DIFFERENTIAL GEOMETRY
SENIOR PROJECT — MAY 15, 2009

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Contents

Abstract. A survey of Differential Geometry is presented with emphasis on surfaces in $\mathbb{R}^3$. Differentiation on surfaces and a dual approach to normal, Gauss, and mean curvature involving the Shape Operator and fundamental forms are developed. Gauss's Theorem Egregium is proved and a supporting discussion of diffeomorphic and isomorphic surfaces is included. Minimal surfaces are examined and Aleksandrov's "Soap Bubble" Theorem regarding compact surfaces of constant mean curvature is proved.

1. Introduction

1.1. **Motivation.** When we contemplate the physical world we often fixate on the things in it. What affect does this or that have on us as humans? How does time change the observable aspects of our surroundings? How can we characterize the structure or shape of objects? Inquiry with intent to satisfy these curiosities represents a vast human effort, namely the development of scientific thought. However, what if the "this or that" in these questions is the physical world itself? Beautiful branches of mathematics, physics, and even astronomy have developed to address this specific subset of questions. These areas of study probe fundamental properties regarding the physicality, such as shape, of not only the things in the space around us, but also the space itself.

Columbus said the world is round. In the absence of massive objects Einstein says the universe is locally flat. Even in our everyday lives it seems rare to deviate from roundness and flatness. If that is true, then what do these aforementioned, "beautiful" fields of interest consist of? Of course, we could argue, and argue successfully, the existence of objects that are neither round nor flat (the canopied rooftop of Denver International Airport for instance), but we find more complexity and beauty when comparing and contrasting shapes. Even in the somewhat restrictive context of round versus flat there exists much to talk about. A very simple example motivates the wealth of questions that arise.

Let us consider a line, not so much the picture of a line itself, but the process one performs to construct a line. We equip ourselves with a sturdy meter-stick, place it flat on the ground, and draw a straight line. Once we run out of straight edge we pick the meter-stick up and carefully align it with our previous line and draw again. Assuming we could continue doing this without major obstacles like curbs or the Indian Ocean, what would happen? Columbus knew. We would eventually place our meter-stick and see a line already drawn on the pavement. We return to where we began! Figure 1.1 shows our resulting "line." The shape of the space we inhabit, the Earth,

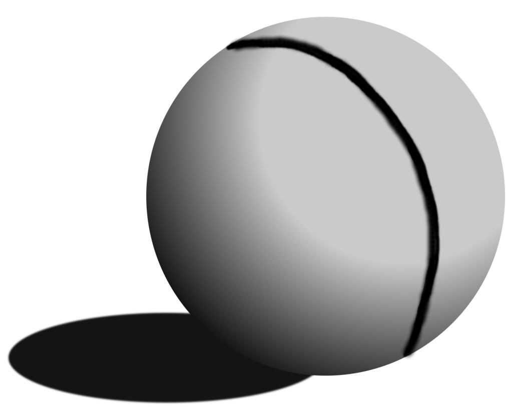


Figure 1. A line curves on a curved space. On this sphere our method of constructing a line by continuous drawing results in a finite, closed contour.

has fundamentally affected our simple drawing of a line. What we drew is not infinite, as true lines ought to be, and is arguably more like a circle than any sort of line. How can we possibly resolve this ambiguity? We cannot even decide if we drew a line or not! This basic example illustrates the issues we confront when carefully characterizing the shape of space itself.

To say the least, the problems evident in this example are troubling. For instance, they call the entire theory of Calculus into question with all its attempts to assign mathematical rigor to shape. Suppose we set about calculating the derivative of our Earth-bound line. From Calculus we know the derivative applied to a line should reveal a constant slope. Should we set about calculating the derivative on meter-stick scales where a constant slope seems plausible? Or would it be best to orient ourselves so that we are looking at the globe where the circle-like properties are evident? These questions attest to discrepancies when we consider local versus global properties of a curved surface. Luckily, the beautiful field we have been referring to steps in to salvage this situation. Differential Geometry addresses these very fundamental issues; it develops a high level of mathematical rigor to apply to the shape of space.

1.2. **Synopsis.** We highlight some of the key notions in basic Differential Geometry. Our intent is to develop enough background to prove Aleksandrov's "Soap Bubble" Theorem and, perhaps more importantly, understand its significance. To orient ourselves and develop some key mathematical tools we first discuss differentiation on arbitrary surfaces in $\mathbb{R}^3$, thus addressing the dilemma outlined in Section 1.1. With the methods of differentiation firmly in place we begin our investigation of shape by studying the various types of curvature defined on a surface, such as the principal, Gauss, and mean curvatures. Next, we demonstrate some computational techniques so we can draw on curvature to formulate some rudimentary properties of surfaces. We observe two markedly different approaches to Differential Geometry: the Shape Operator and the fundamental forms. This exposition reconciles them and draws from the strengths of each set of ideas for the more advanced theorems. We lead into our more specific characterizations, Gauss's Theorem Egregium and ultimately Aleksandrov's Theorem, by defining and exploring the features of minimal and other constant mean curvature surfaces.

2. Differentiation on Surfaces

This section presents a thorough introduction to surfaces and shows how to differentiate both scalar functions and vector fields on a surface. We explore tangent planes and the unit normal in Section 2.3. Sections 2.4, 2.5, and 2.6 develop the primary surface derivatives that we use in this paper. We introduce the torus, a surface that serves as an example for many of our theoretical results, providing a common thread.

2.1. **Surface Definition.** Our first task is to assign mathematical rigor to the notion of surface. Much of our discussion of differentiation mirrors that found in John Oprea's text: Differential Geometry and Its Applications. [1] Oprea's exposition of differentiation is geared towards undergraduates and is highly cohesive with our work involving

[1] J.Oprea. *Differential Geometry and Its Applications.* The Mathematical Association of America, Pearson Education, Inc., Washington D.C., 2007.

the geometry of surfaces in $\mathbb{R}^3$ in particular.

Although a surface may inhabit three dimensional space, it possesses an inherent two dimensionality. If we imagine moving about on a surface, that is on a sphere, soap film, or inner tube, there is always a dimension in which we are forbidden to move without leaving the surface. On the surface of the Earth, this is the familiar direction "up." This observation is advantageous because we seek to develop Calculus techniques on a surface and the topic of Calculus is very familiar to us in two dimensions.

Let D be an open set in $\mathbb{R}^2$ and let

$$\mathbf{x} : D \mapsto \mathbb{R}^3$$

be a mapping from D to three dimensional space. Here and throughout, bold face type denotes a vector, specifically $\mathbf{x}$ is a vector-valued function. We routinely use the parameters u and v in the context of an ordered pair $(u, v) \in D$. Hence, a parallel notation for the mapping specified by $\mathbf{x}$ is

$$(u, v) \mapsto (x^1(u, v), x^2(u, v), x^3(u, v))$$

where the component functions $x^i : D \mapsto \mathbb{R}$ specify the components of the vector $\mathbf{x}(u, v)$. In this paper we assume the component functions x^i are continuous and twice differentiable.

Reader beware! The x^i is not the scalar function x raised to the i^{th} power. We use superscripts to denote separate component functions. In $\mathbb{R}^3$ we are familiar with to the coordinates (x, y, z), but let us adopt the more general notation (x_1, x_2, x_3). This notation affords easy generalization to higher dimensions (where we quickly run out of letters) and allows us to use summation notation where applicable. So the function $x^1(u, v)$ specifies the x_1 coordinate of the mapping, the function $x^2(u, v)$ gives the x_2 coordinate, and the function $x^3(u, v)$ gives x_3.

The parameters u and v can be made to vary together or independently in D to generate parameter curves in $\mathbb{R}^3$. For instance, if we let $v = v_0$ and vary u, then $\mathbf{x}(u, v_0)$ is called a ***u*-parameter curve**. A ***v*-parameter curve** is defined analogously. Notice, in $\mathbb{R}^3$, the u- and v-parameter curves meet at $\mathbf{x}(u_0, v_0)$. That the u- and v-parameter curves meet in $\mathbb{R}^3$ is of crucial importance because we utilize them to develop information about the behavior of the vector-valued function $\mathbf{x}$. We often desire to generate the tangent vectors to the parameter curves at a given point. We obtain such tangent vectors by applying partial derivatives to the component functions x^i,

$$\frac{\partial \mathbf{x}}{\partial u} = \mathbf{x}_u = \left(\frac{\partial x^1}{\partial u}, \frac{\partial x^2}{\partial u}, \frac{\partial x^3}{\partial u} \right)$$

and

$$\frac{\partial \mathbf{x}}{\partial v} = \mathbf{x}_v = \left(\frac{\partial x^1}{\partial v}, \frac{\partial x^2}{\partial v}, \frac{\partial x^3}{\partial v} \right).$$

Hence, given a point $(u_0, v_0) \in D$, tangent vectors to the u- and v-parameter curves, which we know intersect at $\mathbf{x}(u_0, v_0) \in \mathbb{R}^3$, are $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$ respectively.

As the reader may have surmised, the primary goal is to produce vector-valued functions $\mathbf{x} : D \subseteq \mathbb{R}^2 \mapsto M \subseteq \mathbb{R}^3$, where M is a surface. Considering u- and v-parameter curves as well as the associated partial derivatives $\mathbf{x}_u$ and $\mathbf{x}_v$ we notice the two dimensionality of a surface we mentioned earlier manifests strongly in the parameters u and

v. If we hope to use the structure of vector-valued functions to define a surface, then we must introduce some condition to maintain the linear independence of the tangent vectors $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$. The following Lemma 2.1 proves the condition we need is $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$. This is called **regularity**.

Lemma 2.1. *The tangent vectors $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$ are linearly dependent if and only if $\mathbf{x}_u \times \mathbf{x}_v = \mathbf{0}$.*

Proof. Suppose $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly dependent, that is there exists a scalar $c \in \mathbb{R}$ such that $\mathbf{x}_u = c\mathbf{x}_v$. Then

$$\mathbf{x}_u \times \mathbf{x}_v = (c\mathbf{x}_v) \times \mathbf{x}_v = c(\mathbf{x}_v \times \mathbf{x}_v) = \mathbf{0}.$$

Suppose $\mathbf{x}_u \times \mathbf{x}_v = \mathbf{0}$, then $\mathbf{x}_u$ and $\mathbf{x}_v$ are parallel meaning $\mathbf{x}_u = \hat{c}\mathbf{x}_v$ for some $\hat{c} \in \mathbb{R}$. Thus, $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly dependent. $\square$

A **coordinate patch** (or simply **patch**) is a one-to-one mapping $\mathbf{x} : D \mapsto \mathbb{R}^3$ such that given any $(u_0, v_0) \in D$, $\mathbf{x}_u(u_0, v_0) \times \mathbf{x}_v(u_0, v_0) \neq \mathbf{0}$. If $M \subseteq \mathbb{R}^3$ and each point $(x_1, x_2, x_3) \in M$ has a neighborhood, also in M, that is the image of some coordinate patch $\mathbf{x} : D \mapsto \mathbb{R}^3$, then M is called a **surface**. The name "patch" is actually quite descriptive because there are numerous examples of surfaces that require more than one patch to be completely parametrized. There may be overlap among the patches, but for M to be a surface there cannot be any "bare spots" where no coordinate patch can be defined. The stipulation that each coordinate patch in this covering be **regular**, that is $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$, is incredibly important because it prohibits creases. Furthermore, the coordinate patches that cover a surface M are one-to-one and thus inverse mappings exist for each patch. For instance, the mapping $\mathbf{y^{-1}} : \mathbf{M} \mapsto D$ takes a region on the surface M back to a subset of the original domain D in a one-to-one fashion. We may now form mappings from D to itself that move first to the surface M and then back. What does this look like in terms of component functions? Suppose

$$\mathbf{x} : (u, v) \mapsto \left(x^1(u, v), x^2(u, v), x^3(u, v)\right),$$

then we can find functions $\bar{u}$ and $\bar{v}$ such that

$$\mathbf{x}^{-1} : (x_1, x_2, x_3) \mapsto (\bar{u}(x_1, x_2, x_3), \bar{v}(x_1, x_2, x_3)).$$

Notice that the functions $\bar{u}$ and $\bar{v}$ are not the inverses of the component functions x^i. This fact is often masked when we write $\mathbf{x}^{-1} \circ \mathbf{y} : D \mapsto D$ because we can express $\mathbf{x}^{-1} \circ \mathbf{y} = (r(u, v), s(u, v))$, where $r, s : D \mapsto \mathbb{R}$.

Take any two patches $\mathbf{x}$ and $\mathbf{y}$ for a surface M with coordinate functions x^i and y^i. Suppose for each i and each $n \in \mathbb{N}$, the four partial derivatives:

$$\frac{\partial^n x^i}{\partial u^n}, \quad \frac{\partial^n x^i}{\partial v^n}, \quad \frac{\partial^n y^i}{\partial u^n}, \quad \text{and} \quad \frac{\partial^n y^i}{\partial v^n}$$

exist and are continuous. We say $\mathbf{x}^{-1} \circ \mathbf{y} : D \mapsto D$ is differentiable. Furthermore, as $\mathbf{x}$ and $\mathbf{y}$ are arbitrary patches for M, this indicates M is **differentiable** or **smooth**. All surfaces we examine are smooth. The stipulations of regularity ($\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$) and smoothness ensure that a differentiable surface has no creases and varies reasonably nicely in $\mathbb{R}^3$. Let us introduce such a surface, the torus.

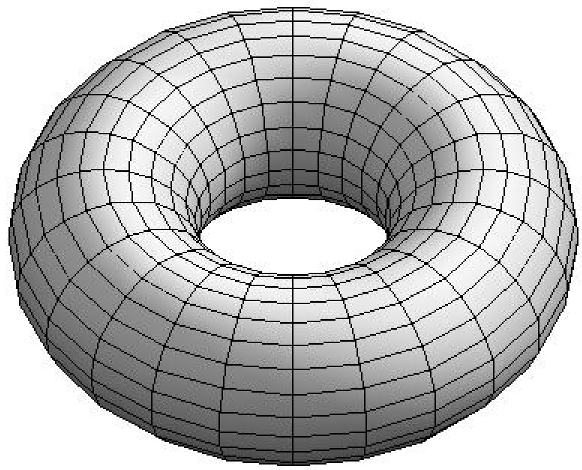


FIGURE 2. A torus with $R = 2r$.

Example: A Smooth Surface, the Torus

Many examples of smooth surfaces can be created by rotating curves or regions in two dimensions about some fixed axis allowing a three dimensional shape to “drag” along through a complete or partial revolution. We can create a torus or inner tube surface in this manner. Begin with a circle in the x_1x_3-plane with center $(R, 0, 0)$ and radius r. Revolution about the x_3-axis produces the torus surface, which can be completely described with the parameters u and v. All points on the torus can be represented as

$$((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)),$$

where $0 \leq u < 2\pi$ and $0 \leq v < 2\pi$. In this expression u traces around the tubular body of the torus and v traces the position about the x_3-axis. To better visualize this imagine we have a pivoting stem of length r attached to the circle through the middle of the torus; that is attached to the circle that is formed by revolving the point $(R, 0, 0)$ about the x_3-axis. The parameter u refers to the angle of the pivot above or below the x_1x_2-plane and the parameter v refers to the angle of the pivot's point of attachment measured from the positive x_1-axis. Figure 2 shows an example of a torus with $R = 2r$.

Appealing to the geometry of the torus as a surface of revolution in this way, we can define the coordinate patch

$$\mathbf{T}(u, v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u)). \tag{1}$$

To show $\mathbf{T}$ is regular, compute

$$\mathbf{T}_u(u, v) = (-r\sin u\cos v, -r\sin u\sin v, r\cos u)$$

and

$$\mathbf{T}_v(u, v) = (R + r\cos u)(-\sin v, \cos v, 0)$$

by taking the partial derivatives of the component functions with respect to u and v respectively. Notice, this patch is regular because for all

$$\mathbf{T}_u \times \mathbf{T}_v = -r(R + r\cos u)(\cos u \cos v, \cos u \sin v, \sin u) \neq \mathbf{0}.$$

So $\mathbf{T}$ is indeed a coordinate patch and covers the entire surface. The domain of u and v specifies the region D, the square $[0, 2\pi) \times [0, 2\pi)$. Furthermore, the torus is smooth because all derivatives of the sine and cosine terms in the component functions exist and are continuous. We return to the torus, with its very familiar inner-tube shape, often because it is a strong example of the theoretical concepts we discuss.

2.2. **Curves on a Surface.** Given a surface M and an open subset $D \subseteq \mathbb{R}^2$, a coordinate patch $\mathbf{x} : D \mapsto \mathbb{R}^3$ maps two dimensional space to the surface in three dimensional space. Remember the surface has a constraint meaning although it exists in $\mathbb{R}^3$, it really only possesses two dimensions or degrees of freedom. This concept is clear when considering the domain of the coordinate patch is a subset of $\mathbb{R}^2$, not $\mathbb{R}^3$. Suppose we draw a curve $\boldsymbol{\alpha}$ on a surface M. Although $\boldsymbol{\alpha}$ exists in $\mathbb{R}^3$, it has only one degree of freedom or a one-dimensionality. Just as for a surface and its associated coordinate patches, we can define a mapping describing this curve that reflects the limiting of dimensionality we are discussing. We think of the curve $\boldsymbol{\alpha}$ as a mapping $\boldsymbol{\alpha} : I \mapsto M$ where $I \subseteq \mathbb{R}$. A more suggestive notation is $\boldsymbol{\alpha}(t) = (\alpha^1(t), \alpha^2(t), \alpha^3(t))$ for $t \in I$ and component scalar functions $\alpha^i : I \to \mathbb{R}$.

In most cases we have difficulty building functions of the form $\boldsymbol{\alpha}(t)$ directly. The solution is to use $\mathbb{R}^2$ as an intermediate step and make use of the coordinate patch $\mathbf{x}$ to step from $\mathbb{R}^2$ up to M. A curve $\boldsymbol{\alpha}$ is differentiable if for each patch $\mathbf{x} : D \mapsto M$, the composite function $\mathbf{x}^{-1} \circ \boldsymbol{\alpha} : \boldsymbol{\alpha}^{-1}(\mathbf{x}(D)) \mapsto D$ is differentiable. In other words, $\boldsymbol{\alpha}$ is differentiable when the mapping that takes the portion of I to the surface M and back to the domain D of any coordinate patch $\mathbf{x}$ is differentiable. So instead of seeking an appropriate function on $\mathbb{R}$ to map out the curve $\boldsymbol{\alpha}$ on M, we tie properties of the curve to the coordinate patch so we can tackle the somewhat easier task of defining parametric equations in D to map $\boldsymbol{\alpha}$ on M. The following existence result ensures we can always do this when $\boldsymbol{\alpha}$ is differentiable.

Lemma 2.2. *Let M be a surface. Suppose $\boldsymbol{\alpha}$ is a differentiable curve on M contained in the image of some coordinate patch $\mathbf{x} : D \mapsto M$, that is for some subset $I \subseteq \mathbb{R}$, $\boldsymbol{\alpha} : I \mapsto \mathbf{x}(D) \subseteq M$. Then there exist unique, differentiable functions $u, v : I \mapsto \mathbb{R}$ such that $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$.*

Proof. The curve $\boldsymbol{\alpha}$ is differentiable so by definition $\mathbf{x}^{-1} \circ \boldsymbol{\alpha} : I \mapsto D$ is differentiable. This composition is a differentiable, vector-valued function so its component functions must also be differentiable. That is, there exists differentiable scalar functions u and v such that

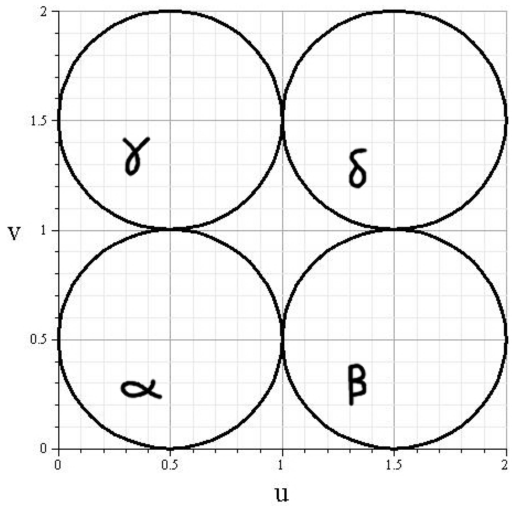
$$\mathbf{x}^{-1} \circ \boldsymbol{\alpha}(t) = (u(t), v(t)).$$

Compose each side of this expression with the coordinate patch $\mathbf{x}$ to find

$$\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t)).$$

Now suppose there were two different scalar functions $\tilde{u}$ and $\tilde{v}$ such that $\boldsymbol{\alpha}(t) = \mathbf{x}(\tilde{u}(t), \tilde{v}(t))$. Then

$$(u(t), v(t)) = \mathbf{x}^{-1} \circ \boldsymbol{\alpha}(t) = \mathbf{x}^{-1} \circ \mathbf{x}(\tilde{u}(t), \tilde{v}(t)) = (\tilde{u}(t), \tilde{v}(t)).$$



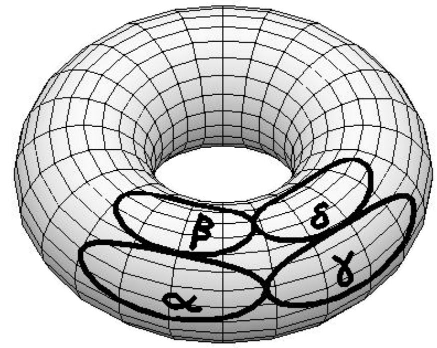


FIGURE 3. Closed circles in D map to closed contours on the surface of the torus.

Hence u and v are unique. □

As with any parametric equations we can always re-parametrize u and v to suit our current problem, but Lemma 2.2 shows all such parametrizations must be related.

Example: Surface Curves on the Torus

Let us consider some surface curves on a the torus in Figure 2. Recall from Section 2.1 the coordinate patch that covers the inner-tube shape of the torus is $\mathbf{T}$: $[0, 2\pi) \times [0, 2\pi) \mapsto \mathbb{R}^3$. Two basic families of curves on the torus are u- and v-parameter curves. We fix $v = v_0$ and see the u-parameter curves trace circles about the body of the inner-tube. Fixing $u = u_0$ leads to v-parameter curves that trace circles about the central z-axis on the torus. We study these two families of curves when we consider the Gauss map in Section 3.2.

Perhaps more interesting are the images of circles in $[0, 2\pi) \times [0, 2\pi)$ under the mapping $\mathbf{T}$, shown in Figure 3. Just as in Section 1.1, where the shape of the Earth affected our ideal of a line, observe the circles in D are distorted when mapped to the surface of the torus under $\mathbf{T}$. We calculate the circumference of these distorted torus-circles in Section 3.3.

2.3. **The Tangent Plane and Unit Normal.** We have already hinted at the usefulness of adopting regularity, that is $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$ for coordinate patch $\mathbf{x}$, but we cannot emphasize enough the importance of Lemma 2.1 in terms of describing the geometry of a surface.

In Calculus, the tangent line serves as a local approximation to a curve in $\mathbb{R}^2$. In the Calculus applied to surfaces, a tangent plane serves as a local approximation to a surface in $\mathbb{R}^3$. Notice again the dimensionality analogies; it takes a single-dimensional object to approximate a two-dimensional object so naturally, a two-dimensional object approximates a three-dimensional object. Suppose $\boldsymbol{\alpha}$ is a differentiable curve on a surface M. Fix t_0 in the domain of $\boldsymbol{\alpha}$ and notice $\boldsymbol{\alpha}(t_0)$ specifies some point $\mathbf{p} \in M$. What is the significance of $\dot{\boldsymbol{\alpha}}(t_0)$ (where the dot denotes differentiation with respect to the parameter t)? In terms of component functions α^i,

$$\dot{\boldsymbol{\alpha}}(t) = \left(\frac{d\alpha^1}{dt}, \frac{d\alpha^2}{dt}, \frac{d\alpha^3}{dt} \right).$$

From vector calculus we know $\dot{\boldsymbol{\alpha}}(t_0)$ denotes a vector pointing in a direction tangent to $\boldsymbol{\alpha}$ at the point $\mathbf{p}$. Notice the direction specified by $\dot{\boldsymbol{\alpha}}(t_0)$ is also tangent to the surface M because $\boldsymbol{\alpha}$ is constrained to move on the surface. In fact, we use this notion to define what it means to be tangent to a surface. A given vector $\mathbf{v}$ is **tangent to** M at $\mathbf{p} \in M$ if there exists some curve $\boldsymbol{\alpha}$ on M such that $\boldsymbol{\alpha}(t_0) = \mathbf{p}$ and $\dot{\boldsymbol{\alpha}}(t_0) = \mathbf{v}$. Through re-parametrization of $\boldsymbol{\alpha}$ we can force $t_0 = 0$ so that is typically how we encounter this definition: $\boldsymbol{\alpha}(0) = \mathbf{p}$ and $\dot{\boldsymbol{\alpha}}(0) = \mathbf{v}$.

Recall there exists exactly two unit vectors tangent to a curve in $\mathbb{R}^2$ at a point, one in each direction along the line of tangency. In the case of a surface, there are infinitely many unit tangent vectors at a point $\mathbf{p}$, one for each curve on the surface (and there can be a curve through $\mathbf{p}$ in every direction). We collect the unit tangent vectors to form the tangent plane

$$T_{\mathbf{p}}(M) = \{\mathbf{v} \mid \mathbf{v} \text{ is tangent to M at } \mathbf{p}\}.$$

Notice there is a serious ambiguity as to which tangent vector we are referring to because $T_{\mathbf{p}}(M)$ is an infinite collection of vectors. Here is where the linear independence of the $\mathbf{x}_u$ and $\mathbf{x}_v$ patch derivatives becomes useful. Notice the point $\mathbf{p}$ is the image under patch $\mathbf{x} : D \mapsto M$ of some point $(u_0, v_0) \in D$. Recall from Section 2.1 the u- and v-parameter curves intersect at $\mathbf{x}(u_0, v_0) = \mathbf{p}$. Lemma 2.3 shows $\mathbf{x}_u$ and $\mathbf{x}_v$ form a basis for $T_{\mathbf{p}}(M)$.

Lemma 2.3. *The vector $\boldsymbol{v}$ is in $T_{\boldsymbol{p}}(M)$ if and only if $\boldsymbol{v}$ is a linear combination of the vectors $\boldsymbol{x}_u(u_0, v_0)$ and $\boldsymbol{x}_v(u_0, v_0)$.*

Proof. Suppose $\mathbf{v} \in T_{\mathbf{p}}(M)$. By definition there exists a curve $\boldsymbol{\alpha} : I \mapsto M$ such that $\boldsymbol{\alpha}(0) = \mathbf{x}(u_0, v_0) = \mathbf{p}$ and $\dot{\boldsymbol{\alpha}}(0) = \mathbf{v}$. By Lemma 2.2 there exist differentiable functions $u, v : I \mapsto \mathbb{R}$ such that

$$\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t)).$$

We know $u(0) = u_0$ and $v(0) = v_0$ because $\boldsymbol{\alpha}(0) = \mathbf{x}(u(0), v(0)) = \mathbf{x}(u_0, v_0)$. Taking the derivative with respect to t we have

$$\frac{d}{dt}\left(\boldsymbol{\alpha}(t)\right) = \frac{d}{dt}\left(\mathbf{x}(u(t), v(t))\right),$$

which according to the vector calculus chain rule is

$$\dot{\boldsymbol{\alpha}}(t) = \dot{u}\,(t)\mathbf{x}_u(u(t), v(t)) + \dot{v}\,(t)\mathbf{x}_v(u(t), v(t)).$$

When evaluating this last expression at $t = 0$ we see

$$\dot{\boldsymbol{\alpha}}(0) = \dot{u}(0)\,\mathbf{x}_u(u(0), v(0)) + \dot{v}(0)\,\mathbf{x}_v(u(0), v(0)),$$

or

$$\mathbf{v} = \dot{u}(0)\,\mathbf{x}_u(u_0, v_0) + \dot{v}(0)\,\mathbf{x}_v(u_0, v_0).$$

Therefore, $\mathbf{v}$ is a linear combination of $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$, the respective weights are given by $\dot{u}(0)$ and $\dot{v}(0)$.

Now suppose for constants $\lambda_1, \lambda_2 \in \mathbb{R}$, $\mathbf{v} = \lambda_1 \mathbf{x}_u(u_0, v_0) + \lambda_2 \mathbf{x}_v(u_0, v_0)$. Consider the function $\boldsymbol{\alpha}(t) = \mathbf{x}(u_0 + \lambda_1 t, v_0 + \lambda_2 t)$. Note $\boldsymbol{\alpha}(0) = \mathbf{x}(u_0, v_0) = \mathbf{p}$. Also,

$$\begin{aligned}
\dot{\boldsymbol{\alpha}}(t) &= \frac{d}{dt}\left(\mathbf{x}(u_0 + \lambda_1 t, v_0 + \lambda_2 t)\right) \\
&= \left(\frac{d}{dt}(u_0 + \lambda_1 t)\right) \mathbf{x}_u(u_0 + \lambda_1 t, v_0 + \lambda_2 t) \\
&+ \left(\frac{d}{dt}(v_0 + \lambda_2 t)\right) \mathbf{x}_v(u_0 + \lambda_1 t, v_0 + \lambda_2 t) \\
&= \lambda_1 \ \mathbf{x}_u(u_0 + \lambda_1 t, v_0 + \lambda_2 t) + \lambda_2 \ \mathbf{x}_v(u_0 + \lambda_1 t, v_0 + \lambda_2 t)
\end{aligned}$$

So $\dot{\boldsymbol{\alpha}}(0) = \mathbf{v}$. Thus, vector $\mathbf{v}$ is tangent to M by definition. □

Given a surface M and a point $\mathbf{p}$, the tangent plane $T_{\mathbf{p}}(M)$ serves two very important purposes. One, it is a local approximation to M at $\mathbf{p}$, so we can approximate even the most bizarre surfaces by forming tangent planes. Two, we can easily construct the unit normal direction to M at $\mathbf{p}$ by finding the unit normal direction to the tangent plane at $\mathbf{p}$ as

$$\hat{\mathbf{U}} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}.$$

This new quantity $\mathbf{U}$ is one of the many keys that allow us to characterize the shape of surfaces. Again, we see why regularity ($\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$) is included in the very definition of surface. As a result, the existence of $\hat{\mathbf{U}}$ is never a concern so it is an exceedingly robust tool.

Example: $\mathbf{T}_{\mathbf{p}}$ and $\hat{\mathbf{U}}$ on the Torus

Refer to Section 2.1 for the form of the torus patch $\mathbf{T}$. Using $\mathbf{T}_u$ and $\mathbf{T}_v$ we compute

$$\hat{\mathbf{U}} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{|\mathbf{T}_u \times \mathbf{T}_v|} = -(\cos u \cos v, \cos u \sin v, \sin u).$$

Note the unit normal does not depend on the inner radius R nor the tube radius r. In order to develop a concrete example of the unit normal and tangent plane pick a point on the torus in Figure 2 (where $R = 2r$). Say

$$\mathbf{p} = \mathbf{T}\left(\frac{\pi}{3}, \frac{7\pi}{6}\right) = r\left(-\frac{5\sqrt{3}}{4}, -\frac{5}{4}, \frac{\sqrt{3}}{2}\right).$$

Compute the unit normal

$$\hat{\mathbf{U}}(\mathbf{p}) = \left(-\frac{\sqrt{3}}{4}, -\frac{1}{4}, \frac{\sqrt{3}}{2}\right).$$

Finally, form the tangent plane $T_{\mathbf{p}}(\mathbf{T})$ represented by a function of $(x_1, x_2, x_3) \in \mathbb{R}^3$ as

$$((x_1, x_2, x_3) - \mathbf{p}) \cdot \hat{\mathbf{U}}(\mathbf{p}) = 0.$$

Figure 4 shows the torus, the point $\mathbf{p}$, and the tangent plane together.

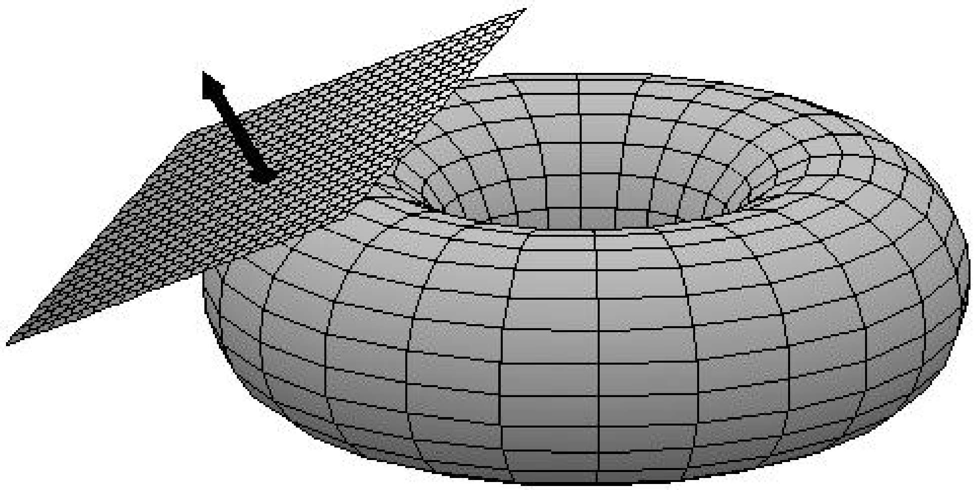


FIGURE 4. On this torus $R = 2r$. The tangent plane $T_{\mathbf{p}}(\mathbf{T})$ is a local approximation of the torus.

2.4. **Directional Derivatives.** We are finally ready to introduce the concept of derivative on a surface. There are many flavors of differentiation that make sense on a surface in $\mathbb{R}^3$, but share a similar motivation, namely the rate of change of some quantity on the surface. Of course, we start with the basics and develop the familiar directional derivative, which translates easily from vector calculus to a surface.

As expected, the directional derivative applies to scalar functions defined on a surface M; $\bar{u}$ and $\bar{v}$ from Section 2.1 are such functions. Suppose $g : M \mapsto \mathbb{R}$. If, for each coordinate patch $\mathbf{x}$ defined on M, the composition $g \circ \mathbf{x} : D \mapsto \mathbb{R}$ is differentiable, then we say g is differentiable on M. Note, g is a function of points $\mathbf{p} \in M$ so we denote $g = g(x_1, x_2, x_3)$. When we discuss the rate of change of g on the surface we must specify a direction of travel. Calculus students are accustomed to considering direction when computing derivatives, think about elevation functions and the slope of a trail depending on which direction we hike. We set about computing the derivative of g along some curve $\boldsymbol{\alpha} : I \mapsto M$ with component functions $\alpha^i(t)$ for $i = 1, 2, 3$. As in Section 2.2 $\boldsymbol{\alpha}$ is parametrized with t. Consider the composition $g \circ \boldsymbol{\alpha} : I \mapsto \mathbb{R}$. The curve $\boldsymbol{\alpha}$ specifies the way in which the coordinates $(x_1(t), x_2(t), x_3(t))$ change with respect to the parameter t:

$$\frac{dx_i}{dt} = \frac{d\alpha^i}{dt}$$

for $i = 1, 2, 3$. Using an index such as i allows us to adopt summation notation for organization and clarity. Using the vector calculus chain rule we compute

$$\begin{aligned}
\frac{d}{dt}(g(\boldsymbol{\alpha}(t))) &= \sum_{i=1}^{3} \frac{\partial g}{\partial x_i}\frac{dx_i}{dt} \\
&= \sum_{i=1}^{3} \frac{\partial g}{\partial x_i}\frac{d\alpha^i}{dt} \\
&= \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}\right) \cdot \left(\frac{d\alpha^1}{dt}, \frac{d\alpha^2}{dt}, \frac{d\alpha^3}{dt}\right)
\end{aligned}$$

where $\cdot$ denotes the dot product. Notice, the vector on the right is $\dot{\boldsymbol{\alpha}}(t)$. The left hand vector is recognizable as the gradient of g, in fact, we adopt the familiar vector calculus definition

$$\nabla \doteq \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right).$$

With this notation we can write the derivative

$$\frac{d}{dt}(g(\boldsymbol{\alpha}(t))) = \nabla g(\boldsymbol{\alpha}(t)) \cdot \dot{\boldsymbol{\alpha}}(t).$$

Recall from Section 2.3 the notion of direction on M is completely analogous to the directions on associated tangent planes $T_{\mathbf{p}}(M)$. So, what is the rate of change of g in some direction $\mathbf{v} \in T_{\mathbf{p}}(M)$? We simply define a curve $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha}(0) = \mathbf{p}$ and $\dot{\boldsymbol{\alpha}}(0) = \mathbf{v}$ and evaluate the above expression for $t = 0$:

$$\frac{d}{dt}(g(\boldsymbol{\alpha}(0))) = \frac{d}{dt}(g(\mathbf{p})) = \nabla g(\mathbf{p}) \cdot \mathbf{v}.$$

The derivative of g in the direction of $\mathbf{v}$ is the projection of the gradient of g onto $\mathbf{v}$. We use the directional derivative so often that it warrants a special notation

$$\mathbf{v}[g](\mathbf{p}) \doteq \nabla g(\mathbf{p}) \cdot \mathbf{v}. \tag{2}$$

Read equation (2) as "$\mathbf{v}$ bracket g at $\mathbf{p}$" and notice we can also use summation notation

$$\mathbf{v}[g](\mathbf{p}) = \sum_{\mathrm{i}=1}^{3} \frac{\partial g}{\partial x_i}(\mathbf{p})\, v_i, \tag{3}$$

where $\mathbf{v} = (v_1, v_2, v_3)$. This is called the **directional** or **bracket derivative**.

Lemma 2.4 (Leibniz's Product Rule). *Suppose* $\boldsymbol{v} = (v_1, v_2, v_3) \in T_{\boldsymbol{p}}(M)$ *and* $g, h : M \mapsto \mathbb{R}$*, then* $\boldsymbol{v}[g\,h](\boldsymbol{p}) = \boldsymbol{v}[g](\boldsymbol{p})\, h(\boldsymbol{p}) + g(\boldsymbol{p})\, \boldsymbol{v}[h](\boldsymbol{p})$.

Proof. This result follows from the summation form of the directional derivative and the chain rule of ordinary calculus. We temporarily drop the point of evaluation $\mathbf{p}$ for clarity,

$$\begin{aligned}
\mathbf{v}[gh] &= \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(g\,h)v_i = \sum_{i=1}^{3}\left(\frac{\partial g}{\partial x_i}h + g\frac{\partial h}{\partial x_i}\right)v_i \\
&= \sum_{i=1}^{3}\frac{\partial g}{\partial x_i}h\,v_i + \sum_{i=1}^{3} g\frac{\partial h}{\partial x_i}v_i \\
&= h\sum_{i=1}^{3}\frac{\partial g}{\partial x_i}v_i + g\sum_{i=1}^{3}\frac{\partial h}{\partial x_i}v_i \\
&= \mathbf{v}[g]\,h + g\,\mathbf{v}[h]
\end{aligned}$$

Hence, $\mathbf{v}[g\,h](\mathbf{p}) = \mathbf{v}[g](\mathbf{p})\,h(\mathbf{p}) + g(\mathbf{p})\,\mathbf{v}[h](\mathbf{p})$. □

We note in passing that the directional derivative is a linear operator, in other words

$$\mathbf{v}[\lambda_1 g + \lambda_2 h] = \lambda_1\mathbf{v}[g] + \lambda_2\mathbf{v}[h]$$

for constants $\lambda_1, \lambda_2 \in \mathbb{R}$.

Lemma 2.5 shows how the directional derivative behaves when applied in the directions $\mathbf{x}_u$ and $\mathbf{x}_v$.

Lemma 2.5. *Suppose $\boldsymbol{p} \in M$ is in the image of some coordinate patch $\boldsymbol{x}$, which has u- and v-parameter derivatives $\boldsymbol{x}_u$ and x_v. Let $g : M \mapsto \mathbb{R}$ be a function, then*

$$\boldsymbol{x}_u[g](\boldsymbol{p}) = \frac{\partial(g\circ\boldsymbol{x})}{\partial u}(\boldsymbol{p}) \quad \text{and} \quad \boldsymbol{x}_v[g](\boldsymbol{p}) = \frac{\partial(g\circ\boldsymbol{x})}{\partial v}(\boldsymbol{p}).$$

Proof. Suppose $\mathbf{x}(u_0, v_0) = \mathbf{p}$ and recall the u-parameter curve on M is $\mathbf{x}(u, v_0)$. In this light, we are in a familiar situation, the parameter is now u:

$$\mathbf{x}_u[g](\mathbf{p}) = \frac{d}{du}(g(\mathbf{x}(u, v_0)))|_{u=u_0} = \frac{\partial(g\circ\mathbf{x})}{\partial u}(u_0, v_0) = \frac{\partial(g\circ\mathbf{x})}{\partial u}(\mathbf{p}).$$

Proceed similarly to find the $\mathbf{x}_v$ bracket derivative. Instead, use the v-parameter curve $\mathbf{x}(u_0, v)$. □

Lemma 2.5 allows us to adopt the shorthand

$$\mathbf{x}_u[g] = \frac{\partial g}{\partial u} \quad \text{and} \quad \mathbf{x}_v[g] = \frac{\partial g}{\partial v}.$$

The u and v partials really only make sense for the composition $g \circ \mathbf{x}$, but Lemma 2.5 tells us it is acceptable to leave out the composition in practice.

2.5. **Covariant Derivatives.** Section 2.4 introduces the bracket derivative as a method of studying the rate of change of scalar functions defined on a surface M, but what about vector fields defined on M? Through the development in this section keep in mind that the vector field on M we eventually want to study is the collection of unit normals $\hat{\mathbf{U}} : M \mapsto \mathbb{R}^3$.

For now we maintain generality by assuming a generic vector field $\mathbf{Y} : M \mapsto \mathbb{R}^3$. So for points $\mathbf{p} \in M$ we can also express $\mathbf{Y}$ as

$$\mathbf{Y}(\mathbf{p}) = (y^1(\mathbf{p}), y^2(\mathbf{p}), y^3(\mathbf{p}))$$

with component functions $y^i : M \mapsto \mathbb{R}$. Another organizational technique that permits summation notation involves the unit vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$. These form a basis for $\mathbb{R}^3$ pointing along the x_1, x_2, and x_3 axes respectively. We write

$$\mathbf{Y}(\mathbf{p}) = \sum_{k=1}^{3} y^k(\mathbf{p})\hat{\mathbf{e}}_k$$

when convenient.

Puzzling out the rate of change of $\mathbf{Y}$, a vector field, over a three-dimensional surface M requires a bit more thought than a basic directional derivative. In Section 2.4, we observe the directional derivative essentially depends on three other derivatives, those in each of the coordinate directions (wrapped up in the gradient ∇). A covariant derivative expands on this and depends on nine other derivatives: three coordinate directions for each of the three components of the field vectors. Picture standing on some surface (held down by gravity or some sort of force) with a magic divining rod that always points up (our very own unit normal $\hat{\mathbf{U}}$). As we navigate the terrain of our surface, the rod wiggles around maintaining the "up" direction. Not only can the rod move in three directions, its orientation depends on the direction we travel, of which we have three degrees of freedom (in the context of $\mathbb{R}^3$, the two dimensional constraint still applies). There are the nine derivatives!

We apply the bracket derivative to the scalar component functions y^i to define the covariant derivative of $\mathbf{Y}$ in the direction of some vector $\mathbf{v} \in T_{\mathbf{p}}(M)$,

$$\nabla_{\mathbf{v}}\mathbf{Y} \doteq (\mathbf{v}[y^1], \mathbf{v}[y^2], \mathbf{v}[y^3]) = \sum_{k=1}^{3} \mathbf{v}[y^k]\hat{\mathbf{e}}_k.$$

This new derivative is also linear because the bracket derivative is. The covariant derivative leads to our first method of studying the shape of surfaces themselves: the Shape Operator and the associated Gauss Map.

Example: Differentiation on the Torus

In Section 2.3 we calculated the unit normal for the torus,

$$\hat{\mathbf{U}} = -(\cos u \cos v, \cos u \sin v, \sin u).$$

Figure 4 shows $\hat{\mathbf{U}}(\mathbf{p})$ where $\mathbf{p} = \mathbf{T}(\pi/3, 7\pi/6)$, but $\hat{\mathbf{U}}$ is really a vector field defined for all points on the torus. Using our new covariant derivative, we can differentiate it over the whole torus. Let $\mathbf{p}$ be an arbitrary point on the torus now and suppose we want to know the derivative of $\hat{\mathbf{U}}$ for some direction $\mathbf{v} \in T_{\mathbf{p}}(\mathbf{T})$. Lemma 2.3 shows that $\mathbf{v}$ is a linear combination of $\mathbf{T}_u$ and $\mathbf{T}_v$ (the basis vectors) so we apply the covariant derivative in the basis vector directions and observe

$$\nabla_{\mathbf{v}}\hat{\mathbf{U}} = \nabla_{(\lambda_1\mathbf{T}_u+\lambda_2\mathbf{T}_v)}\hat{\mathbf{U}} = \lambda_1\nabla_{\mathbf{T}_u}\hat{\mathbf{U}} + \lambda_2\nabla_{\mathbf{T}_v}\hat{\mathbf{U}}.$$

We have

$$\begin{aligned}\nabla_{\mathbf{T}_u}\hat{\mathbf{U}} &= (\mathbf{T}_u[-\cos u\cos v], \mathbf{T}_u[-\cos u\sin v], \mathbf{T}_u[-\sin u]) \\ &= \left(\frac{\partial}{\partial u}(-\cos u\cos v), \frac{\partial}{\partial u}(-\cos u\sin v), \frac{\partial}{\partial u}(-\sin u)\right) \\ &= (\sin u\cos v, \sin u\sin v, -\cos u)\end{aligned}$$

and similarly $\nabla_{\mathbf{T}_v}\hat{\mathbf{U}} = (\cos u\sin v, -\cos u\cos v, 0)$. Notice the use of Lemma 2.5 as we take u and v partials for the bracket derivatives.

2.6. **Second Derivatives.** In Sections 2.4 and 2.5 we present two notion of "first derivative" on a surface: the directional derivative, a scalar value, and the covariant derivative, a vector value. Although we present two types of second derivatives in this section, both are scalar-valued. Vector-valued second derivatives exist, indeed they are of great interest, but we do not use them here.

Again we consider a scalar function $g : M \mapsto \mathbb{R}$. Our first type of second derivative is induced from the definition of the directional derivative using our bracket notation (see equations (2) and (3)),

$$\mathbf{v}[\mathbf{v}[g]](\mathbf{p}) \doteq \nabla(\nabla g(\mathbf{p})\cdot\mathbf{v})\cdot\mathbf{v} = \sum_{j=1}^{3}\sum_{i=1}^{3}\frac{\partial^2 g}{\partial x_j \partial x_i}(\mathbf{p})\, v_i\, v_j,$$

where $\mathbf{v} = (v_1, v_2, v_3)$. We call this the **second directional derivative** because it gives the second derivative of g is a specific direction $\mathbf{v}$ at a point $\mathbf{p} \in M$.

We also use the Laplacian, another scalar-valued second derivative. Suppose $W \subseteq \mathbb{R}^3$, we start by defining the **Laplacian** for a scalar function $G : W \mapsto \mathbb{R}^3$ as

$$\Delta\, G \doteq \nabla\cdot\nabla G = \sum_{i=1}^{3}\frac{\partial^2 G}{\partial x_i^2}. \tag{4}$$

The function G is certainly related the function g we have used before, g can be thought of as the **restriction** of G to M.

There is a subtle, yet important, comment that is specific to this type of differentiation on surfaces in $\mathbb{R}^3$. While $\Delta\, G$ is the Laplacian in $\mathbb{R}^3$, a different Laplacian is required when we think about the restrictions of a two dimensional surface because once we restrict G to M, as g, x_1, x_2, and x_3 are constrained. A similar phenomenon occurs in vector calculus when computing the second derivative along a curve; we must take curvature into account. Since we have yet to begin our discussion about the curvature of a surface, we make the following definition and return to this problem in Section 5.3. Take $\mathbf{p} \in M$ and let $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_{\pi/2}$ be curves on M that pass through $\mathbf{p}$ and are perpendicular to each other at $\mathbf{p}$. Let s and $\tilde{s}$ denote arc length along $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_{\pi/2}$ respectively. If we adopt arc length parametrizations, then the scalar function g can be differentiated on M with respect to s or $\tilde{s}$ along either $\boldsymbol{\alpha}$ or $\boldsymbol{\alpha}_{\pi/2}$. This derivative is different than our familiar directional derivative because it is taken with respect to arc length. The **surface Laplacian** of g at $\mathbf{p} \in M$ is

$$\Delta_M\, g\,(\mathbf{p}) \doteq \frac{\partial^2 g}{\partial s^2}\,(\mathbf{p}) + \frac{\partial^2 g}{\partial \tilde{s}^2}\,(\mathbf{p}). \tag{5}$$

Equation (5) is of theoretical use only (we never use it to actually compute a surface Laplacian) because so much about the curvature of the underlying surface is hidden in the arc length parameters s and $\tilde{s}$. The problem is really computational because equation (4) is difficult to use because of the restrictions on x_1, x_2, and x_3 and equation (5) has arc length, which requires the curvature of M.

For now, we consider a basic example. This example comes from the introduction to Robert Reilly's 1982 paper Mean Curvature, The Laplacian, and Soap Bubbles, which we discuss in greater depth in Section 5.3 because it provides an elegant proof of Aleksandrov's Theorem. [2] Suppose M is a sphere. The Laplacian in spherical coordinates (r, θ, ϕ) is

$$\Delta G = \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 G}{\partial \phi^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial G}{\partial \theta} \right) + \frac{2}{r} \frac{\partial G}{\partial r} + \frac{\partial^2 G}{\partial r^2}$$

and the surface Laplacian is

$$\Delta_M g = \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 g}{\partial \phi^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial g}{\partial \theta} \right).$$

Notice the r derivatives just drop out for the surface Laplacian because a sphere has fixed radius. Once again, this reflects the two dimensionality of a spherical surface, although it sits in three dimensional space.

3. Curvature

We apply the theory of differentiation from Section 2 to explore the curvature of surfaces. Sections 3.1 and 3.2 sample Oprea's text, while Sections 3.3 and 3.4 fundamental forms, which mirrors Andrew Pressley's approach in his text Elementary Differential Geometry. [3] Oprea's and Pressley's approaches are notably disjoint reflecting the two distinct methods of building the fundamentals of Differential Geometry. On one hand, there is the Shape Operator approach, which emphasizes differentiation and gives the discussion a Calculus-like feel. On the other hand, there are fundamental forms and a matrix-based approach. We draw from the strengths of both methods and show the differences are more notational than theoretical by reconciling the Shape Operator with the fundamental form matrices in Lemmas 3.6 and 3.7.

3.1. **The Shape Operator.** Two rudimentary ways to characterize the shape of a surface M are to watch how the unit normal $\hat{\mathbf{U}}$ behaves as we move around (recall the divining stick example of Section 2.5) and to compare M to a sphere. The former of these methods is accomplished using the Shape Operator and the latter using the Gauss Map.

Define the **Shape Operator** of M at a point $\mathbf{p} \in M$ in the direction of $\mathbf{v} \in T_{\mathbf{p}}(M)$ as

$$\mathcal{S}_{\mathbf{p}}(\mathbf{v}) = -\nabla_{\mathbf{v}} \hat{\mathbf{U}}.$$

[2] R. Reilly. Mean curvature, the Laplacian, and Soap Bubbles. *Amer. Math. Monthly* **89** (1982), no. 3, 180-188+197-198.

[3] A. Pressley. *Elementary Differential Geometry.* Springer Undergraduate Mathematics Series, Springer, London, U. K., 2008.

What does this mean? Over the surface M, we consider the unit normal $\hat{\mathbf{U}} : M \mapsto \mathbb{R}^3$ to be a vector field with component functions $u^i : M \mapsto \mathbb{R}$. Let $\boldsymbol{\alpha}$ be a curve on M with $\boldsymbol{\alpha}(0) = \mathbf{p} \in M$ and $\boldsymbol{\alpha}'(0) = \mathbf{v} \in T_{\mathbf{p}}(M)$. By the definitions of the bracket (directional) and covariant derivatives

$$\nabla_{\mathbf{v}} \hat{\mathbf{U}} = \sum_{k=1}^{3} \mathbf{v}[u^k]\, \hat{\mathbf{e}}_k = \sum_{k=1}^{3} \frac{d}{dt}(u^k(\boldsymbol{\alpha}(t)))|_{t=0}\, \hat{\mathbf{e}}_k.$$

So we are examining the component functions of $\hat{\mathbf{U}}$ along $\boldsymbol{\alpha}$. The curve $\boldsymbol{\alpha}$ is parametrized by t, and $\nabla_{\mathbf{v}} \hat{\mathbf{U}}$ involves the partial derivative with respect to t. Put these ideas together and we see $\nabla_{\mathbf{v}} \hat{\mathbf{U}}$ shows how the unit normal $\hat{\mathbf{U}}$ changes in the direction of $\boldsymbol{\alpha}$, a direction also specified by the vector $\mathbf{v}$.

The Shape Operator does exactly what we hinted at: $\mathcal{S}_{\mathbf{p}}$ describes how the unit normal changes over the surface. This alone is somewhat unsatisfying because we are still left to puzzle out the covariant derivative, but $\mathcal{S}_{\mathbf{p}}$'s utility extends much further. We can recast $\mathcal{S}_{\mathbf{p}}$ as a transformation of the vector space $T_{\mathbf{p}}(M)$. After all, $\nabla_{\mathbf{v}}$ is simply a special type of vector-mapping. So the interpretation of $\mathcal{S}_{\mathbf{p}}(\mathbf{v})$ as some representative matrix (call it $\underline{\mathcal{S}_{\mathbf{p}}}$) mapping, under matrix multiplication, vectors in $T_{\mathbf{p}}(M)$ to $\mathbb{R}^3$ is entirely reasonable and moreover will prove very illuminating. Our intuition tells us a mapping of vectors in $\mathbb{R}^3$ requires a 3×3 transformation matrix. However, the Shape Operator acts on vectors $\mathbf{v} \in T_{\mathbf{p}}(M)$ and even though $T_{\mathbf{p}}(M)$ exists in three dimensional space (see Figure 4), Lemma 2.3 demonstrates only two vectors form a basis for $T_{\mathbf{p}}(M)$, namely $\mathbf{x}_u$ and $\mathbf{x}_v$. Hence, $\underline{\mathcal{S}_{\mathbf{p}}}$ will be a 2×2 matrix and act on the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$.

Recall from basic linear algebra a transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is called linear if for vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ and any constants $c_1, c_2 \in \mathbb{R}$,

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2).$$

Also, T is symmetric if for all $\mathbf{v}_1\mathbf{v}_2 \in \mathbb{R}^2$,

$$T(\mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot T(\mathbf{v}_2).$$

Lemma 3.1. *The Shape Operator $\mathcal{S}_{\mathbf{p}}$ is a symmetric, linear transformation mapping $T_{\mathbf{p}}(M)$ to itself.*

Proof. Consider vectors $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}(M)$ for some point $\mathbf{p} \in M$ and constants $c_1, c_2 \in \mathbb{R}$. Suppose $\hat{\mathbf{U}} = (u^1, u^2, u^3)$ is normal to the surface at $\mathbf{p}$. Linearity follows from the linearity of the bracket derivative of Section 2.4 (or more fundamentally the linearity of all the partial derivatives represented in the covariant derivative):

$$\begin{aligned}
\mathcal{S}_{\mathbf{p}}(c_1\mathbf{v} + c_2\mathbf{w}) &= -\nabla_{(c_1\mathbf{v}+c_2\mathbf{w})}\hat{\mathbf{U}} = -\sum_{k=1}^{3}(c_1\mathbf{v} + c_2\mathbf{w})[u^k]\hat{\mathbf{e}}_k \\
&= -c_1 \sum_{k=1}^{3} \mathbf{v}[u^k]\hat{\mathbf{e}}_k - c_2 \sum_{k=1}^{3} \mathbf{w}[u^k]\hat{\mathbf{e}}_k = c_1\mathcal{S}_{\mathbf{p}}(\mathbf{v}) + c_2\mathcal{S}_{\mathbf{p}}(\mathbf{w}).
\end{aligned}$$

To show $\mathcal{S}_{\mathbf{p}}$ maps $T_{\mathbf{p}}(M)$ to itself first consider

$$\begin{aligned}\mathbf{v}[\hat{\mathbf{U}}\cdot\hat{\mathbf{U}}] &= \mathbf{v}\left[\sum_{i=1}^{3}\left(u^i\right)^2\right] = 2\sum_{i=1}^{3}u^i\mathbf{v}[u^i] \\ &= 2(u^1,u^2,u^3)\cdot(\mathbf{v}[u^1],\mathbf{v}[u^2],\mathbf{v}[u^3]) \\ &= 2\;\hat{\mathbf{U}}\cdot(\nabla_v\hat{\mathbf{U}}) = -2\;\hat{\mathbf{U}}\cdot\mathcal{S}_{\mathbf{p}}(\mathbf{v}).\end{aligned}$$

But $\hat{\mathbf{U}}\cdot\hat{\mathbf{U}}=1$, as $\hat{\mathbf{U}}$ is a unit normal so $\mathbf{v}[\hat{\mathbf{U}}\cdot\hat{\mathbf{U}}]=\mathbf{v}[1]=0$. This shows $\hat{\mathbf{U}}\cdot\mathcal{S}_{\mathbf{p}}(\mathbf{v})=0$, which means $\mathcal{S}_{\mathbf{p}}(\mathbf{v})$ is perpendicular to $\hat{\mathbf{U}}$ and therefore $\mathcal{S}_{\mathbf{p}}(\mathbf{v})$ must lie in $T_{\mathbf{p}}(M)$. Note this technique of applying the derivative to a constant dot product, we use it repeatedly in Differential Geometry.

All that remains is to show $\mathcal{S}_{\mathbf{p}}$ is symmetric. Recall Lemma 2.3 shows $\{\mathbf{x}_u,\mathbf{x}_v\}$ form a basis for $T_{\mathbf{p}}(M)$ so it is sufficient to demonstrate symmetry for the basis vectors, that is show

$$\mathcal{S}_{\mathbf{p}}(\mathbf{x}_v)\cdot\mathbf{x}_u=\mathbf{x}_v\cdot\mathcal{S}_{\mathbf{p}}(\mathbf{x}_u).$$

Again, consider the dot product identity $\hat{\mathbf{U}}\cdot\mathbf{x}_u=0$, which holds because $\hat{\mathbf{U}}$ is perpendicular to $T_{\mathbf{p}}(M)$. On one hand, we have $\mathbf{x}_v[\hat{\mathbf{U}}\cdot\mathbf{x}_u]=\mathbf{v}[0]=0$ and on the other, compute

$$\begin{aligned}\mathbf{x}_v[\hat{\mathbf{U}}\cdot\mathbf{x}_u] &= \mathbf{x}_v\left[\sum_{i=1}^{3}u^i\frac{\partial x^i}{\partial u}\right]=\sum_{i=1}^{3}\mathbf{x}_v\left[u^i\frac{\partial x^i}{\partial u}\right] \\ &= \sum_{i=1}^{3}\left(u^i\mathbf{x}_v\left[\frac{\partial x^i}{\partial u}\right]+\mathbf{x}_v[u^i]\frac{\partial x^i}{\partial u}\right) \\ &= \sum_{i=1}^{3}\left(u^i\frac{\partial}{\partial v}\left(\frac{\partial x^i}{\partial u}\circ\mathbf{x}\right)+\mathbf{x}_v[u^i]\frac{\partial x^i}{\partial u}\right) \\ &= \sum_{i=1}^{3}\left(u^i\frac{\partial^2 x^i}{\partial v\partial u}\right)+\sum_{i=1}^{3}\left(\mathbf{x}_v[u^i]\frac{\partial x^i}{\partial u}\right) \\ &= \hat{\mathbf{U}}\cdot\mathbf{x}_{uv}+(\nabla_{\mathbf{x}_v}\hat{\mathbf{U}})\cdot\mathbf{x}_u=\hat{\mathbf{U}}\cdot\mathbf{x}_{uv}-\mathcal{S}_{\mathbf{p}}(\mathbf{x}_v)\cdot\mathbf{x}_u.\end{aligned}$$

Notice we have used the product rule of Lemma 2.4 and the result of Lemma 2.5 in some of the steps of this computation. Hence, $\mathcal{S}_{\mathbf{p}}(\mathbf{x}_v)\cdot\mathbf{x}_u=\hat{\mathbf{U}}\cdot\mathbf{x}_{uv}$. Starting with the dot product identity $\hat{\mathbf{U}}\cdot\mathbf{x}_v=0$ an analogous computation shows $\mathcal{S}_{\mathbf{p}}(\mathbf{x}_u)\cdot\mathbf{x}_v=\hat{\mathbf{U}}\cdot\mathbf{x}_{vu}$. Mixed partials are equal ($\mathbf{x}_{uv}=\mathbf{x}_{vu}$) therefore $\mathcal{S}_{\mathbf{p}}(\mathbf{x}_v)\cdot\mathbf{x}_u=\mathbf{x}_v\cdot\mathcal{S}_{\mathbf{p}}(\mathbf{x}_u)$ as required. □

Lemma 3.1 has two important corollaries that we use in Sections 3.3 and 4.1.

Corollary 3.2. *The following identities hold:*

$$\mathcal{S}_{\mathbf{p}}(\mathbf{x}_u)\cdot\mathbf{x}_u=\hat{\mathbf{U}}\cdot\mathbf{x}_{uu}\quad\textit{and}\quad\mathcal{S}_{\mathbf{p}}(\mathbf{x}_v)\cdot\mathbf{x}_v=\hat{\mathbf{U}}\cdot\mathbf{x}_{vv}.$$

Proof. Apply analogous calculations from the proof of Lemma 3.1 starting with $\mathbf{x}_u[\hat{\mathbf{U}}\cdot\mathbf{x}_u]$ and $\mathbf{x}_v[\hat{\mathbf{U}}\cdot\mathbf{x}_v]$. □

Corollary 3.3. *The Shape Operator has real eigenvalues.*

Proof. From basic linear algebra we know a symmetric, linear 2×2 transformation has real eigenvalues. ☐

These basic results regarding the Shape Operator are quite technical and we have not mentioned much about why the symmetric, linear transformation $\mathcal{S}_{\mathbf{p}}(M)$ is even called the "Shape" Operator.

Suppose $\mathbf{v} = \dot{u}\,\mathbf{x}_u + \dot{v}\,\mathbf{x}_v \in T_{\mathbf{p}}(M)$. Lemma 2.3 shows the weights are $\dot{u}$ and $\dot{v}$ for some function u and v. Once we incorporate matrix machinery the concept of a basis becomes paramount. Specifically, the **standard basis** for $T_{\mathbf{p}}(M)$ is

$$\{\mathbf{x}_u, \mathbf{x}_v\}.$$

As a vector in the standard basis we can write

$$(\mathbf{v})_s = \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}_s, \ (\mathbf{x}_u)_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_s, \text{ and } (\mathbf{x}_{\mathrm{v}})_{\mathrm{s}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\mathrm{s}}, \tag{6}$$

where the subscript s indicates the standard basis. Notice $(\mathbf{x}_u)_s \cdot (\mathbf{x}_v)_s = 0$, but in terms of our x_1, x_2, x_3 coordinate system $\mathbf{x}_u$ and $\mathbf{x}_v$ need not be orthogonal. In fact, the standard basis reflects the notion of two dimensionality on the surface because even though $\mathbf{v}, \mathbf{x}_u, \mathbf{x}_v \in \mathbb{R}^3$, we can use a basis with two elements to describe them.

By the part of Lemma 3.1 that shows $\mathcal{S}_{\mathbf{p}}$ is a mapping from $T_{\mathbf{p}}(M)$ to itself, we know for some constants $a, b, c, d \in \mathbb{R}$,

$$\mathcal{S}_{\mathbf{p}}(\mathbf{x}_u) = -\nabla_{\mathbf{x}_u}\hat{\mathbf{U}} = \begin{pmatrix} a \\ b \end{pmatrix}_s \quad \text{and} \quad \mathcal{S}_{\mathbf{p}}(\mathbf{x}_{\mathrm{v}}) = -\nabla_{\mathbf{x}_{\mathrm{v}}}\hat{\mathbf{U}} = \begin{pmatrix} c \\ d \end{pmatrix}_{\mathrm{s}}.$$

The linearity of $\mathcal{S}_{\mathbf{p}}$ implies

$$\begin{aligned} \mathcal{S}_{\mathbf{p}}(\mathbf{v}) &= \mathcal{S}_{\mathbf{p}}(\dot{u}\,\mathbf{x}_u + \dot{v}\,\mathbf{x}_v) = \dot{u}\,\mathcal{S}_{\mathbf{p}}(\mathbf{x}_u) + \dot{v}\,\mathcal{S}_{\mathbf{p}}(\mathbf{x}_v) \\ &= \dot{u}\,(a\mathbf{x}_u + b\mathbf{x}_v) + \dot{v}\,(c\mathbf{x}_u + d\mathbf{x}_v) \\ &= (\dot{u}\,a + \dot{v}\,c)\mathbf{x}_u + (\dot{u}\,b + \dot{v}\,d)\mathbf{x}_v. \end{aligned}$$

Two observations follow. We deduce the form of the Shape Operator's representative matrix

$$(\underline{\mathcal{S}_{\mathbf{p}}})_s(\mathbf{v})_s = \begin{pmatrix} a & c \\ b & d \end{pmatrix}_s \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}_s = \begin{pmatrix} \dot{u}\,a + \dot{v}\,c \\ \dot{u}\,b + \dot{v}\,d \end{pmatrix}_s.$$

We also see how $\mathcal{S}\mathbf{p}$ involves the rate of change of $\hat{\mathbf{U}}$ in the basis $\mathbf{x}_u$ and $\mathbf{x}_v$ directions. For small deviations from $\mathbf{p}$, $\hat{\mathbf{U}}$ will "twitch" in amounts given by the weights in the matrix $\underline{\mathcal{S}_{\mathbf{p}}}$. The coefficients a, b, c, and d are far from arbitrary and we discover their specific meanings in Section 3.5.

Example: The Shape Operator Applied to the Torus

Most of the computation for the Shape Operator acting on the torus can be found in Section 2.5. For $\mathbf{p} \in \mathbf{T}$ and $\mathbf{v} = \lambda_1 \mathbf{T}_u + \lambda_2 \mathbf{T}_v \in T_{\mathbf{p}}(\mathbf{T})$,

$$\begin{aligned}\mathcal{S}_{\mathbf{p}}(\mathbf{T}_u) &= -\nabla_{\mathbf{T}_u}\hat{\mathbf{U}} = (-\sin u\cos v, -\sin u \sin v, \cos u) = \frac{1}{r}\mathbf{T}_u \\ \mathcal{S}_{\mathbf{p}}(\mathbf{T}_v) &= -\nabla_{\mathbf{T}_v}\hat{\mathbf{U}} = (-\cos u \sin v, \cos u \cos v, 0) = \frac{\cos u}{R + r\cos u}\mathbf{T}_v.\end{aligned}$$

To be more specific let us consider the point from the torus example in Section 2.3, $\mathbf{p} = \mathbf{T}(\pi/3, 7\pi/6) = r(-5\sqrt{3}/4, -5/4, \sqrt{3}/2)$, where $\mathcal{S}_{\mathbf{p}}(\mathbf{T}_u) = (3/4, \sqrt{3}/4, 1/2)$ and $\mathcal{S}_{\mathbf{p}}(\mathbf{T}_v) = (1/4, -\sqrt{3}/4, 0)$. Notice in accordance with Lemma 3.1, which says $\mathcal{S}_{\mathbf{p}}$ maps $T_{\mathbf{p}}(\boldsymbol{T})$ to itself, both $\mathcal{S}_{\mathbf{p}}(\mathbf{T}_u)$ and $\mathcal{S}_{\mathbf{p}}(\mathbf{T}_v)$ are orthogonal to $\hat{\mathbf{U}}(\mathbf{p}) = (-\sqrt{3}/4, -1/4, \sqrt{3}/2)$ (as given in Section 2.3).

3.2. **The Gauss Map.** Much of the rich, qualitative information the Shape Operator provides is captured by the Gauss Map. As we mention in Section 2.3 the discussion of the shape of a surface M often begins with the unit normal $\hat{\mathbf{U}}$. Regularity ($\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$) guarantees a unit normal $\hat{\mathbf{U}}(\mathbf{p})$ exists for every point $\mathbf{p} \in M$. How can we characterize the collection of all unit normals defined on M?

To get an idea of the information held in $\hat{\mathbf{U}}$ consider the plane, infinite cylinder, and sphere of Figure 5. These surfaces are called **orientable** because given any surface curve we can define a continuous unit normal direction. For instance, we orient the plane by defining the $\hat{\mathbf{U}}$ direction to always point to the same side. Thus, we never encounter a point where the unit normal suddenly flips direction (by 180° to be specific) or is discontinuous. In this light, the plane has only one possible unit normal direction. In contrast, the cylinder has many unit normal directions: there is a band of outward unit normals about the body of the cylinder. By the symmetry along the cylinder's infinite dimension, we only need one such band to get all possible directions. The sphere has a unit normal in every possible direction. If we seek a standard to use universally when describing the collection of unit normal directions on an arbitrary surface M, then the sphere is the ideal candidate. Specifically, we elect to use the surface of a unit sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$.

We accomplish this comparison by defining the Gauss Map $\mathcal{G} : M \mapsto S^2$ as $\mathcal{G}(\mathbf{p}) = \hat{\mathbf{U}}(\mathbf{p})$ for all $\mathbf{p} \in M$. Given an arbitrary surface M, the image under the Gauss Map $\mathcal{G}(M)$ corresponds to an area on S^2. The image of a plane under the Gauss Map is really just a point and the infinite cylinder maps to a great circle on S^2 (both are regions of zero area). In fact, the plane and infinite cylinder are fundamental examples of how the Gauss Map reveals similarities between surfaces. Notice, a plane can be transformed into a cylinder quite easily. It is not a coincidence that they both have the same area under the Gauss Map!

Example: The Gauss Map of the Torus

How does a torus behave under this mapping? Figures 6, 7, and 8 demonstrate how the Gauss Map is helpful, especially as a basic characterizing tool for surfaces. For

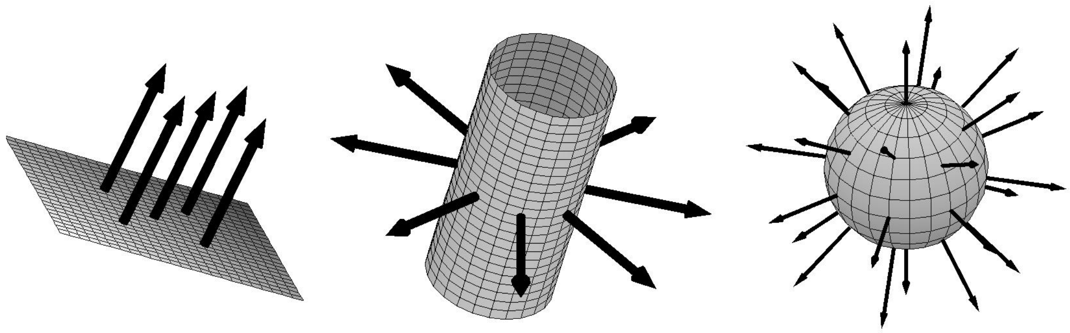


FIGURE 5. The Gauss Map $\mathcal{G} : M \mapsto S^2$ concerns the collection of all unit normals defined on a surface. A cylinder and a plane have the same area on the unit sphere under $\mathcal{G}$.

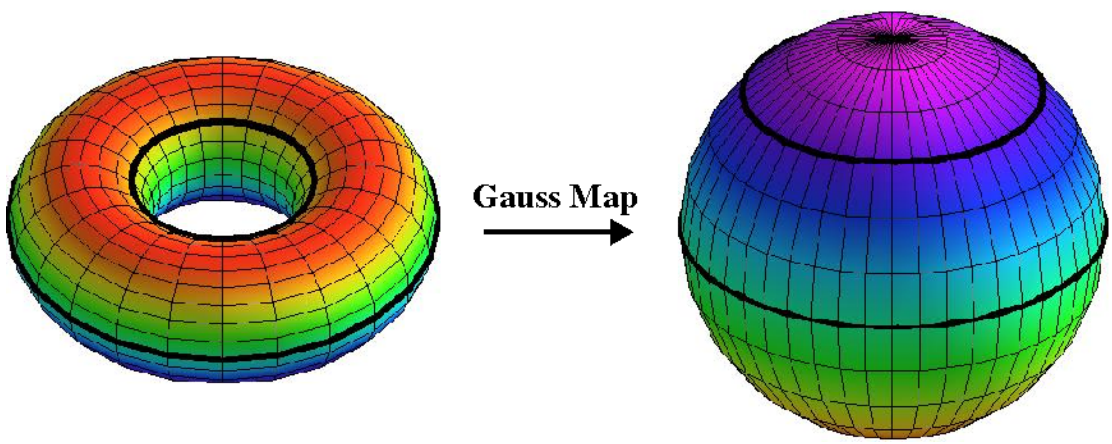


FIGURE 6. Apply $\mathcal{G}$ to v-parameter curves on the torus and we see lines of latitude on the sphere.

instance, in Figure 6 we start with a v-parameter curve on the torus and see that it maps to a line of latitude on the sphere. Likewise, in Figure 7, u-parameter curves map to lines of latitude. Taken together, this indicates $\mathcal{G}$ applied to the torus S^2 covers the entire surface area of the sphere. Thus, a torus, like a sphere, has all available unit normal directions. Contrast this to a cylinder in particular, which only maps to a single band of unit normals under $\mathcal{G}$. We encounter a further difference between a torus and cylinder in Section 4.4.

3.3. **The First Fundamental Form.** The Gauss Map represents a useful application of the Shape Operator, but the utility is somewhat limited because many different classes of surfaces have the same area under the Gauss Map. Our hope is to compare surfaces in more detail and we begin by investigating the mathematical objects that the Shape Operator is based on: fundamental forms. Here is where our discussion veers away from Oprea's presentation and draws more heavily from Pressley's text.

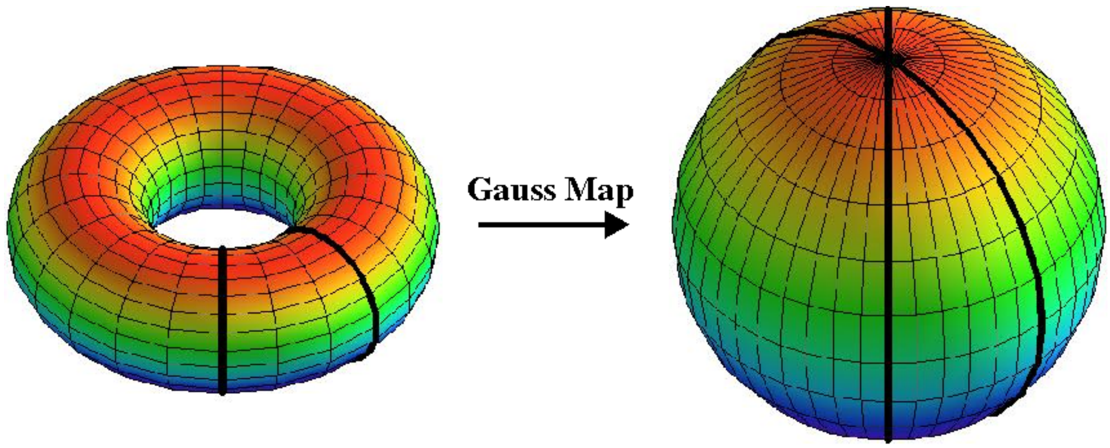


FIGURE 7. Apply $\mathcal{G}$ to u-parameter curves on the torus and we see lines of longitude on the sphere.

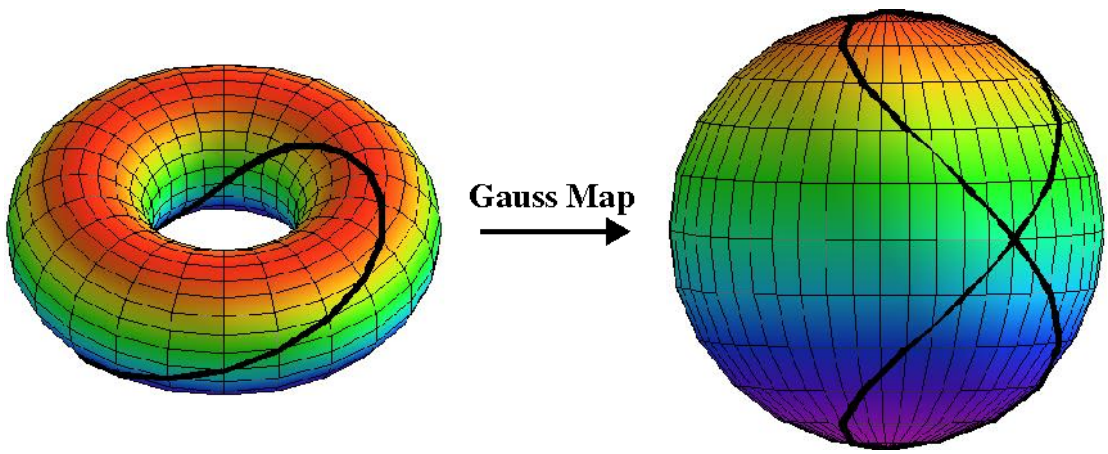


FIGURE 8. Apply $\mathcal{G}$ to any surface curve on the torus and we see a corresponding curve on the sphere. We can learn about the shape of the torus by using the sphere as a clear description of the torus's unit normals.

Recall our meter-stick construction of Section 1.1. Locally, we are able to lay our meter-stick flat on the ground, but what if we use a rigid measurement device of length on the order of, say the radius of the Earth? We find the curvature of Earth forbids us from ever laying this device "flat." So measurement of length is clearly surface dependent. We begin by discussing arc-length computations.

Suppose the curve $\boldsymbol{\alpha}$ is on a surface M. Suppose $\boldsymbol{\alpha}(t)$, where t is a parameter, is in the image of some coordinate patch $\mathbf{x}$. Lemma 2.2 ensures $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$ for some scalar functions u and v. Compute the arc length along $\boldsymbol{\alpha}$ from t_1 to t_2 using an integral:

$$L = \int_{t_1}^{t_2} ||\dot{\boldsymbol{\alpha}}(t)||\, dt.$$

We use $\dot{\boldsymbol{\alpha}}, \dot{u}$, and $\dot{v}$ to denote derivatives with respect to t and $||\dot{\boldsymbol{\alpha}}(t)||$ is the usual Euclidean norm of $\dot{\boldsymbol{\alpha}}$. We compute $||\dot{\boldsymbol{\alpha}}||^2$ explicitly using the Chain Rule:

$$\begin{aligned} ||\dot{\boldsymbol{\alpha}}||^2 &= \left|\left|\frac{d}{dt}(\mathbf{x}(u,v))\right|\right|^2 = \left|\left|\frac{\partial \mathbf{x}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{x}}{\partial v}\frac{dv}{dt}\right|\right|^2 \\ &= ||\mathbf{x}_u \dot{u} + \mathbf{x}_v \dot{v}||^2 = (\mathbf{x}_u \dot{u} + \mathbf{x}_v \dot{v}) \cdot (\mathbf{x}_u \dot{u} + \mathbf{x}_v \dot{v}) \\ &= (\mathbf{x}_u \cdot \mathbf{x}_u)\, \dot{u}^2 + 2\,(\mathbf{x}_u \cdot \mathbf{x}_v)\, \dot{u}\dot{v} + (\mathbf{x}_v \cdot \mathbf{x}_v)\, \dot{v}^2. \end{aligned}$$

The norm of $\dot{\boldsymbol{\alpha}}$ squared is related to the **first fundamental form** of the surface M:

$$(\mathbf{x}_u \cdot \mathbf{x}_u)\, du^2 + 2(\mathbf{x}_u \cdot \mathbf{x}_v)\, du\, dv + (\mathbf{x}_v \cdot \mathbf{x}_v)\, dv^2.$$

The three dot products in the coefficients involving the partial derivatives of patch $\mathbf{x}$ are of enormous importance in Differential Geometry and get their own names:

$$E \doteq \mathbf{x}_u \cdot \mathbf{x}_u, \quad F \doteq \mathbf{x}_u \cdot \mathbf{x}_v, \quad \text{and} \quad \mathrm{G} \doteq \mathbf{x}_\mathrm{v} \cdot \mathbf{x}_\mathrm{v}. \tag{7}$$

So the first fundamental form is

$$E\, du^2 + 2\, F\, du\, dv + G\, dv^2. \tag{8}$$

With this new notation the arc length integral can be written

$$L = \int_{t_1}^{t_2} \sqrt{E\dot{u}^2 + 2\, F\dot{u}\dot{v} + G\dot{v}^2}\, dt. \tag{9}$$

Using formal manipulations of the differential element dt, such as $\dot{u}\ dt = (du/dt)\, dt = du$, we rewrite the integrand

$$\sqrt{(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{u})\ dt^2} = \sqrt{Edu^2 + 2F\, dudv + Mdv^2} = \sqrt{dl^2} = dl.$$

We interpret this integrand as a differential element of length dl along the curve $\boldsymbol{\alpha}$.

The first fundamental form is a profound computational notation and affords a method of integrating curves on arbitrary surfaces. We will not delve extensively into these integration properties, but two examples show the elegance of the first fundamental form.

Consider a sphere with radius r. This sphere has coordinate patch

$$\mathbf{x}(u,v) = (r \sin u \cos v, r \sin u \sin v, r \cos u). \tag{10}$$

Compute the u and v partials:

$$\mathbf{x}_u = (r \cos u \cos v, r \cos u \sin v, -r \sin u)\ ,\ \mathbf{x}_v = (-r \sin u \sin v, r \sin u \cos v, 0)\ , \tag{11}$$

and further

$$E = \mathbf{x}_u \cdot \mathbf{x}_u = r^2 \cos^2 u \cos^2 v + r^2 \cos^2 u \sin^2 v + r^2 \sin^2 u = r^2 \tag{12}$$

$$F = \mathbf{x}_u \cdot \mathbf{x}_v = r^2 \sin u \cos u \sin v \cos v - r^2 \sin u \cos u \sin v \cos v = 0 \tag{13}$$

$$G = \mathbf{x}_v \cdot \mathbf{x}_v = r^2 \sin^2 u \sin^2 v + r^2 \sin^2 u \cos^2 v = r^2 \sin^2 u. \tag{14}$$

Thus, the first fundamental form is $dl^2 = r^2\, du^2 + r^2(\sin^2 u)\, dv^2$ and the integral

$$L = \int_{t_1}^{t_2} \sqrt{r^2\dot{u}^2 + r^2(\sin^2 u)\dot{v}^2}\, dt$$

gives arc length from t_1 to t_2 for a curve $\boldsymbol{\alpha}$ on the sphere with the parameter functions $u(t)$ and $v(t)$.

What is the first fundamental form of a plane? A patch for a plane through the origin with orthonormal basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ is $\mathbf{x}(u, v) = u\mathbf{b}_1 + v\mathbf{b}_2$ (all linear combinations of the basis vectors). Hence, $\mathbf{x}_u = \mathbf{b}_1$ and $\mathbf{x}_v = \mathbf{b}_2$ so $E = \mathbf{x}_u \cdot \mathbf{x}_u = 1, F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$, and $G = \mathbf{x}_v \cdot \mathbf{x}_v = 1$. The first fundamental form is $dl^2 = du^2 + dv^2$ as expected.

We collect the coefficients E, F, and G into a 2×2 symmetric matrix called the **first fundamental form matrix**,

$$\mathcal{F}_1 = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

The matrix $\mathcal{F}_1$ is used to move vectors in $T_{\mathbf{p}}(M)$ to the standard basis $\{\mathbf{x}_u, \mathbf{x}_v\}$.

Lemma 3.4. *Suppose $\mathcal{F}_1$ is the first fundamental form matrix for a surface M with coordinate patch $\boldsymbol{x}$. Given a point $\boldsymbol{p} \in M$ and a vector $\boldsymbol{v} \in T_{\boldsymbol{p}}(M)$,*

$$(\boldsymbol{v})_s = \mathcal{F}^{-1} \begin{pmatrix} \boldsymbol{v} \cdot \boldsymbol{x}_u \\ \boldsymbol{v} \cdot \boldsymbol{x}_v \end{pmatrix}.$$

Proof. Suppose $\mathbf{v} = \dot{u}\,\mathbf{x}_u + \dot{v}\,\mathbf{x}_v$ and examine

$$\begin{pmatrix} \mathbf{v} \cdot \mathbf{x}_u \\ \mathbf{v} \cdot \mathbf{x}_v \end{pmatrix} = \begin{pmatrix} \dot{u}\,\mathbf{x}_u \cdot \mathbf{x}_u + \dot{v}\,\mathbf{x}_v \cdot \mathbf{x}_u \\ \dot{u}\,\mathbf{x}_u \cdot \mathbf{x}_v + \dot{v}\,\mathbf{x}_v \cdot \mathbf{x}_v \end{pmatrix} = \begin{pmatrix} \dot{u}\,E + \dot{v}\,F \\ \dot{u}\,F + \dot{v}\,G \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}.$$

Hence,

$$(\mathbf{v})_s = \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}_s = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{v} \cdot \mathbf{x}_u \\ \mathbf{v} \cdot \mathbf{x}_v \end{pmatrix}.$$

□

Lemma 3.4 enables us to find the decomposition $\mathbf{v} = \dot{u}\,\mathbf{x}_u + \dot{v}\,\mathbf{x}_v$ using properties of the surface only, that is without ever defining a curve $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$ through $\mathbf{p}$.

Example: The First Fundamental Form of the Torus

Given the partial derivatives $\mathbf{T}_u$ and $\mathbf{T}_v$ from Section 2.1 we can compute the coefficients E, F, and G for the torus

$$E = \mathbf{T}_u \cdot \mathbf{T}_u = r^2(\sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u) = r^2 \tag{15}$$

$$F = \mathbf{T}_u \cdot \mathbf{T}_v = r(R + \cos u)(\sin u \sin v \cos v - \sin u \sin v \cos v) = 0 \tag{16}$$

$$G = \mathbf{T}_v \cdot \mathbf{T}_v = (R + r\cos u)^2(\sin^2 v + \cos^2 v) = (R + r\cos u)^2. \tag{17}$$

Putting E, F, and G into the first fundamental form of the torus we find

$$dl^2 = r^2 du^2 + (R + r\cos u)^2 dv^2. \tag{18}$$

Contrast this to the sphere's first fundamental form, both r and R are required to compute arc length on the torus.

As promised in Section 2.2, we now compute the circumference of the torus-circle marked $\boldsymbol{\alpha}$ in Figure 3. This problem, which is baffling without the first fundamental form, becomes a single, albeit tricky, integral computation. In the uv-plane the closed curve $\boldsymbol{\alpha}$ has parametrization $u(t) = 1/2 + \sin t$ and $v(t) = 1/2 + \cos t$ for $t \in [0, 2\pi)$. Recall $R = 2r$ for the torus we are studying. We get

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{E\dot{u}^2 + 2\,F\dot{u}\dot{v} + G\dot{v}^2}\,dt \\ &= \int_0^{2\pi} \left(r^2\cos^2 t + (R + r\cos u)^2 \sin^2 t\right)^{1/2} dt \\ &= r\int_0^{2\pi} \left(\cos^2 t + (2 + \cos(u(t)))^2 \sin^2 t\right)^{1/2} dt \end{aligned}$$

This integral is not easy to evaluate exactly because of the nested trigonometric functions, but numeric integration yields an approximate value of $11.85\,r$. Notice the circumference of the torus-circle not only depends on the original radius in the uv-plane, but also r, which is part of the torus geometry. The circumferences for circles marked $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, and $\boldsymbol{\delta}$ in Figure 3 are about $9.90\,r$, $11.85\,r$, and $9.90\,r$ respectively. These circumference examples reflect torus symmetries; the length of $\boldsymbol{\alpha}$ matches $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ matches $\boldsymbol{\delta}$ due to the relative positions of their centers along v-parameter curves.

3.4. **Normal Curvature.** Thus far, our work with surface curves $\boldsymbol{\alpha} : I \mapsto M$ $(I \subseteq \mathbb{R})$ has involved $\dot{\boldsymbol{\alpha}}$ or $\partial\boldsymbol{\alpha}/\partial t$, where t is the parameter for $\boldsymbol{\alpha}(t)$. Recall from vector calculus $\ddot{\boldsymbol{\alpha}}$ or $\partial^2\boldsymbol{\alpha}/\partial t^2$ is related to the curvature κ of $\boldsymbol{\alpha}$, and hence, the curvature of M as well because $\boldsymbol{\alpha}$ is constrained to move on the surface. Specifically, if we picture $\ddot{\boldsymbol{\alpha}}$ as a vector at $\mathbf{p} \in M$, then $\kappa = ||\ddot{\boldsymbol{\alpha}}||$.

How does $\ddot{\boldsymbol{\alpha}}$ relate to $\dot{\boldsymbol{\alpha}}$ and $\hat{\mathbf{U}}$? Figure 9 is adapted from Pressley's book (Section 6.2) and provides a visualization of this decomposition of κ. The vector $\mathbf{w} \in T_{\mathbf{p}}(M)$ is called the binormal vector and $\mathbf{w} = \hat{\mathbf{U}} \times \dot{\boldsymbol{\alpha}}$. By definition, $\hat{\mathbf{U}}$ is perpendicular to $T_{\mathbf{p}}(M)$ so we can express $\ddot{\boldsymbol{\alpha}}$ as a linear combination

$$\ddot{\boldsymbol{\alpha}} = \lambda_1\hat{\mathbf{U}} + \lambda_2\mathbf{w}$$

for some scalars $\lambda_1, \lambda_2 \in \mathbb{R}$. Notice, the square of the curvature $\kappa^2 = ||\ddot{\boldsymbol{\alpha}}||^2 = \lambda_1^2 + \lambda_2^2$. The constants λ_1 and λ_2 are projections of $\ddot{\boldsymbol{\alpha}}$ onto the normal and binormal directions respectively. We call λ_1 the **normal curvature** κ_n. According to the projection, $\kappa_n = \ddot{\boldsymbol{\alpha}} \cdot \hat{\mathbf{U}}$.

To this point we have not addressed the length of tangent vectors $\dot{\boldsymbol{\alpha}} = \mathbf{v} \in T_{\mathbf{p}}(M)$. Looking at Figure 9 we see κ_n will depend on both the direction of $\ddot{\boldsymbol{\alpha}}$ and its magnitude, which in turn depend upon $\dot{\boldsymbol{\alpha}} = \mathbf{v}$. To discuss *the* normal curvature in a given direction we must assume $||\mathbf{v}|| = 1$. We are free to choose an arc length parametrization for $\boldsymbol{\alpha}$ so we may assume $||\mathbf{v}|| = 1$, or that $\boldsymbol{\alpha}$ is a **unit speed curve**.

Notice κ_n requires us to perform computations involving the curve $\boldsymbol{\alpha}$, while this is acceptable, we can find κ_n using coordinate patches instead, which is preferable because we already use patch functions for so much. Thus, when prompted for κ_n at a point $\mathbf{p} \in M$ we can sidestep the tedium of finding a curve through $\mathbf{p}$ and differentiating it. As in Section 3.3, we use Lemma 2.2, which says $\boldsymbol{\alpha} = \mathbf{x}(u(t), v(t))$ for unique functions

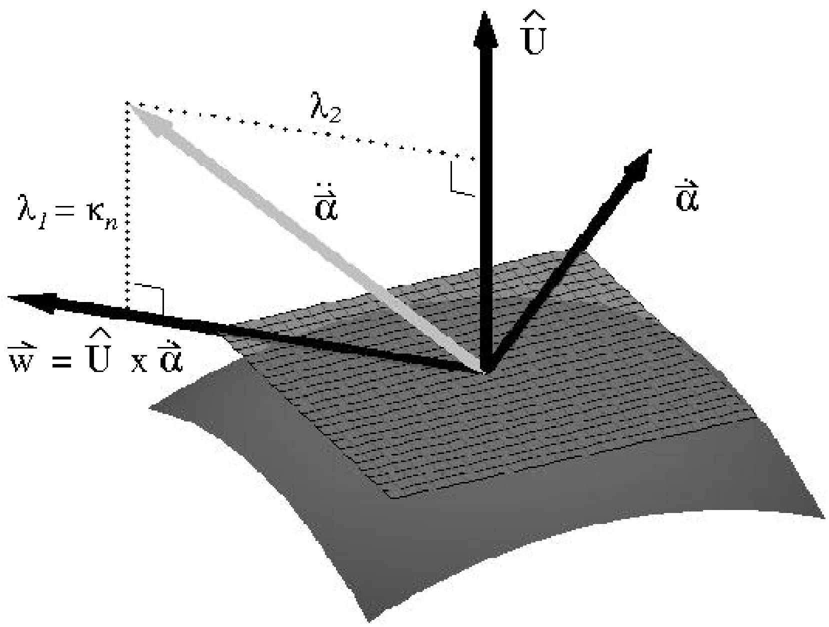


FIGURE 9. Projecting $\ddot{\boldsymbol{\alpha}}$ onto $\hat{\mathbf{U}}$ yields normal curvature κ_n.

u and v to write

$$\begin{aligned}
\kappa_n &= \hat{\mathbf{U}}\cdot\ddot{\boldsymbol{\alpha}} = \hat{\mathbf{U}}\cdot\left(\frac{d^2}{dt^2}(\mathbf{x}(u,v))\right) = \hat{\mathbf{U}}\cdot\left(\frac{d}{dt}(\mathbf{x}_u\dot{u}+\mathbf{x}_v\dot{v})\right)\\
&= \hat{\mathbf{U}}\cdot(\mathbf{x}_u\ddot{u}+(\mathbf{x}_{uu}\dot{u}+\mathbf{x}_{uv}\dot{v})\dot{u}+\mathbf{x}_v\ddot{v}+(\mathbf{x}_{vu}\dot{u}+\mathbf{x}_{vv}\dot{v})\dot{v})\\
&= (\hat{\mathbf{U}}\cdot\mathbf{x}_{uu})\,\dot{u}^2+2(\hat{\mathbf{U}}\cdot\mathbf{x}_{uv})\,\dot{u}\dot{v}+(\hat{\mathbf{U}}\cdot\mathbf{x}_{vv})\,\dot{v}^2\\
\kappa_n &= l\,\dot{u}^2+2m\,\dot{u}\dot{v}+n\,\dot{v}^2, \qquad (19)
\end{aligned}$$

where

$$l \doteq \hat{\mathbf{U}}\cdot\mathbf{x}_{uu}, \quad m \doteq \hat{\mathbf{U}}\cdot\mathbf{x}_{uv}, \quad \text{and} \quad n \doteq \hat{\mathbf{U}}\cdot\mathbf{x}_{vv}. \qquad (20)$$

The terms containing $\hat{\mathbf{U}}\cdot\mathbf{x}_u$ and $\hat{\mathbf{U}}\cdot\mathbf{x}_v$ are zero because $\hat{\mathbf{U}}$ is perpendicular to $T_{\mathbf{p}}(M)$. Just as we gave the coefficients E, F, and G their own names in equation (7), we define the coefficients l, m, and n in equation (20). Furthermore, these coefficients appear in the **second fundamental form** of surface M,

$$l\,du^2+2\,m\,du\,dv+n\,dv^2.$$

Equipped with the coefficients E, F, G, l, m, and n we can characterize any surface in $\mathbb{R}^3$, perhaps one of the most remarkable aspects of Differential Geometry.

For a basic example of the second fundamental form, take the sphere of radius r with patch $\mathbf{x}(u,v) = (r\sin u\cos v, r\sin u\sin v, r\cos u)$ and derivatives

$$\mathbf{x}_u = (r\cos u\cos v, r\cos u\sin v, -r\sin u)$$

and

$$\mathbf{x}_v = (-r\sin u\sin v, r\sin u\cos v, 0).$$

Compute the unit normal

$$\hat{\mathbf{U}} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = (\sin u\cos v, \sin u\sin v, \cos u)$$

and second derivatives

$$\begin{aligned} \mathbf{x}_{uu} &= -(r\sin u\cos v, r\sin u\sin v, r\cos u) \\ \mathbf{x}_{uv} &= (-r\cos u\sin v, r\cos u\cos v, 0) \\ \mathbf{x}_{vv} &= -(r\sin u\cos v, r\sin u\sin v, 0). \end{aligned}$$

Hence, the coefficients l, m, and n are given by

$$l = \hat{\mathbf{U}} \cdot \mathbf{x}_{uu} = -r \tag{21}$$

$$m = \hat{\mathbf{U}} \cdot \mathbf{x}_{uv} = 0 \tag{22}$$

$$n = \hat{\mathbf{U}} \cdot \mathbf{x}_{vv} = -r\sin^2 u, \tag{23}$$

which implies the second fundamental form is $-(r\,du^2 + r\sin^2 u\,dv^2)$.

While it is not yet intuitive how to succinctly and rigorously indicate the connection between the Shape Operator $\mathcal{S}_\mathbf{p}$ and our current, fundamental form-based approach, these concepts are intertwined. To attain this rigor collect these new coefficient terms into the 2×2, symmetric **second fundamental form matrix**,

$$\mathcal{F}_2 = \begin{pmatrix} l & m \\ m & n \end{pmatrix}.$$

In Lemma 3.4 we observe the first fundamental form matrix $\mathcal{F}_1$ is useful when changing to the standard basis. The second fundamental matrix $\mathcal{F}_2$ is used to compute this unique normal curvature κ_n.

Lemma 3.5. *Suppose $\mathcal{F}_2$ is the second fundamental form matrix for a surface M with coordinate patch $\boldsymbol{x}$. Given a point $\boldsymbol{p} \in M$ and vector $\boldsymbol{v} \in T_{\boldsymbol{p}}(M)$, the normal curvature in the $\boldsymbol{v}$-direction is $\kappa_n = \mathcal{F}_2(\boldsymbol{v})_s \cdot (\boldsymbol{v})_s$.*

Proof. A simple calculation shows

$$\begin{aligned} \mathcal{F}_2(\mathbf{v})_s \cdot (\mathbf{v})_s &= \begin{pmatrix} l & m \\ m & n \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}_s \cdot \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}_s \\ &= \begin{pmatrix} l\,\dot{u} + m\,\dot{v} \\ m\,\dot{u} + n\,\dot{v} \end{pmatrix} \cdot \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}_s = l\,\dot{u}^2 + 2\,m\,\dot{u}\dot{v} + n\,\dot{v}^2 = \kappa_n. \end{aligned}$$

□

Example: The Second Fundamental Form of the Torus

Notice to compute the coefficients l, m, and n for the torus we can use $\mathcal{S}_{\mathbf{p}}(\mathbf{T}_u)$, $\mathcal{S}_{\mathbf{p}}(\mathbf{T}_v)$, $\mathbf{T}_u$, and $\mathbf{T}_v$ from Sections 2.1 and 3.1 or $\hat{\mathbf{U}}$, $\mathbf{T}_{uu}$, $\mathbf{T}_{uv}$, and $\mathbf{T}_{vv}$ from Sections 2.3 and 2.6. We elect the latter and find

$$l = \hat{\mathbf{U}} \cdot \mathbf{T}_{uu} = (r\cos^2 u \cos^2 v + r\cos^2 u \sin^2 v + r \sin^2 u) = r \tag{24}$$

$$m = \hat{\mathbf{U}} \cdot \mathbf{T}_{uv} = -(r \sin u \cos u \sin v \cos v - r \sin u \cos u \sin v \cos v) = 0 \tag{25}$$

$$n = \hat{\mathbf{U}} \cdot \mathbf{T}_{vv} = (R + r\cos u)(\cos u \cos^2 v + \cos u \sin^2 v) = (R + r\cos u)\cos u. \tag{26}$$

Thus, the second fundamental form of the torus is

$$l\, du^2 + 2m\, du\, dv + n\, dv^2 = r\, du^2 + (R + r\cos u)\cos u\, dv^2.$$

Compare this to the sphere's second fundamental form $-(r\, du^2 + r\sin^2 u\, dv^2)$. Notice the similarities specifically in the powers of r and sinusoidal dependence on dv^2. Of course, the main contrast is the second fundamental form of the torus incorporates both r and R.

3.5. **The Weingarten Equations.** Section 3.4 hints at the connection between the Shape Operator $\mathcal{S}_{\mathbf{p}}$ and the fundamental forms (both first and second) of a surface M. There are many ways to discover this connection but the strongest is the derivation of the Weingarten equations established by German mathematician Julius Weingarten. While Oprea does not name them such, his text presents a similar derivation. Our familiarity with forms gives us a slightly deeper insight into what appears to be just algebraic manipulation. The overall task is quite simple: compute $\mathcal{S}_{\mathbf{p}}(\mathbf{x}_u)$ and $\mathcal{S}_{\mathbf{p}}(\mathbf{x}_v)$ using the form coefficients E, F, G, l, m, and n. These coefficients are (in summary from equations (7) and (20)):

$$E = \mathbf{x}_u \cdot \mathbf{x}_u\ ,\ F = \mathbf{x}_u \cdot \mathbf{x}_v\ ,\ G = \mathbf{x}_v \cdot \mathbf{x}_v\ ,$$
$$l = \mathcal{S}_{\mathbf{p}}(\mathbf{x}_u) \cdot \mathbf{x}_u = \hat{\mathbf{U}} \cdot \mathbf{x}_{uu}\ ,$$
$$m = \mathcal{S}_{\mathbf{p}}(\mathbf{x}_u) \cdot \mathbf{x}_v = \mathcal{S}_{\mathbf{p}}(\mathbf{x}_v) \cdot \mathbf{x}_u = \hat{\mathbf{U}} \cdot \mathbf{x}_{uv}\ ,\ \text{and}$$
$$n = \mathcal{S}_{\mathbf{p}}(\mathbf{x}_v) \cdot \mathbf{x}_v = \hat{\mathbf{U}} \cdot \mathbf{x}_{vv}.$$

Given $\mathbf{v} \in T_{\mathbf{p}}(M)$, we can apply the Shape Operator to the standard basis vectors $\mathbf{x}_u$ and $\mathbf{x}_v$,

$$\mathcal{S}_{\mathbf{p}}(\mathbf{x}_u) = -\nabla_{\mathbf{x}_u}\hat{\mathbf{U}} = -\hat{\mathbf{U}}_u = a\mathbf{x}_u + b\mathbf{x}_v$$

and

$$\mathcal{S}_{\mathbf{p}}(\mathbf{x}_v) = -\nabla_{\mathbf{x}_v}\hat{\mathbf{U}} = -\hat{\mathbf{U}}_v = c\mathbf{x}_u + d\mathbf{x}_v,$$

where a, b, c, and d populate the representative matrix of $\mathcal{S}_{\mathbf{p}}$. The only new introduction is the shorthand $\hat{\mathbf{U}}_u$ and $\hat{\mathbf{U}}_v$ to represent the covariant derivatives of $\hat{\mathbf{U}}$ in the u and v directions respectively. Making careful use of the form coefficients we find the following system of four equations in the four unknowns a, b, c, and d:

$$l = \mathcal{S}_{\mathbf{p}}(\mathbf{x}_u) \cdot \mathbf{x}_u = -\hat{\mathbf{U}}_u \cdot \mathbf{x}_u = aE + bF$$
$$m = \mathcal{S}_{\mathbf{p}}(\mathbf{x}_u) \cdot \mathbf{x}_v = -\hat{\mathbf{U}}_u \cdot \mathbf{x}_v = aF + bG$$
$$m = \mathcal{S}_{\mathbf{p}}(\mathbf{x}_v) \cdot \mathbf{x}_u = -\hat{\mathbf{U}}_v \cdot \mathbf{x}_u = cE + dF$$

$$n = \mathcal{S}_{\mathbf{p}}(\mathbf{x}_v) \cdot \mathbf{x}_v = -\hat{\mathbf{U}}_v \cdot \mathbf{x}_v = cF + dG.$$

This system completely determines a, b, c, and d:

$$a = \frac{lG - mF}{EG - F^2}\ ,\ b = \frac{mE - lF}{EG - F^2}\ ,\ c = \frac{mG - nF}{EG - F^2}\ ,\ d = \frac{nE - mF}{EG - F^2}.$$

Do not be deceived by the simple letters in these expressions, they disguise a mess of partial derivatives and dot products! The **Weingarten equations** follow

$$\mathcal{S}_{\mathbf{p}}(\mathbf{x}_u) = a\mathbf{x}_u + b\mathbf{x}_v = \frac{lG - mF}{EG - F^2}\mathbf{x}_u + \frac{mE - lF}{EG - F^2}\mathbf{x}_v \tag{27}$$

and

$$\mathcal{S}_{\mathbf{p}}(\mathbf{x}_v) = c\mathbf{x}_u + d\mathbf{x}_v = \frac{mG - nF}{EG - F^2}\mathbf{x}_u + \frac{nE - mF}{EG - F^2}\mathbf{x}_v. \tag{28}$$

At this juncture we can puzzle out the form of the Shape Operator's representative matrix in terms of the form matrices $\mathcal{F}_1$ and $\mathcal{F}_2$.

Lemma 3.6. *Suppose $EG \neq F^2$, that is $\mathcal{F}_1$ is non-singular. The representative matrix of the Shape Operator $\mathcal{S}_{\boldsymbol{p}}$ is*

$$\underline{\mathcal{S}_{\boldsymbol{p}}} = \mathcal{F}_1^{-1}\mathcal{F}_2.$$

Proof. Compute

$$\mathcal{F}_1^{-1} = \frac{1}{\det(\mathcal{F}_1)}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix} = \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}.$$

Perform matrix multiplication to see

$$\begin{aligned} \mathcal{F}_1^{-1}\mathcal{F}_2 &= \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} l & m \\ m & n \end{pmatrix} \\ &= \frac{1}{EG - F^2}\begin{pmatrix} lG - mF & mG - nF \\ -lF + mE & -mF + nE \end{pmatrix}. \end{aligned}$$

Compare these to the Weingarten Equations by projecting $\mathcal{F}_1^{-1}\mathcal{F}_2$ onto the standard basis vectors $\mathbf{x}_u$:

$$\frac{1}{EG - F^2}\begin{pmatrix} lG - mF & mG - nF \\ mE - lF & nE - mF \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}_s = \frac{1}{EG - F^2}\begin{pmatrix} lG - mF \\ mE - lF \end{pmatrix}_s$$

and $\mathbf{x}_v$:

$$\frac{1}{EG - F^2}\begin{pmatrix} lG - mF & mG - nF \\ mE - lF & nE - mF \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}_s = \frac{1}{EG - F^2}\begin{pmatrix} mG - nF \\ nE - mF \end{pmatrix}_s.$$

These match the Weingarten Equations so $\underline{\mathcal{S}_p} = \mathcal{F}_1^{-1}\mathcal{F}_2$. □

We now have the matrix form of the Shape Operator that can act on any vector in $T_{\mathbf{p}}(M)$:

$$\underline{\mathcal{S}_{\mathbf{p}}} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \frac{1}{EG - F^2}\begin{pmatrix} lG - mF & mG - nF \\ mE - lF & nE - mF \end{pmatrix}.$$

Notice we have not specified that $\mathcal{S}_{\mathbf{p}}$ is in the standard basis; it certainly acts on vectors in the standard basis as a 2×2 matrix. If we are to claim an operator is in the standard basis of equation (6), then we introduce some ambiguity because the coefficients E, F,

G, l, m, and n are defined using $\mathbf{x}_u$, $\mathbf{x}_v$, $\hat{\mathbf{U}}$, $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$, and $\mathbf{x}_{vv}$, which have different forms in the standard basis. For instance

$$(E)_s = (\mathbf{x}_u)_s \cdot (\mathbf{x}_u)_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_s \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}_s = 1,$$

$$(F)_s = (\mathbf{x}_u)_s \cdot (\mathbf{x}_v)_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_s \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}_s = 0,$$

and

$$(G)_s = (\mathbf{x}_v)_s \cdot (\mathbf{x}_v)_s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_s \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}_s = 1,$$

which implies

$$(\mathcal{F}_1)_s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_s .$$

Taken together, Lemmas 3.4, 3.5, and 3.6 suggest an algorithm for computing the normal curvature in a given direction $\mathbf{v}$. First, project $\mathbf{v}$ into the standard basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ using Lemma 3.4

$$(\mathbf{v})_s = \mathcal{F}_1^{-1} \begin{pmatrix} \mathbf{v} \cdot \mathbf{x}_u \\ \mathbf{v} \cdot \mathbf{x}_v \end{pmatrix}.$$

Next, use Lemma 3.5 to compute

$$\kappa_n = \mathcal{F}_2 (\mathbf{v})_s \cdot (\mathbf{v})_s = \mathcal{F}_2 \mathcal{F}_1^{-1} \begin{pmatrix} \mathbf{v} \cdot \mathbf{x}_u \\ \mathbf{v} \cdot \mathbf{x}_v \end{pmatrix} \cdot \mathcal{F}_1^{-1} \begin{pmatrix} \mathbf{v} \cdot \mathbf{x}_u \\ \mathbf{v} \cdot \mathbf{x}_v \end{pmatrix}. \tag{29}$$

Equation (29) represents a combination of many of our ideas thus far concerning the fundamental forms, standard basis, and tangent plane. It gives the normal curvature at a point in any given direction $\mathbf{v} \in T_{\mathbf{p}}(M)$, without the tedium of defining surface curves.

There is another way to compute normal curvature. We demonstrate it here not because one way is more useful than another, but because it serves as yet another concrete tie between fundamental forms and the Shape Operator. The key concept is that the dot product of two vectors $\mathbf{v}, \mathbf{w} \in T_{\mathbf{p}}(M)$ involves $\mathcal{F}_1$. Write $\mathbf{v}$ and $\mathbf{w}$ in the standard basis as $\mathbf{v} = v_1\mathbf{x}_u + v_2\mathbf{x}_v$ and $\mathbf{w} = w_1\mathbf{x}_u + w_2\mathbf{x}_v$ and compute the dot product

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= v_1 w_1 (\mathbf{x}_u \cdot \mathbf{x}_u) + v_1 w_2 (\mathbf{x}_u \cdot \mathbf{x}_v) + v_2 w_1 (\mathbf{x}_v \cdot \mathbf{x}_u) + v_2 w_2 (\mathbf{x}_v \cdot \mathbf{x}_v) \\ &= v_1 w_1 E + v_1 w_2 F + v_2 w_1 F + v_2 w_2 G \\ &= \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}^T \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \mathbf{v}^T \mathcal{F}_1 \mathbf{w}. \end{aligned}$$

This realization is troublesome considering the typical method for computing dot products component-wise in an introductory vector calculus course. On an arbitrary surface with form coefficients E, F, and G the dot product must be defined as

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathcal{F}_1 \mathbf{w} \tag{30}$$

because the standard basis is not always orthonormal. In introductory vector calculus, we assume an orthonormal basis ($\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ for instance) so $E = G = 1$ and $F = 0$, in which case $\mathcal{F}_1$ is just the identity and we need not include it. So we must approach the dot products very carefully and always check the basis before performing a component-wise dot product.

Lemma 3.7. *Given a surface M, the normal curvature κ_n at a point $\boldsymbol{p} \in M$ in the direction $\boldsymbol{v} \in T_{\boldsymbol{p}}(M)$ is*

$$\kappa_n(\boldsymbol{v}) = \mathcal{S}_{\boldsymbol{p}}(\boldsymbol{v}) \cdot \boldsymbol{v}.$$

Proof. Express $\mathbf{v}$ in the standard basis $(\mathbf{v})_s$. By Lemma 3.1 $\mathcal{S}_{\mathbf{p}}(\mathbf{v}) \in T_{\mathbf{p}}(M)$ as well so we can express it in the standard basis too:

$$\mathcal{S}_{\mathbf{p}}(\mathbf{v}) = \underline{\mathcal{S}_{\mathbf{p}}}(\mathbf{v})_s = (\underline{\mathcal{S}_{\mathbf{p}}}(\mathbf{v})_s)_s = (\mathcal{F}_1^{-1}\mathcal{F}_2(\mathbf{v})_s)_s.$$

With this piece of information compute

$$\mathcal{S}_{\mathbf{p}}(\mathbf{v}) \cdot \mathbf{v} = (\mathcal{F}_1^{-1}\mathcal{F}_2(\mathbf{v})_s)_s \cdot (\mathbf{v})_s.$$

Notice this is a dot product in the standard basis so we must use equation(30) to compute

$$(\mathcal{F}_1^{-1}\mathcal{F}_2(\mathbf{v})_s)_s \cdot (\mathbf{v})_s = (\mathcal{F}_1^{-1}\mathcal{F}_2(\mathbf{v})_s)_s^T \mathcal{F}_1(\mathbf{v})_s.$$

Since $(AB)^T = B^T A^T$ for all matrices A and B, we have

$$\mathcal{S}_{\mathbf{p}}(\mathbf{v}) \cdot \mathbf{v} = (\mathbf{v})_s^T \mathcal{F}_2^T (\mathcal{F}_1^{-1})^T \mathcal{F}_1(\mathbf{v})_s = (\mathbf{v})_s^T \mathcal{F}_2 \mathcal{F}_1^{-1} \mathcal{F}_1(\mathbf{v})_s.$$

The last equality is true because $\mathcal{F}_1^{-1}$ and $\mathcal{F}_2$ are symmetric matrices. Cancel $\mathcal{F}_1^{-1}\mathcal{F}_1$ and finally, by Lemma 3.5

$$\mathcal{S}_{\mathbf{p}}(\mathbf{v}) \cdot \mathbf{v} = (\mathbf{v})_s^T \mathcal{F}_2(\mathbf{v})_s = \mathcal{F}_2(\mathbf{v})_s \cdot (\mathbf{v})_s = \kappa_n(\mathbf{v}).$$

Let us stop to ask why we did not need to insert $\mathcal{F}_1$ for this final dot product step. The answer: this dot product is not performed in the standard basis. □

Oprea uses the Shape Operator method from Lemma 3.7 in his computations while Pressley uses the method in equation (29). Both yield the normal curvature.

At a given point $\mathbf{p} \in M$ there is a normal curvature associated with every direction of travel from $\mathbf{p}$. Imagine standing at $\mathbf{p}$ with some sort of normal curvature measuring tool. You would start by pointing the tool in some direction and could obtain a continuous reading as you rotate through the 360° or 2π radians of directions of travel. Given a patch for the surface, we can do the "measurement" mathematically using (29) and a clever choice of $(\mathbf{v})_s$ to represent all directions. Let

$$(\mathbf{v}(t))_s = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}_s,$$

where $t \in [0, 2\pi)$. This way we have $(\mathbf{v}(0))_s = (1,0)_s = (\mathbf{x}_u)_s$ and $(\mathbf{v}(\pi/2))_s = (0,1)_s = (\mathbf{x}_v)_s$, as well as unit tangent vectors $||\mathbf{v}|| = 1$ for each t. With this set-up

$$\begin{aligned} \kappa_n(t) &= \begin{pmatrix} l & m \\ m & n \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}_s \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}_s \\ &= \begin{pmatrix} l\cos t + m \sin t \\ m \cos t + n \sin t \end{pmatrix}_s \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}_s \\ \kappa_n(t) &= l\cos^2 t + 2m \sin t \cos t + n \sin^2 t. \end{aligned} \tag{31}$$

This agrees with the definition of κ_n in (19) with $u(t) = \sin t$ and $v(t) = -\cos t$.

Example: Normal Curvature of the Torus

Using the coefficients l, m, and n for the torus from equations (24), (25), and (26) we can compute normal curvature κ_n at any point $\mathbf{p} = \mathbf{T}(u_0, v_0) \in \mathbf{T}$ as a function of the direction of travel t. Remember, in our example $R = 2r$ so we write equation (31) as

$$\kappa_n(t) = r\cos^2 t + (R + r\cos u_0)\cos u_0 \sin^2 t = r(\cos^2 t + (2 + \cos u_0)\cos u_0 \sin^2 t). \tag{32}$$

One interesting feature is the lack of v_0 in equation (32), which is not surprising as the torus is a surface of revolution so there is a symmetry in the v parameter. Also, observe the varying magnitude and distinct maxima and minima of $\kappa_n(t)$ as this is the topic of Section 4.1.

Example: $\mathcal{F}_1$, $\mathcal{F}_2$, and $\underline{\mathcal{S}_\mathbf{p}}$ of the Torus

We compute the form coefficients E, F, G, l, m, and n for the torus in equations (15), (16), (17), (24), (25), (26) respectively. We can organize these into the form matrices

$$\mathcal{F}_1 = \begin{pmatrix} r^2 & 0 \\ 0 & (R + r\cos u)^2 \end{pmatrix} \quad \text{and} \quad \mathcal{F}_2 = \begin{pmatrix} r & 0 \\ 0 & (R + r\cos u)\cos u \end{pmatrix}.$$

Using Lemma 3.6 we can use $\mathcal{F}_1^{-1}$ and $\mathcal{F}_2$ to define the Shape Operator matrix

$$\begin{aligned} \underline{\mathcal{S}_\mathbf{p}} &= \mathcal{F}_1^{-1}\mathcal{F}_2 \\ &= \frac{1}{r^2(R + r\cos u)^2} \begin{pmatrix} (R + r\cos u)^2 & 0 \\ 0 & r^2 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & (R + r\cos u)\cos u \end{pmatrix} \\ &= \frac{1}{r^2(R + r\cos u)^2} \begin{pmatrix} r(R + r\cos u)^2 & 0 \\ 0 & r^2(R + r\cos u)\cos u \end{pmatrix} \end{aligned}$$

$$\underline{\mathcal{S}_\mathbf{p}} = \begin{pmatrix} 1/r & 0 \\ 0 & \cos u/(R + r\cos u) \end{pmatrix}. \tag{33}$$

In the previous example we compute normal curvature of the torus from the definition. We get the same result when we solve the problem with the matrix approach from Lemma 3.7. Recall the dot product definition from equation (30) because we need it for this calculation. Compute

$$\begin{aligned} \mathcal{S}_\mathbf{p}(\mathbf{v}(t)) \cdot \mathbf{v}(t) &= (\mathbf{v}(t))_s^T \mathcal{F}_1 \underline{\mathcal{S}_\mathbf{p}} (\mathbf{v}(t))_s \\ &= \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}_s^T \begin{pmatrix} r & 0 \\ 0 & (R + r\cos u_0)\cos u_0 \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}_s \\ &= r\cos^2 t + (R + r\cos u_0)\cos u_0 \sin^2 t \\ &= r\,(\cos^2 t + (2 + \cos u_0)\sin^2 t), \end{aligned}$$

where the last step follows for our specific torus ($R = 2r$). Notice, we get the same answer for normal curvature regardless of our approach.

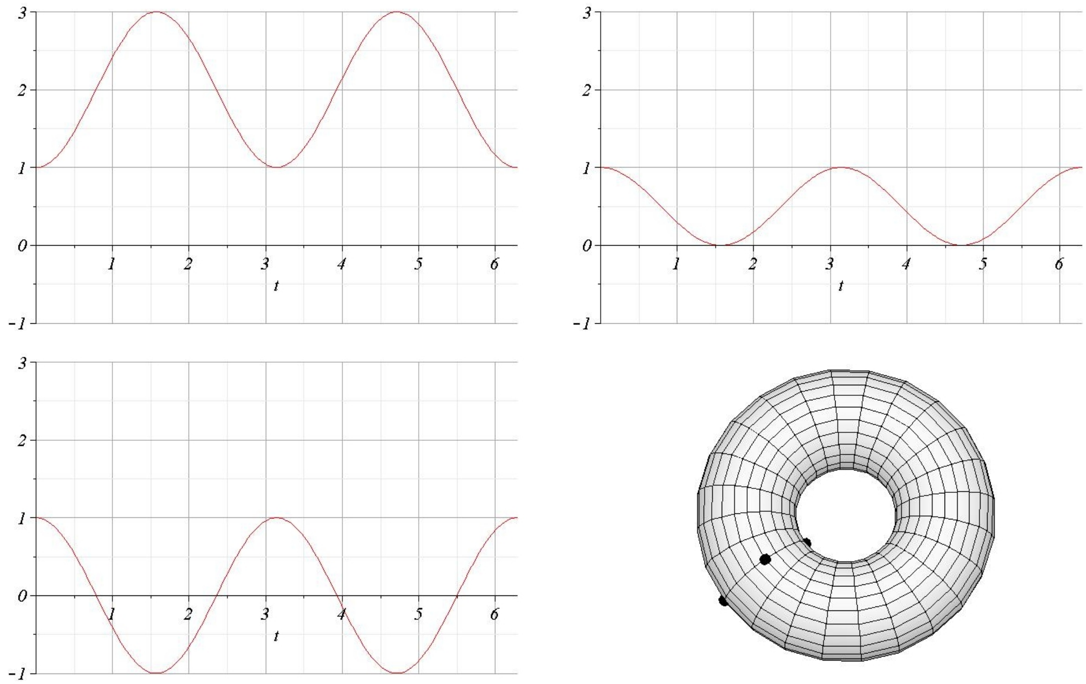


FIGURE 10. For this torus $R = 2r$. The bottom right plot shows three points for which we compute the normal curvature in the other plots. The top left shows normal curvature over all directions of travel 0 to 2π for the point on the outside of the torus. Notice $\kappa_n > 0$ for all t and there is a clear dependence on direction. The top right shows the normal curvature over all directions for the point on the top of the torus. Of particular interest are the directions where $\kappa_n = 0$ because they are easy to find on the torus. If we start at the top point, then there are two flat directions in which we can move. The bottom left plot shows the normal curvature over all directions for the point on the inside of the torus. At this point there are directions of both positive and negative curvature.

4. CURVATURE II

Now that we have reconciled the Shape Operator approach (of Oprea's text) and fundamental form approach (of Pressley's text), we can let the differences move from the forefront of our discussion and focus on some theorems. In Section 4.1 we define the principal curvatures κ_1 and κ_2 and use them to define Gauss curvature K and mean curvature H. Lemma 4.3 demonstrates the utility of the form coefficients in calculating K and H. In Section 4.3 we explore parallel surfaces and show a nice theoretical result involving transforming between surfaces of constant Gauss curvature and surfaces of constant mean curvature. Section 4.4 provides a proof and explanation of Gauss's Theorem Egregium, one of the cornerstone theorems of basic Differential Geometry.

4.1. **Principal Curvatures.** In the example of Section 3.5 we observe a varying normal curvature at a given point on the surface of a torus. Normal curvature κ_n is a function of unit tangent directions given by $\mathbf{v}$. If we seek to characterize the overall "curviness" of a surface, then we need to pay attention to the extreme values of κ_n. Let the **principal curvatures** κ_1 and κ_2 be the maximum and minimum normal curvatures at a given point

$$\kappa_1 = \kappa_n(\mathbf{v}_1) = \max_{\mathbf{v}} \kappa_n(\mathbf{v}) \quad \text{and} \quad \kappa_2 = \kappa_n(\mathbf{v}_2) = \min_{\mathbf{v}} \kappa_n(\mathbf{v}).$$

Vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are called **principal vectors**. Recall we assume $||\mathbf{v}|| = 1$ for all $\mathbf{v} \in T_{\mathbf{p}}(M)$ so $\mathbf{v}_1$ and $\mathbf{v}_2$ are unit vectors. Look at Figure 10 and we see two directions for which the maximum normal curvature is attained. So there is some flexibility in our choice of κ_1, but as we demonstrate in Lemma 4.1 once we pick κ_1, then we also determine κ_2. Of course, there are points on surfaces where $\kappa_1 = \kappa_2$, these points are called **umbilic**. A plane is an umbilic surface because every point is umbilic. Just as any other vector in $T_{\mathbf{p}}(M)$ we can express the principal vectors in the standard basis $(\mathbf{v}_1)_s$ and $(\mathbf{v}_2)_s$. Lemma 3.7, which parallels a theorem from Oprea's book, reconciles the principal curvatures with the matrix formalism we have been developing. In short, the principal curvatures are eigenvalues of the Shape Operator.

Lemma 4.1. *Let M be a surface and suppose $\boldsymbol{v} \in T_{\boldsymbol{p}}(M)$.*

1. *If $\boldsymbol{p} \in M$ is umbilic, then $\underline{\mathcal{S}_{\boldsymbol{p}}}(\boldsymbol{v})_s = \kappa(\boldsymbol{v})_s$, where $\kappa = \kappa_1 = \kappa_2$.*
2. *If $\boldsymbol{p} \in M$ is not umbilic, then $\underline{\mathcal{S}_{\boldsymbol{p}}}$ has exactly two perpendicular, unit eigenvectors, the principal vectors $(\boldsymbol{v}_1)_1$ and $(\boldsymbol{v}_2)_s$, and eigenvalues κ_1 and κ_2 respectively.*

Proof. We show the eigenvectors and eigenvalues of $\underline{\mathcal{S}_{\mathbf{p}}}$ must be as stated. Suppose $\mathcal{S}_{\mathbf{p}}(\boldsymbol{\xi}_1) = \lambda_1 \xi_1$ and $\boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_1 = 1$, that is $\boldsymbol{\xi}_1$ is a unit eigenvector of $\underline{\mathcal{S}_{\mathbf{p}}}$ with eigenvalue λ_1. Construct the unit vector $\boldsymbol{\xi}_2$ by rotating counterclockwise by $\pi/2$ from $\boldsymbol{\xi}_1$ in $T_{\mathbf{p}}(M)$. Notice $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\}$ is an orthonormal basis for $T_{\mathbf{p}}(M)$. From Lemma 3.1 we know, for some real scalars c_1 and c_2, $\mathcal{S}_{\mathbf{p}}(\boldsymbol{\xi}_2) = c_1\boldsymbol{\xi}_1 + c_2\boldsymbol{\xi}_2$. Remember Lemma 3.1 also states $\mathcal{S}_{\mathbf{p}}$ is a symmetric linear transformation. Compute

$$c_1 = \mathcal{S}_{\mathbf{p}}(\boldsymbol{\xi}_2) \cdot \boldsymbol{\xi}_1 = \mathcal{S}_{\mathbf{p}}(\boldsymbol{\xi}_1) \cdot \boldsymbol{\xi}_2 = \lambda_1 \boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2 = 0.$$

Hence, $\mathcal{S}_{\mathbf{p}}(\boldsymbol{\xi}_2) = c_2\boldsymbol{\xi}_2$, meaning $\boldsymbol{\xi}_2$ is an eigenvector of $\underline{\mathcal{S}_{\mathbf{p}}}$ with eigenvalue $\lambda_2 = c_2$. The eigenvectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are perpendicular by construction.

Next, use Lemma 3.7 to compute normal curvatures in the directions $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$:

$$\kappa_n(\boldsymbol{\xi}_1) = \mathcal{S}_{\mathbf{p}}(\boldsymbol{\xi}_1) \cdot \boldsymbol{\xi}_1 = \lambda_1 \boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_1 = \lambda_1$$

and

$$\kappa_n(\boldsymbol{\xi}_2) = \mathcal{S}_{\mathbf{p}}(\boldsymbol{\xi}_2) \cdot \boldsymbol{\xi}_2 = \lambda_2 \boldsymbol{\xi}_2 \cdot \boldsymbol{\xi}_2 = \lambda_2.$$

This shows the eigenvalues λ_1 and λ_2 are normal curvatures.

We take the case $\lambda_1 = \lambda_2 = \lambda$ first. For any $\boldsymbol{\xi} \in T_{\mathbf{p}}(M)$ $\mathcal{S}_{\mathbf{p}}(\boldsymbol{\xi}) = \lambda\boldsymbol{\xi}$ so $\mathbf{p}$ is umbilic.

So suppose, without loss of generality, $\lambda_2 < \lambda_1$. Let $\boldsymbol{\xi}$ be any unit vector in $T_{\mathbf{p}}(M)$ and write $\boldsymbol{\xi} = \cos\theta\boldsymbol{\xi}_1 + \sin\theta\boldsymbol{\xi}_2$, where θ is the angle between $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_1$. Compute the normal curvature using Lemma 3.7 making careful use of the linearity of $\mathcal{S}_{\mathbf{p}}$

$$\begin{aligned}
\kappa_n(\theta) &= \mathcal{S}_{\mathbf{p}}(\boldsymbol{\xi}) \cdot \boldsymbol{\xi} = \mathcal{S}_{\mathbf{p}}(\cos\theta\,\boldsymbol{\xi}_1 + \sin\theta\,\boldsymbol{\xi}_2) \cdot (\cos\theta\,\boldsymbol{\xi}_1 + \sin\theta\,\boldsymbol{\xi}_2) \\
&= (\cos\theta\,\mathcal{S}_{\mathbf{p}}(\boldsymbol{\xi}_1) + \sin\theta\,\mathcal{S}_{\mathbf{p}}(\boldsymbol{\xi}_2)) \cdot (\cos\theta\,\boldsymbol{\xi}_1 + \sin\theta\,\boldsymbol{\xi}_2) \\
&= \cos^2\theta\,\lambda_1\,\boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_1 + \sin\theta\cos\theta\,(\lambda_1\,\boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2 + \lambda_2\,\boldsymbol{\xi}_2 \cdot \boldsymbol{\xi}_1) + \sin^2\theta\,\lambda_2\,\boldsymbol{\xi}_2 \cdot \boldsymbol{\xi}_2 \\
&= \cos^2\theta\,\lambda_1 + \sin^2\theta\,\lambda_2.
\end{aligned}$$

From this we discern the maximum normal curvature (principal curvature κ_1 and principal vector $\mathbf{v}_1$) occurs at $\theta = 0$ because $\sin^2\theta$, $\cos^2\theta \geq 0$ and $\lambda_1 > \lambda_2$. When $\theta = 0$, $\boldsymbol{\xi} = \boldsymbol{\xi}_1$ meaning $\kappa_1 = \lambda_1$ and $\mathbf{v}_1 = \boldsymbol{\xi}_1$. Similarly the minimum normal curvature (principal curvature κ_2 and principal vector $\mathbf{v}_2$) occurs when $\theta = \pi/2$, so $\kappa_2 = \lambda_2$ and $\mathbf{v}_2 = \boldsymbol{\xi}_2$. □

The proof of 4.1 implies yet another way to compute the normal curvature in a given direction, specifically when the principal curvatures κ_1 and κ_2 are known.

Corollary 4.2 (Euler's Formula). *Given a surface M, point $\boldsymbol{p} \in M$, and $\boldsymbol{v} \in T_{\boldsymbol{p}}(M)$. Let κ_1 and κ_2 be the principal curvatures of M at $\boldsymbol{p}$, with non-zero principal vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Then*

$$\kappa_n(\boldsymbol{v}) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta, \tag{34}$$

where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{v}_1$.

These results are really quite remarkable. All the information about normal curvature lies in the Shape Operator, in particular its eigenvalues. Most striking is the orthogonality of the resulting eigenvectors, the principal vectors. This means if we stand at any point on a surface and find the direction of maximum normal curvature, the direction of minimum normal curvature will be 90° away! Regardless of how exotic a surface we conjure up, as long as it satisfies our regular surface definition this relationship holds. Suddenly, the enormous variety of surfaces seems less complex because of this common property.

Before we move onto the Gauss and mean curvatures, we examine how the principal curvatures relate to the form matrices. Lemma 3.7 shows the principal curvatures are the eigenvalues of the Shape Operator, which we know in terms of the form matrices (see Lemma 3.6). Hence, the normal curvatures are the roots of the equation

$$\det(\mathcal{F}_1^{-1}\mathcal{F}_2 - \kappa I) = 0,$$

where I is the 2×2 identity matrix. We can manipulate this equation as follows:

$$\begin{gathered}
\det(\mathcal{F}_1)\det(\mathcal{F}_1^{-1}\mathcal{F}_2 - \kappa I) = 0, \\
\det(\mathcal{F}_1(\mathcal{F}_1^{-1}\mathcal{F}_2 - \kappa I)) = 0, \\
\det(\mathcal{F}_2 - \kappa\mathcal{F}_1) = 0.
\end{gathered}$$

In terms of the form coefficients, we find the principal curvatures by solving the quadratic expression

$$\det \begin{pmatrix} l - \kappa E & m - \kappa F \\ m - \kappa F & n - \kappa G \end{pmatrix} = 0. \tag{35}$$

Example: The Principal Curvatures of the Torus

The form coefficients for the torus are given in equations (15), (16), (17), (24), (25), and (26). In summary,

$$E = r^2, \quad F = 0, \quad G = (R + r\cos u)^2,$$

$$l = r, \quad m = 0, \quad \text{and} \quad n = (R + r\cos u)\cos u,$$

where R is in the inner radius and r is the radius of the inner-tube portion of the torus. Equation (35) gives the quadratic equation that we must solve for the principal curvatures:

$$\det\begin{pmatrix} r - \kappa r^2 & 0 \\ 0 & (R + r\cos u)\cos u - \kappa(R + r\cos u)^2 \end{pmatrix} = 0.$$

The roots of the resulting quadratic are

$$\frac{1}{r} \quad \text{and} \quad \frac{\cos u}{R + r\cos u}. \tag{36}$$

One of these is the the maximum normal curvature κ_1 and the other is the minimum κ_2. For our torus we set $R = 2r$ and conclude

$$\kappa_1 = \frac{1}{r} \quad \text{and} \quad \kappa_2 = \frac{\cos u}{r(2 + \cos u)}.$$

Neither principal curvature depends on the parameter v, which is a consequence of the symmetry of the torus as a surface of revolution.

Example: A Sphere is Umbilic

Equation (10) gives the coordinate patch for a sphere and equation (11) gives the associated partials. We compute the set of coefficients E, F, G, l, m, and n in equations (12), (13), (14), (21), (22), and (23) respectively. In summary,

$$E = r^2, \quad F = 0, \quad G = r^2\sin^2 u,$$

$$l = -r, \quad m = 0, \quad \text{and} \quad n = -r\sin^2 u,$$

where r is the radius of the sphere. Given these form coefficients we search for the roots of equation (35):

$$\det\begin{pmatrix} -r - \kappa r^2 & 0 \\ 0 & -r\sin^2 u - \kappa r^2 \sin^2 u \end{pmatrix} = 0$$

$$r^2\sin^2 u(1 + \kappa r)^2 = 0,$$

which has repeated root $\kappa = -1/r$. The sphere is umbilic by definition because for all points the principal curvatures are equal: $\kappa_1 = \kappa_2 = -1/r$. Of course, κ exhibits neither u nor v dependence so the normal curvature is really $-1/r$ at every point on the sphere and in any direction. We say the sphere has constant curvature, a notion we will explore further in Section 4.3

4.2. **Gauss and Mean Curvatures.** While both the Gauss and mean curvatures are based on the principal curvatures of a surface M, they are slightly more geometrically accessible than the principal curvatures.

Let κ_1 and κ_2 be the principal curvatures of M. The **Gauss curvature** (or **Gaussian curvature**) of M is

$$K = \kappa_1 \kappa_2.$$

The **mean curvature** of M is

$$H = \frac{1}{2}(\kappa_1 + \kappa_2).$$

The name "mean" curvature is indicative of its significance as the average curvature at a given point. Recall κ_1 is the maximum normal curvature and κ_2 is the minimum normal curvature, so H literally is the average. Gauss curvature is named in honor of Carl Freidrich Gauss who proved one of the most "remarkable" theorems in Differential Geometry, the Theorem Egregium. Egregium actually translates as "remarkable" and the contents of Gauss's result are certainly that. He proved that Gauss curvature K is preserved under isometries, which we discuss further in Section 4.4.

Lemma 4.3 presents the explicit formulas for K and H in terms of the form coefficients E, F, G, l, m, and n.

Lemma 4.3. *Let M be a surface with first fundamental form $E\,du^2 + 2\,F\,du\,dv + G\,dv^2$ and second fundamental form $l\,du^2 + 2\,m\,du\,dv + n\,dv^2$, then*

1. *Gauss curvature*
$$K = \frac{ln - m^2}{EG - F^2},$$
2. *Mean curvature*
$$H = \frac{lG - 2mF + nE}{2(EG - F^2)},$$
3. *and the principal curvatures are $H \pm \sqrt{H^2 - K}$.*

Proof. Start with equation (35) to see

$$\det \begin{pmatrix} l - \kappa E & m - \kappa F \\ m - \kappa F & n - \kappa G \end{pmatrix} = 0$$

$$(l - \kappa E)(n - \kappa G) - (m - \kappa F)^2 = 0$$

$$(EG - F^2)\kappa^2 - (lG - 2mF + nE)\kappa + lN - M^2 = 0. \tag{37}$$

The principal curvatures κ_1 and κ_2 are the roots of this equation, which is of the form $A\kappa^2 + B\kappa + C = 0$. We can sidestep computing the roots by remembering for an equation of this form, the sum of the roots is $-B/A$ and the product of the roots is C/A (think completing the square). Hence,

$$K = \kappa_1 \kappa_2 = \frac{ln - m^2}{EG - F^2}$$

and

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\frac{lG - mF + nE}{EG - F^2}.$$

Using equation (37) we have

$$\kappa^2 - H\kappa + K = 0,$$

which indicates the principal curvatures are $H \pm \sqrt{H^2 - K}$. □

The Gauss and mean curvatures are so important and useful because they are associated with the intrinsic geometry of a surface. Oprea even calls them "two computable 'invariants' of a surface." Oprea then points out the tie between these geometric invariants and linear algebra ideas, we do the same. In Lemma 3.1 we prove the Shape Operator is a linear transformation. From Linear Algebra we know two invariants associated with a linear transformation are the sum of its eigenvalues, the **trace**, and the product of its eigenvalues, the **determinant**. In this light,

$$K = \kappa_1\kappa_2 = \det(\mathcal{S}_{\mathbf{p}}) \quad \text{and} \quad H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}\text{trace}(\mathcal{S}_{\mathbf{p}}).$$

We now introduce another basis for $T_{\mathbf{p}}(M)$. The **principal vector basis** uses the orthonormal principal vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ and is denoted by a p subscript. This basis is useful because the principal vectors are eigenvectors of the Shape Operator, which indicates

$$(\underline{\mathcal{S}_{\mathbf{p}}})_p(\mathbf{v}_1)_p = (\underline{\mathcal{S}_{\mathbf{p}}})_p \begin{pmatrix} 1 \\ 0 \end{pmatrix}_p = \kappa_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$(\underline{\mathcal{S}_{\mathbf{p}}})_p(\mathbf{v}_2)_p = (\underline{\mathcal{S}_{\mathbf{p}}})_p \begin{pmatrix} 0 \\ 1 \end{pmatrix}_p = \kappa_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, in the principal vector basis, the Shape Operator is

$$(\underline{\mathcal{S}_{\mathbf{p}}})_p = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}_p.$$

Now the Linear Algebra "invariant" analogy is clear because Gauss curvature is obviously the determinant and mean curvature is the trace of the Shape Operator matrix in the principal vector basis.

Example: Gauss and Mean Curvature for the Torus

We can compute the Gauss and mean curvature for the torus by referring to equation (36)

$$K = \frac{\cos u}{r(R + r\cos u)} \quad \text{and} \quad H = \frac{R + 2r\cos u}{2r(R + r\cos u)}.$$

4.3. **Constant Curvatures.** In this section we present an interesting correspondence between surfaces with constant mean curvature H and constant, positive Gaussian curvature $K > 0$. Specifically, one can obtain a constant Gaussian curvature surface starting with a surface of constant mean. The converse is also true. Pressley presents a particularly concise exposition of this idea.

Given a surface M we have studied tangent plane approximations $T_{\mathbf{p}}(M)$ at points $\mathbf{p} \in M$ in great detail. We now investigate another type of approximation to M.

Suppose M has coordinate patch $\mathbf{x}$ and let $\varepsilon \in \mathbb{R}$ be some constant scalar. The **parallel surface** M^ε to M is given by

$$\mathbf{x}^\varepsilon = \mathbf{x} + \varepsilon \hat{\mathbf{U}}.$$

Geometrically speaking to get parallel surface to M, we displace each point $\mathbf{p} \in M$ a distance ε in the direction of $\hat{\mathbf{U}}(\mathbf{p})$. This process is easy to visualize if M is a sphere. If $\varepsilon > 0$, then M^ε is a sphere of larger radius. If $\varepsilon < 0$, then M^ε is a sphere of smaller radius. If the radius of sphere M is R, then the radius of sphere M^ε is $R + \varepsilon$.

Lemma 4.4. *Suppose the surface M has patch $\boldsymbol{x}$. Let κ_1 and κ_2 be the principal curvatures of M and suppose there exists constant $c \in \mathbb{R}$ such that $|\kappa_1|$, $|\kappa_2| \leq c$. Take $\varepsilon \in \mathbb{R}$ such that $|\varepsilon| \leq 1/c$ and construct the parallel surface M^ε to M.*

1. *The mapping $\boldsymbol{x}^\varepsilon : D \mapsto M$ where $D \subseteq \mathbb{R}^2$ and $\boldsymbol{x}^\varepsilon = \boldsymbol{x} + \varepsilon \hat{\mathbf{U}}$ is a regular, coordinate patch.*
2. *At any point $(u, v) \in D$, M and M^ε have the same unit normal, that is $\hat{\mathbf{U}}^\varepsilon = \hat{\mathbf{U}}$.*

Proof. To show 1 we compute the u and v partial derivatives of $\mathbf{x}$. Recall from Section 3.1 taking partial derivatives of $\hat{\mathbf{U}}$ means applying the Shape Operator to the vector $\hat{\mathbf{U}}$ and, as a matrix, the Shape Operator is

$$\underline{\mathcal{S}_\mathbf{p}} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Although we already know a, b, c, and d (see Section 3.5), their specific values are not important for this proof. Compute

$$\begin{aligned} \mathbf{x}_u^\varepsilon &= \mathbf{x}_u + \varepsilon \hat{\mathbf{U}}_u = \mathbf{x}_u - \varepsilon \mathcal{S}_\mathbf{p}(\mathbf{x}_u) \\ &= \mathbf{x}_u - \varepsilon(a\mathbf{x}_u + b\mathbf{x}_v) \\ \mathbf{x}_u^\varepsilon &= (1 - \varepsilon a)\mathbf{x}_u - \varepsilon b \mathbf{x}_v \end{aligned} \tag{38}$$

$$\begin{aligned} \mathbf{x}_v^\varepsilon &= \mathbf{x}_v + \varepsilon \hat{\mathbf{U}}_v = \mathbf{x}_v - \varepsilon \mathcal{S}_\mathbf{p}(\mathbf{x}_v) \\ &= \mathbf{x}_v - \varepsilon(c\mathbf{x}_u + d\mathbf{x}_v) \\ \mathbf{x}_v^\varepsilon &= -\varepsilon c \mathbf{x}_u + (1 - \varepsilon d)\mathbf{x}_v. \end{aligned} \tag{39}$$

Next, compute the cross product

$$\begin{aligned} \mathbf{x}_u^\varepsilon \times \mathbf{x}_v^\varepsilon &= (1 - \varepsilon(a + d) + \varepsilon^2(ad - bc))\, \mathbf{x}_u \times \mathbf{x}_v \\ &= (1 - \varepsilon \operatorname{trace}(\underline{\mathcal{S}_\mathbf{p}}) + \varepsilon^2 \det(\underline{\mathcal{S}_\mathbf{p}}))\, \mathbf{x}_u \times \mathbf{x}_v \\ &= (1 - \varepsilon(\kappa_1 + \kappa_2) + \varepsilon^2(\kappa_1 \kappa_2))\, \mathbf{x}_u \times \mathbf{x}_v \\ &= (1 - \varepsilon\kappa_1)(1 - \varepsilon\kappa_2)\, \mathbf{x}_u \times \mathbf{x}_v \end{aligned}$$

This shows $\mathbf{x}_u^\varepsilon \times \mathbf{x}_v^\varepsilon$ and $\mathbf{x}_u \times \mathbf{x}_v$ are parallel. By choice $|\varepsilon| < 1/c$ and $|\kappa_1|, |\kappa_2| \leq c$, so $|\varepsilon\kappa_1|, |\varepsilon\kappa_2| < 1$, which implies $(1 - \varepsilon\kappa_1)(1 - \varepsilon\kappa_2) > 0$. Thus, $\mathbf{x}_u^\varepsilon \times \mathbf{x}_v^\varepsilon \neq \mathbf{0}$ so $\mathbf{x}^\varepsilon$ is regular by definition (see Section 2.1). To show part 2 calculate

$$\hat{\mathbf{U}}^\varepsilon = \frac{\mathbf{x}_u^\varepsilon \times \mathbf{x}_v^\varepsilon}{|\mathbf{x}_u^\varepsilon \times \mathbf{x}_v^\varepsilon|} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = \hat{\mathbf{U}}.$$

□

Lemma 4.4 motivates the use of parallel surfaces to explore properties of a given surface. Notice how simple it is to obtain a new patch $\mathbf{x}^\varepsilon$ and additionally M^ε has all the same unit normals as M. The next lemma show the curvatures of M and M^ε are also closely related.

Lemma 4.5. *Suppose κ_1 and κ_2 are the principal curvatures (with principal vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$) of a surface M and $|\kappa_1|, |\kappa_2| \leq c$. Let K, H be the Gauss and mean curvatures of M. Take $\varepsilon < c$ and construct the parallel surface M^ε.*

1. *The principal curvatures of M are*

$$\kappa_1^\varepsilon = \frac{\kappa_1}{1 - \varepsilon\kappa_1} \quad \text{and} \quad \kappa_2^\varepsilon = \frac{\kappa_2}{1 - \varepsilon\kappa_2}.$$

2. *The principal vectors of M^ε are the same as those for M, that is*

$$\boldsymbol{v}_1^\varepsilon = \boldsymbol{v}_1 \quad \text{and} \quad \boldsymbol{v}_2^\varepsilon = \boldsymbol{v}_2.$$

3. *The Gauss and mean curvatures of M^ε are*

$$K^\varepsilon = \frac{K}{1 - 2\varepsilon H + \varepsilon^2 K} \quad \text{and} \quad H^\varepsilon = \frac{H - \varepsilon K}{1 - 2\varepsilon H + \varepsilon^2 K}.$$

Proof. Our goal might be to write the Weingarten equations for $\mathcal{S}_{\mathbf{p}}(\mathbf{x}_u^\varepsilon)$ and $\mathcal{S}_{\mathbf{p}}(\mathbf{x}_v^\varepsilon)$, but as Section 3.5 shows this will involve tedious algebra computing all the six form coefficients. If we were to do this, then we would eventually write the matrix $\underline{\mathcal{S}_{\mathbf{p}}^\varepsilon}$ and find its eigenvalues. According to Lemma 4.1 these will be the principal curvatures. However, our work so far allows us to prove this at a higher level than the basic algebra and we can find $\underline{\mathcal{S}_{\mathbf{p}}^\varepsilon}$ without the Weingarten equations.

First, observe the $T_{\mathbf{p}}^\varepsilon(M^\varepsilon)$, the tangent plane to the parallel surface, has its own standard basis $\{\mathbf{x}_u^\varepsilon, \mathbf{x}_v^\varepsilon\}$. The Shape Operator will map $T_{\mathbf{p}}^\varepsilon(M^\varepsilon)$ to itself and so for some constants a^ε, b^ε, c^ε, and d^ε

$$\begin{aligned} \mathcal{S}_{\mathbf{p}}(\mathbf{x}_u^\varepsilon) &= a^\varepsilon \mathbf{x}_u^\varepsilon + b^\varepsilon \mathbf{x}_v^\varepsilon \\ \mathcal{S}_{\mathbf{p}}(\mathbf{x}_v^\varepsilon) &= c^\varepsilon \mathbf{x}_u^\varepsilon + d^\varepsilon \mathbf{x}_v^\varepsilon. \end{aligned}$$

The coefficients tell us how to move from $\mathbf{x}_u^\varepsilon$ and $\mathbf{x}_v^\varepsilon$ to $\mathcal{S}_{\mathbf{p}}(\mathbf{x}_u^\varepsilon)$ and $\mathcal{S}_{\mathbf{p}}(\mathbf{x}_v^\varepsilon)$. Notice, from the proof of Lemma 4.4 equations (38) and (39) tell us how to move from $\mathbf{x}_u$ and $\mathbf{x}_v$ to $\mathbf{x}_u^\varepsilon$ and $\mathbf{x}_v^\varepsilon$. As a matrix we can write this as

$$I - \varepsilon\underline{\mathcal{S}_{\mathbf{p}}^T} = \begin{pmatrix} 1 - \varepsilon a & -\varepsilon b \\ -\varepsilon c & 1 - \varepsilon d \end{pmatrix},$$

where I is the 2×2 identity matrix. From Lemma 4.4 we know $\hat{\mathbf{U}}^\varepsilon = \hat{\mathbf{U}}$, which means we can use $\underline{\mathcal{S}_{\mathbf{p}}}$ to move from $\mathbf{x}_u$ and $\mathbf{x}_v$ to $\mathcal{S}_{\mathbf{p}}(\mathbf{x}_u^\varepsilon)$ and $\mathcal{S}_{\mathbf{p}}(\mathbf{x}_u^\varepsilon)$. The following diagram summarizes these observations:

$$\begin{tikzcd} & \{\mathbf{x}_u \text{ and } \mathbf{x}_v\} \arrow[dl, "I - \varepsilon\underline{\mathcal{S}_{\mathbf{p}}^T}"'] \arrow[dr, "\underline{\mathcal{S}_{\mathbf{p}}}"] & \\ \{\mathbf{x}_u^\varepsilon \text{ and } \mathbf{x}_v^\varepsilon\} & & \{\mathcal{S}_{\mathbf{p}}(\mathbf{x}_u^\varepsilon) \text{ and } \mathcal{S}_{\mathbf{p}}(\mathbf{x}_v^\varepsilon)\} \end{tikzcd}$$

Hence,

$$\underline{\mathcal{S}_{\mathbf{p}}^\varepsilon} = (I - \varepsilon\underline{\mathcal{S}_{\mathbf{p}}^T})^{-1}\underline{\mathcal{S}_{\mathbf{p}}}.$$

It follows that

$$\underline{\mathcal{S}_{\mathbf{p}}^{\varepsilon}}\mathbf{v}_1 = (I - \varepsilon\underline{\mathcal{S}_{\mathbf{p}}^{T}})^{-1}\underline{\mathcal{S}_{\mathbf{p}}}\mathbf{v}_1$$

The eigenvalues of $\underline{\mathcal{S}_{\mathbf{p}}}$ are κ_1 and κ_2 so the eigenvalues of $\underline{\mathcal{S}_{\mathbf{p}}^{\varepsilon}}$ are

$$\kappa_1^{\varepsilon} = \frac{\kappa_1}{1 - \varepsilon\kappa_1} \quad \text{and} \quad \kappa_2^{\varepsilon} = \frac{\kappa_2}{1 - \varepsilon\kappa_2}, \tag{40}$$

as stated in part 1.

The eigenvectors of $\underline{\mathcal{S}_{\mathbf{p}}}$ are the same as those of $\underline{\mathcal{S}_{\mathbf{p}}^{\varepsilon}}$ so $\mathbf{v}_1^{\varepsilon} = \mathbf{v}_1$ and $\mathbf{v}_2^{\varepsilon} = \mathbf{v}_2$, as stated in part 2.

Lastly, straightforward algebra using the expressions in (40) proves part 3. □

With expressions K^{ε} and H^{ε} in hand in terms of K, H, and ε, creating parallel surfaces with constant mean curvature from surfaces with constant Gauss curvature (and vice-verse) is an algebraic, rather than geometric, problem.

Lemma 4.6. *Suppose M is a surface with constant mean curvature $H \neq 0$, then for $\varepsilon = 1/(2H)$ the parallel surface M^{ε} has constant Gauss curvature. Conversely, suppose M has constant Gauss curvature $K > 0$, then for $\varepsilon = 1/\sqrt{K}$ the parallel surfaces M^{ε} and $M^{-\varepsilon}$ have constant mean curvature.*

Proof. Let M be a surface of constant mean curvature $H \neq 0$ and let $\varepsilon = 1/(2H)$. Compute

$$K^{\varepsilon} = \frac{K}{1 - 2\varepsilon H + \varepsilon^2 K} = \frac{K}{1 - 2\left(\frac{1}{2H}\right)H + \left(\frac{1}{2H}\right)^2 K} = 4H^2.$$

Hence, M^{ε} has constant Gauss curvature $4H^2$.

Now, let M be a surface of constant Gauss curvature $K > 0$ and let $\varepsilon = 1/\sqrt{K}$. Compute

$$\begin{aligned} H^{\varepsilon} &= \frac{H - \varepsilon K}{1 - 2\varepsilon H + \varepsilon^2 K} = \frac{H - \left(\frac{1}{\sqrt{K}}\right)K}{1 - 2\left(\frac{1}{\sqrt{K}}\right)H + \left(\frac{1}{\sqrt{K}}\right)^2 K} \\ &= \frac{H - \sqrt{K}}{(2\sqrt{K} - 2H)\left(\frac{1}{\sqrt{K}}\right)} = -\frac{1}{2}\sqrt{K}. \end{aligned}$$

Hence, M^{ε} has constant mean curvature $-\sqrt{K}/2$. A similar calculation shows $M^{-\varepsilon}$ has constant mean curvature $\sqrt{K}/2$. □

We might initially expect a wealth of examples that demonstrate this theorem, but this is a false hope. The problem is obtaining surfaces with constant $K > 0$ or $H \neq 0$ to start with. Furthermore, to apply Lemma 4.6 it becomes necessary to find patch functions for these rare surfaces in order to compute the form coefficients, which is also difficult. We can rest slightly easier by examining a trivial case. A sphere has constant Gauss curvature $K = 1/r^2$ and constant mean curvature $H = -1/r$ (see Section 4.1). Indeed, when we form parallel surfaces to the sphere, we obtain spheres of different radii and they too have constant curvatures. Oprea coauthors a work, with Mariana

Hadzhilazova and Ivaïlo M. Mladenov on a class of surfaces called unduloids, which are less trivial examples of the result from Lemma 4.6. [4]

4.4. **Gauss's Theorem Egregium.** We are now ready to state and prove Gauss's "Remarkable" Theorem. The proof itself is not particularly remarkable, as it mostly takes patient and careful organization. We present Oprea's proof here because it agrees most closely with our notation, but Pressley presents a similar one. We start by introducing the useful shorthand to denote the partial derivatives of the form coefficients E and G:

$$E_v = \frac{\partial}{\partial v}E = \frac{\partial}{\partial v}(\mathbf{x}_u \cdot \mathbf{x}_u) \quad \text{and} \quad G_u = \frac{\partial}{\partial u}G = \frac{\partial}{\partial u}(\mathbf{x}_v \cdot \mathbf{x}_v).$$

In this section we assume $F = 0$, that is $\mathbf{x}_u \cdot \mathbf{x}_v = 0$. The formulas we prove are thus a special case of a larger framework, but all the important ideas are present in our proofs.

Lemma 4.7. *Gauss curvature depends only on the form coefficients E, F, and G. Suppose M is a surface with $F = 0$, then the Gauss curvature is given by*

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right) \tag{41}$$

Proof. From Lemma 4.3 we know the Gauss curvature is given by

$$K = \frac{ln - m^2}{EG},$$

when $F = 0$. Lemma 4.4 implies l, m, and n can somehow be written in terms of E, G, E_v, G_u. Remember $l = \mathbf{x}_{uu} \cdot \hat{\mathbf{U}}$, $m = \mathbf{x}_{uv} \cdot \hat{\mathbf{U}}$, and $n = \mathbf{x}_{vv} \cdot \hat{\mathbf{U}}$, so we search for a way to express the second derivatives in terms of $\mathbf{x}_u$, $\mathbf{x}_v$, and $\hat{\mathbf{U}}$. To this end we write

$$\begin{aligned}
\mathbf{x}_{uu} &= \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v + l\,\hat{\mathbf{U}} \\
\mathbf{x}_{uv} &= \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v + m\,\hat{\mathbf{U}} \\
\mathbf{x}_{vv} &= \Gamma^u_{vv}\mathbf{x}_u + \Gamma^v_{vv}\mathbf{x}_v + n\,\hat{\mathbf{U}}.
\end{aligned}$$

Remember $\mathbf{x}_u$ and $\mathbf{x}_v$ are in $T_{\mathbf{p}}(M)$ and $\hat{\mathbf{U}}$ is perpendicular to $T_{\mathbf{p}}(M)$ so l, m, and n appear as the coefficients of $\hat{\mathbf{U}}$ because $\mathbf{x}_{uu} \cdot \hat{\mathbf{U}} = l$ and so forth. The Γ coefficients are called **Christoffel symbols** and our goal is to solve for them. First compute

$$\begin{aligned}
\mathbf{x}_{uu} \cdot \mathbf{x}_u &= \Gamma^u_{uu}\mathbf{x}_u \cdot \mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v \cdot \mathbf{x}_u + l\hat{\mathbf{U}} \cdot \mathbf{x}_u \\
&= \Gamma^u_{uu}E + \Gamma^v_{uu}F = \Gamma^u_{uu}E,
\end{aligned}$$

which implies

$$\Gamma^u_{uu} = \frac{\mathbf{x}_{uu} \cdot \mathbf{x}_u}{E}.$$

Notice,

$$E_u = \frac{\partial}{\partial u}(\mathbf{x}_u \cdot \mathbf{x}_u) = \mathbf{x}_{uu} \cdot \mathbf{x}_u + \mathbf{x}_u \cdot \mathbf{x}_{uu} = 2\,\mathbf{x}_{uu} \cdot \mathbf{x}_u$$

so

$$\Gamma^u_{uu} = \frac{E_u}{2E}.$$

[4] M. Hadzhilazova et. al. Unduloids and Their Geometry (English summary). *Arch. Math (Brno)* **43** (2007), no. 5, 417-429.

Performing a similar trick with $\mathbf{x}_{vv} \cdot \mathbf{x}_v$ and the coefficient G we find

$$\Gamma_{vv}^{v} = \frac{\mathbf{x}_{vv} \cdot \mathbf{x}_v}{G} = \frac{G_v}{2G}.$$

Next, we compute

$$\begin{aligned} \mathbf{x}_{uu} \cdot \mathbf{x}_v &= \Gamma_{uu}^{u} \mathbf{x}_u \cdot \mathbf{x}_v + \Gamma_{uu}^{v} \mathbf{x}_v \cdot \mathbf{x}_v + l \hat{\mathbf{U}} \cdot \mathbf{x}_v \\ &= \Gamma_{uu}^{u} F + \Gamma_{uu}^{v} G = \Gamma_{uu}^{v} G, \end{aligned}$$

and

$$\begin{aligned} \mathbf{x}_{uv} \cdot \mathbf{x}_u &= \Gamma_{uv}^{u} \mathbf{x}_u \cdot \mathbf{x}_u + \Gamma_{uv}^{v} \mathbf{x}_v \cdot \mathbf{x}_u + l \hat{\mathbf{U}} \cdot \mathbf{x}_u \\ &= \Gamma_{uv}^{u} E + \Gamma_{uv}^{v} F = \Gamma_{uv}^{u} E, \end{aligned}$$

which imply

$$\Gamma_{uu}^{v} = \frac{\mathbf{x}_{uu} \cdot \mathbf{x}_v}{G} \quad \text{and} \quad \Gamma_{uv}^{u} = \frac{\mathbf{x}_{uv} \cdot \mathbf{x}_u}{E}.$$

We need a slightly different strategy here so start with $\mathbf{x}_u \cdot \mathbf{x}_v = F = 0$ and take the partial with respect to u

$$0 = \frac{\partial}{\partial u}(0) = \frac{\partial}{\partial u}(\mathbf{x}_u \cdot \mathbf{x}_v) = \mathbf{x}_{uu} \cdot \mathbf{x}_v + \mathbf{x}_u \cdot \mathbf{x}_{uv},$$

which implies $\mathbf{x}_{uu} \cdot \mathbf{x}_v = -\mathbf{x}_{uv} \cdot \mathbf{x}_u$. Now take the partial of the coefficient E with respect to v to find

$$E_v = \frac{\partial}{\partial v}(\mathbf{x}_u \cdot \mathbf{x}_u) = \mathbf{x}_{uv} \cdot \mathbf{x}_u + \mathbf{x}_u \cdot \mathbf{x}_{uv} = 2\,\mathbf{x}_{uv} \cdot \mathbf{x}_u = -2\,\mathbf{x}_{uu} \cdot \mathbf{x}_v.$$

Hence,

$$\Gamma_{uu}^{v} = \frac{\mathbf{x}_{uu} \cdot \mathbf{x}_v}{G} = -\frac{E_v}{2G} \quad \text{and} \quad \Gamma_{uv}^{u} = \frac{\mathbf{x}_{uv} \cdot \mathbf{x}_u}{E} = \frac{E_v}{2E}.$$

The last two Christoffel symbols are obtained by applying a v partial derivative to the identity $\mathbf{x}_u \cdot \mathbf{x}_v = F = 0$ to find $-\mathbf{x}_{uv} \cdot \mathbf{x}_v = \mathbf{x}_{vv} \cdot \mathbf{x}_u$. Also, compute

$$G_u = \frac{\partial}{\partial u}(\mathbf{x}_v \cdot \mathbf{x}_v) = \mathbf{x}_{uv} \cdot \mathbf{x}_v + \mathbf{x}_v \cdot \mathbf{x}_{uv} = 2\mathbf{x}_{uv} \cdot \mathbf{x}_v = -2\,\mathbf{x}_{vv} \cdot \mathbf{x}_u.$$

Finally, we obtain

$$\Gamma_{uv}^{v} = \frac{\mathbf{x}_{uv} \cdot \mathbf{x}_v}{G} = \frac{G_u}{2G} \quad \text{and} \quad \Gamma_{vv}^{u} = \frac{\mathbf{x}_{vv} \cdot \mathbf{x}_u}{E} = -\frac{G_u}{2E}.$$

We also use formulas for $\hat{\mathbf{U}}_u$ and $\hat{\mathbf{U}}_v$, which can be computed for the $F = 0$ case from the Weingarten equations (27) and (28) from Section 3.5:

$$\hat{\mathbf{U}}_u = -\mathcal{S}_{\mathbf{p}}(\mathbf{x}_u) = -\frac{l}{E}\mathbf{x}_u - \frac{m}{G}\mathbf{x}_v$$

and

$$\hat{\mathbf{U}}_v = -\mathcal{S}_{\mathbf{p}}(\mathbf{x}_v) = -\frac{m}{E}\mathbf{x}_u - \frac{n}{G}\mathbf{x}_v.$$

Taken together, Oprea calls the $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$, $\mathbf{x}_{vv}$, $\hat{\mathbf{U}}_u$, and $\hat{\mathbf{U}}_v$ equations the **fundamental acceleration formulas** for the $F=0$ case. In summary,

$$\begin{aligned}
\mathbf{x}_{uu} &= \frac{E_u}{2E}\mathbf{x}_u - \frac{E_v}{2G}\mathbf{x}_v + l\,\hat{\mathbf{U}} && (42)\\
\mathbf{x}_{uv} &= \frac{E_v}{2E}\mathbf{x}_u + \frac{G_u}{2G}\mathbf{x}_v + m\,\hat{\mathbf{U}} && (43)\\
\mathbf{x}_{vv} &= -\frac{G_u}{2E}\mathbf{x}_u + \frac{G_v}{2G}\mathbf{x}_v + n\,\hat{\mathbf{U}}\\
\hat{\mathbf{U}}_u &= -\frac{l}{E}\mathbf{x}_u - \frac{m}{G}\mathbf{x}_v\\
\hat{\mathbf{U}}_v &= -\frac{m}{E}\mathbf{x}_u - \frac{n}{G}\mathbf{x}_v.
\end{aligned}$$

As with many of our previous proofs, we start with a seemingly trivial identity and implicitly derive the relationship we seek. Our formula will come from third derivatives of the patch $\mathbf{x}$. We know the order of partial differentiation does not matter so $\mathbf{x}_{uuv} = \mathbf{x}_{uvu}$ or $\mathbf{x}_{uuv} - \mathbf{x}_{uvu} = \mathbf{0}$. Compute these partials starting with equation (42)

$$\begin{aligned}
\mathbf{x}_{uuv} &= \frac{\partial}{\partial v}\mathbf{x}_{uu} = \frac{\partial}{\partial v}\left(\frac{E_u}{2E}\mathbf{x}_u - \frac{E_v}{2G}\mathbf{x}_v + l\,\hat{\mathbf{U}}\right)\\
&= \left(\frac{E_u}{2E}\right)_v \mathbf{x}_u + \frac{E_u}{2E}\mathbf{x}_{uv} - \left(\frac{E_v}{2G}\right)_v \mathbf{x}_v - \frac{E_v}{2G}\mathbf{x}_{vv} + l_v\hat{\mathbf{U}} + l\,\hat{\mathbf{U}}_v
\end{aligned}$$

and equation (43)

$$\begin{aligned}
\mathbf{x}_{uvu} &= \frac{\partial}{\partial u}\mathbf{x}_{uv} = \frac{\partial}{\partial u}\left(\frac{E_v}{2E}\mathbf{x}_u + \frac{G_u}{2G}\mathbf{x}_v + m\,\hat{\mathbf{U}}\right)\\
&= \left(\frac{E_v}{2E}\right)_u \mathbf{x}_u + \frac{E_v}{2E}\mathbf{x}_{uu} - \left(\frac{G_u}{2G}\right)_v \mathbf{x}_v - \frac{G_u}{2G}\mathbf{x}_{uv} + m_u\hat{\mathbf{U}} + m\,\hat{\mathbf{U}}_u.
\end{aligned}$$

According to the fundamental acceleration formulas, we can write these third derivatives solely in terms of $\mathbf{x}_u$, $\mathbf{x}_v$, and $\hat{\mathbf{U}}$. For this proof we are only interested in the $\mathbf{x}_v$ term so we examine it the closest with an understanding that we can compute the other terms in a similar fashion. We have

$$\mathbf{x}_{uuv} = (\ldots)\mathbf{x}_u + \left(\frac{E_uG_u}{4EG} - \left(\frac{E_v}{2G}\right)_v - \frac{E_vG_v}{4G^2} - \frac{ln}{G}\right)\mathbf{x}_v + (\ldots)\hat{\mathbf{U}}$$

and

$$\mathbf{x}_{uvu} = (\ldots)\mathbf{x}_u + \left(-\frac{E_vG_v}{4EG} + \left(\frac{G_u}{2G}\right)_u + \frac{G_uG_u}{4G^2} - \frac{m^2}{G}\right)\mathbf{x}_v + (\ldots)\hat{\mathbf{U}}$$

Subtracting these highlighted $\mathbf{x}_v$ terms we have

$$0 = \frac{E_uG_u}{4EG} - \frac{E_vG_v}{4EG} - \left(\frac{E_v}{2G}\right)_v - \left(\frac{G_u}{2G}\right)_u - \frac{G_uG_u}{4G^2} - \frac{E_vG_v}{4G^2} - \frac{ln-m^2}{G}.$$

Notice (the last term) we have built an expression for K in terms of E, G, and the associated partials

$$K = \frac{ln-m^2}{EG} = \frac{E_uG_u - E_vG_v}{4E^2G} - \frac{1}{E}\left(\frac{E_v}{2G}\right)_v - \frac{1}{E}\left(\frac{G_u}{2G}\right)_u - \frac{G_uG_u + E_vG_v}{4EG^2}.$$

Some careful algebra, which is not illuminating as it is mostly careful book-keeping, shows this is indeed equivalent to equation (41). $\square$

At this point the reader should be looking for some convincing evidence that Lemma 4.7 is actually amazing. All we did was demonstrate a new way of computing Gauss curvature. However, the "remarkable" tag is justified when we think about this result in terms of mappings between different surfaces. The discussion is certainly more abstract than the proof of Lemma 4.7 and we delve primarily into Pressley's text.

Every time we use a patch function, we are mapping from two dimensional domains to surfaces. We have yet to discuss mappings between surfaces, which are not only possible, but also interesting and useful. Two surfaces M_1 and M_2 are said to be **diffeomorphic** if there exists a differentiable, one-to-one, and onto function $\mathbf{\Omega} : M_1 \mapsto M_2$, where $\mathbf{\Omega}^{-1} : M_2 \mapsto M_1$ is also smooth. The vector-valued function $\mathbf{\Omega}$ is called a **diffeomorphism**. Lemma 4.8 shows us how coordinate patches for M_1 and M_2 relate.

Lemma 4.8. *Let M_1 and M_2 be diffeomorphic surfaces with diffeomorphism $\mathbf{\Omega} : M_1 \mapsto M_2$. Let D_1 and D_2 be open subsets of $\mathbb{R}^2$ and suppose $\boldsymbol{x} : D_1 \mapsto M_1$ and $\boldsymbol{y} : D_2 \mapsto M_2$ are regular coordinate patches for M_1 and M_2 respectively. Then $\mathbf{\Omega} \circ \boldsymbol{x} : D_1 \mapsto M_2$ is a regular patch for M_2 and $\mathbf{\Omega} \circ \boldsymbol{y} : D_2 \mapsto M_1$ is a regular patch for M_1.*

Proof. Let the coordinates of D_2 be $\tilde{u}$ and $\tilde{v}$. Let $\tilde{\mathbf{x}} = \mathbf{\Omega} \circ \mathbf{x}$, we must show $\tilde{\mathbf{x}}_{\tilde{u}} \times \tilde{\mathbf{x}}_{\tilde{v}} \neq 0$. Using the chain rule compute

$$\tilde{\mathbf{x}}_{\tilde{u}} = \frac{\partial u}{\partial \tilde{u}}\mathbf{x}_u + \frac{\partial v}{\partial \tilde{u}}\mathbf{x}_v$$

and

$$\tilde{\mathbf{x}}_{\tilde{v}} = \frac{\partial u}{\partial \tilde{v}}\mathbf{x}_u + \frac{\partial v}{\partial \tilde{v}}\mathbf{x}_v,$$

which gives

$$\tilde{\mathbf{x}}_{\tilde{u}} \times \tilde{\mathbf{x}}_{\tilde{v}} = \left(\frac{\partial u}{\partial \tilde{u}}\frac{\partial v}{\partial \tilde{v}} - \frac{\partial u}{\partial \tilde{v}}\frac{\partial v}{\partial \tilde{u}} \right) \mathbf{x}_u \times \mathbf{x}_v. \tag{44}$$

Thus, we must show the scalar in equation (44) is non-zero, for we know $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$.

Since $\mathbf{\Omega}$ is a diffeomorphism

$$\mathbf{\Omega} \circ \mathbf{x}(u, v) = \mathbf{\Omega}(\mathbf{x}(u, v)) = \mathbf{y}(G(u, v)),$$

where $G : D_1 \mapsto D_2$ is smooth and bijective, with $G^{-1} : D_2 \mapsto D_1$ smooth as well. Notice $(u, v) = G(\tilde{u}, \tilde{v})$ because G is bijective. The scalar term in equation (44) is the determinant of the Jacobian matrix of G^{-1}, which is given by

$$J(G^{-1}) = \begin{pmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{u}} \\ \\ \frac{\partial u}{\partial \tilde{v}} & \frac{\partial v}{\partial \tilde{v}} \end{pmatrix}.$$

From Calculus we know the following about the Jacobian of G and G^{-1}:

$$I_2 = J(G^{-1} \circ G) = J(G^{-1})\, J(G),$$

where I_2 is the 2×2 identity matrix. Hence, $J(G^{-1})^{-1} = J(G)$,which implies $J(G^{-1})$ is an invertible matrix with non-zero determinant, so $\tilde{\mathbf{x}}_{\tilde{u}} \times \tilde{\mathbf{x}}_{\tilde{v}} \neq \mathbf{0}$.

An analogous proof is valid for $\tilde{\mathbf{y}} = \mathbf{\Omega} \circ \mathbf{y}$. $\square$

Lemma 4.8 proves patch functions for diffeomorphic surfaces relate nicely by function composition.

Next, we investigate isometric surfaces. As a general comment "isometric" is more strict than "diffeomorphic" in terms of surface mappings. Suppose $\mathbf{\Omega} : M_1 \mapsto M_2$ is a diffeomorphism that takes curves in M_1 to curves *of the same length* in M_2, then M_1 and M_2 are said to be **isomorphic**. If $\mathbf{\Omega}$ preserves lengths in this way, then we call it an **isometry**. Given two surfaces we check for such an isometry using their first fundamental forms.

Lemma 4.9. *Diffeomorphic surfaces, M_1 and M_2, are isometric if and only if they have the same first fundamental form.*

Proof. Suppose $\mathbf{\Omega} : M_1 \mapsto M_2$ is a diffeomorphism. Suppose $D_1 \subseteq \mathbb{R}^2$ and let $\mathbf{x}_1 : D_1 \mapsto M_1$ be a coordinate patch for M_1.

First, suppose M_1 and M_2 have the same first fundamental form, equation (8) says this common first fundamental form is $E\,du^2 + 2\,F\,du\,dv + G\,dv^2$. For $I \subseteq \mathbb{R}$ let $\boldsymbol{\alpha}_1 : I \mapsto M_1$ such that for $t \in I$, $\boldsymbol{\alpha}_1(t) = \mathbf{x}_1(u(t), v(t))$ be a surface curve on M_1. By Lemma 4.8 $\mathbf{\Omega} \circ \mathbf{x}$ is a patch for M_2 so the image curve $\boldsymbol{\alpha}_2$ on M_2 can be found by computing

$$\mathbf{\Omega}(\boldsymbol{\alpha}_1(t)) = \mathbf{\Omega}(\mathbf{x}_1(u(t), v(t))) = \mathbf{\Omega} \circ \mathbf{x}_1(u(t), v(t)) = \boldsymbol{\alpha}_2(t).$$

The arc length formula given in equation (9) is the same for $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ because not only are the form coefficients the same, the functions u and v are the same. So for any $t_1, t_2 \in I$

$$L(\boldsymbol{\alpha}_1) = \int_{t_1}^{t_2} \sqrt{E\,\dot{u}^2 + 2\,F\,\dot{u}\,\dot{v} + G\dot{v}^2}\,dt = L(\boldsymbol{\alpha}_2).$$

Thus, $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ have the same length, which means M_1 and M_2 are isomorphic by definition.

Now suppose $\mathbf{\Omega}$ is an isometry. For $t \in I$ suppose the functions $u(t)$ and $v(t)$ parametrize a curve in D_1. The curves $\boldsymbol{\alpha}_1(t) = \mathbf{x}_1(u(t), v(t))$ and $\boldsymbol{\alpha}_2(t) = \mathbf{\Omega} \circ \mathbf{x}_1(u(t), v(t))$ have equal length because $\mathbf{\Omega}$ is an isometry. According to equation (9) for any $t_1, t_2 \in I$ we have

$$\begin{aligned} L(\boldsymbol{\alpha}_1) &= \int_{t_1}^{t_2} \sqrt{E_1\,\dot{u}^2 + 2\,F_1\,\dot{u}\,\dot{v} + G_1\dot{v}^2}\,dt \\ &= L(\boldsymbol{\alpha}_2) = \int_{t_1}^{t_2} \sqrt{E_2\,\dot{u}^2 + 2\,F_2\,\dot{u}\,\dot{v} + G_2\dot{v}^2}\,dt, \end{aligned}$$

where E_1, F_1, and G_1 are the form coefficients of M_1 and E_2, F_2, and G_2 are the form coefficients of M_2. The integrands on these expressions must be equal, that is (after squaring)

$$E_1\,\dot{u}^2 + 2\,F_1\,\dot{u}\,\dot{v} + G_1\dot{v}^2 = E_2\,\dot{u}^2 + 2\,F_2\,\dot{u}\,\dot{v} + G_2\dot{v}^2. \tag{45}$$

This expression is true for any parameter functions $u(t)$ and $v(t)$, so it is true for three specific choices of u and v. Fix $t_1 \in I$ and for each choice, let $u_1 = u(t_1)$ and $v_1 = v(t_1)$:

- $u(t) = u_1 + t - t_1$ and $v(t) = v_1$ so that $\dot{u}(t) = 1$ and $\dot{v}(t) = 0$, which implies $E_1 = E_2$ in equation (45).
- $u(t) = u_1$ and $v(t) = v_1 + t - t_1$ so that $\dot{u}(t) = 0$ and $\dot{v}(t) = 1$, which implies $G_1 = G_2$.

- $u(t) = u_1 + t - t_1$ and $v(t) = v_1 + t - t_1$ so that $\dot{u} = 1$ and $\dot{v} = 1$, which implies $E_1 + 2\,F_1 + G_1 = E_2 + 2\,F_2 + G_2$ in equation (45). From what we have already learned this in turn shows $F_1 = F_2$.

Hence, M_1 and M_2 have the same first fundamental form. □

Perhaps now we see more clearly why Gauss's result is so remarkable. Corollary 4.10 puts together the abstract result of Lemmas 4.8 and 4.9 with the Gauss curvature formula from equation (41) in Lemma 4.7.

Corollary 4.10 (Gauss's Theorem Egregium)**.** *Isometric surfaces have the same Gauss curvature.*

Proof. By Lemma 4.9 isometric surfaces have the same first fundamental form $E\,du^2 + 2\,F\,du\,dv_G\,dv^2$. Since we can compute Gauss curvature K from the first fundamental form coefficients only (see equation (41)), isometric surfaces have the same Gauss curvature. □

This result indicates an enormous bound in our characterization of the huge number of surfaces. We can categorize isometric surfaces into a single class because they all have the same Gauss curvature.

Example: Diffeomorphic, Non-Isometric Surfaces

The torus and a cylinder of height 2π and unit radius are diffeomorphic, but not isomorphic. We use a plane as an intermediary in the process. Recall our example in Section 3.3 and Figure 3 where we compute the circumference of torus circles. We observed that the circumferences were different than in flat space. We are now ready to show this discrepancy is not a coincidence because the torus is not isomorphic to the plane.

First, recall the coordinate patch for the torus is (equation (1))

$$\mathbf{T}(u, v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u))\,.$$

The domain in $\mathbb{R}^2$ is $D = [0, 2\pi) \times [0, 2\pi)$ so that we get the whole torus shape and a cylinder of height 2π and unit radius. A coordinate patch for the cylinder is $\mathbf{C} : D \mapsto \mathbb{R}^3$ such that

$$\mathbf{C}(u, v) = (\cos u, \sin u, v).$$

To define a diffeomorphism $\boldsymbol{\Omega} : \mathbf{C} \mapsto \mathbf{T}$ label the cylinder coordinates (c_1, c_2, c_3). Notice one way to define $\mathbf{C}^{-1}$, which maps from the cylinder back to the domain D is

$$\mathbf{C}^{-1}(c_1, c_2, c_3) = (\arccos c_1, c_3).$$

Thus,

$$\boldsymbol{\Omega}(c_1, c_2, c_3) = \mathbf{T}(\mathbf{C}^{-1}(c_1, c_2, c_3)) = ((R + rc_1)\cos c_3, (R + rc_1)\sin c_3, rc_3)$$

and the torus and this particular cylinder are diffeomorphic.

Recall from Section 3.3, specifically equation (18), the first fundamental form of the torus is

$$r^2 du^2 + (R + r\cos u)^2 dv^2.$$

For the cylinder we compute

$$\mathbf{C}_u(u,v) = (-\sin u, \cos u, 0)$$

and

$$\mathbf{C}_v(u,v) = (0,0,1)$$

so that

$$\begin{aligned} E &= \mathbf{C}_u \cdot \mathbf{C}_u = 1 \\ F &= \mathbf{C}_u \cdot \mathbf{C}_v = 0 \\ G &= \mathbf{C}_v \cdot \mathbf{C}_v = 1, \end{aligned} \tag{46}$$

which implies the first fundamental form of the cylinder is different than that of the torus: $du^2 + dv^2$. Note, this is the same first fundamental form as a plane so a cylinder *is* isomorphic to a plane.

This example is particularly nice because the visualizations of the required transformations to get from the planar domain D to the cylinder to the torus are easily visualized. We wrap D upon itself, match up the $u = 0$ with the $u = 2\pi$ edge, and the result is a cylinder. Furthermore, we do not change the Gauss curvature K in this process. To make the torus however, we must align the top of the cylinder with the bottom forming the inner-tube. We can imagine doing this experiment with paper and we would find it necessary to crinkle the cylinder to form the torus. This reflects that the cylinder and torus are not isomorphic and there is a change in Gauss curvature when we try to force one to become the other.

5. Minimal Surfaces

5.1. **Minimal Surface Definition.** Before we define minimal surface, we present Lemma 5.1, which not only justifies the name "mean curvature," but also calibrates our intuition about the quantity H. Recall we have already entertained the idea of mean curvature as an average because by definition

$$H = \frac{1}{2}(\kappa_1 + \kappa_2),$$

where κ_1 and κ_2 are the principal curvatures. The next two Lemmas develop our intuition about mean curvature, both follow directly from Euler's formula of Corollary 4.2.

Lemma 5.1. *Suppose M is a surface. At the point $\mathbf{p} \in M$ the mean curvature H is the average normal curvature over all possible directions of travel, that is*

$$H = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\theta)\, d\theta. \tag{47}$$

Proof. From Euler's formula (equation (34) of Corollary 4.2) we know

$$\kappa_n(\theta) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.$$

Basic integration proves equation (47);

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\kappa(\theta)\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi}(\kappa_1\cos^2\theta+\kappa_2\sin^2\theta)\,d\theta\\ &= \frac{\kappa_1}{2\pi}\pi+\frac{\kappa_2}{2\pi}\pi=\frac{1}{2}(\kappa_1+\kappa_2)=H\end{aligned}$$

□

Lemma 5.2. *Suppose M is a surface and let $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ be perpendicular unit vectors in $T_{\boldsymbol{p}}(M)$. Then,*

$$H=\frac{1}{2}(\kappa_n(\boldsymbol{\xi}_1)+\kappa_n(\boldsymbol{\xi}_2)). \tag{48}$$

Proof. Let ϕ be the angle from the principal vector $\mathbf{v}_1$ to $\boldsymbol{\xi}_1$ so the angle from v_1 to $\boldsymbol{\xi}_2$ is $\phi+\pi/2$ because $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are perpendicular. Also, work in the principal vector basis so that the coordinate axes align with the principal vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Again, by Euler's formula of equation (34) we have

$$\begin{aligned}\kappa_n(\boldsymbol{\xi}_1)+\kappa_n(\boldsymbol{\xi}_2) &= \kappa_n(\phi)+\kappa_n\left(\phi+\frac{\pi}{2}\right)\\ &= \kappa_1\cos^2\phi+\kappa_2\sin^2\phi+\kappa_1\cos^2\left(\phi+\frac{\pi}{2}\right)+\kappa_2\sin^2\left(\phi+\frac{\pi}{2}\right)\\ &= \kappa_1(\cos^2\phi+\sin^2\phi)+\kappa_2(\sin^2\phi+\cos^2\phi)\\ &= \kappa_1+\kappa_2=2\,H\end{aligned}$$

Hence, equation (48) is valid. □

Given this result we may imagine being on a surface and actually computing mean curvature. Have two options. We could perform a survey of the normal curvature in many directions of travel $0°$ to $360°$ and then average them to approximate the integral in equation (47) and thus, the mean curvature H. Alternately, we could sample the normal curvature in one direction, spin $90°$, sample normal curvature, and average the measurements using equation (48) to get H.

A **minimal surface** M has zero mean curvature at all points, that is $H=0$ for all $\mathbf{p}\in M$. Lemma 5.1 helps to motivate just how special minimal surfaces are. Suppose we stand at a point on our surface and measure a mean curvature $H=0$. This scenario is entirely feasible and, in fact, easy to picture. At this special point of zero mean curvature we scan the surface (again $0°$ to $360°$) and see it sloping up just as much as it slopes down. For a minimal surface, this is true *at all points*!

The trivial minimal surface is a plane. A plane has no curvature whatsoever so it must be minimal. A sphere is not minimal because, see Section 4.1, it has constant mean curvature $-1/r$, where r is its radius. Likewise a torus is not minimal. Even the common examples of minimal surfaces (many are covered in Oprea's book and some in Pressley's) are not intuitive. In fact, it is for this reason that minimal surfaces are still an interesting subfield of Differential Geometry. For instance, the Hoffman surface and an associated family of minimal surfaces were discovered in the latter half of the 20^{th} century. [5] We must incorporate a vast number of techniques from partial differential

[5] T. Hern et. al. Looking at Order of Integration and a Minimal Surface. *The College Mathematics Journal* **29** (1998), no. 2, 128-133.

equations, topology, and other fields to study minimal surfaces. Likewise, minimal surfaces appear unexpectedly in some fields of mathematics. Additionally, minimal surfaces have strong physical relevance because a minimal configuration turns out to minimize the energy of various real-world systems. Soap films, for instance, assume minimal surface configurations when they form across fixed boundaries. We explore the physicality of minimal surfaces in Sections 5.2 and 5.3.

Given a surface M in $\mathbb{R}^3$ whose graph can be given by a scalar function of two variables $x_3 = f(x_1, x_2)$, we can form a coordinate patch

$$\mathbf{x}(u, v) = (u, v, f(u, v)).$$

This is called a **Monge patch**. For this special case, Oprea presents a concise summary of what we have developed so far, which we reproduce here. The partial derivatives are

$$\mathbf{x}_u = (1, 0, f_u)\,, \quad \mathbf{x}_v = (0, 1, f_v)\,,$$

$$\mathbf{x}_{uu} = (0, 0, f_{uu})\,, \quad \mathbf{x}_{uv} = (0, 0, f_{uv})\,, \quad \text{and} \quad \mathbf{x}_{vv} = (0, 0, f_{vv}).$$

From these, compute the unit normal

$$\hat{\mathbf{U}} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} = \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}.$$

The form coefficients follow

$$E = 1 + f_u^2\,, \quad F = f_u f_v\,, \quad G = 1 + f_v^2\,,$$

$$l = \frac{f_{uu}}{\sqrt{1 + f_u^2 + f_v^2}}\,, \quad m = \frac{f_{uv}}{\sqrt{1 + f_u^2 + f_v^2}}\,, \quad \text{and} \quad n = \frac{f_{vv}}{\sqrt{1 + f_u^2 + f_v^2}}.$$

Finally, we compute Gauss and mean curvature according to Lemma 4.3:

$$K = \frac{ln - m^2}{EG - F^2} = \frac{f_{uu} f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2}$$

and

$$H = \frac{(1 + f_v^2) f_{uu} + (1 + f_u^2) f_{vv} - 2\, f_u f_v f_{uv}}{2\,(1 + f_u^2 + f_v^2)^{3/2}}. \tag{49}$$

Equation (49) implies a condition, albeit a complicated one, on f that will make M minimal. The surface M is minimal if and only if

$$(1 + f_v^2) f_{uu} + (1 + f_u^2) f_{vv} - 2\, f_u f_v f_{uv} = 0. \tag{50}$$

Equation (50) is called the **minimal surface equation**; notice, it is a second order, partial differential equation that couples u and v.

To find examples we use various techniques; here we present four minimal surfaces:

- a catenoid,
- a helicoid,
- Scherk's surface,
- Enneper's surface.

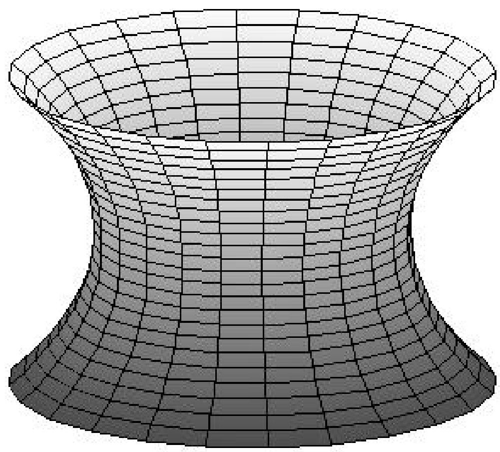


FIGURE 11. This catenoid is a minimal surface. Physicists use the catenoid to model the optimal configuration of a soap film between two concentric rings separated by a given distance along an axis through their centers.

These examples reflect the slightly bizarre history of minimal surfaces. The only known minimal surfaces until 1835 were the catenoid and helicoid. Then in 1835 Polish-born mathematician Heinrich Ferdinand Scherk discovered a new class of minimal surface (named in his honor). Since, many minimal surfaces have been discovered, but they arose from various mathematical contexts, not just modifications of Scherk's or any one category of ideas. Our fourth example was discovered by German mathematician Alfred Enneper in 1863.

Example: A Catenoid

Like the torus, the catenoid is a surface of revolution. We obtain a catenoid surface by rotating the curve $x_2 = \cosh(x_1)$ about the x_3 axis. A patch for the catenoid is

$$\mathbf{x}(u, v) = (\cosh u \cos v, \cosh u \sin v, u),$$

where $v \in [0, 2\pi)$. Figure 11 shows a rendering of the catenoid. The catenoid solves a basic problem in the mathematics and physics of soap films. If we separate two rings by a fixed distance and orient them so they face each other, then a soap film with the rings as a boundary will assume the shape of a catenoid. This is a specific example of soap films as minimal surfaces, we argue this is the case in general in Section 5.2.

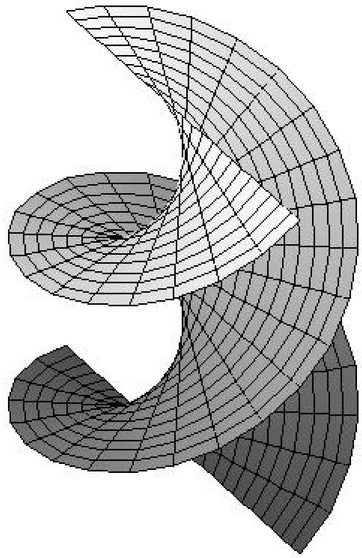


FIGURE 12. A helicoid is formed by filling the space between a helix and its central axis.

Example: A Helicoid

The helicoid shape is the other minimal surface known during the 1700s. To construct the helicoid we begin with a helix, which is a space curve given by $\boldsymbol{\alpha}(u) = (\cos u, \sin u, u)$, and sweep out a surface from the central axis of the helix to the curve itself. Figure 12 shows a helicoid.

Example: Scherk's Surface

Some minimal surfaces are obtained by searching for solutions to the minimal surface equation (50). If we look for solution of the form $f(u, v) = g(u) + h(v)$, then equation (50) becomes

$$\frac{d^2g}{du^2}\left(1 + \left(\frac{dh}{dv}\right)^2\right) + \frac{d^2h}{dv^2}\left(1 + \left(\frac{dg}{du}\right)^2\right) = 0.$$

This can be solved by separation of variables, as it is no longer a partial differential equation, to find

$$f(u, v) = \ln(\cos u) - \ln(\cos v) = \ln\left(\frac{\cos u}{\cos v}\right),$$

is a solution. Figure 13 displays part of Scherk's surface. This is a particularly interesting minimal surface because it is periodic. Pressley quantifies the periodicity by noting

$$\frac{\cos(u + n\pi)}{\cos(v + m\pi)} = \frac{\cos u}{\cos v},$$

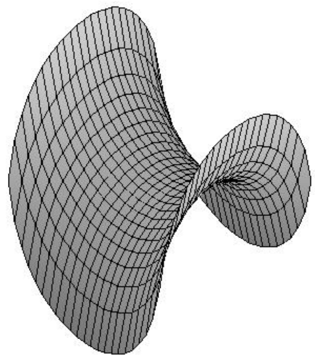


FIGURE 13. Scherk's surface for the domain $(u, v) \in [0, 2\pi) \times [0, 2\pi)$. The surface is actually infinite and periodic defined on all squares in uv-space such that the center of the square is at $(m\pi, n\pi)$, where m and n are integers and $m + n$ is even.

for integers m and n such that $m+n$ is even. Hence, for such m and n, Scherk's surface is defined for any square domain in uv-space with center at $(m\pi, n\pi)$.

Example: Enneper's Surface

Enneper's surface has patch

$$\mathbf{x}(u, v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + vu^2, u^2 - v^2\right).$$

Enneper's surface is a remarkable minimal surface for two reasons. One, its patch only involves the product of u and v as well as integer powers of u and v. It is interesting that such a surface would have zero mean curvature at every point. Two, Enneper's surface is self-intersecting. Figure 14 shows Enneper's surface intersecting itself.

5.2. **The Laplace-Young Equation.** In this Section we provide a physical justification of why soap bubbles should assume minimal (mean curvature $H = 0$) configurations. Both Oprea and Pressley provide justifications, but the demonstration is particularly clear in The Science of Soap Films and Soap Bubbles by Cyril Isenberg. [6]

Figure 15 shows the basic physical set-up. We take a small portion of surface area on some given surface and study the physical work required to expand the surface. We can represent the expansion by an outward displacement δu when the surface is subject to a pressure P.

The dimensions of the smaller surface are measured by x and y and we assume the sides come together at right angles, so the shape is approximately rectangular. This bit of surface area is curved and we approximate the curving by letting x and y be chord lengths on circles of radius R_x and R_y respectively. The centers of these circles would

[6] C. Isenberg. *The Science of Soap Films and Soap Bubbles.* Tieto Ltd., Cleveland, Avon, England, 1978.

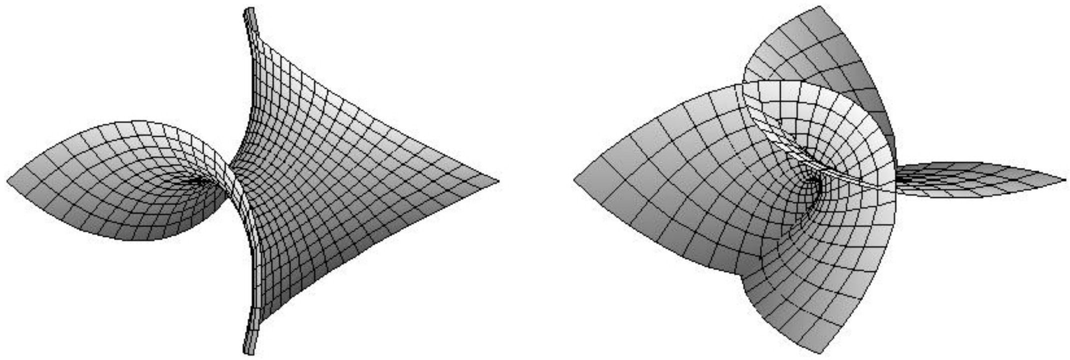


FIGURE 14. At left we show the general shape of Enneper's surface near $(u, v) = (0, 0)$. At right we see the self-intersection of Enneper's surface.

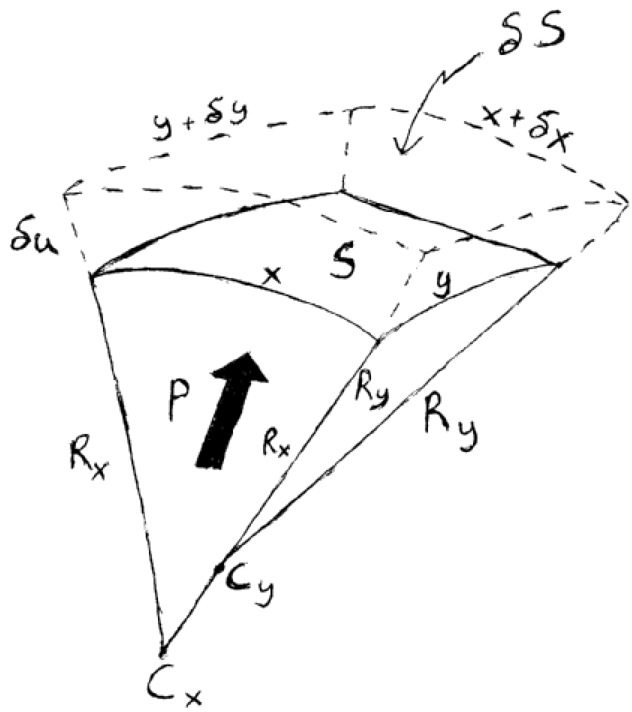


FIGURE 15. The set-up for the derivation of Laplace-Young equation.

be at points C_x and C_y respectively. To form the expanded piece we picture these radii extended by δu to yield a surface area with dimensions $x + \delta x$ and $y + \delta y$. Hence, the change of surface area

$$\delta S \approx (x + \delta x)(y + \delta y) - xy. \tag{51}$$

Note, to compute δS we approximate the surfaces pieces as rectangles, so that the area is simply the product of their dimensions.

Based on the geometry of our set-up, we derive expressions relating the side lengths of two pairs of similar triangles

$$\frac{x+\delta x}{R_x+\delta u}=\frac{x}{R_x} \quad \text{and} \quad \frac{y+\delta y}{R_y+\delta u}=\frac{y}{R_y},$$

which imply

$$x+\delta x = x\left(1+\frac{\delta u}{R_x}\right) \quad \text{and} \quad y+\delta y = y\left(1+\frac{\delta u}{R_y}\right).$$

Plugging these into equation (51) gives

$$\begin{aligned}\delta S &= x\left(1+\frac{\delta u}{R_x}\right) y\left(1+\frac{\delta u}{R_y}\right) - xy \\ &= xy\,\delta u\left(\frac{1}{R_x}+\frac{1}{R_y}\right) + xy\frac{(\delta u)^2}{R_x R_y}\end{aligned}$$

We ignore the $(\delta u)^2$ term because δu is assumed to be a small displacement. Hence, the change of surface area is

$$\delta S \approx xy\,\delta u\left(\frac{1}{R_x}+\frac{1}{R_y}\right). \tag{52}$$

Now we turn our attention to the physics of this set-up. The work done by the pressure P to expand the film must go to increase the surface tension T, that is

$$W_P = W_T, \tag{53}$$

where W_P is the work done by the pressure and W_T is the work done by surface tension. We can rationalize this by considering a molecular model for a soap film as charged particles. A film is held together by the intermolecular forces between these charged particles so it requires work to separate them, which we do upon expanding the film. For our purposes it is sufficient to approximate the surface tension with T (which has units force per unit length), so the work done separating the molecules is just

$$W_T = T\,\delta S \approx T\,xy\,\delta u\left(\frac{1}{R_x}+\frac{1}{R_y}\right),$$

where the approximation comes from equation (52). Pressure P is force per unit area and work is force times displacement so we have

$$W_P = P\,xy\,\delta u.$$

Setting $W_P = W_T$ as in equation (53) gives the **Laplace-Young Equation**:

$$P \approx T\left(\frac{1}{R_x}+\frac{1}{R_y}\right). \tag{54}$$

Recall that the normal curvature of a sphere is $1/R$, where R is the radius of curvature. With our approximations, $1/R_x$ and $1/R_y$ can be thought of as normal curvatures in perpendicular directions so Lemma 5.2 applies and

$$P \approx T(2\,H). \tag{55}$$

Physically, we see the pressure and surface tension balancing according to the mean curvature.

If we specifically consider a soap film help stationary on some boundary, then $P = 0$ as the air pressure on one side equals the air pressure on the other. Hence, soap bubbles must assume zero mean curvature, $H = 0$, configurations. Once we see soap films are minimal, then all the associated mathematical theory applies these physical systems. For example, the integral curvature condition of equation (47) must apply, that is

$$\int_0^{2\pi} \kappa_n(\theta)\, d\theta = 0,$$

and any equation f that models a soap film must satisfy the minimal surface equation (50).

5.3. **Aleksandrov's "Soap Bubble" Theorem.** What if the soap film is closed? That is suppose we wish to study soap bubbles rather than films on frames. Russian mathematician Pavel Sergeevich Aleksandrov (1896 to 1982) had a long and successful career in topology and we examine one of his results in this section. Aleksandrov's Theorem is often called the "Soap Bubble" Theorem because it shows soap bubbles are spherical! Throughout the exposition here, keep in mind how a simple physical property as "being spherical" requires quite difficult mathematics to concretely prove. There are many ways to organize this proof, but we adopt the overall structure of Robert C. Reilly's proof of Aleksandrov's Theorem, who presents a fantastic exposition in his 1982 (the year of Aleksandrov's death) paper "Mean Curvature, The Laplacian, and Soap Bubbles." [7]

In Section 5.2, we see soap films have mean curvature $H = 0$. While the Laplace-Young equation (54) applies to soap bubbles, we can no longer say $P = 0$ because a bubble is closed. The overall effect of this is to give a soap bubble constant, albeit nonzero, mean curvature. If we consider equation (55), then we have

$$H \approx \frac{P}{2T}.$$

Aleksandrov's Theorem is sufficient to show soap bubbles are spherical because it essentially says compact surfaces of constant mean curvature must be spherical. A surface is **compact** if it is closed and bounded.

Reilly's version of the proof rests on eight propositions, which are from markedly different areas of undergraduate-level (with one exception) mathematics. We present most of the propositions without proof and in some cases refer the reader to sources where proofs can be found.

Proposition A (The Divergence Theorem). *Let M be a compact surface that encloses $W \subseteq \mathbb{R}^3$. Assume W includes M. If $F : W \mapsto \mathbb{R}$ is a differentiable function on W, then*

$$\int\int\int_W \Delta\, F\, dx_1\, dx_2\, dx_3 = \int\int_M \hat{\mathbf{U}}[F]\, dA, \tag{56}$$

[7] R. Reilly. Mean curvature, the Laplacian, and Soap Bubbles. *Amer. Math. Monthly* **89** (1982), no. 3, 180-188+197-198.

where $\hat{\mathbf{U}}$ *is the unit normal of* M.

Proof. This is slightly different than the typical vector calculus Divergence Theorem, which states for some vector-valued function $\mathbf{G} : W \mapsto \mathbb{R}^3$
$$\int\int\int_W \nabla \cdot \mathbf{G}\, dx_1\, dx_2\, dx_3 = \int\int_M \mathbf{G} \cdot \hat{\mathbf{U}}\, dA.$$
To get equation (56) simply let $\mathbf{G} = \nabla F$. □

Proposition B (Newton's Inequality). *Suppose* A *is an* $n \times n$ *matrix with elements* $a_{ij} \in \mathbb{R}$ *for* $1 \leq i, j \leq n$. *Define the norm* $||A||$ *by*
$$||A||^2 \doteq \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}^2,$$
this is called the ***Frobenius matrix norm***. *Let* trace(A) *denote the sum of the diagonal elements of* A, *that is*
$$\text{trace}(A) = \sum_{i=1}^{n} a_{ii}.$$
Then
$$||A||^2 \geq \frac{1}{n}(\text{trace}(A))^2.$$
Equality occurs if and only if for some $c \in \mathbb{R}$ $A = c\, I_n$, *where* I_n *is the* $n \times n$ *identity matrix.*

Proposition C (The Cauchy-Schwarz Inequality for Integrals). *Given a surface* M, *for integrable functions* f *and* g *we have*
$$\left(\int\int_M f^2\, dA\right)\left(\int\int_M g^2\, dA\right) \geq \left(\int\int_M f\, g\, dA\right)^2.$$

Proposition D is an existence theorem, which, as with many other such theorems, has a simple result, but an incredibly complicated proof.

Proposition D. *Suppose compact surface* M *encloses* $W \subseteq \mathbb{R}^3$. *Assume* W *includes* M. *Then there exists a differentiable, scalar function* $F : W \mapsto \mathbb{R}$ *such that*
$$\Delta F = 1 \quad \text{and} \quad F|_M = 0, \tag{57}$$
where $F|_M$ *is the restriction of* F *to* M.

Proof. The proof can be found in Sections 6.3 and 6.4 of the book Elliptic Partial Differential Equations of the Second Order by Gilbarg and Trudinger. [8] □

[8] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, New York, 1977.

Notice, this is essentially a statement about the existence of solutions to a certain boundary value problem, given in equation (57).

Propositions E and F refer to the Laplacian Δ and surface Laplacian Δ_M of Section 2.6.

Proposition E. *Let $G : W \mapsto \mathbb{R}$ be a twice differentiable function on $W \subseteq \mathbb{R}^3$. Suppose the surface $M \subseteq W$. Denote the restriction of G to M with g. Take $\boldsymbol{p} \in M$ and define $\hat{\mathbf{U}}(\boldsymbol{p})$, the unit normal at $\boldsymbol{p}$, as well as the mean curvature at $\boldsymbol{p}$, $H(\boldsymbol{p})$. Then*

$$\Delta\, G(\boldsymbol{p}) = \Delta_M\, g(\boldsymbol{p}) - 2\, H(\boldsymbol{p})\, \hat{\mathbf{U}}[G](\boldsymbol{p}) + \hat{\mathbf{U}}[\hat{\mathbf{U}}[G]](\boldsymbol{p}). \tag{58}$$

Proof. We refer the reader to Reilly's paper for a proof of this result. □

Proposition F. *Suppose for surface M that $g : M \mapsto \mathbb{R}$ is a twice differentiable function.*

- *If g is constant, then $\Delta_M\, g = 0$.*
- *If M is compact, then $\int\int_M \Delta_M\, g\, dA = 0$.*

Proof. If g is constant, then $\Delta_M\, g = 0$ by definition, found in equation (5). A proof of the second part can be found in the appendix of Reilly's paper. □

Reilly introduces the **Minkowski support function** for a surface as the dot product between $\mathbf{p} \in M$ and the unit normal at $\mathbf{p}$, $\hat{\mathbf{U}}(\mathbf{p})$, that is

$$P(\mathbf{p}) \doteq \mathbf{p} \cdot \hat{\mathbf{U}}(\mathbf{p}) = (x_1, x_2, x_3) \cdot \hat{\mathbf{U}}.$$

Formally, the support function $P : M \mapsto \mathbb{R}$ is a natural function to define, but it has little physical relevance in our context. Reilly uses at a tool in his proof based on Propositions G and H.

Proposition G. *Suppose that M is a compact surface and $P : M \mapsto \mathbb{R}$ is the Minkowski support function. Then*

$$\int\int_M (H\, P + 1)\, dA = 0,$$

where H is the mean curvature of M.

Proof. Define $G : M \mapsto \mathbb{R}$, such that $G(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2)/2$ so that $\Delta\, G(\mathbf{p}) = 3$ for all $\mathbf{p} \in M$. Let $\hat{\mathbf{U}}(\mathbf{p}) = (\hat{u}^1(\mathbf{p}), \hat{u}^2(\mathbf{p}), \hat{u}^3(\mathbf{p}))$ and compute

$$\hat{\mathbf{U}}[G](\mathbf{p}) = \nabla G(\mathbf{p}) \cdot \hat{\mathbf{U}}(\mathbf{p}) = (x_1, x_2, x_3) \cdot \hat{\mathbf{U}}(\mathbf{p}) = P(\mathbf{p})$$

and

$$\begin{aligned}
\hat{\mathbf{U}}[\hat{\mathbf{U}}[G(\mathbf{p})]] &= \hat{\mathbf{U}}[x_1\, u^1(\mathbf{p}) + x_2\, u^2(\mathbf{p}) + x_3\, u^3(\mathbf{p})] \\
&= \nabla(x_1\, u^1(\mathbf{p}) + x_2\, u^2(\mathbf{p}) + x_3\, u^3(\mathbf{p})) \cdot \hat{\mathbf{U}}(\mathbf{p}) \\
&= (u^1(\mathbf{p}), u^2(\mathbf{p}), u^3(\mathbf{p})) \cdot \hat{\mathbf{U}}(\mathbf{p}) = \hat{\mathbf{U}}(\mathbf{p}) \cdot \hat{\mathbf{U}}(\mathbf{p}) = 1.
\end{aligned}$$

Putting these expressions into equation (58) yields

$$3 = \Delta_M\, g - 2\, H\, P + 1,$$

which we rearrange to find

$$H\,P + 1 = \frac{1}{2}\Delta_M\, g.$$

Integrate both sides of the surface and apply the second part of Proposition F (since M is compact) to find

$$\int\int_M (H\,P+1)\,dA = \frac{1}{2}\int\int \Delta_M\, g\,dA = 0.$$

□

Proposition H. *Suppose M encloses $W \subseteq \mathbb{R}^3$ and let $P : M \mapsto \mathbb{R}$ be the support function, then*

$$\int\int_M P\,dA = 3\,V,$$

where V is the volume of W.

Proof. Let $G : M \mapsto \mathbb{R}$ be defined as in Proposition G, that is $G(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2)/2$. Also, recall from the proof of Proposition G that $\hat{\mathbf{U}}[G] = P$ and $\Delta\,G = 3$. By the Divergence theorem in Proposition A

$$\int\int_M P\,dA = \int\int_M \hat{\mathbf{U}}[G]\,dA = \int\int\int_W \Delta\,G\,dV = \int\int\int_W 3\,dV = 3\,V.$$

□

We are now ready to state and give a proof of Aleksandrov's Theorem using Propositions A through H.

Lemma 5.3 (Aleksandrov's Theorem)**.** *Suppose a surface M encloses a domain $W \subseteq \mathbb{R}^3$. If M has constant mean curvature, then M is the surface of a sphere and W is a ball.*

Proof. By Proposition D there exists a function $F : W \mapsto \mathbb{R}$ such that

$$\Delta\,F = 1 \quad \text{and} \quad F|_M = 0.$$

By the first part of Proposition F and the fact that F is constant when restricted to M, we have $\Delta_M\,F = 0$. Putting the function F in equation (58) gives

$$1 = -2H(\mathbf{p})\,\hat{\mathbf{U}}[F](\mathbf{p}) + \hat{\mathbf{U}}[\hat{\mathbf{U}}[F]](\mathbf{p}) \tag{59}$$

for all $\mathbf{p} \in M$. Multiply both sides of (59) by $\hat{\mathbf{U}}[F](\mathbf{p})$ and integrate over the entire surface

$$\int\int_M \hat{\mathbf{U}}[F]\,dA = \int\int_M -2\,H\,(\hat{\mathbf{U}}[F])^2\,dA + \int\int_M (\hat{\mathbf{U}}[F])\,(\hat{\mathbf{U}}[\hat{\mathbf{U}}[F]])\,dA. \tag{60}$$

We investigate each of these terms separately.

Take the left-hand side first. By the Divergence Theorem of Proposition A, $\Delta\,F = 1$ implies

$$\int\int_M \hat{\mathbf{U}}[F]\,dA = \int\int\int_W \Delta\,F\,dV = \int\int\int 1\,dV = V, \tag{61}$$

where V is the volume of W.

Next, consider the first term on the right-hand side of equation (60). According to Proposition G

$$0 = \int\int_M (H\,P)\,dA = \int\int_M 1\,dA + H\int\int_M P\,dA = A + H\int\int_M P\,dA,$$

where A is the surface area of M and we use the fact that M has constant mean curvature H. Rearranging we find

$$H = -\frac{A}{\int\int_M P\,dA} = -\frac{A}{3\,V}$$

by Proposition H. Notice, this is an interesting expression for the mean curvature of the surface as a ratio of surface area to volume enclosed. Next, we use the Cauchy-Schwarz inequality from Proposition C to put a lower bound on the surface integral of $(\hat{\mathbf{U}}[F])^2$. Observe

$$\left(\int\int_M 1^2\,dA\right)\left(\int\int_M (\hat{\mathbf{U}}[F])^2\,dA\right) \geq \left(\int \in t_M(1)\,(\hat{\mathbf{U}}[F])\right)^2,$$

which implies

$$\int\int_M (\hat{\mathbf{U}}[F])^2\,dA \geq \frac{\left(\int\int\int_W \Delta\,F\,dV\right)^2}{\int\int_M dA} = \frac{V^2}{A} \tag{62}$$

The last inequality in (62) follows from Proposition A, our version of the Divergence Theorem. Combining the equality in equation (61) with the inequality in equation (62) we write

$$\int\int_M -2\,H\,(\hat{\mathbf{U}}[F])^2\,dA \geq \left(\frac{2\,A}{3\,V}\right)\left(\frac{V^2}{A}\right) = \frac{2}{3}V. \tag{63}$$

We now turn our attention to the second term on the right-hand side of (60). Rewrite the integrand using Leibniz's product rule for the bracket derivative (see Lemma 2.4) as

$$(\hat{\mathbf{U}}[F])\,(\hat{\mathbf{U}}[\hat{\mathbf{U}}[F]]) = \hat{\mathbf{U}}\left[\frac{1}{2}(\hat{\mathbf{U}}[F])\,(\hat{\mathbf{U}}[F])\right] = \hat{\mathbf{U}}\left[\frac{1}{2}(\hat{\mathbf{U}}[F])^2\right]. \tag{64}$$

Hence, we can use the Divergence Theorem in Proposition A yet again to write

$$\int\int_M \hat{\mathbf{U}}\left[\frac{1}{2}(\hat{\mathbf{U}}[F])^2\right]dA = \int\int\int_W \Delta\left(\frac{1}{2}(\hat{\mathbf{U}}[F])^2\right)dV. \tag{65}$$

Reilly's next step is to rewrite the integrand using the gradient of F (∇F) so we can use summation notation more easily. The following, somewhat arbitrary manipulation achieves this end

$$\begin{aligned}
\Delta\left(\frac{1}{2}(\hat{\mathbf{U}}[F])^2\right) &= \Delta\left(\frac{1}{2}(\hat{\mathbf{U}}[F])^2\,\hat{\mathbf{U}}\cdot\hat{\mathbf{U}}\right)\\
&= \Delta\left(\frac{1}{2}(\hat{\mathbf{U}}[F])\,\hat{\mathbf{U}}\cdot(\hat{\mathbf{U}}[F])\hat{\mathbf{U}}\right) = \Delta\left(\frac{1}{2}\left|\hat{\mathbf{U}}[F]\,\hat{\mathbf{U}}\right|^2\right).
\end{aligned}$$

Now it becomes clear how to use the summation notation

$$\Delta\left(\frac{1}{2}\left|\hat{\mathbf{U}}[F]\,\hat{\mathbf{U}}\right|^2\right) = \Delta\left(\frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial F}{\partial x_i}\right)^2\right) = \sum_{j=1}^{3}\frac{\partial^2}{\partial x_j^2}\left(\frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial F}{\partial x_i}\right)^2\right)$$

We carry out this differentiation to get

$$\sum_{j=1}^{3}\frac{\partial^2}{\partial x_j^2}\left(\frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial F}{\partial x_i}\right)^2\right) = \sum_{j=1}^{3}\sum_{i=1}^{3}\left(F_{jii}\,\frac{\partial F}{\partial x_j} + \left(\frac{\partial^2 F}{\partial x_j \partial x_i}\right)^2\right),$$

where F_{jii} is shorthand for third partial derivatives of F. Mixed third partial derivatives are equal so

$$\sum_{i=1}^{3} F_{jii} = \sum_{i=1}^{3} F_{iij} = \frac{\partial}{\partial \mathbf{x}_j}(\Delta\, F) = \frac{\partial}{\partial \mathbf{x}_j}(1) = 0$$

because $\Delta\, F = 1$ by assumption (from Proposition D). Hence,

$$\Delta\left(\frac{1}{2}\left|\hat{\mathbf{U}}[F]\,\hat{\mathbf{U}}\right|^2\right) = \sum_{j=1}^{3}\sum_{i=1}^{3}\left(\frac{\partial^2 F}{\partial x_j \partial x_i}\right)^2. \tag{66}$$

Now the right-hand side is the Frobenius matrix norm of the 3×3 matrix with the nine second partial derivatives of F as its entries. We apply Newton's Inequality of Proposition B to see

$$\sum_{j=1}^{3}\sum_{i=1}^{3}\left(\frac{\partial^2 F}{\partial x_j \partial x_i}\right)^2 \geq \frac{1}{3}\left(\sum_{j=1}^{3}\frac{\partial^2 F}{\partial x_j^2}\right)^2 = \frac{1}{3}(\Delta\, F)^2 = \frac{1}{3}(1)^2 = \frac{1}{3}.$$

Thus, by equations (64), (65), and (66) we see

$$\int\int_M (\hat{\mathbf{U}}[F])\,(\hat{\mathbf{U}}[\hat{\mathbf{U}}[F]])\,dA \geq \frac{1}{3}V \tag{67}$$

and

$$\int\int_M (\hat{\mathbf{U}}[F])\,(\hat{\mathbf{U}}[\hat{\mathbf{U}}[F]])\,dA = \frac{1}{3}V \tag{68}$$

if and only if

$$\frac{\partial^2 F}{\partial x_j \partial x_i} = \frac{1}{3}\delta_{ij} \tag{69}$$

for $1 \leq i, j \leq 3$, where δ_{ij} is the usual Kronecker delta function. The if and only if portion follows from Proposition C.

Look back at equation (60). Using equations (61), (63), and (67) we have

$$\begin{aligned} V &= \int\int_M \hat{\mathbf{U}}[F]\,dA \\ &= \int\int_M -2\,H\,(\hat{\mathbf{U}}[F])^2\,dA + \int\int_M (\hat{\mathbf{U}}[F])\,(\hat{\mathbf{U}}[\hat{\mathbf{U}}[F]])\,dA \\ &\geq \frac{2}{3}V + \frac{1}{3}V = V. \end{aligned} \tag{70}$$

Notice equation (70) shows that equality must hold, which means equation (68) must be true. This implies equation (69) is valid. Equation (69) is a system of nine partial differential equations, only three of which are non-zero. So we have

$$\frac{\partial^2 F}{\partial x_1^2} = \frac{1}{3}, \quad \frac{\partial^2 F}{\partial x_2^2} = \frac{1}{3}, \quad \text{and} \quad \frac{\partial^2 F}{\partial x_3^2} = \frac{1}{3},$$

a system that we can integrate easily (all are separable) to find

$$F(x_1, x_2, x_3) = \frac{1}{6}(x_1^2 + x_2^2 + x_3^2) + a_1\, x_1 + a_2\, x_2 + a_3\, x_3 + b,$$

for some real scalar constants a_1, a_2, a_3 and b. By assumption, from Proposition D again, $F|_M = 0$ so for all $(x_1, x_2, x_3) \in M$

$$0 = \frac{1}{6}(x_1^2 + x_2^2 + x_3^2) + a_1\, x_1 + a_2\, x_2 + a_3\, x_3 + b.$$

This is the explicit form of a sphere. Hence M is a sphere. □

Aleksandrov's theorem represents an amazing physical manifestation of the basic Differential Geometry, and other pieces of supporting mathematics, we discuss throughout this paper. Notice, such a simple observation about the shape of a soap bubble requires a significant body of mathematical foundation to prove! Yet, our mathematics works, that is we get an actual *proof.* Is this a reflection of our mathematical development or is there some inherent math-based structure in nature?

6. Conclusion

Our introductory survey into Differential Geometry builds a strong foundation, but only serves to scratch the surface of this intriguing field of mathematics. Despite being only a scratch, the material we present here represents deep and powerful mathematics by itself. Section 2 presents a fairly complete outline of differentiation on surfaces, concepts on which much of this paper rest. Our primary topic of study is curvature of surfaces in $\mathbb{R}^3$ and we introduce three types: the normal curvature κ_n in Section 3.4 as well as the Gauss K and mean H curvatures in Section 4.2. Throughout our exposition we refer repeatedly to two texts; one is by John Oprea and the other by Andrew Pressley. While both are undergraduate level books they take remarkably dissimilar approaches. Oprea uses the Shape Operator, to which we also devote Section 3.1, and makes no mention of fundamental forms. Pressley builds exclusively from the first and second fundamental forms and only introduces the Shape Operator once much of the theory is already in place. We reconcile the Shape Operator with the fundamental forms throughout our theoretical build-up, in particular Lemma 3.7 joins the two ideas. Regardless of approach, perhaps the most basic ideas of Differential Geometry lie in the coefficients E, F, G, l, m, and n. As we note in Section 3.4 one of the most amazing aspects of this field is just how much information is contained in or can be derived using these coefficients. Two major results we present are Gauss's Theorem Egregium in Corollary 4.10 and Aleksandrov's Soap Bubble Theorem in Section 5.3. Each result has an important physical manifestation.

There are a wealth of directions which to take the study of Differential Geometry if the material here has caught the reader's interest. Basically all of what we present in terms of differentiation on surfaces in $\mathbb{R}^3$ generalizes to **manifolds** in $\mathbb{R}^n$. The study of manifolds is relevant in many areas of mathematics (both pure and applied) and theoretical physics. Of course, one can continue to develop results in a similar vein to those we have already discussed. For instance, both Pressley's and Oprea's books have compelling chapters concerning **geodesics**, that is distance-minimizing paths that connect two points on a surface. For a reader familiar with basic topology, there are many sources that approach Differential Geometry in that light. One relatively recent development, the discovery of an entire family of minimal surfaces called **Hoffman surfaces**, serves as a great springboard into the aspects of Differential Geometry grounded in topology. For applied mathematicians and physicists there is much room to explore minimal surfaces in the context of material interfaces and soap films. The theory of soap bubbles in particular lends itself nicely to Differential Geometry, in fact Oprea authors a book called The Mathematics of Soap Films: Explorations with Maple. On the theoretical physics side, much of the supporting mathematics for Einstein's theory of general relativity can be understood in terms of curvature of manifolds and geodesic equations.

Differential Geometry, and certainly the portion of it we present, begins to answer some of the questions we mention in Section 1.1 about the fundamental shape and nature of space. Little in our physical universe is more basic than shape and although it is surprisingly difficult to get a mathematical handle on such general concepts, the pursuit is exceedingly beautiful.

7. References

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