{Y} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & -2 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ \dot{X} \\ \dot{Y} \end{pmatrix}.$$

Our equations are now in the form $\dot{\mathbf{X}} = \mathbf{AX}$. We have successfully linearized our equations of motion.

We shall now linearize the system by another method in order to verify that the previous method works. This method is described in Boyce and DiPrima's *Elementary Differential Equations* [2].

Equations (9) and (10) are two second-order differential equations that describe the motion of the system. However, we can convert to the system to four first-order differential equations. Define:

$$u_1 = x$$
$$u_2 = \dot{x}$$
$$u_3 = y$$

and

$$u_4 = \dot{y}.$$

Combining these results with Equations (9) and (10), we obtain

$$\dot{u_1} = u_2 \tag{37}$$

$$\dot{u_2} = \frac{\partial U}{\partial u_1} + 2nu_4 \tag{38}$$

$$\dot{u_3} = u_4 \tag{39}$$

and

$$\dot{u_4} = \frac{\partial U}{\partial u_3} - 2nu_2. \tag{40}$$

The next step is to compute the partial derivatives of Equations (37), (38), (39), and (40) with respect to u_1, u_2, u_3, and u_4; the results are displayed in Table 2. The horizontal quantities represent the partial derivatives of the vertical quantities with respect to a particular quantity. Note that the scalar function U is a function of both u_1 and u_3 (or of x and y).

We put these values into a matrix and convert back to x and y to express the linearized system as

	$\frac{\partial}{\partial u_1}$	$\frac{\partial}{\partial u_2}$	$\frac{\partial}{\partial u_3}$	$\frac{\partial}{\partial u_4}$
$\dot{u_1}$	0	1	0	0
$\dot{u_2}$	$\frac{\partial^2 U}{\partial u_1^2}$	0	$\frac{\partial^2 U}{\partial u_1 \partial u_3}$	2
$\dot{u_3}$	0	0	0	1
$\dot{u_4}$	$\frac{\partial^2 U}{\partial u_1 \partial u_3}$	-2	$\frac{\partial^2 U}{\partial u_3^2}$	0

TABLE 2. The partial derivatives

$$\begin{pmatrix} \dot{x} \\ \ddot{x} \\ \dot{y} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\partial^2 U}{\partial x^2} & 0 & \frac{\partial^2 U}{\partial x \partial y} & 2 \\ 0 & 0 & 0 & 1 \\ \frac{\partial^2 U}{\partial x \partial y} & -2 & \frac{\partial^2 U}{\partial y^2} & 0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{pmatrix}.$$

The second derivatives are evaluated at a critical point because we are linearizing the system about a critical point. Using Maple, we can evaluate these partial derivatives of U at the chosen equilibrium. We have now forced our system into the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

and thus have successfully linearized the system. The result is the same as when we used a Taylor expansion.

5.3. **Uncoupling the Equations of Motion.** Unfortunately, our linearized equations of motion are still coupled. Because of this, we must either solve the equations simultaneously or we must transform the system into an uncoupled system where each equation depends on only one unknown. Suppose that there is some matrix, $\mathbf{B}$, that transforms the coupled system $\mathbf{X}$ into an uncoupled system $\mathbf{Y}$. Then $\mathbf{Y} = \mathbf{B}\mathbf{X}$ implies that $\mathbf{X} = \mathbf{B}^{-1}\mathbf{Y}$, assuming the matrix $\mathbf{B}$ is invertible (we shall check the validity of this assumption momentarily). Differentiating with respect to time, $\dot{\mathbf{X}} = \mathbf{B}^{-1}\dot{\mathbf{Y}}$. Putting all of this together, we see that

$$\mathbf{B}^{-1}\dot{\mathbf{Y}} = \mathbf{A}\mathbf{B}^{-1}\mathbf{Y}$$

or

$$\dot{\mathbf{Y}} = \mathbf{B}\mathbf{A}\mathbf{B}^{-1}\mathbf{Y}.$$

We now define $\mathbf{\Lambda} = \mathbf{B}\mathbf{A}\mathbf{B}^{-1}$, where Λ is a diagonal matrix whose entries are the eigenvalues of $\mathbf{A}$ and we see that $\mathbf{B}$ must be the matrix whose columns are the linearly independent eigenvectors of $\mathbf{A}$.[10] We then write

[10] This decomposition is allowed if $\mathbf{A}$ is an $n \times n$ matrix with n linearly independent eigenvectors, which we shall assume to be true. In our system, A is a 4×4 matrix, which is certainly a square

$$\dot{\mathbf{Y}} = \mathbf{\Lambda Y}.$$

Each component of this system is a first-order linear differential equation. The solutions are

$$Y_i = c_i e^{\lambda_i t} \quad \text{where} \quad i = 1, 2, 3, 4.$$

Therefore, we can write

$$\mathbf{X} = \mathbf{B}^{-1} \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_3 t} \\ c_4 e^{\lambda_4 t} \end{pmatrix}.$$

A complete analysis of the system requires that we determine the eigenvectors. For our purposes (determining the stability of the L4 and L5 Lagrange points), it is not necessary to determine the eigenvectors. Instead, we simply note that x, $\dot{x}$, y, and $\dot{y}$ can be expressed as

$$x = \sum_{j=1}^{4} \bar{\alpha}_j e^{\lambda_j t},$$

$$\dot{x} = \sum_{j=1}^{4} \bar{\alpha}_j \lambda_j e^{\lambda_j t}$$

$$y = \sum_{j=1}^{4} \bar{\beta}_j e^{\lambda_j t},$$

and

$$\dot{y} = \sum_{j=1}^{4} \bar{\beta}_j \lambda_j e^{\lambda_j t}$$

where the $\bar{\alpha}_j$ and $\bar{\beta}_j$ are constants. Following the analysis in Section 5.1, our next step is to calculate the eigenvalues.

5.4. **Finding the Eigenvalues.** In order to calculate the eigenvalues of our matrix $\mathbf{A}$, we need to solve the equation

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

which means we need to solve

matrix. Furthermore, we also see that since the columns of $\mathbf{B}$ are linearly independent and since $\mathbf{B}$ is a square matrix, $\mathbf{B}$ must be invertible, supporting our previous assumption.

$$\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ U_{xx} & -\lambda & U_{xy} & 2 \\ 0 & 0 & -\lambda & 1 \\ U_{xy} & -2 & U_{yy} & -\lambda \end{vmatrix} = 0$$

for λ. Note that we have abbreviated the partial second derivatives using the subscript notation—also recall that these second derivatives are evaluated at a critical point. Evaluating the determinant, we see that

$$\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ U_{xx} & -\lambda & U_{xy} & 2 \\ 0 & 0 & -\lambda & 1 \\ U_{xy} & -2 & U_{yy} & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & U_{xy} & 2 \\ 0 & -\lambda & 1 \\ -2 & U_{yy} & -\lambda \end{vmatrix} - \begin{vmatrix} U_{xx} & U_{xy} & 2 \\ 0 & -\lambda & 1 \\ U_{xy} & U_{yy} & -\lambda \end{vmatrix}$$

$$= \lambda^2 \begin{vmatrix} -\lambda & 1 \\ U_{yy} & -\lambda \end{vmatrix} + 2\lambda \begin{vmatrix} U_{xy} & 2 \\ -\lambda & 1 \end{vmatrix} - U_{xx} \begin{vmatrix} -\lambda & 1 \\ U_{yy} & -\lambda \end{vmatrix} - U_{xy} \begin{vmatrix} U_{xy} & 2 \\ -\lambda & 1 \end{vmatrix}$$

$$= \lambda^2(\lambda^2 - U_{yy}) + 2\lambda(U_{xy} + 2\lambda) - U_{xx}(\lambda^2 - U_{yy}) - U_{xy}(U_{xy} + 2\lambda)$$

$$= \lambda^4 - \lambda^2 U_{yy} + 2\lambda U_{xy} + 4\lambda^2 - U_{xx}\lambda^2 + U_{xx}U_{yy} - Uxy^2 - 2\lambda U_{xy}$$

$$= \lambda^4 + (4 - U_{yy} - U_{xx})\lambda^2 + (U_{xx}U_{yy} - U_{xy}^2).$$

Since this determinant must equal zero, we obtain the biquadratic equation

$$\lambda^4 + (4 - U_{yy} - U_{xx})\lambda^2 + (U_{xx}U_{yy} - U_{xy}^2) = 0 \tag{41}$$

Using the quadratic formula on Equation (41), we can solve for λ^2:

$$\lambda^2 = \frac{1}{2}\left[-4 + U_{yy} + U_{xx} \pm \sqrt{(4 - U_{yy} - U_{xx})^2 - 4(U_{xx}U_{yy} - U_{xy}^2)} \right].$$

From this, we see that the four eigenvalues are

$$\lambda_{1,2} = \pm \left[\frac{1}{2}(-4 + U_{yy} + U_{xx}) + \frac{1}{2}\sqrt{(U_{yy} + U_{xx} - 4)^2 - 4(U_{xx}U_{yy} - U_{xy}^2)} \right]^{1/2}$$

and

$$\lambda_{3,4} = \pm \left[\frac{1}{2}(-4 + U_{yy} + U_{xx}) - \frac{1}{2}\sqrt{(U_{yy} + U_{xx} - 4)^2 - 4(U_{xx}U_{yy} - U_{xy}^2)} \right]^{1/2}.$$

The general form of the eigenvalues is $\lambda_{1,2} = \pm(a_1 + b_1 i)$ and $\lambda_{3,4} = \pm(a_2 - b_2 i)$. Looking at our general solution for x, we see that because we have four eigenvalues with four real parts, two positive and two negative, there will always be some positive real term in the exponential term. Therefore, the solution will grow exponentially if $j \neq 0$. Section 5.1 shows that solutions that grow exponentially are unstable. Thus, the only stable solutions are ones where the eigenvalues are purely imaginary.

Recall that $x = 1/2 - \mu_2$ and $y = \pm\sqrt{3}/2$ at the L4 and L5 Lagrange points. Using Maple to determine the partial derivatives U_{xx}, U_{xy}, and U_{yy} and evaluate them at the L_4 and L_5 points, we get

$$U_{xx} = \frac{3}{4}$$
$$U_{yy} = \frac{9}{4}$$

and

$$U_{xy} = \pm\frac{3\sqrt{3}\,(1-2\mu_2)}{4}.$$

Using these values in our equations for the eigenvalues, we see that

$$\lambda_{1,2} = \pm\frac{\sqrt{-1-\sqrt{1-27(1-\mu_2)\mu_2}}}{\sqrt{2}}$$

and

$$\lambda_{3,4} = \pm\frac{\sqrt{-1+\sqrt{1-27(1-\mu_2)\mu_2}}}{\sqrt{2}}.$$

In order for the solution to be purely imaginary,

$$1 - 27(1-\mu_2)\mu_2 \geq 0$$

or

$$\mu_2 \leq \frac{27-\sqrt{621}}{54} \approx 0.0385.$$

Recall that in our system of units

$$\begin{aligned}\mu_2 &= Gm_2 \\ &= \frac{Gm_2}{G(m_1+m_2)} \\ &= \frac{m_2}{m_1+m_2}\end{aligned}$$

and $\mu_1 = 1 - \mu_2$. Since the mass of Saturn is $m_1 = 568.47e24$ kg and the mass of Janus is $m_2 = 1.98e18$ kg, we can say that $\mu_2 = 3.48e-9$ and $\mu_1 = 1 - \mu_2$ for the Saturn-Janus-Epimetheus system. We see that this value of μ_2 satisfies our requirements for a stable system.

6. The Initial Conditions

The previous sections have sought to determine positions where a particle remains stationary in the rotating frame of reference. The L_4 and L_5 Lagrange points are stable equilibrium points in the rotating frame of reference—if the particle receives a small displacement from one of these points it should oscillate about that point. We wish to create a computer animation of this trajectory for the Saturn-Janus-Epimetheus system. From observations, we know that in a rotating frame where Janus and Saturn are stationary, Epimetheus oscillates about both the L_4 and L_5 points in a type of orbit known as a *horseshoe* orbit (see Figure 10). In order to animate this orbit, however, we must have some initial conditions for the system, both for position and velocity.

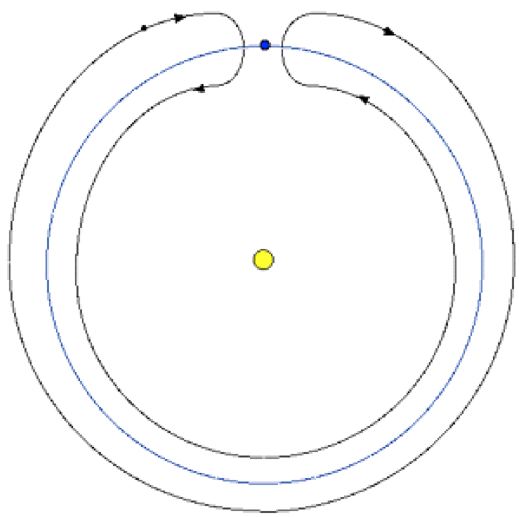


Figure 10. An example of a horseshoe orbit. The large and small dots are m_1 and m_2 respectively. This trajectory occurs in the rotating frame, where m_1 and m_2 are fixed (image from www.exo.net [4]).

6.1. **The Rotating Frame.** Recall that on the zero-velocity curves, we have $2U = C_J$, or

$$(x^2 + y^2) + 2\left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right) = C_J. \tag{42}$$

Using a series expansion on these terms, inserting the appropriate values of r_1, r_2, and x (Equations (23), (24), (25), (26), (27), (28), (30), (31), and (32)), and disregarding terms with higher powers than $\mu_2^{\frac{1}{2}}$, we can write the values of C_J at the Lagrange points:

$$C_{L_1} \approx 3 + 3^{4/3}\mu_2^{2/3} \tag{43}$$

$$C_{L_2} \approx 3 + 3^{4/3}\mu_2^{2/3} \tag{44}$$

$$C_{L_3} \approx 3 + \mu_2 \tag{45}$$

$$C_{L_4} \approx 3 - \mu_2 \tag{46}$$

and

$$C_{L_5} \approx 3 - \mu_2. \tag{47}$$

Now consider a generic horseshoeing orbit. Such an orbit periodically oscillates about the L_3 point as it moves between the L_4 and L_5 points. When the orbit is as elongated as possible, then the "massless" particle reaches the L_1 and L_2 points. A real horseshoe orbit will orbit between these two extremes. We can thus parametrize the Jacobi constant for a horseshoeing system as

$$C_J = 3 + \zeta\mu_2^{2/3} \tag{48}$$

where $0 \leq \zeta \leq 3^{4/3}$. The lowest possible value of ζ corresponds to the Jacobi constant at the L_3 point, while the highest possible value corresponds to the Jacobi constant at the L_1 and L_2 points.

We wish to consider locations where the velocity of the particle is nonzero. Notice that the positions of the L_4 and L_5 points (Equations (20) and (21)) lie on the unit circle. Consider a small displacement δr from the unit circle centered on the center of mass of the system. Also assume that that the orbit is circular where the radius of the orbit is $r = 1 + \delta r$. The motion of a circular body in this situation should be "near-Keplerian," which implies that $v \approx -\frac{3}{2}\delta r$, from Kepler's third law. Again using series expansions, we determine that

$$2U = 3 + 3\delta r^2 + \mu_2\left(\frac{2}{r_2} + r_2^2 - 4\right) = C_J.$$

Using a series of algebraic manipulations, we can solve for the small displacement δr in terms of ζ and μ_2:

$$\delta r = 2\left(\frac{\zeta}{3}\right)^{1/2}\mu_2^{1/3}.$$

An observable property of the Saturn-Janus-Epimetheus system is the value of δr. If we assume that $\delta r = 3e-4$, then

$$\zeta = 3\left(\frac{\delta r}{2\mu_2^{1/3}}\right)^2 \tag{49}$$

$$= 1.0715e-5. \tag{50}$$

Using this value of ζ, we now calculate the value of the Jacobi constant, C_J. Then, picking a position, we determine U and therefore determine a value for v. Using this method, we determine possible initial conditions with which we can animate the system. For example suppose we choose an initial position on the $-x$-axis, at $x = -1 + \delta r = -1 + 3e - 4 = -0.9997$ and $y = 0$. We know that $\mu_2 = 3.48 \times 10^{-9}$ and $\mu_1 = 1 - \mu_2$. We can then calculate

$$r_1 = \sqrt{(x + \mu_2)^2 + y^2} \approx 0.9997$$

and

$$r_2 = \sqrt{(x - \mu_1)^2 + y^2} \approx 1.9997.$$

The value of the scalar function U is therefore

$$U = \frac{1}{2}(x^2) + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} \approx 1.5000.$$

The value of ζ is

$$\zeta = 3\left(\frac{3 \times 10^{-4}}{2\mu_2^{1/3}}\right)^2 \approx 0.02308$$

which means that the value of the Jacobi constant is

$$C_J = 3 + \zeta\mu_2^{2/3} \approx 3.00000.$$

Our initial velocity is therefore

$$v = \sqrt{2U - C_J} \approx 4.5559 \times 10^{-4}.$$

6.2. **The Inertial Frame.** Using the same sort of process as in the rotating frame, we can animate the trajectory of Epimetheus in the inertial frame. However, the trajectory of the particle in the inertial frame depends on the position of Janus, which is also moving in time. We know that Janus orbits about Saturn (which for all intents and purposes we can consider to be stationary, since it is so massive) with an angle t between Janus, Saturn, and the original location of Janus, where t is a unitless, arbitrary quantity of time. Therefore, Janus moves about the origin in a circle, with the equations of motion

$$x = \cos(t), \quad y = \sin(t), \quad \text{and} \quad z = 0$$

and with an initial position of $[1, 0]$ and a constant velocity of 1. Epimetheus' initial velocity then is the same value as before, but with the velocity of Janus subtracted. We also take care that they are moving in the same direction. Therefore, Epimetheus must have an initial velocity in the $-y$ direction while Janus must have an initial velocity in the $+y$ direction.

In the inertial frame, Epimetheus does not undergo this horseshoe motion. Rather, its radius changes as it swaps the inner and outer positions with Janus.

7. Numerical Integration

Recall that the equations of motion, Equations (6), (7), and (8), are coupled, nonlinear, second order differential equations. We cannot analytically solve these equations. In order to animate the system with respect to time we must numerically integrate the equations of motion using MATLAB. To begin, we consider a simpler case than the Saturn-Janus-Epimetheus system. Suppose we have the differential equation

$$y' = f(t, y) \tag{51}$$

with the initial condition $y(t_0) = y_0$. Suppose we wish to solve this differential equation over the interval $[t_0, M]$. The following are several methods for numerically integrating Equation (51).

7.1. **Euler's Method.** The basic idea of Euler's Method is to divide the interval $[a, b]$ into M equal subintervals, so that the stepsize is $h = (b - a)/M$. We then calculate the derivative at the beginning of each subinterval and use that slope to extrapolate forward to determine the next point. Thus, Euler's Method is an iterative process. We start with the initial point, $y(t_0) = y_0$. The derivative at y_0 is $y' = f(t_0, y_0)$. We can then write

$$y_1 = y_0 + hf(t_0, y_0).$$

For the next point we can write

$$y_2 = y_1 + hf(t_1, y_1)$$

where $f(t_1, y_1)$ is the slope of the function at the point (t_1, y_1). Therefore,

$$e_k = y(t_k) - y_k \quad \text{for } k = 1, 2, ..., M.$$

We define the global discretization error, e_k, to be the difference between the actual solution and the approximate solution:

$$e_k = y(t_k) - y_k \quad \text{for } k = 1, 2, ..., M.$$

To sum up this process, we use the differential equation and the initial conditions to create a series of line segments that approximate the solution (see Figure 11).

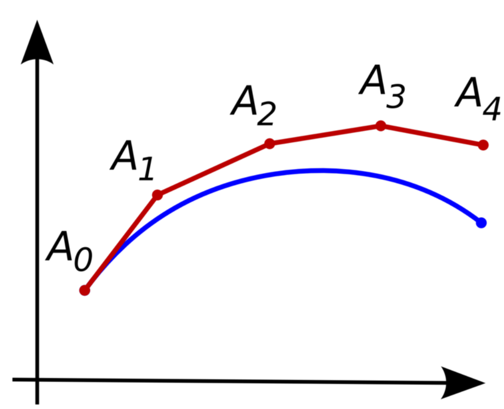


FIGURE 11. An example of Euler's Method (image from Wikipedia [9]).

Is Euler's Method accurate or efficient? The following theorem examines the precision of Euler's Method.

Theorem 7.1. *Assume that $y(t)$ is the solution to the initial value problem $y' = f(t, y)$ over the interval $[t_0, t_M]$ with $y(t_0) = y_0$. If $y(t) \in C^2[t_0, b]$, where $C^2[t_0, b]$ is the set of all twice-differentiable functions whose second derivatives are continuous on the interval $[t_0, b]$, and $(t_k, y_k)_{k=0}^M$ is the sequence of approximations generated by Euler's method, then*

$$|e_k| = |y(t_k) - y_k| = \boldsymbol{O}(h)$$

is the error accumulated in each step of the iteration. The error at the end of the interval is called the final global error*, or F.G.E.:*

$$E(y(t_M), h) = |y(T_M) - y_M| = \boldsymbol{O}(h)$$

where $\boldsymbol{O}(h)$ represents terms of order h.

The F.G.E. is proportional to h. Thus, large step sizes will cause large errors. This agrees with intuition—smaller step sizes will keep the approximation from deviating too much from the actual solution. Smaller step sizes, however, could be computationally intensive.

7.2. **The Taylor Method.** Recall Taylor's Theorem:

Theorem 7.2 (Taylor's Theorem)**.** *Assume that $y(t) \in C^{N+1}[t_0, b]$. Then $y(t)$ has a Taylor series expansion of order N about the fixed value $t = t_k \in [t_0, b]$:*

$$y(t_k + h) = y(t_k) + hT_N(t_k, y(t_k)) + \boldsymbol{O}(h^{N+1})$$

where

$$T_N(t_k, y(t_k)) = \sum_{j=1}^{N} \frac{f^{(j-1)}(t, y(t))}{j!} h^{j-1}.$$

In order to obtain an approximation to $y(t)$, we find a solution to the differential equation by finding the Taylor series representation of $y(t)$ on each subinterval $[t_k, t_{k+1}]$. We can write out the Taylor Series

$$y_{k+1} = y_k + hy'(t_k) + h^2 \frac{y''(t_k)}{2!} + h^3 \frac{y'''(t_k)}{3!} + \cdots + \frac{h^N y^{(N)}(t_k)}{N!}$$

for each step $k = 0, 1, ..., M$. The final global error is given by

$$E(y(b), h) = |y(b) - y_M| = \mathbf{O}(h^N).$$

If we are using a fourth order Taylor series with a step size of $h = 1/2$, the F.G.E. will be $\mathbf{O}(1/16)$. Compared to Euler's Method, which would give an F.G.E. $\mathbf{O}(1/2)$, the Taylor Method is more precise for this step size. In fact, the Taylor Method will be more precise as long as $N > 1$ and $h < 1$. It is possible to choose N as small as is necessary to minimize error while retaining efficiency.

Despite the advantages of the Taylor Method, there are some drawbacks. We need to have some idea of what value of N to use before beginning our computations.

We also have to compute higher derivatives, which may be quite tedious. The next method is more suited for our purpose.

7.3. **The Runge-Kutta Method.** The basic concept of the Runge-Kutta Method is similar to Euler's Method. Whereas Euler's Method uses one derivative at the beginning of the interval to extrapolate forward, the Runge-Kutta method uses derivatives from N different locations in each subinterval. If $N = 4$, for example, we use a derivative from the beginning, the end, and two from the middle (see Figure 12).

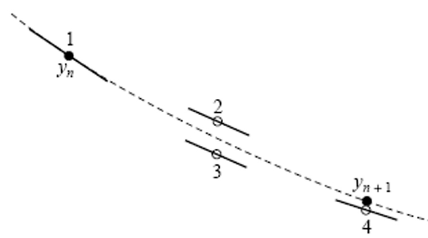


FIGURE 12. The slopes used in the RK4 method (image from Press et al. [7]).

The main advantage to the Runge-Kutta Method is that it has the same F.G.E. order, $\mathbf{O}(h^N)$, as the Taylor Method, meaning that we can choose a specific N to minimize the error; generally a common choice is $N = 4$. This provides four derivatives, which will help to make the approximation more accurate; however, it does not use more derivatives than necessary so as to save computing time.

Like we did with Euler's Method, we wish to create a series of line segments to approximate the solution. To use the Runge-Kutta Method, we compute

$$y_{k+1} = y_k + w_1k_1 + w_2k_2 + w_3k_3 + w_4k_4$$

where w_1, w_2, w_3, w_4 are constants and

$$\begin{aligned}
k_1 &= hf(t_k, y_k)\\
k_2 &= hf(t_k + a_1h, y_k + b_1k_1)\\
k_3 &= hf(t_k + a_2h, y_k + b_2k_1 + b_3k_2)\\
k_4 &= hf(t_k + a_3h, y_k + b_4k_1 + b_5k_2 + b_6k_3).
\end{aligned}$$

Comparing these coefficients with the Taylor series coefficients, we determine that

$$
\begin{aligned}
&b_1 = a_1\\
&b_2 + b_3 = a_2\\
&b_4 + b_5 + b_6 = a_3\\
&w_1 + w_2 + w_3 + w_4 = 1\\
&w_2 a_1 + w_3 a_2 + w_4 a_3 = \frac{1}{2}\\
&w_2 a_1^2 + w_3 a_2^2 + w_4 a_3^2 = \frac{1}{3}\\
&w_2 a_1^3 + w_3 a_2^3 + w_4 a_3^3 = \frac{1}{4}\\
&w_3 a_1 b_3 + w_4(a_1 b_5 + a_2 b_6) = \frac{1}{6}\\
&w_3 a_1 a_2 b_3 + w_4 a_3(a_1 b_5 + a_2 b_6) = \frac{1}{8}\\
&w_3 a_1^2 b_3 + w_4(a_1^2 b_5 + a_2^2 b_6) = \frac{1}{12}\\
&w_4 a_1 b_3 b_6 = \frac{1}{24}
\end{aligned}
$$

If we choose $a_1 = 1/2$ and $b_2 = 0$, then $a_2 = 1/2$, $a_3 = 1$, $b_1 = 1/2$, $b_3 = 1/2$, $b_4 = 0$, $b_5 = 0$, $b_6 = 1$, $w_1 = 1/6$, $w_2 = 1/3$, $w_3 = 1/3$, and $w_4 = 1/6$. Using these values, we can write

$$y_{k+1} = y_k + \frac{h(f_1 + 2f_2 + 2f_3 + f_4)}{6}$$

where

$$
\begin{aligned}
f_1 &= f(t_k, y_k),\\
f_2 &= f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f_1\right),\\
f_3 &= f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f_2\right),
\end{aligned}
$$

and

$$f_4 = f(t_k + h, y_k + h f_3).$$

7.4. **A Comparison of the Euler and Runge-Kutta Methods.** [11]

Consider the initial value problem

$$y' = 1 - t + 4y, \quad \text{where } y(0) = 1.$$

Suppose we select a stepsize of $h = 0.2$. Then we can calculate

[11] This example is from *Elementary Differential Equations, 8th Edition* [2]

$$f_1 = f(0,1) = 5,$$
$$f_2 = f(0+0.1, 1+0.5) = 6.9,$$
$$f_3 = f(0+0.1, 1+0.69) = 7.66,$$

and

$$f_4 = f(0+0.2, 1+1.532) = 10.928.$$

Therefore,

$$y_1 = 1 + \frac{0.2}{6}\left(5 + 2(6.9) + 2(7.66) + 10.928\right) = 2.5016.$$

We produce a table comparing the numerical approximations using various methods:

	Euler	Runge-Kutta			Exact
t	h=0.025	h=0.2	h=0.1	h=0.05	
0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.6079462		1.6089333	1.6090338	1.6090418
0.2	2.5020619	2.5016000	2.5050062	2.5053060	2.5053299
0.3	3.8228282		3.8294145	3.8300854	3.301388
0.4	5.7796888	5.7776358	5.7927853	5.7941197	5.7942260
0.5	8.6849039		8.7093175	8.7118060	8.7120041
1.0	64.497931	64.441579	64.858107	64.894875	64.897803
1.5	474.83402		478.81928	479.22674	479.25919
2.0	3496.6702	3490.5574	3535.8667	3539.8804	3540.2001

The table clearly shows that the Runge-Kutta Method is far more precise than Euler's Method, even when comparing a step size of $h = 0.025$ with Euler to a step size of $h = 0.1$ with Runge-Kutta.

7.5. **An Example of Numerical Integration: The Lorenz Equations.** In order to become familiar with MATLAB and its ability to solve differential equations, we examine a system of equations that is simpler than the equations of motion for the restricted three-body problem. The Lorenz Equations were developed by Edward N. Lorenz, an American meteorologist who wished to study the Earth's atmosphere. The temperature of the atmosphere is not constant with altitude; this difference in temperature causes warm air to rise and low air to sink, an effect known as convection. For large temperature differences the convection is complex and turbulent.

To model this effect, Lorenz developed his equations:

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = rx - y - xz$$
$$\frac{dz}{dt} = -bz + xy$$

where x, y, and z are spatial coordinates and the terms σ, r, and b are constants.[12] These equations are nonlinear, first order differential equations. We know σ and b fairly accurately for the Earth, since they depend on the composition and geometrical properties of the atmosphere. In this example, we let $\sigma = 10$ and $b = 8/3$. The constant r, however, depends on the temperature difference in the various layers of the atmosphere, which enables us to choose different values.

We can solve the Lorenz equations using numerical approximations. Suppose we wish to use the Runge-Kutta method. MATLAB has a special command called **ode45**; this command uses the fourth order Runge-Kutta method with a given step size and compares it to the fifth order Runge-Kutta method with the same step size. If the accuracy is below a set desirability, then the methods are performed again with smaller step sizes.

Suppose we choose a solution interval from $t = 0$ to $t = 22$ and let the initial conditions be $x = 5$, $y = 5$, and $z = 5$. We wish to plot the solution for x versus t. As is evident in Figure 13, the solution is rather erratic. It jumps between positive and negative values, without any evident pattern.

Now suppose we use a different initial condition, say $x = 5.01$, $y = 5$, and $z = 5$. We compare the two results in Figure 14. Until approximately $t = 12$, the two paths follow a similar pattern. After $t = 12$, they become extremely different. It is interesting to note that such a large change becomes evident from such a small difference in initial position.

Now suppose we wish to adjust the value of r. To do so we can create another function and generate more solutions. If we wish to compare the graphs, we can plot them up together, as in Figure 15.

Now suppose we wish to examine the behavior of x, y, and z together, as time evolves. The natural way to do this is with an animation. We animate the system using Euler's Method (see the Appendix). The result is shown in Figure 16. Our animation agrees well with the built-in **Lorenz** script in MATLAB (see Figure 17).

When animating the Saturn-Janus-Epimetheus system, we will use the Runge-Kutta Method through the **ode45** command.

[12]The constant σ is the Prandtl number, a "dimensionless number approximating the ratio of momentum diffusivity and thermal diffusivity" [1] while r is the Rayleigh number, "associated with buoyancy driven flow (also known as free convection or natural convection)" [11] and b is a physical proportion.

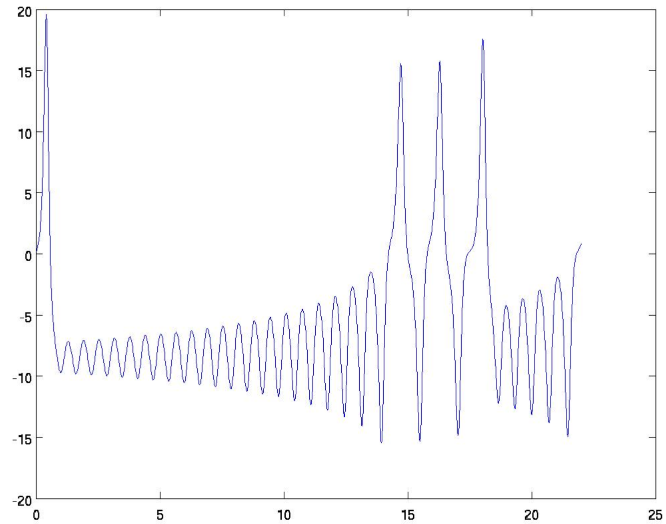


Figure 13. A graph of x vs. t, with $r = 28$ (image created in MATLAB).

8. The Animations

In the previous section we use the Lorenz Equations to test the animation capabilities of MATLAB. Using Euler's Method, we generate an animation that agrees well with MATLAB's built-in Lorenz animation. This animation is a "live" animation, meaning it numerically integrates to determine the next point in real time as it animates. This process is not very efficient, especially given the large amounts of data we are working with in the Saturn-Janus-Epimetheus system. For this reason, we use a built-in MATLAB command called **comet**. The **comet** command displays a three-dimensional plot of a curve that goes through certain points at certain times. This creates an animation of the particle that moves forward in time. Each successive point is calculated beforehand and the particle heads to that point. We use this tool to model the Saturn-Janus-Epimetheus system (see the MATLAB code in the Appendix).

A still image of the animation in the rotating frame is given in Figure 18. Do the results make sense? In the animation we see that Epimetheus clearly traces out a horseshoe orbit. However, the minimum distance between Janus and Epimetheus is extremely small compared to the distance between Saturn and Janus, making it difficult to see the horseshoe pattern in a still image. If we increase the mass of Janus, we get a much clearer picture of what is going on (see Figure 19). Now the horseshoe orbit is evident. Furthermore, we see that Epimetheus swaps orbits with Janus, alternating between being interior and being exterior. We also see libration in the looped pattern that Epimetheus makes as it moves through its horseshoe orbit. This is to be expected as Epimetheus does not lie exactly at the epicenter of

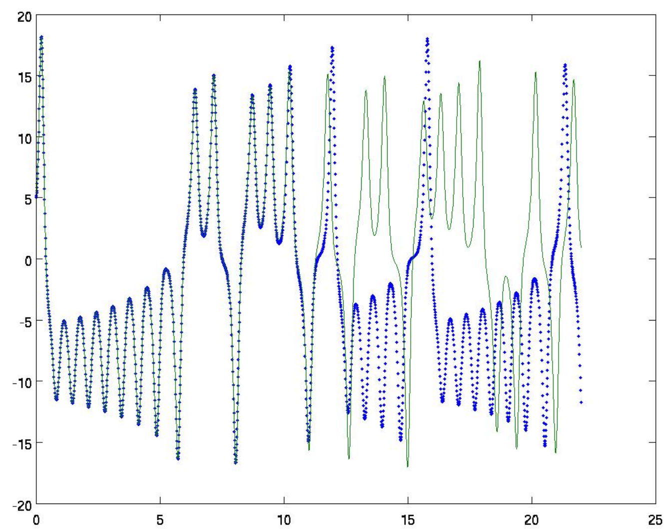


FIGURE 14. A graph of x vs. t, with $r = 28$. The dotted line corresponds to an initial position of $(5, 5, 5)$ while the solid line corresponds to an initial position of $(5.01, 5, 5)$ (image created in MATLAB).

its orbit. Also, we see that Saturn and Janus are fixed, as expected in the rotating frame.

We now turn to the inertial frame of reference. In the inertial frame of reference we see that the system no longer produces a visible horseshoe pattern (see Figure 20); however, the same behavior still occurs. Over the course of the animation it is evident that Epimetheus appears to catch up to Janus. Then, just before the two meet, Epimetheus seems to fall behind and slow down. Just as Janus is about to crash into Epimetheus, Epimetheus speeds up and moves away from Janus. This is precisely the horseshoe behavior that we would expect.

9. Conclusion

We have successfully animated an idealized version of the Saturn-Janus-Epimetheus system. During this process we made two important assumptions. We assumed that Saturn and Janus were in circular orbits about their common center of mass and that Epimetheus was massless. Clearly the second assumption is false—Epimetheus does have mass, enough to have a slight effect on the motion of Janus. As it turns out the first assumption is not valid either; the orbits of Janus and Epimetheus are both slightly eccentric, as determined by observation. These facts require modifications to our simple model of the circular restricted three-body problem. However, the basic concept of the horseshoe orbit still applies to the system. In January 2006, Janus and Epimetheus swapped orbital positions, Janus moving to the inner orbit

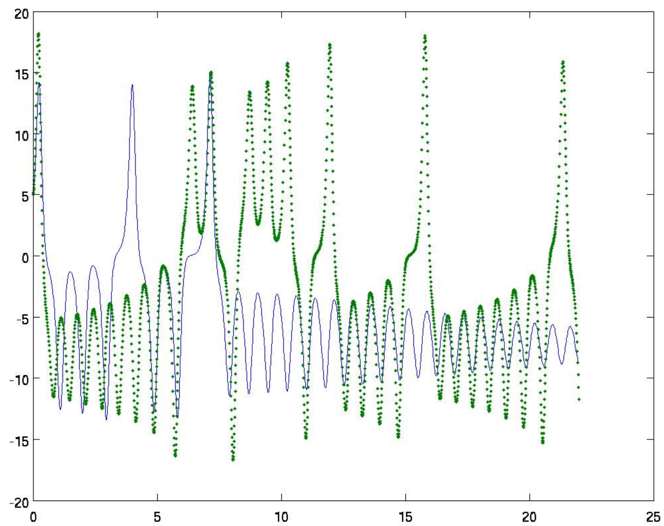


FIGURE 15. A graph of x vs. t, with the initial position $5, 5, 5$. The solid line corresponds to $r = 21$ while the dotted line corresponds to $r = 28$ (image created in MATLAB).

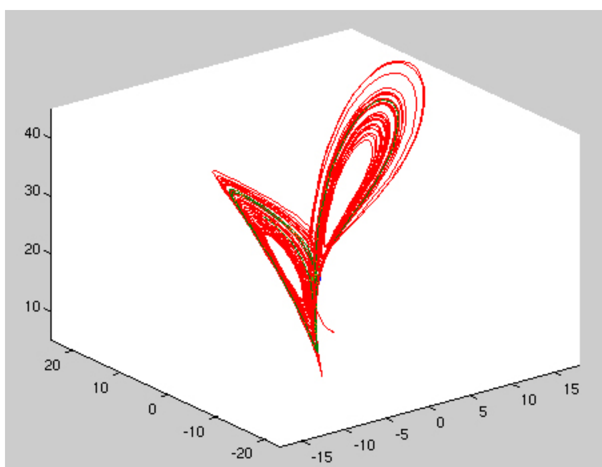


FIGURE 16. A still shot of an animation of the Lorenz Equations, using Euler's Method with $r = 28$ and with the initial position $[5, 5, 5]$ (image created in MATLAB).

and Epimetheus to the outer orbit. This shows that our simple model qualitatively predicts the exchange fairly accurately.

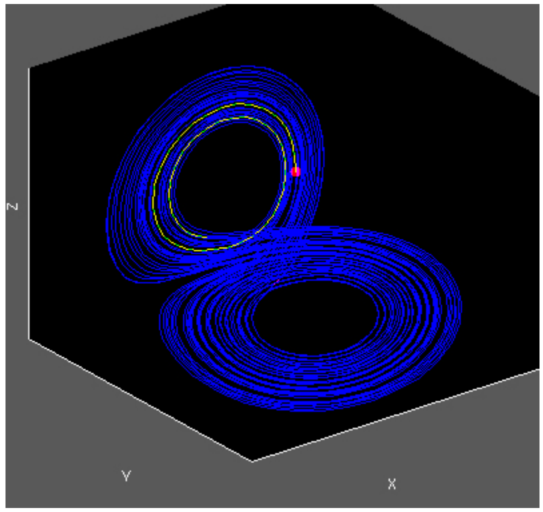


Figure 17. MATLAB's built-in animation of the Lorenz Equations (image created in MATLAB).

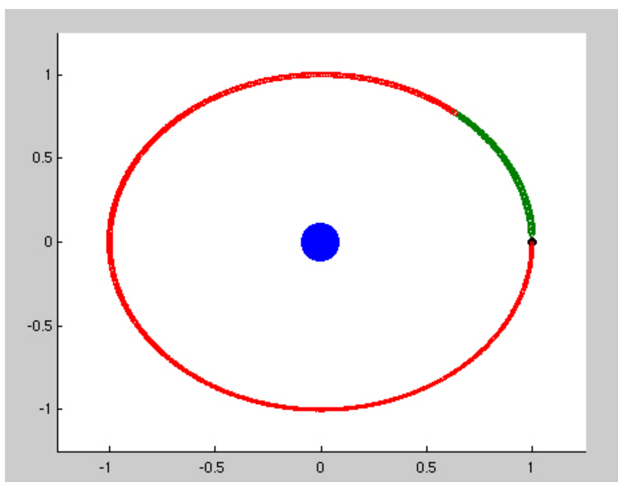


Figure 18. A still frame from an animation of the horseshoeing Saturn-Janus-Epimetheus system in the rotating frame of reference. Saturn is the larger circle at the origin, while the smaller circle is Janus (image created in MATLAB).

The Saturn-Janus-Epimetheus system is not the only system we know of that has a horseshoe orbit. Several asteroids have similar interactions with the Sun-Earth system, including 3753 Cruithne, 54509 YORP, (85770) 1998 UP1, 2002 AA29, and 2003 YN107. As more extrasolar planets are discovered, it is possible that one day we will discover two planets in horseshoe orbits about their common star. For

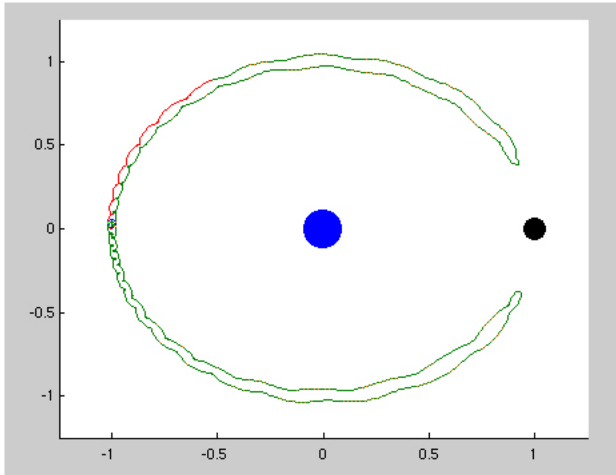


FIGURE 19. A still frame from an animation of the horseshoeing Saturn-Janus-Epimetheus system in the rotating frame of reference with a larger value of $\mu_2 = 5e - 4$. The representations of Saturn and Janus are similar to Figure 18 (image created in MATLAB).

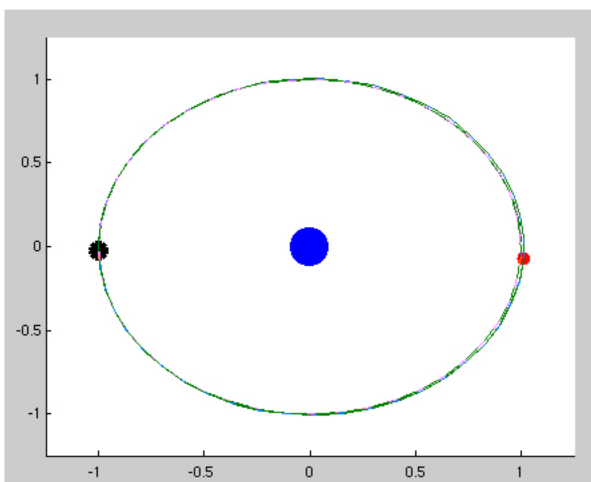


FIGURE 20. A still frame from an animation of the horseshoeing Saturn-Janus-Epimetheus system in the inertial frame of reference. The largest circle is Saturn, the second largest is Janus, and the smallest is Epimetheus (image created in MATLAB).

more information about the three-body problem and horseshoe orbits in general, see Murray and Dermott's *Solar System Dynamics* [6].

10. Appendix

The following is the MATLAB script we use to animate the Lorenz equations.

```
A=[-8/3 0 0; 0 -10 10; 0 28 -1];
y=[35 -10 -7]';
h=0.01;
p=plot3(y(1),y(2),y(3),'.','EraseMode','none','MarkerSize',5);
axis([0 50 -25 25 -25 25]);
hold on
for i=1:4000
A(1,3)=y(2);
A(3,1)=-y(2);
ydot=A*y;
y=y+h*ydot;
set(p,'XData',y(1),'YData',y(2),'ZData',y(3))
drawnow
end
```

The following is the MATLAB function we used to define the equations of motion for the Saturn-Janus-Epimetheus system in the rotating frame of reference:

```
function du=eqns1(t,u)
  du=zeros(6,1);
format long
m2=5e-9;
m1=1.0-m2;
n=1;
du(1)=u(2);
du(2)=-(m1*(u(1)+m2)/((u(1)+m2)^2+(u(3))^2+(u(5))^2)^1.5 +...
   m2*(u(1)-m1)/((u(1)-m1)^2+(u(3))^2+(u(5))^2)^1.5)+2*n*u(4)+n^2*u(1);
du(3)=u(4);
du(4)=-(m1*u(3)/((u(1)+m2)^2+(u(3))^2+(u(5))^2)^1.5 +...
   m2*u(3)/((u(1)-m1)^2+(u(3))^2+(u(5))^2)^1.5) - 2*n*u(2)+n^2*u(3);
du(5)=u(6);
du(6)=-(m1*u(5)/((u(1)+m2)^2+(u(3))^2+(u(5))^2)^1.5 +...
   m2*u(5)/((u(1)-m1)^2+(u(3))^2+(u(5))^2)^1.5);
```

We then call the **eqns1** function in the following MATLAB script.

```
options=odeset('RelTol',1e-4,'AbsTol',[1e-4 1e-4 1e-4 1e-4 1e-2 1e-2]);
[T,Y]=ode45(@eqns1,[0 5000],[-0.9997 0 0 4.55585096332178e-4 0 0],options);
axis([-1.25 1.25 -1.25 1.25])
hold on
plot(-5e-4,0,'bo','LineWidth',3,'MarkerEdgeColor','b',...
    'MarkerFaceColor','b','MarkerSize',25)
plot(1,0,'ko','LineWidth',3,'MarkerEdgeColor','k',...
    'MarkerFaceColor','k','MarkerSize',5)
```

```
comet(Y(:,1),Y(:,3))
hold off
```

To animate the Saturn-Janus-Epimetheus system in the inertial frame of reference, we use the following function and script.

```
function du=eqns2(t,u)
  du=zeros(8,1);
m1=1;
m2=5e-4;
n=1;
du(1)=u(2);
du(2)=m1*(-u(1))/(u(1)^2+u(3)^2+u(5)^2)^1.5 + m2*(cos(t)- ...
u(1))/((cos(t)-u(1))^2+(sin(t)-u(3))^2+u(5)^2)^1.5;
du(3)=u(4);
du(4)=m1*(-u(3))/(u(1)^2+u(3)^2+u(5)^2)^1.5 +...
m2*(sin(t)-u(3))/((cos(t)-u(1))^2+(sin(t)-u(3))^2+u(5)^2)^1.5;
du(5)=u(6);
du(6)=m1*(-u(5))/(u(1)^2+u(3)^2+u(5)^2)^1.5 +...
m2*(-u(5))/((cos(t)-u(1))^2+(sin(t)-u(3))^2+u(5)^2)^1.5;
du(7)=-sin(t);
du(8)=cos(t);
```

```
options=odeset('RelTol',1e-4,'AbsTol',[1e-4 1e-4 1e-4 1e-4 1e-2 ...
1e-2 1e-2 1e-2]);
[T,Y1]=ode45(@eqns2,[0 1500],[-0.99967 0 0 -(4.55585096332178e-4+1) ...
0 0 1 0],options);
t=[0:100];
axis([-1.25 1.25 -1.25 1.25])
hold on
plot(-5e-4,0,'bo','LineWidth',3,'MarkerEdgeColor','b',...
'MarkerFaceColor','b','MarkerSize',25)
mycomet(Y1(:,7),Y1(:,8),Y1(:,1),Y1(:,3))
hold off
```

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