ENUMERATION BY ALGEBRAIC COMBINATORICS

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Abstract. Pólya's theorem can be used to enumerate objects under permutation groups. Using group theory, combinatorics, and many examples, Burnside's theorem and Pólya's theorem are derived. The examples used are a hexagon, cube, and tetrahedron under their respective dihedral groups. Generalizations using more permutations and applications to graph theory and chemistry are looked at.

1. Introduction

A jeweler sells six-beaded necklaces in his shop. Given that there are two different colors of beads, how many varieties of necklaces does he need to create in order to have every possible permutation of colors?

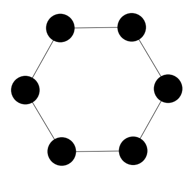


Figure 1. A six-beaded necklace.

Consider the case of three beads of each color. Mathematically, there are $\binom{6}{3} = \frac{6!}{3!3!} = 20$ ways to string three light gray beads and three dark gray beads onto a fixed necklace. However, we can rotate or reflect the three necklaces in Figure 2 to create all 20.

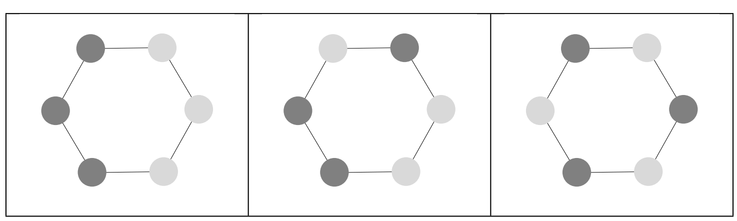


Figure 2. The set of necklaces with 3 dark beads and 3 light beads that are distinct under rotation, reflection, and inversion of the colors. In other words, these necklaces are equivalent when the colors are permuted by groups D_12 and S_2.

Given 6 beads we have 2 choices of color per bead, so there are 64 ways to color an unmovable beaded necklace. However, if we are given the freedom to rotate and reflect it, there are only 13 distinct varieties. Furthermore, if the colors are interchangeable (i.e. a completely light necklace is equivalent to a completely dark necklace), then there are only 8 distinct arrangements.

How can we get from 64 down to 13 or 8? We might guess that it has something to do with symmetry. The hexagon has 12 symmetries: rotation by 0, 60, 120, 180, 240, or 300 degrees, three reflections through opposite vertices, and three reflections through opposite sides. There is no obvious relationship between the number of possibilities for a fixed necklace, the number of symmetries, and the number of distinct varieties, but surely there is one. In this paper, we use combinatorics and group theory to work through the problem of the six-beaded necklace and others like it.

2. Burnside's Theorem

We begin by examining the number of possible n-colored necklaces and cubes under symmetry. We solve simple enumeration problems by observation, claim that the results hold for more complicated examples, then prove our claim algebraically. The result that we observe by counting cubes was popularized by William Burnside in 1911, though he quotes it from a paper published by George Frobenius in 1887 [3]. The theorem was also known to Augustin Cauchy in an obscure form by 1845. [4]

2.1. **Basic Abstract Algebra.**

First, we give abstract algebra background necessary to describe an enumeration problem solvable by Burnside's theorem.

Definition 2.1. (Binary Operation [3]) Let G be a set. A *binary operation* on G is a function that assigns each ordered pair of elements of G an element of G.

Definition 2.2. (Group [3]) Let G be a nonempty set together with a binary operation that assigns to each ordered pair (a, b) of elements of G an element in G denoted by ab. We say that G is a *group* under this operation if the following three properties are satisfied.

(1) *Associativity.* The operation is associative; that is, $(ab)c = a(bc)$ for all $a, b, c \in G$.

(2) *Identity.* There is an element e (called the *identity*) in G such that $ae = ea = a$ for all $a \in G$.

(3) *Inverses.* For each element $a \in G$, there is an element $b \in G$ (called an *inverse* of a) such that $ab = ba = e$.

Example 2.1. A simple example of a group is the integers under addition modulo 4. In this case,

$$\mathbb{Z}_4 = \{0, 1, 2, 3\}.$$

Addition mod 4 is a binary operation (the group is closed) since $0+0=0$, $0+1=1$, $0+2=2$, $0+3=3$, $1+1=2$, $1+2=3$, $1+3=0$, $2+2=0$, $2+3=1$, and $3+3=2$. We need not check the operations in the other direction since addition is commutative. The group has an identity of 0, and each element has a unique inverse: $0^{-1}=0$, $1^{-1}=3$, $2^{-1}=2$, and $3^{-1}=1$.

2.2. **6-Beaded Necklace.** We describe the symmetry of a 6-beaded necklace, or hexagon. Symmetry is the property that an object is invariant under certain transformations. The 6-beaded necklace is symmetric in 12 ways as shown in Figures 3, 4, and 5.

- The rotations of the necklace counterclockwise about its center by 0^o, 60^o, 120^o, 180^o, 240^o, or 300^o (the rotation by 0^o is the identity), denoted I, R^{60}, R^{120}, R^{180}, R^{240}, and R^{300},

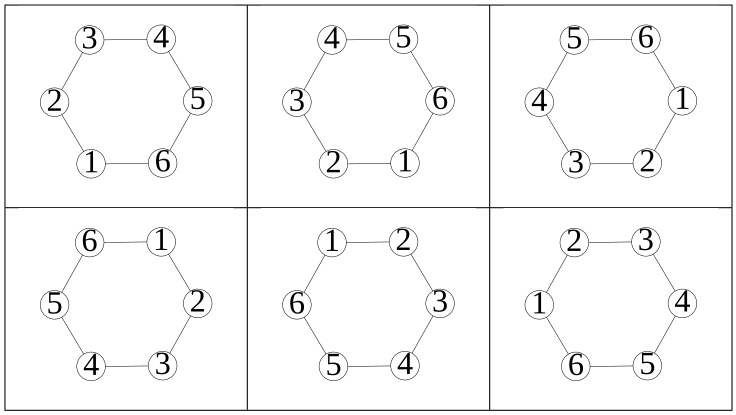


FIGURE 3. The rotations of D_12.

- The reflections across the diagonal through each of the 3 pairs of opposite vertices, denoted D_1, D_2, and D_3,

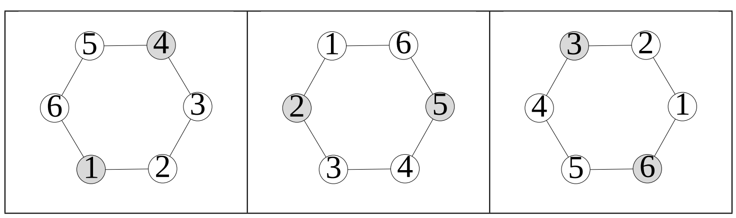


FIGURE 4. The reflections across the diagonals of D_12.

- The reflections across the side bisector for each of the 3 pairs of opposite edges, denoted E_1, E_2, and E_3.

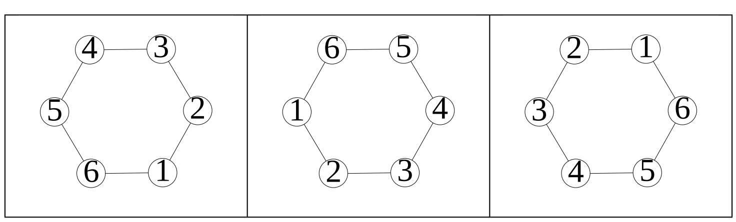


FIGURE 5. The reflections across the side bisectors of D_12.

D_12	I	R^{60}	R^{120}	R^{180}	R^{240}	R^{300}	D_1	D_2	D_3	E_1	E_2	E_3
I	I	R^{60}	R^{120}	R^{180}	R^{240}	R^{300}	D_1	D_2	D_3	E_1	E_2	E_3
R^{60}	R^{60}	R^{120}	R^{180}	R^{240}	R^{300}	I	E_2	E_3	E_1	D_1	D_2	D_3
R^{120}	R^{120}	R^{180}	R^{240}	R^{300}	I	R^{60}	D_2	D_3	D_1	E_2	E_3	E_1
R^{180}	R^{180}	R^{240}	R^{300}	I	R^{60}	R^{120}	E_3	E_1	E_2	D_2	D_3	D_1
R^{240}	R^{240}	R^{300}	I	R^{60}	R^{120}	R^{180}	D_3	D_1	D_2	E_3	E_1	E_2
R^{300}	R^{300}	I	R^{60}	R^{120}	R^{180}	R^{240}	E_1	E_2	E_3	D_3	D_1	D_2
D_1	D_1	E_1	D_3	E_3	D_2	E_2	I	R^{240}	R^{120}	R^{60}	R^{300}	R^{180}
D_2	D_2	E_2	D_1	E_1	D_3	E_3	R^{120}	I	R^{240}	R^{180}	R^{60}	R^{300}
D_3	D_3	E_3	D_2	E_2	D_1	E_1	R^{240}	R^{120}	I	R^{300}	R^{180}	R^{60}
E_1	E_1	D_3	E_3	D_2	E_2	D_1	R^{300}	R^{180}	R^{60}	I	R^{240}	R^{120}
E_2	E_2	D_1	E_1	D_3	E_3	D_2	R^{60}	R^{300}	R^{180}	R^{120}	I	R^{240}
E_3	E_3	D_2	E_2	D_1	E_1	D_3	R^{180}	R^{60}	R^{300}	R^{240}	R^{120}	I

TABLE 1. The Cayley Table of D_12. Each entry in the table represents an operation in row composed with an operation in a column.

We claim that these transformations form a group, called the dihedral group of order 12, denoted D_12. From the completeness of Table 1, the Cayley table of D_12, D_12 is closed under function composition. Notice that the element I is the identity of D_12 since every element composed with I is equal to itself. Also, every row of the table contains one I, so Theorem **??** holds, or each element has an inverse. Since all of the properties of Definition 2.2 hold, the symmetric transformations of a 6-beaded necklace does indeed form a group.

Next we examine the symmetric transformations on the numbered necklaces and notice that not every group operation affects every coloring of every necklace. For example, consider R^{120}. If we start with the pattern (123456) and apply R^{120}, we get (345612), then applying R^{120} again, we get (561234). If we continue to rotate by 120^o, the pattern repeats. Therefore, this coloring remains the same under R^{120} as long as beads (135) are the same color and beads (246) are the same color. If our necklace can contain two colors of beads, then we say that we have two choices of color for each set of fixed beads, so we have 2^2 possible arrangements fixed under R^{120}.

The colorings fixed under each transformation are described in Table 2. Notice that the number of the total choices under all symmetries of two colored necklaces is 156. When we divide 156 by 12 (the number of symmetries), we get 13. By observation, there are 13 varieties of 2-colored necklaces that are different when the symmetries are applied: three choices where there are 3 beads of each color as shown in Figure 2 and ten more choices shown in Figure 6. In contrast, there are $2^6 = 64$ possibilities for necklaces upon which we cannot apply a symmetric transformation.

2.3. **Cube.** The next question that we tackle is,"given a cube that is painted dark on some faces and light on the others, keeping symmetries in mind, how many different possible varieties are there?" To answer this

Transformation ϕ in D_12	Colorings Fixed ($fix(\phi)$)	Number ($\lvert fix(\phi)\rvert$)
Identity	All 6 separately	$2^6 = 64$
Rotation of 60^o	All 6 as a unit	$2^1 = 2$
Rotation of 120^o	Beads 1,3,5; Beads 2,4,6	$2^2 = 4$
Rotation of 180^o	Beads 1,4; Beads 2,5; Beads 5,6	$2^3 = 8$
Rotation of 240^o	Beads 1,3,5; Beads 2,4,6	$2^2 = 4$
Rotation of 300^o	All 6 as a unit	$2^1 = 2$
Reflection across diagonal 1	Bead 1; Beads 2,6; Beads 3,5; Bead 4	$2^3 = 8$
Reflection across diagonal 2	Bead 2; Beads 1,3; Beads 4,6; Bead 5	$2^3 = 8$
Reflection across diagonal 3	Bead 3; Beads 2,4; Beads 1,5; Bead 6	$2^3 = 8$
Reflection across side bisector 1	Beads 1,6; Beads 2,5; Beads 3,4	$2^4 = 16$
Reflection across side bisector 2	Beads 1,2; Beads 3,6; Beads 4,5	$2^4 = 16$
Reflection across side bisector 3	Beads 1,4; Beads 2,3; Beads 5,6	$2^4 = 16$
	Total Sum	156

TABLE 2. The elements of D_{12}, or the symmetries of a 6-beaded necklace, and the number of colorings fixed by each element of D_{12}.

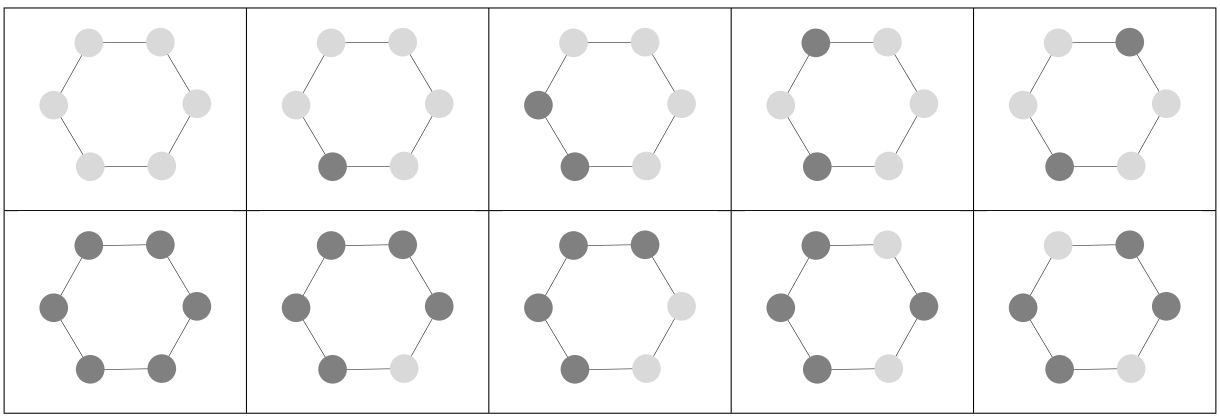


FIGURE 6. The set of necklaces with 0, 1, or 2 beads of one color.

question, we define a possibility as the number of ways to color a fixed cube and a variety as the number of ways to color a cube that is free to rotate along its axes of symmetry. We create models of all the varieties and then explain the results mathematically. The results, by observation, are summarized in Table 3.

Given that a cube has six faces and two possible colors for each face, it is not a surprise that there are $2^6 = 64$ possibilities. How do we apply some function to the cube and get from 64 to 10?

A cube has 24 unique rotational operations about its lines of symmetry as shown in Figures 7, 8, 9, and 10.

- I

  The first is the identity operation, or any rotation of zero degrees.

- $F^{90}_{1,2,3}$, $F^{180}_{1,2,3}$, $F^{270}_{1,2,3}$

  Next are rotations of 90, 180, and 270 degrees about the three lines of symmetry that go from the center of one face to the center of the opposite face.

Light Gray	Dark Gray	Possibilities	Varieties
6	0	$\binom{6}{0} = 1$	1
5	1	$\binom{6}{1} = 6$	1
4	2	$\binom{6}{2} = 15$	2
3	3	$\binom{6}{3} = 20$	2
2	4	$\binom{6}{4} = 15$	2
1	5	$\binom{6}{5} = 6$	1
0	6	$\binom{6}{6} = 1$	1
	Total Sum	64	10

TABLE 3. A comparison of the different possibilities and varieties of a 2-colored cube.

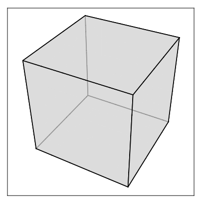


FIGURE 7. The identity of D_{24}, also written as I or R^0.

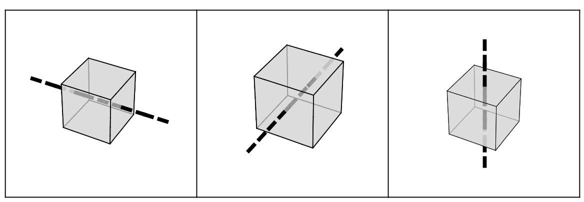


FIGURE 8. The rotations through the centers of the faces of D_{24}: F_1, F_2, F_3.

- $E^{180}_{1,2,3,4,5,6}$

  After that, there are rotations of 180 degrees about the six lines of symmetry that go from the center of one edge to the center of the diagonal edge.

- $C^{120}_{1,2,3,4}$, $C^{240}_{1,2,3,4}$

  Finally, there are four rotations of 120 and 240 degrees about the lines of symmetry from a corner to the diagonal corner.

Definition 2.3. (Multiplication Principle [1]) *Let S be a set of ordered pairs (a, b) of objects, where the first object, a, comes from a set of size p and for each choice of object a, there are q choices for object b. Then the size of S is p times q: $|S| = p \cdot q$.*

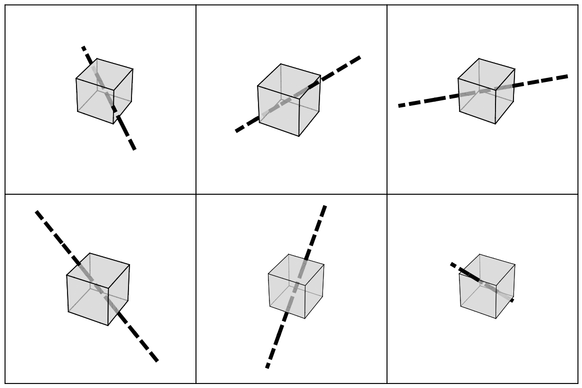


FIGURE 9. The rotations through the edges of D_{24}: E_1, E_2, E_3, E_4, E_5, E_6.

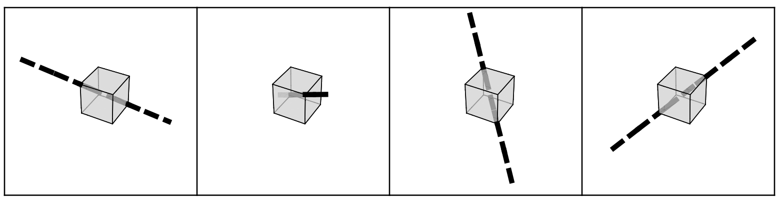


FIGURE 10. The rotations through the corner to corner diagonals of D_{24}: C_1, C_2, C_3, C_4.

There are 24 total unique symmetries, or possible orientations, of a cube. We can count these orientations in several ways. If we consider the cube as a unit cube in the first octant with a vertex at the origin, there 8 possible vertices that can rest at $(0, 0, 0)$. For each vertex at the origin, there are 3 vertices that can be at $(0, 0, 1)$. By the multiplication principle, there are 24 orientations. Similarly, there are 6 faces that can rest on the xy-plane and for each such face there can be 4 faces on the yz-plane, so there are 24 symmetries as we expect. These symmetries form the dihedral group D_{24} under the operation of function composition.

Transformations ϕ in D_{24}	Possibilities Fixed ($fix(\phi)$)	Number ($\lvert fix(\phi)\rvert$)
I	All 6 separately	$2^6 = 64$
F_1^{90}	left, right, 4 other sides	$2^3 = 8$
F_2^{90}	front, back, 4 other sides	$2^3 = 8$
F_3^{90}	top, bottom, 4 other sides	$2^3 = 8$
F_1^{180}	left, right, 2 opp. sides, 2 opp. sides	$2^4 = 16$
F_2^{180}	front, back, 2 opp. sides, 2 opp. sides	$2^4 = 16$
F_3^{180}	top, bottom, 2 opp. sides, 2 opp. sides	$2^4 = 16$
F_1^{270}	left, right, 4 other sides	$2^3 = 8$
F_2^{270}	front, back, 4 other sides	$2^3 = 8$
F_3^{270}	top, bottom, 4 other sides	$2^3 = 8$
$E_{1,2,3,4,5,6}^{180}$	top and bottom, front and left, back and right	$6 \cdot 2^3 = 6 \cdot 8 = 48$
$C_{1,2,3,4}^{120}$	3 sides w/ common vertex, other 3 sides	$4 \cdot 2^2 = 4 \cdot 4 = 16$
$C_{1,2,3,4}^{240}$	3 sides w/ common vertex, other 3 sides	$4 \cdot 2^2 = 4 \cdot 4 = 16$
	Total Sum	240

TABLE 4. The number of possibilities fixed under each operation in a cube, D_{24}.

Next, we will examine the number of sides fixed by each transformation. These are detailed in Table 4. We can see that for a cube, the sum over all elements of D_{24} of the number of possible colorings fixed by each transformation is equal to the size of D_{24} times the number of varieties. In this case, $240 = 24 \cdot 10$. From this data and that from the 6-beaded necklace, we claim that the number of varieties is equal to the order of the dihedral group times the total number of choices of sets of sides fixed by each transformation.

2.4. **The Proof of Burnside's Theorem.** Now, we prove our conjecture after introducing more math.

Definition 2.4. (Equivalence Relation [3]) An *equivalence relation* on a set S is a set R of ordered pairs of elements of S such that

(1) $(a, a) \in R$ for all $a \in S$ (reflexive property)
(2) $(a, b) \in R$ implies $(b, a) \in R$ (symmetric property)
(3) $(a, b) \in R$ and $(b, c) \in R$ imply $(a, c) \in R$ (transitive property)

Definition 2.5. (Partition [3]) A *partition* of a set S is a collection of nonempty disjoint subsets of S whose union is S.

Theorem 2.1. (Equivalence Classes Partition [3]) *The equivalence classes of an equivalence relation on a set S constitute a partition of S. Conversely, for any partition P of S, there is an equivalence relation on S whose equivalence classes are the elements of P.*

Definition 2.6. (Subgroup [3]) If a subset H of a group G is itself a group under the operation of G, we say that H is a *subgroup* of G, denoted $H \leq G$.

Theorem 2.2. (Subgroup Test) *Let G be a group and H a nonempty subset of G. Then, H is a subgroup of G if and only if $ab \in H$ whenever $a, b \in H$, and $a^{-1} \in H$ whenever $a \in H$.* [1]

Proof. If H is a subgroup of G, then H is nonempty since H contains the identity. Whenever $a, b \in H$, $ab \in H$ and $a^{-1} \in H$ since H contains the identity of G and H is closed under multiplication.

Conversely, if H is a nonempty subset of G such that whenever $a, b \in H$, $ab \in H$ and $a^{-1} \in H$ we must show that $H \leq G$. Let a be any element in H. Let $a = b$ and deduce that $e = aa^{-1} \in H$, so H contains the identity of G. Then, since H contains e and a, H contains the element $ea^{-1} = a^{-1} \in H$ and H is closed under taking inverses. If a and b are any two elements of H, then H contains a and b^{-1}, so H contains $a(b^{-1})^{-1} = ab$. Hence H is closed under multiplication, which proves H is a subgroup of G. $\square$

[1] Gallian 61, Dummit 48

Example 2.2. Consider the group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ and its subset $\{0, 2\}$. We will show that $\{0, 2\} \leq \mathbb{Z}_4$ using Theorem 2.2. First, we check for closure under addition: $0 + 0 = 0$, $0 + 2 = 2$, and $2 + 2 = 0$. Next, we check for closure under taking inverses: $0^{-1} = 0$ and $2^{-1} = 2$. Therefore $\{0, 2\}$ is indeed a subgroup of $\mathbb{Z}_4$.

Definition 2.7. (Coset of H in G [3]) Let G be a group and H a subgroup of G. For any $a \in G$, the set $\{ah$ such that $h \in H\}$, denoted by aH, is called the *left coset of H in G containing a.*

Lemma 2.2.1. (Properties of Cosets [3]) *Let H be a subgroup of G, and let a and b belong to G. Then,*

(1) $a \in aH$,

(2) $aH = H$ *if and only if* $a \in H$,

(3) $aH = bH$ *or* $aH \cap bH = \emptyset$,

(4) $aH = bH$ *if and only if* $a^{-1}b \in H$,

(5) $|aH| = |bH|$ *if H is finite,*

(6) $aH = Ha$ *if and only if* $H = aHa^{-1}$,

(7) aH *is a subgroup of G if and only if* $a \in H$.

Theorem 2.3. (Lagrange's Theorem [3]) *If G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$. Moreover, the number of distince left (right) cosets of H in G is $\frac{|G|}{|H|}$.*

Proof. Let $a_1H, a_2H, ..., a_rH$ denote the distinct left cosets of H in G. Then, for each $a \in G$, we have $aH = a_iH$ for some i. By property 1 of cosets, $a \in aH$. Thus, each member of G belongs to one of the cosets a_iH. In symbols, $G = a_1H \cup ... \cup a_rH$. Now, property 3 of cosets shows that this union is disjoint, so $|G| = |a_1H| + |a_2H| + ... + |a_rH|$. Finally, since $|a_iH| = |H|$ for each i, we have $|G| = r|H|$. $\square$

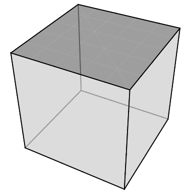
Definition 2.8. (Permutation of a Set S [3]) A *permutation* of a set S is a function from S to S that is both one-to-one and onto.

Definition 2.9. (Permutation Group of S [3]) A *permutation group of a set S* is a set of permutations of S that forms a group under function composition.

Definition 2.10. (Stabilizer of a Point [3]) Let G be a group of permutations of a set S. For each $s \in S$, let $stab_G(s) = \{g \in G$ such that $g(s) = s\}$. We call $stab_G(s)$ the *stabilizer* of $s \in G$.

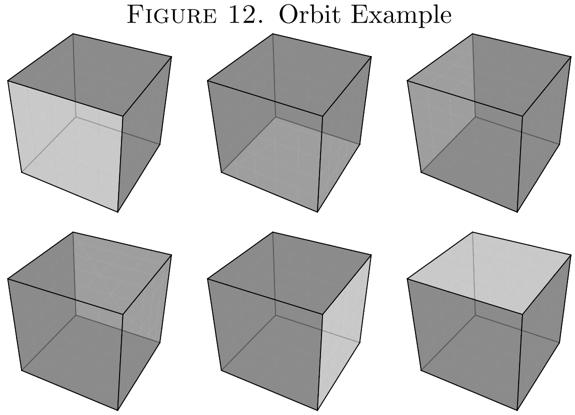
Example 2.3. The stabilizer of the cube in Figure 2.4 consists of I, F_1^{90}, F_1^{180}, and F_1^{270}. These are the rotations about the vertical line of symmetry through the middle of the top and bottom faces.

Figure 11. Stabilizer Example



Definition 2.11. (Orbit of a Point [3]) Let G be a group of permutations of a set S. For each $s \in S$, set $orb_G(s) = \{g(s) \text{ such that } g \in G\}$. The set $orb_G(s)$ is a subset of S called the *orbit* of s under G.

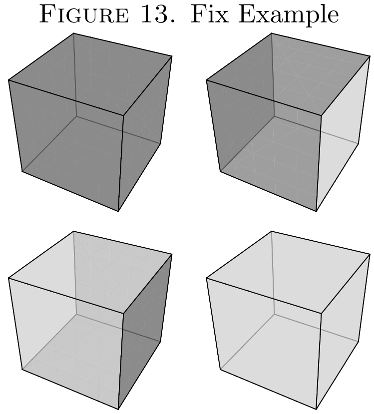
Figure 12. Orbit Example



Example 2.4. The orbit of the cube in Figure 2.4 consists of all orientations of a cube with one light side.

Definition 2.12. (Elements Fixed by g [3]) For any group G of permutations on a set S and any $g \in G$, we let $fix(g) = \{s \in S \text{ such that } g(s) = s\}$. This set is called the *fix* of g.

Figure 13. Fix Example



Example 2.5. The cubes fixed under C_1, corner-corner diagonal symmetry are pictured in Figure 2.4.

Lemma 2.3.1. *(*Stabilizer is a Subgroup *[3]) The stabilizer of s in G is a subgroup of G.*

Proof. Let $s \in S$ and define $stab_G(s) = \{g \in G$ such that $g(s) = s\}$. The identity element of G, e, is in $stab_G(s)$ since $e(s) = s$. Let $g_1, g_2 \in stab_G(s)$. Then

$$g_1 \circ g_2(s) = g_1(g_2(s)) = g_1(s) = s$$

and $stab_G(s)$ is closed under function composition.

Let $g \in stab_G(s)$ and consider g^{-1}. Given that $e(s) = s$,

$$g^{-1} \circ g(s) = g^{-1}(s) = e(s) = s = g(s) = g \circ g^{-1}(s)$$

so $stab_G(s)$ is closed under inverses. $\square$

Lemma 2.3.2. *(*Orbits Partition *[3]) Given that G is a group of permutations of a set S, the orbits of the members of S constitute a partition of S.*

Proof. To prove that the orbits of the members of S constitute a partition of S, we must show that the orbits are equivalence classes of the set S. Let $s_1, s_2, s_3 \in S$ and $g_1, g_2, g_3 \in G$. Certainly, $s_1 \in orb_G(s_1)$. Now suppose $s_3 \in orb_G(s_1) \cap orb_G(s_2)$. Then $s_3 = g_1(s_1)$ and $s_3 = g_2(s_2)$, and therefore $(g_2^{-1}(g_1(s_1)) = s_2$. So, if $s_3 \in orb_G(s_2)$, then $s_3 = g_3(s_2) = (g_3(g_2^{-1}(g_1(s_1)))$ for some g_3. This proves $orb_G(s_2) \subseteq orb_G(s_1)$. By symmetry, $orb_G(s_1) \subseteq orb_G(s_2)$. $\square$

Theorem 2.4. (Orbit-Stabilizer Theorem [3]) *Let G be a finite group of permutations of a set S. Then, for any $s \in S$, $|G| = |orb_G(s)| \cdot |stab_G(s)|$.*

Proof. By Lagrange's Theorem (Theorem 2.3), $\frac{|G|}{|stab_G(s)|}$ is the number of distinct left (right) cosets of $stab_G(s)$ in G. Thus, it suffices to establish a one-to-one correspondence between the left cosets of $stab_G(s)$ and the elements in the orbit of s. To do this, we define a correspondence T by mapping the coset $g_1 \cdot stab_G(s)$ to $g_1(s)$ under T. To show that T is a well-defined function, we must show that $g_1 \cdot stab_G(s) = g_2 \cdot stab_G(s)$ implies that $g_1(s) = g_2(s)$. But $g_1 \cdot stab_G(s) = g_2 \cdot stab_G(s)$ implies $g_1^{-1}g_2 \in stab_G(s)$, so that $(g_1^{-1}g_2)(s) = s$ and, therefore, $g_2(i) = g_1(i)$. Reversing the argument from the last step to the first step shows that T is also one-to-one. We conclude the proof by showing that T is onto $orb_G(s)$. Let $s_2 \in orb_G(s_1)$. Then $g_1(s_1) = s_2$ for some $g_1 \in G$ and $T(g_1 \cdot stab_G(s_1)) = g_1(s_1) = s_2$, so T is onto. $\square$

Theorem 2.5. (Burnside's Theorem [3]) *If G is a finite group of permutations on a set S, then the number of distinct orbits of G on S is*

$$\frac{1}{|G|}\sum_{g\in G}|fix(g)|. \tag{1}$$

Proof. Let n be the number of pairs (g,s) with $g\in G$ and $s\in S$ such that $g(s)=s$. We can count the pairs in two ways. First, $n=\sum_{g\in G}|fix(g)|$. Second, $n=\sum_{s\in S}|stab_G(s)|$. Therefore,

$$\sum_{g\in G}|fix(g)|=\sum_{s\in S}|stab_G(s)|.$$

Given that the orbits of the members of S constitute a partition of S (Lemma 2.3.2), if s_1 and s_2 are in the same orbit, $orb_G(s_1)=orb_G(s_2)$ and $|stab_G(s_1)|=|stab_G(s_2)|$. Therefore, by the Orbit-Stabilizer Theorem (Theorem 2.4),

$$\sum_{s_2\in orb_G(s_1)}|stab_G(s_2)|=|orb_G(s_1)|\cdot|stab_G(s_1)|=|G|.$$

Finally, we can take the sum over all of the elements of G to get

$$\sum_{g\in G}|fix(g)|=\sum_{s\in S}|stab_G(s)|=|G|\cdot(\text{number of orbits}),$$

and it follows that

$$\frac{1}{|G|}\sum_{\phi\in G}|fix(\phi)|=(\text{number of distinct orbits}).$$

□

Example 2.6. (Necklace) Let S be the set of all possible color arrangements s, such that for a two-colored six-beaded necklace, $|S|=2^6=64$. Let G be the dihedral group D_12. The group of symmetries $g\in G$ which keep a particular coloring the same is $stab_G(s)$. The $orb_G(s)$ is the set of colorings $s\in S$ that can be created by applying a symmetry $g\in G$. The set of colorings $s\in S$ that remain fixed under a particular $g\in G$ are in the set $fix(g)$. The values for $|fix(g)|$ can be found in Table 2. Therefore, according to Theorem 2.5,

$$(\text{number of varieties})=\frac{1}{|G|}\sum_{g\in G}|fix(g)|=\frac{1}{12}\cdot 156=13$$

as we observe in Section 2.3.

Example 2.7. (Cube) Let S be the set of all possible color arrangements s, such that for a two-colored cube, $|S|=2^6=64$. Let the permutation group G be D_{24}. The subgroup of G which keep a particular coloring the same is $stab_G(s)$. The set of colorings in S that can be created from a particular $s\in S$ by applying the permutation $g \in G$ is $orb_G(s)$. The $fix(g)$ is the set of colorings in S that are invariant under $g \in G$. According to Burnside's theorem (Theorem 2.5), The number of orbits of G on S, or the number of colorings distinct under G is

$$\frac{1}{24} \sum_{g \in G} |fix(g)|.$$

The values for $|fix(g)|$ can be found in Table 4, so $\sum_{g \in G} |fix(g)| = 240$. The result is that there are 10 distinct orbits, or varieties, for a two-colored cube.

2.5. **Tetrahedron.** Burnside's theorem does not place a restriction on the number of color choices. For example, examine a tetrahedron colored with k colors using Burnside's theorem (Theorem 2.5). A tetrahedron has four faces so our set $S = \{\text{bottom, left, right, front}\}$ and $|S| = 4$. Let their be k possible colors for each face. Therefore, we count k^4 possible colorings for an immobile tetrahedron. A tetrahedron has 12 different symmetries, so $|G| = 12$ and the group G contains the following transformations.

- I

  The first is the identity operation, or any rotation of zero degrees.
- $V^{120}_{1,2,3,4}$, $V^{240}_{1,2,3,4}$

  Next are rotations of 120 and 240 degrees about the 4 lines of symmetry from each vertex to the center of the opposite side.
- $E^{180}_{1,2,3}$

  Finally, there are rotations of 180 degrees about the three lines of symmetry that go from the center of one edge to the center of the center of the opposite edge.

The $fix(g)$ for each $g \in G$ are represented in Table 5. Substituting these numbers into Theorem 2.5, we get

$$\frac{1}{12} \cdot (k^4 + 11k^2) = (\text{number of distinct k-colored tetrahedrons under } G).$$

If we substitute in $k = 2$ to calculate the number of 2-colored tetrahedrons, we get

$$\frac{1}{12} \cdot (2^4 + 11 \cdot 2^2) = \frac{1}{12} \cdot 60 = 5.$$

We can verify that there are indeed five varieties of two-colored tetrahedrons by observation.

3. Pólya's Theorem

3.1. **Motivation.** In the cube counting problem (Example 2.7), Burnside's theorem (Theorem 2.5) allows us to calculate the number of varieties of two-colored cubes under the group D_{24}, but it does not tell us much about those varieties. Let the colors of the cube be denoted L for the light colored faces and D for

g	$fix(g)$	$\lvert fix(g)\rvert$
I	All 4 separately	k^4
$V^{120}_{1,2,3,4}$	bottom, other three sides	$4 \cdot k^2$
$V^{240}_{1,2,3,4}$	bottom, other three sides	$4 \cdot k^2$
$E^{180}_{1,2,3}$	top and left, bottom and right	$3 \cdot k^2$
	Total Sum	$k^4 + 11k^2$

TABLE 5. The number of possible colorings fixed under each symmetry of a k-colored tetrahedron.

the dark colored faces. We generate a polynomial for some $g \in G$ such that each grouping of sides in $fix(g)$ is represented by the factor $(L^a + D^a)$ where a is the number of sides in the grouping.

For example, the identity element of G is represented by the polynomial $(L + D)^6$ since each face can be colored separately and remain invariant under I for each of the 6 sides that complete our cube. Next, consider F_3^{180}. In this case, the top of the cube can be any color, the bottom can be any color, the front and back must be the same color, and the left and right sides must be the same color. The top and the bottom are both represented by $(L+D)$. The set containing the front and back is represented by (L^2+D^2). We take the product of all four terms, so our polynomial representing F_3^{180} is $(L + D)^2(L^2 + D^2)^2$. The polynomials for the other terms are represented in Table 6.

If we take the sum of all of the polynomials (each polynomial represents an element of G) for a two-colored cube, we get

$$24(L^6 + L^5D + 2L^4D^2 + 2L^3D^3 + 2L^2D^4 + LD^5 + D^6). \tag{2}$$

Notice that in Equation 2 that after factoring out $|G|$ (24 in this case), the coefficients of the terms are informative. The coefficient of $LD^5 = 1$ and there is exactly 1 variety with 1 light side and 5 dark sides. The coefficient of $L^3D^3 = 2$ and there are exactly two varieties of cubes with 3 light sides and 3 dark sides. We did not start with any more information than we did for Burnside's theorem (Theorem 2.5), but we learn more about the colorings using polynomials.

$\phi \in D_{24}$	Polynomial	Expansion
I	$(L + D)^6$	$L^6 + 6L^5D + 15L^4D^2 + 20L^3D^3 + 15L^2D^4 + 6LD^5 + D^6$
$F^{90}_{1,2,3}$	$3 \cdot (L + D)^2(L^4 + W^4)$	$3 \cdot (L^6 + 2L^5D + L^4D^2 + L^2D^4 + 2LD^5 + D^6)$
$F^{180}_{1,2,3}$	$3 \cdot (L + D)^2(L^2 + D^2)^2$	$3 \cdot (L^6 + 2L^5D + 3L^4D^2 + 4L^3D^3 + 3L^2D^4 + 2LD^5 + D^6)$
$F^{270}_{1,2,3}$	$3 \cdot (L + D)^2(L^4 + W^4)$	$3 \cdot (L^6 + 2L^5D + L^4D^2 + L^2D^4 + 2LD^5 + D^6)$
$E^{180}_{1,2,3,4,5,6}$	$6 \cdot (L^2 + D^2)^3$	$6 \cdot (L^6 + 3L^4D^2 + 3L^2D^4 + D^6)$
$C^{120}_{1,2,3,4}$	$4 \cdot (L^3 + D^3)^2$	$4 \cdot (L^6 + 2L^3D^3 + D^6)$
$C^{240}_{1,2,3,4}$	$4 \cdot (L^3 + D^3)^2$	$4 \cdot (L^6 + 2L^3D^3 + D^6)$
Sum	$24(L^6 + L^5D + 2L^4D^2 + 2L^3D^3 + 2L^2D^4 + LD^5 + D^6)$	

TABLE 6. The polynomials for each symmetry of a 2-colored cube in D_{24}.

3.2. **Cycle Index of a Permutation Group.** We begin our build up to Pólya's theorem with more discussion of permutations. The cycle index of a permutation group is a polynomial used in Pólya's theorem and many of its extensions.

Theorem 3.1. *(*Products of Disjoint Cycles [3]*) Every permutation of a finite set can be written as a cycle or as a product of disjoint cycles.*

Proof. Let S be a finite set where $|S| = m$ and let g be a permutation of S. To write g in disjoint cycle form, choose any $s_1 \in S$ and let $s_2 = g(s_1)$, $s_3 = g(g(s_1)) = g^2(s_1)$, ... until $s_1 = \pi^l(s_1)$ for some l. We know that such an l exists because the sequence must be finite. Say $g^i(s_1) = g^j(s_1)$ for some i and j with $i < j$. Then $s_1 = g^l(s_1)$, where $l = j - i$. We express this relationship among s_1, s_2, ..., s_l as $g = (s_1, s_2, ..., s_l)$.... It is possible that we have not exhausted the set S in this process. So, we pick any $t_1 \in S$ not appearing in the first cycle and generate a new cycle. That is, we let $t_2 = g(t_1)$ and so on until we reach $t_1 = g^k(t_1)$ for some k. This new cycle has no elements in common with the previously constructed cycle. For, if it did, then $g^i(s_1) = g^j(t_1)$ for some i and j. But then, $g^{i-j}(s_1) = t_1$, and therefore $t_1 = s_r$ for some r. This is a contradiction since t_1 is not contained in cycle s. Continuing this process until we run out of elements of S, our permutation will appear as $g = (s_1, s_2, ..., s_l)(t_1, ..., t_k)...(c_1, ..., c_r)$. In this way, every permutation can be written as a product of disjoint cycles. ☐

If g is a permutation of a finite set S, we can split S into disjoint *cycles*, or subsets of S cyclically permuted by g. Given a cycle of length l, the elements of the cycle are s, $g(s)$, $g^2(s)$, ..., $g^{l-1}(s)$.

Definition 3.1. (Type [2]) If permutation g splits S into b_1 cycles of length 1, b_2 cycles of length 2, etc., g is of *type* $\{b_1, b_2, b_3, ...\}$.

We can see that $b_i = 0$ for $i > |S|$. Also, $b_i = 0$ for all but at most a finite number of i's. Also, $|S| = b_1 + 2b_2 + 3b_3 + ...$

Definition 3.2. (Cycle Index [2]) If G is a permutation group, for each $g \in G$, consider the product ${x_1}^{b_1}{x_2}^{b_2} \cdots {x_m}^{b_m}$ where g is of type $\{b_1, b_2, ...\}$. The *cycle index* of G is the polynomial

$$P_G(x_1, x_2, ..., x_m) = \frac{1}{|G|}\sum_{g\in G} {x_1}^{b_1}{x_2}^{b_2} \cdots {x_m}^{b_m} \tag{3}$$

Example 3.1. Let $G = \{e\}$, or the identity permutation of S. We can see that e is of type $\{m, 0, 0, ...\}$ since it has m cycles of length 1, so its cycle index is equal to x_1^m.

Example 3.2. Let G be the permutation group generated by $(123)(456)(78)$. The each element of group G is of type $\{0, 1, 2\}$ and therefore

$$P_G(x_1, x_2, x_3) = \frac{1}{6}\sum_{g \in G} x_2 x_3^2 = x_2 x_3^2.$$

Example 3.3. [2]Let S be the set of cube faces and G be the permutation group of S. Using our previously defined notation and descriptions of the symmetries of a cube, we can visualize

- I is of type $\{6, 0, 0, ...\}$.
- $F^{180}_{1,2,3}$ are of type $\{2, 2, 0, 0, ...\}$.
- $F^{90,270}_{1,2,3}$ are of type $\{2, 0, 0, 1, 0, ...\}$.
- $E^{180}_{1,2,3,4,5,6}$ are of type $\{0, 3, 0, 0, ...\}$.
- $V^{120,240}_{1,2,3,4}$ are of type $\{0, 0, 2, 0, 0, ...\}$.

Since $|G| = 24$ and $1 + 3 + 6 + 6 + 8 = 24$, our list of symmetries is exhaustive. We calculate the cycle index of G to be

$$P_G = \frac{1}{24}(x_1^6 + 3x_1^2x_2^2 + 6x_1^2x_4 + 6x_2^3 + 8x_3^2). \tag{4}$$

The following theorems and definitions help us understand the next example: the cycle index of the Cayley representation of a general finite set.

Definition 3.3. *(Group Isomorphism) An* isomorphism ϕ *from a group* G *to a group* $\bar{G}$ *is a one-to-one and onto mapping from* G *to* $\bar{G}$ *that preserves the group operation. That is,*

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2) \text{ for all } g_1, g_2 \in G. \tag{5}$$

Theorem 3.2. *(Cayley's Theorem [3]) Every group is isomorphic to a group of permutations.*

Proof. Let G be any group. We will find a group $\bar{G}$ of permutations that is isomorphic to G. For any $g \in G$, define function $T_g : G \mapsto G$ by $T_g(x) = gx$ for all $x \in G$. To prove that T_g is a permutation on the set of elements in G, we show that $T_g : G \mapsto G$ is one-to-one and onto. For all $g \in G$, $T_g(x) = gx \in G$ by the group property of closure. Given $gx_1 = gx_2$, by cancellation $x_1 = x_2$, and thus T_g is one-to-one. Given any $x_1 \in G$, there exists an $x_2 \in G$ such that $T_g(x_2) = gx_2 = x_1$ thus T_g is onto.

For the permutation T_g, we show that that if g runs through G, then the T_g's form a permutation group $\bar{G}$ of G. First, $\bar{G}$ is nonempty since if e is the identity of G. For all $x \in G$, $x = ex = g_e(x) \in \bar{G}$. For all $g_1, g_2, g_3 \in G$, $\bar{G}$ is associative since $(T_{g_1}T_{g_2})T_{g_3}(x) = (g_1g_2)g_3x = g_1(g_2g_3)(x) = T_{g_1}(T_{g_2}T_{g_3})x$. Next we show that $\bar{G}$ is closed under multiplication and taking inverses. If g_1 and g_2 are in group G, g_1g_2 is also in G. From the way we defined T_g, we see that $T_{g_1}T_{g_2}(x) = g_1g_2x = T_{g_1g_2}(x) \in \bar{G}$. The inverse of $T_g = gx$ is $T_g^{-1}(x) = g^{-1}x$ since $T_gT_g^{-1}(x) = gg^{-1}x = ex = x$ for all $x \in G$ and therefore all $T_g \in \bar{G}$. By Theorem 2.2, $\bar{G}$ is a group.

For every $g \in G$, define $\phi(g) = T_g$. If $T_g = T_h$, then $T_g(e) = T_h(e)$ or $ge = he$. Thus, $g = h$ and ϕ is one-to-one. By the way $\bar{G}$ is constructed, ϕ is onto. For some $g_1, g_2 \in G$, $\phi(g_1g_2) = T_{g_1}T_{g_2} = \phi(g_1)\phi(g_2)$. Therefore ϕ is an isomorphism from G to $\bar{G}$. $\bar{G}$ is called the *Cayley representation* of the group G. □

Example 3.4. [2]Let S be any finite group, and let $|S| = m$. If s is a fixed element of S, then for all $x \in S$, the mapping $S \mapsto sx$ is a permutation of S. Denoting this permutation g_s, if s runs through S, the g_s's form a permutation group G. By Cayley's Theorem and its proof, G is isomorphic to S and is the Cayley representation of S. We are interested in its cycle index.

If $s \in S$, let the order of s be $|s| = k(s)$. Permutation g_s splits S into cycles of length $k(s)$, since if $s \in S$, the cycle obtained by g_s is $\{x, sx, s^2x, ..., s^{k(s)}x = x\}$. These cycles are cosets of S. It follows from Lagrange's Theorem (Theorem 2.3) that for each $k(s) \mid m$ and there are $\frac{m}{k(s)}$ cycles of length $k(s)$. Thus, the cycle index of G is

$$P_G = \frac{1}{m}\sum_{s \in S}[x_{k(s)}]^{\frac{m}{k(s)}}. \tag{6}$$

3.3. **Weighted Form of Burnside's Theorem.** Earlier, we discussed Burnside's theorem in which the elements of G are permutations. It is a trivial extension so say that now the elements of G act as permutations. That is, to each $g \in G$ we have attached a permutation of S denoted π_g. The mapping is homomorphic, or $\pi_{gg'} = \pi_g\pi_{g'}$ for all $g, g' \in G$. We say that two elements of S are equivalent, or $s_1 \sim s_2$, if there exists a $g \in G$ such that $\pi_g s_1 = s_2$. The equivalent classes of S are called *transitive sets*. We now state our weighted form of Burnside's theorem.

Theorem 3.3. (Weighted Burnside's Theorem [2]) *Given set S, group G, and permutation group π_G, the number of transitive sets equals*

$$\frac{1}{|G|}\sum_{g \in G}\psi(g), \tag{7}$$

where $|G|$ denotes the number of elements of G, and for each $g \in G$, $\psi(g)$ denotes the number of elements of S that are invariant under π_g.

This is the *weighted* version of Burnside's theorem since the different elements of G need not correspond to different permutations π. Since we are adding up all of the π_g's, some may be counted more than once.

3.4. **Functions and Patterns.** Let D, domain, and R, range, be finite sets. Let the set of all functions from D into R be denoted R^D, where the number of elements in R^D is $|R|^{|D|}$ since for each $d \in D$, we have $|R|$ choices for $f(d)$. Let G be a permutation group of D.

Definition 3.4. (Pattern [2]) Given $f_1, f_2 \in R^D$, we call f_1 and f_2 *equivalent*, or $f_1 \sim f_2$ if there exists a $g \in G$ such that $f_1(gd) = f_2(d)$ for all $d \in D$. These equivalence classes are called *patterns*.

We confirm that $f_1 g = f_2$ is indeed an equivalence relation. The relation is reflexive because $f \sim f$ by the identity permutation in G. The relation is symmetric since $f_1 \sim f_2$ then $f_2 \sim f_1$ since if $g \in G$ implies $g^{-1} \in G$. The relation is transitive since if $f_1 \sim f_2$ and $f_2 \sim f_3$, then $f_1 \sim f_3$ because G is closed under multiplication.

Example 3.5. [2]Let D be the set of six faces of a cube. Let G be the group of permutations of D by symmetry. Let R be the colors *light* and *dark*. The set of all possible colorings of a fixed cube is R^D and the number of colorings is $|R|^{|D|} = 2^6$. The number of varieties of rotatable cubes, also called transitive sets, orbits of G, or patterns, is 10. The method for arriving at these numbers has been explained in Example 2.7.

Example 3.6. [2]Let $D = \{1, 2, 3\}$, let G be the symmetric group of D (the group of all permutation), and let $R = \{x, y\}$. We can see that there are eight functions, but only four patterns as shown in Table 7. Since

Pattern	*Corresponding Functions*
x^3	$f(1) = f(2) = f(3) = x$
x^2y	$f(1) = f(2) = x, f(3) = y$; $f(1) = f(3) = x, f(2) = y$; $f(2) = f(3) = x, f(1) = y$
y^2x	$f(1) = f(2) = y, f(3) = x$; $f(1) = f(3) = y, f(2) = x$; $f(2) = f(3) = y, f(1) = x$
y^3	$f(1) = f(2) = f(3) = y$

Table 7. The functions and patterns when $D = \{1, 2, 3\}$, $R = \{x, y\}$, and $G = S_3$.

the symmetric group does not depend on the order of the factors, functions f_1 and f_2 are equivalent if and only if the products $f_1(1)f_1(2)f_1(3)$ and $f_2(1)f_2(2)f_2(3)$ are identical.

We have stated the weighted version of Burnside's theorem. Now, we define the weights.

3.5. **Weights of Functions and Patterns.**

Definition 3.5. (Ring [3]) A *ring* R is a nonempty set with two binary operations, addition and multiplication, such that for all $a, b, c \in R$:

(1) $a + b = b + a$.

(2) $(a + b) + c = a + (b + c)$.

(3) There is an additive identity 0. That is, there is an element 0 in R such that $a + 0 = a$ for all $a \in R$.

(4) There is an element $-a \in R$ such that $a + (-a) = 0$.

(5) $a(bc) = (ab)c$.

(6) $a(b + c) = ab + ac$ and $(b + c)a = ba + bc$.

A ring is called a *commutative ring* if $ab = ba$ for all $a, b \in R$.

Definition 3.6. (Weight [2]) Given that D and R are finite sets, the set of all functions from D into R is R^D, and G is a permutation group of D. We assign each element in R a *weight*. This weight can be a number, a variable, or generally, an element of a commutative ring containing the rational numbers. The weight for each $r \in R$ is denoted $w(r)$.

Definition 3.7. (Weight of a Function [2]) The weight of a function $f \in R^D$, denoted W(f) is

$$W(f) = \prod_{d \in D} w[f(d)]. \tag{8}$$

If $f_1 \sim f_2$, or the functions belong to the same pattern, then they have the same weight. This is because if $f_1 g = f_2$ for all $g \in G$. Using the fact that the same factors are involved and they are commutative we conclude that

$$\prod_{d \in D} w[f_1(d)] = \prod_{d \in D} w[f_1(g(d))] = \prod_{d \in D} w[f_2(d)].$$

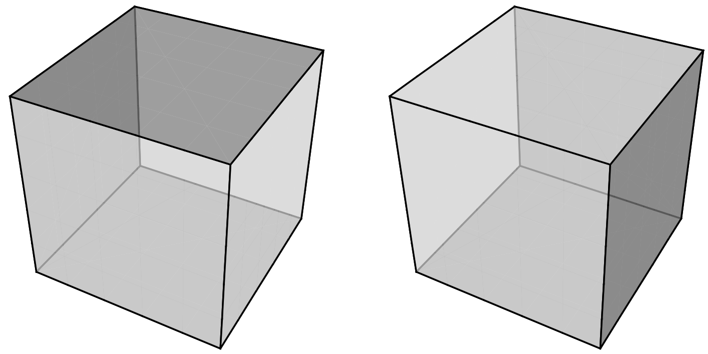
Definition 3.8. (Weight of a Pattern [2]) The weight of a pattern F, denoted $W(F)$, is equal to the weight of any (and every) function in F. Given any $f \in F$,

$$W(F) = \prod_{d \in D} w[f(d)]. \tag{9}$$

Example 3.7. [2]Recall the light gray and dark gray cube from Example 3.5. Let D be the set of faces, $R = \{light, dark\}$, and let G be is the set of symmetries. We set weights, $w(light) = x$ and $w(dark) = y$, the weights for each of the 10 patterns are: $x^6, x^5y, x^4y^2, x^3y^3, x^3, y^3, x^2y^4, xy^5, y^6$. Different patterns need not have different weights. For example, x^3y^3 represents the pattern with three light sides and three dark sides in two ways: three dark sides meet at a single corner, three dark sides that wrap around the cube.

Example 3.8. [2]Recall Example 3.6 where $D = \{1, 2, 3\}$, G is the symmetric group of D, and $R = \{x, y\}$. We set $w(x) = x$ and $w(y) = y$ so x^3, x^2y, xy^2, and y^3 become the pattern weights. In this case, different

Figure 14. Two cubes with the same weight and different patterns.



patterns have different weights. If we set $w(r) = 1$ for all $r \in R$, then $W(f) = 1$ for all $f \in F$, and therefore $W(F) = 1$ for each pattern.

3.6. **Store and Inventory.**

Definition 3.9. (Store [2]) Given that D and R are finite sets and every $r \in R$ has a weight $w(r)$, the set of all functions from D into R is R^D, and G is a permutation group of D. We call R the *store* because it is the set from which we choose function values.

Definition 3.10. (Inventory [2]) Since the values of $w(r)$ must come from a commutative ring, the weights can be added to each other. The *store enumerator*, or *inventory of R* is

$$\text{inventory of R } = \sum_{r \in R} w(r). \tag{10}$$

Example 3.9. *Let us say that a cafe owner has 3 candy bars that cost \$1 each, 2 bottles of pop that cost \$3 each, and 4 hamburgers in stock that cost \$5 each. His* store *is* $R = \{c_1, c_2, c_3, p_1, p_2, h_1, h_2, h_3, h_4\}$. *Depending on the values to which we set the weights, the* inventory of R *will give us different information:*

- *To get complete information about the store, we can set the weight values at* $w(c_1) = c_1, w(c_2) = c_2, ..., w(h_4) = h_4$. *Then the* inventory of $R = c_1 + c_2 + c_3 + p_1 + p_2 + h_1 + h_2 + h_3 + h_4$.
- *We might only be interested in the number of items of each type. In this case, we set the weights as* $w(c_1) = w(c_2) = w(c_3) = c$ *and* $w(p_1) = w(p_2) = p$ *and* $w(h_1) = w(h_2) = w(h_3) = w(h_4) = h$. *In this case, the* inventory of $R = 3c + 2p + 4h$.
- *To get the value of the store, we set the weights as* $w(c_1) = w(c_2) = w(c_3) = 1$ *and* $w(p_1) = w(p_2) = 3$ *and* $w(h_1) = w(h_2) = w(h_3) = w(h_4) = 5$. *In this case, the* inventory of $R = 3 \cdot 1 + 2 \cdot 3 + 4 \cdot 5 = \$29$.
- *To get the number of items in the store, we set the weights as* $w(c_1) = 1, w(c_2) = 1, ..., w(h_4) = 1$. *Then the* inventory of $R = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 9$.

3.7. **Inventory of a Function.** We are given that D and R are finite sets, every $r \in R$ has a weight $w(r)$, the set of all functions from D into R is R^D, every function $f \in R^D$ has weight

$$w(f) = \prod_{d \in D} w[f(d)],$$

and G is a permutation group of D. The inventory of R^D is the $|D|^{\text{th}}$ power of the inventory of R:

$$\text{inventory of } R^D = \prod_f W(f) = \left[\sum_{r \in R} w(r)\right]^{|D|}. \tag{11}$$

We can write the $|D|^{\text{th}}$ power of $\sum_{r \in R} w(r)$ as the product of the $|D|$ factors. In each factor

$$\sum_{r \in R} w(r),$$

pick a term $w(r)$. By taking the product of these terms, we get one term of the full expansion. Since each factor has $|R|$ terms, there are $|R|^{|D|}$ terms in the full expansion. Next, we take $f(d) = r$ which allows us to say that the selection of a term from a factor is a one-to-one mapping f of D into R. To every f there corresponds the term

$$\prod_{d \in D} w[f(d)] \tag{12}$$

of the full expansion. Since the term in (12) is exactly the definition of $W(f)$ in (8), or the weight of a function, the full product is equal to the sum of all $W(f)$, which is the inventory of $|R|^{|D|}$.

[2]Next, we will derive the inventory of S, a certain subset of $|R|^{|D|}$. First, partition D into k disjoint components $D_1, D_2, ...D_k$ such that $|D| = |D_1| + |D_2| + ... + |D_k|$. Let S be the set of all functions f that are constant on each component. They may, but do not have to be different on different components. Let f be expressed as a composition of two functions, ϕ and ψ, such that $f = \phi\psi$. The function ψ maps d onto the index of the component to which d belongs, so we always have $d \in D_{\psi(d)}$. The function ϕ is the mapping from $\{1, 2, ..., k\}$ into R, so there are $|R|^k$ possibilities for ϕ. Therefore,

$$\text{inventory of } S = \prod_{i=1}^{k} \sum_{r \in R} [w(r)]^{|D_i|}. \tag{13}$$

We can see this same result for the *inventory of* S by examining the full expansion of the product. To select one term of the product, we pick a value for ϕ which is a mapping from $\{1, 2, ..., k\}$ into R. This ϕ produces the term

$$\{w[\phi(1)]\}^{|D_1|} \cdot \{w[\phi(2)]\}^{|D_2|} \cdots \{w[\phi(k)]\}^{|D_k|} = \prod_{i=1}^{k} \{w[\phi(i)]\}^{|D_i|}.$$

If $\phi\psi = f$, then this term is exactly $W(f)$ since

$$\{w[\phi(i)]\}^{|D_i|} = \prod_{d \in D_i} w\{\phi[\psi(d)]\},$$

and

$$\prod_{i=1}^{k} \prod_{d \in D_i} w[f(d)] = W(f).$$

In this way, each $f \in S$ is obtained exactly once. Thus the sum of $W(f)$ for all $f \in S$ is equal to the sum of all terms of the expansion of the inventory of S.

Example 3.10. [2]Given m identical counters, we pass them out between three people, P_1, P_2, P_3, under the constraint that you give the same number to P_1 and P_2. How many ways are there to do this?

Let $D = \{P_1, P_2, P_3\}$, $R = \{0, 1, 2, ...\}$, $f(P_1) = f(P_2)$, and $f(P_1) + f(P_2) + f(P_3) = m$. First we separate D into $D_1 = \{P_1, P_2\}$ and $D_2 = \{P_3\}$. Then we set the weights of R such that $0, 1, 2, 3, ...$ have weights $1, x, x^2, x^3, ...$ and the weight of the functions $W(f) = x^m$. If S is the set of all functions constant on each D_i, and by (13), the inventory of the set of all functions constant on each D_i equals

$$(1 + x^2 + x^4 + \cdots)(1 + x + x^2 + \cdots). \tag{14}$$

We are seeking the coefficient of x^m in the expansion (14). Using the formula for the sum of a geometric series,

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i,$$

we can simplify our inventory to

$$\frac{1}{1-x^2} \cdot \frac{1}{1-x}.$$

Then we expand this result using partial fractions to

$$\begin{aligned} \frac{1}{1-x^2} \cdot \frac{1}{1-x} &= \frac{A}{1+x} + \frac{B}{(1-x)^2} + \frac{C}{1-x} \\ &= \frac{A(1-x)^2(1-x) + B(1+x)(1-x) + C(1+x)(1-x)^2}{(1+x)(1-x)^2(1-x)} \end{aligned}$$

Next, substitute, $A = \frac{1}{4}$, $B = \frac{1}{2}$, and $C = \frac{1}{4}$ to get

$$\frac{\frac{1}{4}(1-x)^2(1-x)}{(1+x)(1-x)^2(1-x)} + \frac{\frac{1}{2}(1+x)(1-x)}{(1+x)(1-x)^2(1-x)} + \frac{\frac{1}{2}(1+x)(1-x)^2}{(1+x)(1-x)^2(1-x)}$$
$$= \frac{1}{4}(1+x)^{-1} + \frac{1}{2}(1-x)^{-2} + \frac{1}{4}(1-x)^{-1}.$$

For the number of possible distributions we obtain

$$\frac{1}{2}(m+1) + \frac{1}{4}[1 + (-1)^m] \tag{15}$$

so the number of function is $\frac{1}{2}m + 1$ if m is even and $\frac{1}{2}m + \frac{1}{2}$ if m is odd.

Let $[x^k]$ denote the coefficient of the x^k-term. We will verify (15) result directly for small values of m:

- If $m = 1$, the inventory is $(1+x^2)(1+x)$, and the x-term is $1 \cdot x$ so $[x] = 1$. In words, there is 1 possibility for the distribution of 1 counter and that is 0 to P_1 and P_2 and 1 to P_3.
- If $m = 2$, the inventory is $(1+x^2+x^4)(1+x+x^2)$, and the x^2-term is $1 \cdot x^2 + x^2 \cdot 1$ so $[x^2] = 2$. In words, there are 2 possibilities for the distribution of 2 counters and those are: 1 each to P_1 and P_2 and 0 to P_3; 0 to P_1 and P_2 and 2 to P_3.
- If $m = 3$, the inventory is $(1+x^2+x^4+x^6)(1+x+x^2+x^3)$, and the x^3-term is $1 \cdot x^3 + x^2 \cdot x$ so $[x^3] = 2$. In words, there are 2 possibilities for the distribution of 3 counters: 1 each to P_1, P_2, and P_3; 0 to P_1 and P_2 and 3 to P_3.
- If $m = 4$, the inventory is $(1+x^2+x^4+x^6+x^8)(1+x+x^2+x^3+x^4)$, and the x^4-term is $1 \cdot x^4 + x^2 \cdot x^2 + x^4 \cdot 1$ so $[x^4] = 3$. The possibilities for the distribution of 4 counters are: 2 each to P_1 and P_2 and 0 to P_3; 1 each to P_1 and P_2 and 2 to P_3; 0 each to P_1 and P_2 and 4 to P_3.

These results are concur with (15) and also give us more information on the possible distributions.

3.8. **The Pattern Inventory.**

Theorem 3.4. (Pólya's Fundamental Theorem [2]) We are given finite sets D and R, a permutation group G of D, each $r \in R$ has weight $w(r)$, each function $f \in R^D$ and the patterns F have weights $W(f)$ and $W(F)$. The pattern inventory is

$$\sum_F W(F) = P_G \left\{ \sum_{r \in R} w(r), \sum_{r \in R} [w(r)]^2, \sum_{r \in R} [w(r)]^3, \ldots \right\} \tag{16}$$

where P_G is the cycle index of G.

Proof. Let R and D be finite sets, R^D be the set of all functions from D into R, and G a permutation group of D. Let S be the subset of function $f \in R^D$ such that $W(f(d)) = w$. If $g \in G$, and $f_1(d) = f_2(gd)$, then f_1

and f_2 belong to the same pattern and have the same weight. Therefore, if $f_1 \in S$, then $f_1 g^{-1} \in S$. Thus for each $g \in G$ there is a mapping π_g of S into itself,

$$\pi_g(f(d)) = f(g^{-1}d).$$

This mapping is a permutation since $(\pi_g)^{-1} = \pi_{g^{(-1)}}$. For each $f \in S$, if $g, g' \in G$, for all $d \in D$,

$$\pi_{gg'}(f(d)) = f((gg')^{-1}d) = f(g'^{-1}g^{-1}d) = \pi_g(f(g'^{-1}d)) = \pi_g(\pi_{g'}f(d))$$

and the mapping $g \to \pi_g$ is homomorphic.

We say the f_1 and f_2 are equivalent if and only if

$$f_1(d) = f_2(g^{-1}d) = \pi_g(f_2(d)).$$

Therefore, the patterns contained in S are equivalence classes. According to the weighted Burnside's theorem, the number of patterns contained in S is equal to

$$\frac{1}{|G|}\sum_{g\in G}\psi_w(g),$$

where $\psi_w(g)$ is the number of functions f such that $W(f) = w$. This is equivalent to $f(g^{-1}(d)) = f(d) = \pi_g(f(d))$ or $f(d) = f(g(d))$.

The patterns contained in S all have weight w; therefore if we multiply (7), the weighted Burnside theorem, by w and sum over all possible values of w, we obtain the pattern inventory

$$\sum W(F) = \frac{1}{|G|}\sum_{w}\sum_{g\in G} w\cdot\psi_w(g).$$

We have

$$\sum_{w} w\cdot\psi_w(g) = \sum_{f}^{(g)} W(f),$$

where $\sum_{f}^{(g)}$ means the summation over all $f \in R^D$ that satisfies $f(d) = f(gd)$ for all $d \in D$. It follows that

$$\sum W(F) = \frac{1}{|G|}\sum_{g\in G}\sum_{f}^{(g)} W(f).$$

In order to evaluate $\sum_f^{(g)} W(f)$, we remark that g is a permutation of D, and therefore splits D into cycles that are cyclically permuted by g. The condition $f(d) = f(gd)$ means that

$$f(d) = f(gd) = f(g^2 d) = \cdots ,$$

where f is constant on each cycle of D. Conversely, each f that is constant on every cycle automatically satisfies $f(gd) = f(d)$ since $g(d)$ always belongs to the same cycle as d itself. Thus if the cycles are $D_1, D_2, ..., D_k$, then the sum $\sum_f^{(g)}$ is just the inventory of S from (13).

Let $\{b_1, b_2, ...\}$ be the type of g. This means, among the numbers $|D_1|, ..., |D_k|$, the number 1 occurs b_1 times, the number 2 occurs b_2 times, etc. Consequently, we have

$$\sum_f^{(g)} = \left\{ \sum_{r \in R} w(r) \right\}^{b_1} \cdot \left\{ \sum_{r \in R} [w(r)]^2 \right\}^{b_2} \cdots .$$

The number of factors in finite, but we need not invent a notation for the last one since all b_i are zero from a certain i onward. The expression above can be obtained by the substitution of

$$x_1 = \sum_{r \in R} w(r), \quad x_2 = \sum_{r \in R} [w(r)]^2, \quad x_3 = \sum r \in R[w(r)]^3, ...$$

into the product $x_1^{b_1} x_2^{b_2} x_3^{b_3} \cdots$, which is the term corresponding for g in $|G| \cdot P_G$. Summing with respect to g and dividing by $|G|$, we infer that the value of the pattern index, $\sum W(F)$, is obtained by making the same substitution into $P_G(x_1, x_2, x_3, ...)$, and this proves Pólya's theorem. □

Example 3.11. If all of the weights are chosen to be equal to unity, then we obtain that the number of patterns is equal to $P_g(|R|, |R|, |R|, ...)$. We can show this by directly substituting $w(r) = 1$ for all $r \in R$. The pattern inventory becomes simply the number of pattern which is equal to $P_G(|R|, |R|, |R|, ...)$.

Example 3.12. Consider the light and dark painted cube. Let D bet he set of faces of the cube, G be the set of rotational symmetries, R be the set of colors, $\{light, dark\}$.

Let the weights $w(light) = 1$ and $w(dark) = 1$. Then from Example 3.11,

$$\text{the number of patterns } = P_G(|R|, |R|, |R|, ..)$$

and we obtain

$$\frac{1}{24}(2^6 + 3 \cdot 2^4 + 6 \cdot 2^3 + 6 \cdot 2^3 + 8 \cdot 2^2) = 10.$$

This is the same number of patterns that we discovered by inspection and then derived by Burnside's theorem.

Example 3.13. [2]Now, let the weights $w(light) = x$ and $w(dark) = y$. First, calculate the pattern inventory using Pólya's theorem,

$$\sum_F W(F) = P_G \left\{ \sum_{r \in R} w(r), \sum_{r \in R} [w(r)]^2, \sum_{r \in R} [w(r)]^3, ... \right\}.$$

Recall the the cycle index of G is

$$P_G(x_1, x_2, x_3, x_4) = \frac{1}{24}(x_1^6 + 3x_1^2x_2^2 + 6x_1^2x_4 + 6x_2^3 + 8x_3^2).$$

Next, the we substitute $\sum_{r \in R} [w(r)]^i$ for the x_i's. Therefore, the pattern inventory is

$$\frac{1}{24}[(x+y)^6 + 3(x+y)^2(x^2+y^2)^2 + 6(x+y)^2(x^4+y^4) + 6(x^2+y^2)^3_+ 8(x^3+y^3)^2].$$

Expanding the pattern inventory, we get

$$\frac{1}{24}(x^6 + x^5y + 2x^3y^3 + 2x^2y^4 + xy^5 + y^6).$$

This polynomial can be interpreted to give us complete information about the possible colorings of the cube. For example, we might ask how many patterns have four light faces and two dark faces. The answer is the coefficient of the x^4y^2-term, $[x^4y^2] = 2$. This is consistent with the value that we found in (2). Also, the sum of the coefficients is predictably equal to ten, which is the total number of patterns.

Example 3.14. [2]Let D be a finite set, and let G be a permutation group of D. We call two subsets of D, D_1 and D_2, equivalent if for some $g \in G$, we have $gD_1 = D_2$. In other words, D_2 is the set of all elements gd obtained by letting d run through D_1. We can form classes of equivalent subsets in this manner.

We can create functions, $f_g(d) = gd$, that are in one-to-one correspondence with the subsets. Let $R = \{light, dark\}$, $w(light) = 1$, and $w(dark) = 1$. In this case, the number of subset classes is equal to the pattern inventory, which by Pólya's theorem, is equal to $P_G(2, 2, 2, ...)$.

Now let $w(light) = w$ and $w(dark) = 1$, where w is a variable. Then the subsets of k elements correspond to functions f with $W(f) = w^k$. Therefore, the number of equivalence classes which consist of subsets of k elements is equal to the coefficient of w^k in the pattern inventory, $P_G(1 + w, 1 + w^2, 1 + w^3, ..)$. Summing over all k, we obtain $P_G(2, 2, 2, ...)$ since the sum of the coefficients of a polynomial $p(w)$ is equal to $p(1)$.

This example relates strongly to the cube problem. For example, let a subset D be equal to the set of all cubes with just one side light. There exists some $g \in G$ such that all cubes with one light side are equivalent to all of the other cubes with one light side, but not to any cubes with a different number of light sides.

Then the cycle index is the same as it is for any two colored cube. The pattern inventory is now

$$\frac{1}{24}[(w+1)^6 + 3(w+1)^2(w^2+1)^2 + 6(w+1)^2(w^4+1) + 6(w^2+1)^3 + 8(w^3+1)^2].$$

If we expand the pattern inventory, we get

$$w^6 + w^5 + 2w^3 + 2w^2 + w + 1.$$

We can create the same two colored cubes as before using these simplified calculations simply by painting the complementary faces dark.

4. Generalizations and Extensions of Pólya's Theorem

4.1. **Generalization of Pólya's Theorem.** In Pólya's theorem, G, a permutation group of D induced an equivalence relation on the mappings $f \in R^D$. Now, we consider a second permutation group H of R with a new equivalence relation on the mappings $f \in R^D$ which is defined based on both groups. Let f_1 be equivalent to f_2 if there exists a $g \in G, h \in H$ such that $f_1 g = h f_2$. We now show that

$$f_1 g = h f_2 \text{ or } f_1(g(d)) = h(f_2(d)) \text{ for all } d \in D \tag{17}$$

is an equivalence relation.

(1) If g and h are identity permutations, $f_1(g(d)) = h(f_1(d))$ so $f_1 \sim f_1$ and (17) satisfies the reflexive property.
(2) If $f_1 \sim f_2$, there exists a $g \in G, h \in H$ such that $f_1(g(d)) = h(f_2(d))$ for all $d \in D$. Since $g^{-1}(d) \in D$ since G is a permutation group, $f_1(g(g^{-1}(d))) = h(f_2(g^{-1}(d))$, so $f_2(g^{-1}(d)) = h^{-1}(f_1(d))$ for all d in D so $f_2 \sim f_1$. Therefore (17) satisfies the symmetric property.
(3) If $f_1 \sim f_2$ and $f_2 \sim f_3$, then there exists $g, g' \in G, h, h' \in H$ such that $f_1(g(d)) = h(f_2(d'))$ and $f_2(g'(d)) = h'(f_3(d))$. Let $d' = g'(d)$. Then $f_1(g(g'(d)) = h(f_2(g'(d))) = h(h'(f_3(d)))$ since $g, g' \in G$ and $h, h' \in H$. Therefore $f_1 \sim f_3$ and (17) satisfies the transitive property.

Therefore (17) is an equivalence relation so R^D splits into equivalence classes, or patterns.

We assume that each $f \in R^D$ has a weight $W(f)$ such that the the weights are elements of a commutative ring. We need not assume that the $W(f)$ are obtained from the elements of R, but we do assume that equivalent functions have the same weight, or

$$f_1 \sim f_2 \implies W(f_1) = W(f_2). \tag{18}$$

If F is a pattern, we define its weights $W(F)$ to be the common value of all $W(f)$ with $f \in F$ just as we did for Pólya's theorem (Theorem 3.4). As our first extension of Pólya's theorem, we now calculate the pattern inventory, or the sum of the weights of all patterns, using equivalence relation (17).

Theorem 4.1. (de Bruijn's Extension [2]) *Given finite sets, D and R, functions $f \in R^D$, and permutation groups G on D and H on R, the pattern inventory is*

$$\sum W(F) = \frac{1}{|G| \cdot |H|} \sum_{g \in G} \sum_{h \in H} \sum_{f}^{(g,h)} W(f) \tag{19}$$

where $\sum_{f}^{(g,h)}$ *is the sum of* $W(f)$ *extended over all* f *that satisfy* $fg = hf$.

Example 4.1. Consider a 6-beaded necklace. You care about the pattern, but not the colors of the beads. For example, you are color blind, and able to distinguish between colors but not identify them. In math terminology, the colors are "unlabelled." In this case, D is the let of beads, R is the set of colors $\{light, dark\}$, $G = D_6$, and $H = S_2$. We set all of the weights equal to $w(light) = 1$ and $w(dark) = 1$. By Theorem 4.1, the pattern inventory is

$$\sum W(F) = \frac{1}{|G| \cdot |H|} \sum_{g \in G} \sum_{h \in H} \sum_{f}^{(g,h)} W(f) = \frac{1}{24 \cdot 2} \cdot 288 = 6$$

The 6 cubes are: all one color, 5 sides one color, and 4 sides on color in two ways.

4.2. **The Kranz Group.** Let S and T be finite sets with G and H as permutation groups of finite sets S and T. Consider $S \oplus T$. Choose a $g \in G$ and for each $s \in S$, pick an $h_s \in H$. These elements determine a permutation of $S \oplus T$.

$$(s,t) \rightarrow (g(s), h_s(t)) \text{ such that } s \in S,\ t \in T$$

There are $|G| \cdot |H|^{|S|}$ permutations of the form $(s,t) \rightarrow (g(s), h_s(t))$ such that $s \in S$ and $t \in T$. These permutations form a group called the Kranz group $G[H]$. We will now calculate the cycle-index of this group.

Theorem 4.2. *(*Pólya's Kranz Theorem [2]*) The cycle-index of $G[H]$, is*

$$P_{G[H]}(x_1, x_2, x_3, ...) = P_G[P_H(x_1, x_2, ...), P_H(x_2, x_4, ...), ...]$$

where the right hand side is obtained by substituting

$$y_k = P_H(x_k, x_{2k}, x_{3k}, ...)$$

into

$$P_G(y_1, y_2, y_3, ...).$$

Proof. Given a finite set R such that each $r \in R$ has weight $w(r)$, consider mappings from T into R. These mappings form patterns λ and the equivalence classes induced by H, a permutation group of R. Let Λ be the set of these patterns. That pattern inventory is

$$\sum_{\lambda \in \Lambda} W(\lambda) = P_H \left\{ \sum w(r), \sum [w(r)]^2, ... \right\}.$$

Now, consider mappings of S into Λ that form patterns ψ by means of the equivalence relation defined by G, a permutation group of S. After defining weights $W^*(\psi)$, we obtain the inventory of the ψ's

$$\sum_{\psi} P_G \left\{ W(\lambda), \sum [W(\lambda)]^2, \sum [W(\lambda)]^3, ... \right\}.$$

The sums $\sum [W(\lambda)]^2$, etc. are obtained by applying the pattern inventory of Λ to new weights, $w^2, w^3, ...$ Therefore, our pattern inventory of the ψ's equals the expression that we obtain by substituting

$$x_1 = \sum w(r), x_2 = \sum [w(r)]^2, ...$$

into the right hand side of

$$P_{G[H]}(x_1, x_2, x_3, ...) = P_G[P_H(x_1, x_2, ...), P_H(x_2, x_4, ...), ...].$$

After that, we construct a one-to-one correspondence between the set of ψ's and the patterns that arise from the equivalence introduced by $G[H]$ in the set $R^{(S \oplus T)}$. We can also substitute for the $w(r)$'s in the inventory of Λ. Finally, we get our equation since R and w are arbitrary. $\square$

Example 4.2. [2]Given n cubes whose faces are light or dark. How many ways can we color the cubes when the equivalence classes are defined by permutations of the set of cubes and rotations of the separate cubes?

The Kranz group under consideration is $S_n[G]$, where S_n is the symmetric group of degree n and G is the dihedral group of the rotations of the cube. To calculate the number of patterns, we substitute $x_1 = x_2 = ... = 2$ into the cycle index. Making this substitution into any of $P_G(x_1, x_2, x_3, ...)$, $P_G(x_2, x_4, x_6, ...)$, and $P_G(x_3, x_6, x_9, ...)$, we always get $P_G(2, 2, 2, ...)$, and this is equal to 10 which is the number of varieties of cubes. Thus, the answer to our question is $S_n(10, 10, 10, ...)$. This is the coefficient of w^n in the Taylor expansion of

$$exp(10w + \frac{1}{2} \cdot 10w^2 + \frac{1}{3} \cdot 10w^3 + \cdots) = (1-w)^{-10},$$

and therefore

$$S_n(10, 10, 10, ...) = \frac{(n+9)!}{n!9!}.$$

Example 4.3. [2]How many of the preceding patterns have the property that they do not change if we interchange the colors?

This number if found by substituting

$$x_1 = x_3 = x_5 = \cdots = 0, x_2 = x_4 = x_6 = \cdots = 2$$

into the cycle index. The polynomials become $P_G(0, 2, 0, 2, ...)$ and $P_G(2, 2, 2, ...)$, alternately. As $P_G(0, 2, 0, 2...) = 2$ we obtain

$$P_{S_n[G]}(0, 2, 0, 2, ...) = P_{S_n}(2, 10, 2, 10, ...).$$

This is the coefficient of w^n in

$$exp(2w + \frac{1}{2} \cdot 10w^2 + \frac{1}{3} \cdot 2w^3 + ...)$$

which simplifies to

$$1 + 2w + 7w^2 + 12w^3 + 27w^4 + 42w^5 + 77w^6 + ...$$

so our answer for $n = 5$ is 42 patterns.

5. Applications to Graph Theory

5.1. **Superposition Theorem.** [4]Pólya's theorem and its extensions are related to the superposition theorem. We uses the superposition theorem to solve the same problem as Example 4.1.

We are given an ordered set of k permutation groups of degree n: $G_1, G_2, ..., G_k$. The set of all k-ads $(a_1, a_2, ..., a_k)$ where $a_i \in G_i$. Two k-ads $(a_1, a_2, ..., a_k)$ and $(b_1, b_2, ..., b_k)$ are similar if there exists an $x \in S_n$ and $g_i \in G_i$ such that $b_i = xa_ig_i$ for $i = 1, ..., k$. The superposition theorem gives us a method for calculating the number of equivalence classes.

For a graphical interpretation of this principle, let T_i where $i = 1, ..., k$ be a set of k graphs with p vertices. Let G_i be the automorphism group of T_i, or the graphs obtained by superposing k graphs on the same vertices. We distinguish the edges by coloring the edges of each graph different colors. The isomorphism classes of the superimposed graphs are the equivalence classes for $G_1, G_2, ..., G_k$. The superposition principle give us the number of nonisomorphic superimposed graphs.

We can compute the cycle index of G_i to be

$$Z(G_i) = \frac{1}{|G_i|} \sum_{(j)} A_{(j)}^{(i)} s_1^{j_1} s_2^{j_2} \cdots s_p^{j_p}$$

where $A_{(j)}^{(i)}$ is the number of permutations of cycle type $(j) = (j_1, j_2, ..., j_p)$ in T_i. So the number of equivalent classes is

$$\prod_{(i)} \frac{1}{|G_i|} \sum_{(j)} A_{(j)}^1 A_{(j)}^2 \cdots A_{(j)}^k (1^{j_1} j_1! \cdot 2^{j_2} j_2! \cdots p^{j_p} j_p!)^{k-1}$$

This is more easily seen by defining function $\star$ such that if

$$A = \sum_{(j)} a_{(j)} s_1^{j_1} s_2^{j_2} \cdots s_p^{j_p}$$

and

$$B = \sum_{(j)} b_{(j)} s_1^{j_1} s_2^{j_2} \cdots s_p^{j_p}$$

then

$$A \star B = \sum_{(j)} a_{(j)} b_{(j)} \prod_{(i)} (i^{j_1} j_1!) s_1^{j_1} s_2^{j_2} \cdots s_p^{j_p}$$

such that for each monomial $s_1^{j_1} s_2^{j_2} \cdots s_p^{j_p}$ the coefficient in $A \star B$ is the product of the coefficient of that monomial in A and B times $1^{j_1} j_1! \cdots p^{j_p} j_p!$. So, if A And B are the cycle indices of two graphs, $A \star B$ is the sum of the cycle indices of the automorphism groups of the distinct superimposed graphs.

If C is an expression in $s_1, s_2, ..., s_p$, define $N(C)$ to be the result of setting $s_1 = s_2 = ...s_p = 1$. For each cycle index A, if $N(A) = 1$, then $N(A \star B)$ is equal to the number of superpositions of two graphs. Also, $A \star B$ can be extended to any number of superpositions. If A_i is the cycle index of the automorphism group of T_i, the product of $Z(G_i)$, then the number of superposed graphs is $N(A_1 \star A_2 \star ... \star A_k)$ which we write as $N(G_1 \star G_2 \star ... \star G_k)$.

Example 5.1. [4]Consider the six- beaded necklace. Let D be the 6 beads. Let R be the colors light and dark. Let $G = D_6$ and let $H = S_2$ (interchanges the elements of R). We want to find the number of distinct configurations of two light beads and four dark beads. Let $w(light) = x$ and $w(dark) = y$. Therefore the inventory of R is $x + y$. We can use Pólya's theorem to derive the pattern inventory $P_{D_6}(x + y)$. The number of necklaces with two light beads and four dark beads is the coefficient of the monomial x^2y^4 in the expansion of the pattern inventory.

To solve this using the superposition theorem, consider the graphs in Figure 15. The superposition of the two graphs have a one-to-one correspondence with the necklaces. The number of necklaces can be seen as $N(D_6 \star (S_2 \oplus S_4))$ since the direct product $S_2 \oplus S_4$ is the group automorphism of Figure 16. Since

$$P_{D_6} = \frac{1}{12}(x_1^6 + 2x_6 + 2x_3^2 + 4x_2^3 + 3x_1^2 x_2^2)$$

and

$$P_{S_2 \oplus S_4} = \frac{1}{48}(x_1^6 + 7x_1^4 x_2 + 9x_1^2 x_2^2 + 8x_1^3 x_3 + 6x_1^2 x_3 + 3x_2^2 + 8x_1 x_2 x_3 + 6x_2 x_4)$$

we have

$$N(D_6 \star (S_2 \oplus S_4)) = \frac{1}{12 \cdot 48}(6! + 4 \cdot 3 \cdot 3! \cdot 2^3 + 3 \cdot 9 \cdot 2 \cdot 2^2 \cdot 2) = 3.$$

We can see that this is the correct answer by calculating out the entire polynomial using Pólya's theorem (as in Example 4.1) and by direct observation of the possible arrangements of two light beads and four dark beads as in the introduction to this paper. This is exactly the coefficient of the x^2y^4 term in the pattern index of a 2-color 6-beaded necklace.

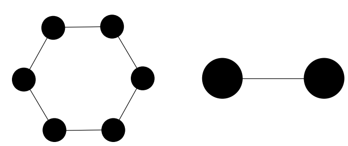


FIGURE 15. The two superimposed graphs in Example 5.1.

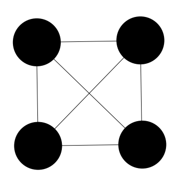


FIGURE 16. The automorphism graph in Example 5.1.

5.2. **Another Generalization.** The next important generalization of Pólya's theorem was published in 1966 by Harary and Palmer [4]. We are given D, R, $f : D \to R \in R^D$ and perform permutations G on D and H on R with the equivalence relations defined as in de Bruijn's generalization. We create a group of permutations of the functions induced by G and H. Given the each function f is in the power set R^D, we now have the power group H^G and we then compute the cycle index of this group.

If the look at the necklace problem again, with D as the six beads, R as the colors light and dark, H interchanges the colors and G permutes the beads. The total number of possibilities (can use unweighted function) is

$$\frac{1}{|H|}\sum_{h\in H} P_G(c_1(h), ..., c_m(h))$$

such that

$$c_k(h) = \sum_{\alpha|k} \alpha j_\alpha(h).$$

Let $G = D_6$ and $H = S_2$. These permutations are of type x_1^2 and x_2 respectively. Therefore x_1^2 corresponds to $c_k(h) = xj_x = 2$ for all k and x_1 corresponds to $c_k(h) = 2$ if k is even. Therefore the configuration generating function is

$$\frac{1}{2}P_{D_6}(2, 2, 2, ...) + P_{D_6}(0, 2, 0, 2, ...) = 8.$$

We can compare this to $P_{D_6}(2, 2, 2, ...) = 13$ when the colors are not interchangeable. This means that there are three necklaces that are now equivalent to themselves when we are able to interchange the colors as in Example 4.1. This happens to be the value of $P_{D_6}(0, 2, 0, 2...)$.

6. Conclusion

In this paper, we begin by counting the number of colored shapes under the dihedral groups. First, we count a cube and hexagon by observation. Then we use algebra to derive Burnside's theorem and Pólya's theorem. Finally, we note some generalizations of Pólya's theorem which are related to our original counting problems.

A major application of Pólya's theorem which we do not cover is the enumeration of graphs and trees. Some further reading on Pólya's theorem and its generalizations and applications are the works of Frank Harary, N.G. de Bruijn, and Pólya's original paper. De Bruijn published a paper after the one used as a source in this paper which further generalizes Pólya's theorem. Harary has several papers which include extensions of Pólya's main theorem and applications to graph theory. Pólya's original paper focusses on the applications to chemical enumeration. [4]

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