MATROID THEORY

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Contents

1. Introduction to Matroids

A matroid is a structure that generalizes the properties of independence. Relevant applications are found in graph theory and linear algebra. There are several ways to define a matroid, each relate to the concept of independence. This paper will focus on the the definitions of a matroid in terms of bases, the rank function, independent sets and cycles.

Throughout this paper, we observe how both graphs and matrices can be viewed as matroids. Then we translate graph theory to linear algebra, and vice versa, using the language of matroids to facilitate our discussion.

Many proofs for the properties of each definition of a matroid have been omitted from this paper, but you may find complete proofs in Oxley[2], Whitney[3], and Wilson[4].

The four definitions of a matroid introduced in this paper are equivalent to each other. However, the proofs are also omitted from this paper. The complete proofs can be found in Whitney[3].

The following subsections are a brief introduction to the basics of graph theory and linear algebra.

1.1. **Basic Graph Theory.**

We first introduce the concept of a graph before we begin to incorporate graphs into the theory of matroids. Robin Wilson provides the following definition of a simple graph.

Definition 1.1. A simple graph G is a non-empty finite set of elements, called <u>vertices</u>, and a finite set of unordered pairs of elements called <u>edges</u>.[4]

In the example shown in Figure 1, the set of vertices, $V(G)$, are $\{1, 2, 3, 4, 5\}$, and the set of edges, $E(G)$ are $\{a, b, c, d, e, f, g\}$. Matroids focus on the properties of independence by using the set of edges, $E(G)$, as the elements of a matroid.

We will use the graph, G, in Figure 1 throughout our discussion of matroids.

Definitions 1.2 and 1.3 describe a walk, a path, and a closed path. These concepts will be useful when discussing independent and dependent sets in graph theory.

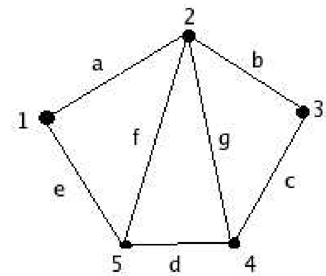

FIGURE 1. Graph, G

Definition 1.2. Given a graph G, a walk in G is a finite sequence of edges of the form $v_0v_1, v_1v_2, ...v_{m-1}v_m$, where v_i is a vertex of G, in which any two consecutive edges are adjacent or identical.

A walk in which all the edges are distinct is a trail. If the vertices $v_0, v_1, ..., v_m$ are distinct (except, possibly, $v_0 = v_m$), then the trail is a path. [4]

Definition 1.3. Given the distinct vertices $v_0, v_1, ...v_m$, a path is closed if $v_0 = v_m$.[4]

A closed path is also known as a cycle in graph theory.

1.2. **Basic Linear Algebra.** A is a 5x8 matrix, and its column vectors are in R^5. The set of column vectors of the matrix A are $\{1, 2, 3, 4, 5, 6, 7, 8\}$. We will focus on the set of column vectors in a matrix as the elements of a matroid.

$$\begin{array}{cc} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \mathbf{A} = & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Now that we have a basic foundation of linear algebra and graph theory, we will begin our introduction of matroids by using the concept of a base.

2. Bases

This section provides one definition of a matroid, as well as demonstrates how our examples from linear algebra and graph theory fit this definition. The following definition is from Hassler Whitney's paper, "On the Abstract Properties of Linear Independence," which is the first published paper that explored the theory of matroids.

Definition 2.1. A matroid M consists of a non-empty finite set E and a non-empty collection B of subsets of E, called bases, satisfying the following properties: [3]

- $B(i)$ no base properly contains another base;
- $B(ii)$ if B_1 and B_2 are bases and if $\{e\}$ is any element of B_1, then there is an element f of B_2 such that $(B_1 - \{e\}) \cup \{f\}$ is also a base.

$B(ii)$ is known as the exchange property.[4] This property states that if an element is removed from B_1, then there exists an element in B_2, such that a new base, B_3, is formed when that element is added to B_1.

We can use the property $B(ii)$ to show that every base in a matroid has the same number of elements.

Theorem 2.1. *Every base of a matroid has the same number of elements.*

Proof. First assume that two bases of a matroid M, B_1 and B_2, contain different number of elements, such that $|B_1| < |B_2|$. Now suppose there is some element, $\{e_1\} \in M$, such that $e_1 \in B_1$, but $e_1 \notin B_2$. If we remove $\{e_1\}$ from B_1, then by $B(ii)$, we know there is some element, $e_2 \in B_2$, but $e_2 \notin B_1$ such that $B_3 = B_1 \setminus (\{e_1\} \cup \{e_2\})$, where B_3 is a base in M. Therefore, $|B_1| = |B_3|$ but $|B_2| \neq |B_1| = |B_3|$.

If we continue the process of exchanging elements, defined by the property $B9ii)$, k number of times, then there will be no element initially in B_1 that is not in the base B_k. Therefore, for all $e \in B_k$, the element e is also in B_2, and thus $B_k \subseteq B_2$.

From $B(i)$, we know that no base properly contains another base. This is a contradiction. Therefore we know that every base has the same number of elements.

$\square$

2.1. **An Example in Linear Algebra.** Recall that in our previous example of the matrix A, the column vectors are in R^5. These columns form a matroid. We will take the base of a matroid to be a maximal linearly independent set that spans the column space (i.e., a basis for the column space). Consider two bases of our matroid:

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\},$$

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Now if we remove the second vector in B_1, then we can replace it with the second vector in B_2 to get a new base, B_3,

$$B_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

For this case, $B(ii)$ is satisfied. We would find the same results if we continued this process with all possible bases of A. It is well known from Linear Algebra that no basis of A properly contains another basis.

2.2. **An Example in Graph Theory.** We will take a base of our matroid to be a spanning tree of G. The following is a definition of a spanning tree.

Definition 2.2. Let G be a graph with n vertices. A spanning tree is a connected subgraph that uses all vertices of G that has $n-1$ edges. [4]

If we refer back to Figure 1, then we can see that the bases of the graph, G, are in Table 2.2.

Bases
$\{a, b, c, d\}$
$\{a, e, d, c\}$
$\{b, c, d, e\}$
$\{b, a, e, d\}$
$\{c, b, a, e\}$
$\{c, b, f, e\}$
$\{c, d, f, a\}$
$\{c, g, a, e\}$
$\{c, g, f, e\}$

Table 2.2, The Spanning Trees of G

By observing the set of bases listed above, we can see that $B(i)$ is satisfied, because no base properly contains another base. We can now demonstrate $B(ii)$ by using this property with two bases. If we choose, $B_1 = \{a, b, c, d\}$ and $B_2 = \{c, g, a, e\}$, then we can see the spanning trees of B_1 and B_2 in Figures 2 and 3.

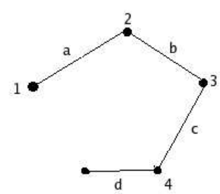

FIGURE 2. The Spanning Tree, B_1

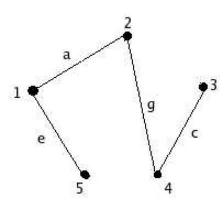

FIGURE 3. The Spanning Tree, B_2

Notice that each spanning tree has 5 vertices and 4 edges. We can demonstrate $B(ii)$ by removing an element $\{a\}$ from B_1, and then there exists an element in B_2 such that a new base is created, $B_3 = (B_1 \setminus \{a\}) \cup \{e\})$. Figure 4 shows the new base, B_3.

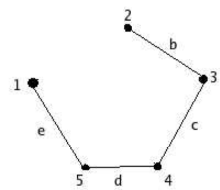

FIGURE 4. B_3

A similar computation works for any choice of bases.

Exercise 9.11 in Robin Wilson's book, Introduction to Graph Theory, explains the exchange axiom for spanning trees.

> Let T_1 and T_2 be spanning trees of a connected graph G. [4]
>
> (i) If e is any edge of T_1, show that there exists an edge f of T_2 such that the graph $(T_1 - \{e\}) \cup \{f\}$ (obtained from T_1 on replacing e by f) is also a spanning tree.
>
> (ii) Deduce that T_1 can be 'transformed' into T_2 by replacing the edges of T_1 one at a time by edges of T_2 in such a way that a spanning tree is obtained at each stage.

Because we take the spanning trees of a graph to be the bases of a matroid, we can conclude that the bases of a matroid have the same number of elements, and by the definition of a spanning tree has $n - 1$ elements (if there are n vertices).

3. Rank Function

In this section, we will continue the discussion of matroids by introducing a new definition of a matroid in terms of its rank function. The following definition of a matroid is from Robin Wilson's book, Introduction to Graph Theory

Definition 3.1. A matroid consists of a non-empty finite set E and an integer-valued function r defined on the set of subset of E, satisfying:

$R(i)$ $0 \leq r(A) \leq |A|$, for each subset A of E;
$R(ii)$ if $A \subseteq B \subseteq E$, then $r(A) \leq r(B)$;
$R(iii)$ for any $A, B \subseteq E$, $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$.

[4]

The property $R(i)$ guarantees that the rank of a subset cannot be negative, nor exceed its size. The second property guarantees that taking a superset does not decrease the rank of a set. The third property is equivalent to the exchange property that was defined in the previous section.

Now that we can use the concept of rank in our discussion of graph theory, we can define loops and parallel elements in Definitions 3.2 and 3.3 .

Definition 3.2. A loop of a matroid M is an element e of E satisfying $r(\{e\}) = 0$. [4]

Definition 3.3. A pair of parallel elements of M is a pair $\{e, f\}$ of E that satisfy $r\{e, f\} = 1$.[4]

3.1. **The Rank Function in Graph Theory.** Recall that we can take the edges of a graph to be the elements of a matroid. For each subgraph, the rank will be the maximal number of edges in the subgraph that do not form a cycle.

We can show how the rank function works in graph theory using the following example. We will let E be the set of edges of the graph in Figure 5. In Figure 6, there are no cycles and the graph is connected. Therefore rank of A is the number of elements in A, so that $r(A) = |A| = 2$. Figure 6 is the subgraph containing $A = \{c, d\}$.

In Figure 7, there are four elements and one cycle. The rank of B is three, because the subsets of B with the maximum number of edges, which do not contain a cycle, are $\{b, c, d\}$, $\{b, c, e\}$, and $\{b, e, d\}$. Therefore, $3 = r(B) < |B| = 4$.

The subset of E found in Figure 8 is a loop, with $r(C) = 0$.

The subset of E in Figure 9 is a set of elements that are parallel elements. Therefore, $r(D) = 1$.

If we take the cycle, $\{c, d, e\}$, and remove any element of the cycle, the rank of the remaining elements will always be two, as shown in Figure 6. Therefore, if we take the cycle with the remaining elements of E, we find that the rank of E is three, which means that the rank of the matroid is also three. The rank of M equals the size of a base of M.

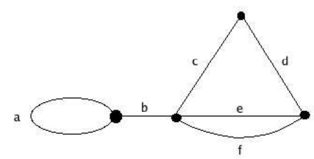

FIGURE 5. The set E

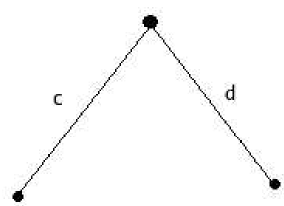

FIGURE 6. The subset $\{c, d\}$ of E

The rank of G, in Figure 1, is 4, and because we take the set of edges of G as the elements of M, the rank of M is also $r(M) = 4$.

We can show an example of the property $R(ii)$ in the graph G, by considering two subsets of G, $A = \{a, b, e\}$ and $B = \{a, b, e, f\}$, so that $A \subseteq B \subseteq E$. In this case, $r(A) = r(B) = 3$. However, if we let $C = \{a, b, d, e\}$, then $3 = r(A) \leq r(B) = 4$. If we continue this example with other subsets, we would come to the same conclusion.

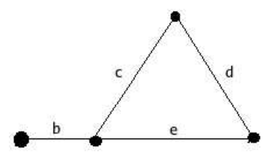

FIGURE 7. The subset $\{b, c, d, e\}$ of E

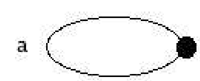

FIGURE 8. The subset C of E

We can demonstrate the property $R(iii)$ by using our previous example with two subsets of M being $A = \{a, b, e\}$ and $B = \{a, b, d, e\}$.

$$\begin{aligned} r(A \cup c) + r(A \cap C) &\leq r(A) + r(C) \\ r(\{a, b, d, e\}) + r(\{a, b.e\}) &\leq r(\{a, b, d, e\}) + r(\{a, b, e\}) \\ 4 + 3 &< 4 + 3 \end{aligned}$$

Therefore, property $R(iii)$ is satisfied in this case.

3.2. **The Rank Function in Linear Algebra.** We define $\text{rank}(A)$ to be the size of a basis for $\text{span}(A)$, or the dimension of the space spanned by A. Because we take the column vectors in a matrix to be the elements in our matroid, define our rank function to be the rank

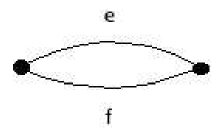

FIGURE 9. The subset D of E

of each subset of M. In our previous example, one basis of the column space of A is

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Because there are four column vectors in this basis, and this is a maximal linearly independent set, the rank of the matrix A is also four.

An example of a loop in a matrix is the zero column vector,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

because the space spanned by $\vec{0}$ is 0 dimensional.

The following is a definition of parallel elements in linear algebra.

Definition 3.4. Two nonzero vectors, $\vec{u}$ and $\vec{v}$, are parallel elements, if $\vec{u} = \lambda\vec{v}$, for some scalar λ.

An example of a set of parallel elements in a matrix is the set $\{e, f\}$, given by

$$e = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$f = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

Because $2e = f$, the rank of the set $\{e, f\}$ is one. Therefore, We say that the set $\{e, f\}$ is a set of parallel elements.

Now we can demonstrate the properties of a matroid in terms of its rank function by examining the matrix A. We can see the property, $R(i)$, by observing the set, C, of column vectors from the matrix A.

$$C = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

In this example, the size of C is five, while the rank of C is four. Therefore, $R(i)$ is satisfied for C.

Now we will show an example of the property $R(ii)$. If we continue with this example, and take,

$$D = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\},$$

so that $D \subseteq C \subseteq A$, we can see that $3 = r(D) < r(C) = 4$. Therefore, property $R(ii)$ is satisfied in this example.

From the definition of a matroid, the equation in $R(iii)$ is satisfied with the two subsets, $C, D \subset E$.

$$\begin{aligned} r(C \cup D) + r(C \cap D) &\leq r(C) + r(D) \\ 4 + 3 &= 4 + 3 \end{aligned}$$

We would come to the same conclusion if we continued to examine various subsets of M.

4. Independent Sets

We will continue our discussion of matroids by introducing a new definition of a matroid in terms of independent sets. A general definition of independence is given by Robin Wilson,

Definition 4.1. A subset of a matroid M is independent if it is contained in a base of a matroid.[4]

Conversely, a subset of M is dependent if it is not independent.

We can also define a matroid in terms of its independent sets. The following definition of a matroid is from Robin Wilson's book, Introduction to Graph Theory.

Definition 4.2. A matroid M consists of a non-empty finite set E and a non-empty collection I of subsets of E (called independent sets) satisfying the following properties: [4]

$I(i)$ any subset of an independent set is independent;

$I(ii)$ if I and J are independent sets with $|J| > |I|$, then there is an element e, contained in J but not in I, such that $I \cup \{e\}$ is independent.

To explain property $I(i)$, we will let K be a subset of a non-empty finite set E. From Definition 4.1, we know that if K is independent, then it is contained in a base. Therefore, any subset of K is independent because the subset is also contained within a base.

Property $I(ii)$ is the equivalent of the exchange axiom, which was defined in the section on bases. This property states that if two independents sets satisfy the inequality $|J| > |I|$, then there exists an element in J, such that the new independent set is formed when that element is added to I.

Now we can see the connection to the previous sections. Moreover, if A is an independent set, then A is contained in some base of M, which implies that $r(A) = |A|$.

4.1. Independent Sets in Graph Theory.

We will take the independent sets of a graph to be the sets of edges in a graph that do not contain a cycle.[4] Recall that in graph theory, a cycle is a closed path. Another definition of independent sets in graph theory uses forests, which are defined by Robin Wilson as,

Definition 4.3. A forest is a graph that contains no cycles. A connected forest is a tree. [4]

We can say that the independent sets of a graph are the edge sets of the forests contained in the graph.[4] Figures 10 and 11 are examples of forests contained in the graph G defined in Figure 1.

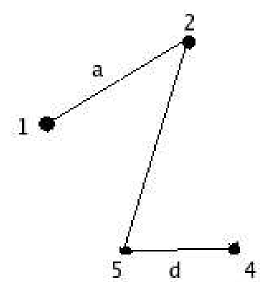

FIGURE 10. An Example of a Forest Contained in G

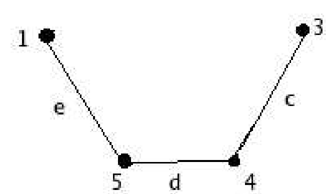

FIGURE 11. Another Example of a Forest Contained in G

The first property $I(i)$, can be shown because a set is independent if it is contained within a base. Therefore independent sets must be contained within a spanning tree of a graph, which means that the rank of an independent set must be less than or equal to the rank of the graph. Table 4.1 lists the forests contained in the graph G defined in Figure 1.

Forests of G
$\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{e\}$
$\{f\}$, $\{g\}$, $\{a,b\}$, $\{b,c\}$
$\{c,d\}$, $\{d,e\}$, $\{e,f\}$, $\{f,g\}$
$\{g,a\}$, $\{a,f\}$, $\{e,f\}$, $\{d,f\}$
$\{b,f\}$, $\{b,g\}$, $\{c,g\}$, $\{d,g\}$
$\{a,b,c\}$, $\{a,b,g\}$, $\{a,e,d\}$
$\{a,f,d\}$, $\{a,g,c\}$, $\{a,g,d\}$
$\{b,c,d\}$, $\{b,g,d\}$, $\{b,f,d\}$
$\{b,f,e\}$, $\{c,d,e\}$, $\{c,g,d\}$
$\{c,d,f\}$, $\{e,f,e\}$, $\{a,f,g\}$
$\{a,b,c,d\}$, $\{a,e,d,c\}$, $\{b,c,d,e\}$
$\{b,a,e,d\}$, $\{c,b,a,e\}$, $\{c,b,f,e\}$
$\{c,d,f,a\}$, $\{c,g,a,e\}$, $\{c,g,f,e\}$

Table 4.1, The Forests of G

From observing the table of forests, we can see that the forests are contained within the spanning trees, which are the bases listed in the last three rows.

Now we will demonstrate why the exchange axiom for independent sets requires that two independent sets, K and L, must satisfy the inequality $|K| > |L|$. Suppose we let the two forests contained in G be the sets K and L_0 shown in Figures 12 and 13. Notice that $|K| = |L| = 3$.

We find that there is no element, e contained in K but not L, such that the set $L \cup \{e\}$ is independent.

However, if we let $L_1 = L_0 \setminus \{c\}$, so that $3 = |K| > |L_1| = 2$, then we necessarily have an element, in this case $\{d\}$, such that $d \in K$ but not in L_1. Therefore, we find the independent set $L_1 \cup \{d\}$.

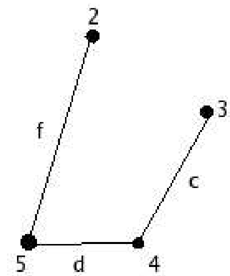

FIGURE 12. The Forest, K

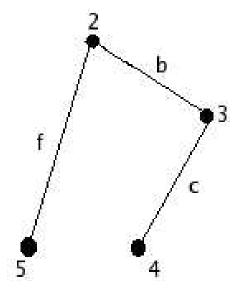

FIGURE 13. The Forest, L_0

4.2. **Independent Sets in Linear Algebra.** We will take the independent sets of a matroid, M, of column vectors. I is independent in M if I is linearly independent.

A more formal definition is given by David C. Lay in his book, Linear Algebra and its Applications.

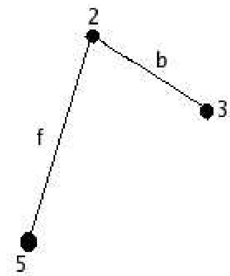

FIGURE 14. The Forest, L_1

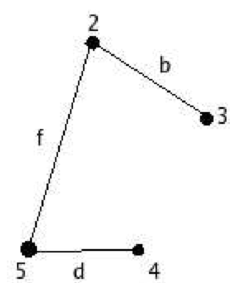

FIGURE 15. The Forest, $L_1 \cup \{d\}$

Definition 4.4. An indexed set of vectors $\{v_1, .., v_p\}$ in R^n is said to be linearly independent if the vector equation

$$x_1v_1 + x_2v_2 + ... + x_pv_p = 0$$

has only the trivial solution. The set $\{v_1..., v_p\}$ is said to be linearly dependent if there exists weights $c_1, ..., c_p$, not all zero, such that

$$c_1v_1 + c_2v_2 + ...c_pv_p = 0$$

[1]

From the previous section, we know that a base is defined as a maximal linearly independent set that spans the column space. Therefore, if we take B_1 as a base of the matrix A, namely

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\},$$

then any combination of the column vectors would create an independent set. To show that B_1 is a linearly independent set of vectors, and we can take the set of vectors, and designate them as the column vectors in the matrix, b_1,

$$b_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now we will take the column vectors as the set of vectors in B_2.

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\},$$

The following examples are subsets of B_2.

$$c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0,$$

$$b_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0.$$

In these two examples, we find that $b_1 = b_2 = 0$ and $c_1 = c_2 = c_3 = 0$ are the only solutions. Therefore, both subsets are linearly independent by Definition 4.3.

To demonstrate $I(i)$ and $I(ii)$, we can take two independent sets of the matrix A to be K and L,

$$K = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\},$$

$$L = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\},$$

so that $|K| > |L|$. From our previous discussion of linear independence in a set of vectors, we can see that any subset of K or L are linearly independent.

The inequality stated in $I(ii)$ ensures that the dimension of the space spanned by K is greater than the dimension of the space spanned by L, which makes it impossible to add an element from K to L. For example, given three vectors that span a space, we can extend a different set of two vectors which span a plane to a set of three vectors which spans a space.

Now we can demonstrate the exchange property by noticing that K and L share a common element,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix};$$

which means that we must choose one vector from the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\},$$

to add to L, so that L is independent. We can see the linear independence in the sets;

$$L_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} and$$

$$L_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

5. Cycles

In this section, we will continue our discussion of matroids by introducing a new definition of a matroid in terms of cycles. We will take a cycle of a matroid, M, to be a minimally dependent subset of elements in M. The following theorem is from Robin Wilson's book, Introduction to Graph Theory

Definition 5.1. A matroid M consists of a non-empty finite set E, and a collection C of non-empty subsets of E (called cycles) satisfying the following properties:[4]

$C(i)$ no cycle properly contains another cycle;
$C(ii)$ if C_1 and C_2 are two distinct cycles each containing an element e, then there exists a cycle in $C_1 \cup C_2$ that does not contain $\{e\}$

Now we can connect a cycle to the concepts introduced in the previous sections. Let A be a cycle. $A - \{e\}$ is in some base for all $e \in A$, which implies that $r(A) = |A| - 1$, and $r(B) = |B|$ for all $B \subset A$. Therefore, A is minimally dependent, which means that if we take any element from A, then the remaining set is linearly independent.

5.1. **Cycles in Graph Theory.** A cycle is a minimally dependent set, which means that any element can be removed from the set, and the set will become independent. This property can be seen in Figures 16 and 17. The graph in Figure 7 shows a set that is dependent, but which is not minimally dependent. There exists an element, $\{b\}$, which can be removed while a cycle still exists in the set.

We can also define a cycle in graph theory in terms of a path, and so we will take a cycle of a matroid, M, to be a closed path of G containing at least one edge.[4] The definition of a closed path is given in Definition 1.5.

The cycles of the graph G in Figure 1 are provided in the table 5.1.

Cycles
$\{a, b, c, d, e\}$
$\{a, e, f\}$
$\{a, e, d, g\}$
$\{d, f, g\}$
$\{b, c, g\}$
$\{b, c, d, f\}$

Table 5.1

Two graphic examples of cycles found in Figure 1 are in the Figures 16 and 17.

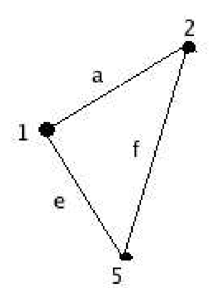

FIGURE 16. The Cycle, C_1

We can see that the property $C(i)$ holds by observing the table of cycles in G.

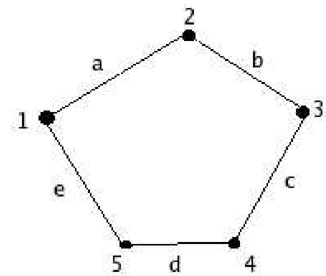

FIGURE 17. The Cycle, C_2

Using our examples of the cycles C_1 and C_2, we can see that the two cycles each contain the elements $\{a\}$ and $\{e\}$. Figure 18 shows the graph of $C_3 = C_1 \cup C_2$. We can see there are three cycles in C_3, which are $\{a, e, f\}$, $\{a, b, c, d, e\}$, and $\{b, c, d, f\}$.

The cycle $\{b, c, d, f\}$ in Figure 19 is a cycle in C_3 which contains neither $\{a\}$ nor $\{e\}$. Therefore, the property $C(ii)$ holds in this case.

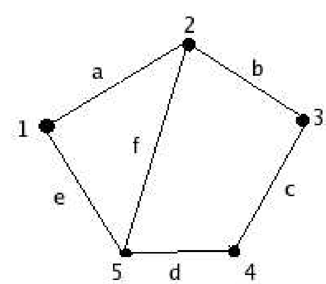

FIGURE 18. The Cycles, C_3

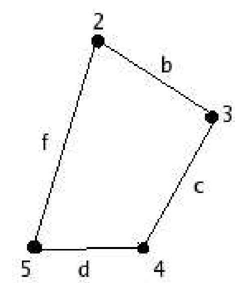

FIGURE 19. The Cycle, C_4

5.2. **Cycles in Linear Algebra.** We will take the cycles of a matrix to be a set of minimally dependent column vectors. To show examples of cycles in linear algebra, we can take L_1 and L_2 to be two cycles in the matrix, A.

$$L_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \textit{and}$$

$$L_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

We will see L_1 and L_2 are a minimally dependent set of column vectors.

$$d_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d_4 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0,$$

$$e_1 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + e_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + e_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + e_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = 0.$$

For L_1, we can let $d_1 = d_2 = d_4 = 1$ and $d_3 = -2$, so that the column vectors of L_1 add to the zero vector. For L_2, we can let $e_1 = e_2 = e_4 = 1$ and $e_3 = -2$, so that the column vectors of L_2 add to the zero vector. By Definition 4.3, L_1 and L_2 are linearly dependent. Notice that if you remove any column vector from L_1 or L_2, then the set is linearly independent. Therefore, L_1 and L_2 are cycles.

To demonstrate $C(ii)$, we will let $L_3 = L_1 \cup L_2$.

$$L_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

The common column vector of L_1 and L_2 is

$$f = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, there is a cycle contained in L_3 that does not contain f, namely the set of column vectors

$$L_4 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Therefore, the property $C(ii)$ is satisfied for this example.

6. Vertex-Edge Incidence Matrix

Thus far, we have seen how both graphs and matrices can be viewed as matroids. Now we will link graph theory and linear algebra by translating a graph to a unique matrix, and vice versa, using the language of matroids to motivate our discussion. The vertex-edge incidence matrix demonstrates the relationship between a matrix and a graph. The following is a formal definition of a vertex-edge incidence matrix given by James Oxley.

Theorem 6.1. *Let G be a graph and A_G be its vertex-edge incidence matrix. When the entries of A_G are viewed modulo(2), its vector matroid $M[A_G]$ has as its independent sets all subsets of $E(G)$ that do not contain the edges of a cycle. Thus $M[A_G] = M(G)$ and every graphic matroid is binary.* [2]

The idea of a matrix being viewed mod(2), means that the entries of the matrix are either 0 or 1. Since the vertex-edge incidence matrix represents a graph, we call the graph binary. Because we have shown that a graph can be viewed as a matroid, we can say that the matroid is also binary.

The following is an example of a vertex-edge incidence matrix using the graph in Figure 20. If an edge and a vertex are incident on a graph, then the corresponding entry in the matrix is 1. Otherwise, if an edge and vertex are not incident, then the corresponding entry in the matrix is zero.

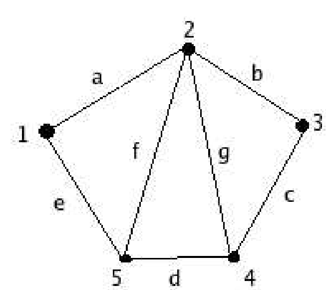

Figure 20. Graph, G

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

We can see the relationship between graphs and matrices in the vertex-edge incidence matrix if we use the set $\{a, e, f\}$ as our example. We can see that the rank of the set $\{a, e, f\}$ is 2, because any subset, containing two elements, does not contain a cycle. In the graph in Figure 20, we can see that the set $\{a, e, f\}$ is a cycle. The sum of the column vectors corresponding to the set of edges in our example is,

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can also see that this set of vectors is minimally dependent. If you take any one vector from the set, the set become an independent set.

The rank of the column vectors corresponding to the set $\{a, e, f\}$ is also two, because any subset of the set of column vectors, containing two elements, does not contain a cycle.

One base of G is the set of edges $\{a, b, c, d\}$. The corresponding set of column vectors are,

$$N = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

We can see that the set of vectors are a maximal independent set, because $|N| = r(N) = 4$. Therefore, the set of vectors, N, is also a base.

Now we can see the link between graph theory and linear algebra, by using the language of matroids to motivate our discussion and to generalize the properties of independence.

References

[1] David Lay. *Linear Algebra and its Applications*. University of Maryland-College Park, 2006.

[2] James Oxley. *Matroid Theory*. Oxford University Press, New York, 1992.

[3] Hassler Whitney. *"On The Abstract Properties of Linear Independence"*. American Journal of Mathematics. Vol 57. John Hopkins University Press, 1935.

[4] Robin Wilson. *Introduction to Graph Theory*. Harlow:Pearson Education Limited, 1996.