A Classification of the Subgroups of the Rationals Under Addition

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1 Abstract

In this paper, we examine one of the most fundamental and interesting algebraic structures, the rational numbers, from the perspective of group theory. We delve into subgroups of the rationals under addition, with the ultimate goal of completely classifying their isomorphisms, gaining insight into general torsion-free groups along the way. We will develop an intuitive and efficient approach to describing a subgroup, namely the height function, and use this description to completely classify subgroup isomorphism. Some time will be dedicated to methods of combining subgroups into new subgroups, one each corresponding to the four main algebraic operations. The reader will come to understand the relationship between the resulting groups of such operations and the groups used to form them. Finally, we will focus attention on an elegant result that gives a way of identifying an arbitrary torsion-free group as a subgroup of the rationals.

2 Introduction

There are many insights that stem from a detailed study of the rational numbers from the viewpoint of group theory. First and foremost, the rational numbers are a construction that is central to modern mathematics. Indeed, students learn of their basic nature very early in education. To improve our understanding of the way rational numbers function is an important study in algebra. Secondly, the rational numbers are a quintessential torsion-free group, meaning a group whose only element of finite order is the identity. Studying the rational numbers can provide important revelations about torsion-free groups as a whole. Furthermore, and of particular interest to this paper, examining the group structure of the rationals can give us specific criteria to determine when an arbitrary torsion-free group is actually isomorphic to a subgroup of the rationals.

The earliest roots of these studies arise from the development of group theory, especially through the Nineteenth Century. However, the fundamentals of this branch of mathematics were never thoroughly applied to the rational numbers in order to answer the question "when are two subgroups of the rationals isomorphic" until the Twentieth Century. In 1937, the German mathematician Reinhold Baer published a 54 page paper that accurately answered this question. The paper also contained many results pertaining to other torsion-free groups. Further work on the subject continued throughout the 1950s, 60s, and 70s, though much of the publishing done on the matter was simply alternate descriptions of Baer's conclusions. Finally, in the early 2000s, the mathematicians Friedrich L. Kluempen and Denise Reboli published a paper titled "When Are Two Subgroups of the Rational Numbers Isomorphic". The purpose of this publication was to broaden the accessibility of the subject so as to allow for undergraduate students to study it. While the approach taken in Kluempen and

Reboli's paper differs from this paper's methods, the fundamentals are the same. Indeed, they are equivalent classifications. While the paper does an excellent job of presenting the material in an understandable fashion, there is obviously a certain level of expected knowledge in order to fully understand the subject.

Accordingly, there are many important definitions that will be used throughout this paper that are essential for the reader to understand intimately. First and foremost, the basics of group theory are taken to be understood by the reader. Any undergraduate who has taken an introductory course to abstract algebra or group theory will have the sufficient tools to grasp the group theory concepts presented in this paper. Secondly, there is a great deal of number theory involved in this topic, and readers should be familiar with basic ideas from this branch of mathematics as well. Unless otherwise noted, the definitions used in this paper will be those presented in Charles C. Pinter's second edition of "A Book of Abstract Algebra".

We will now introduce certain definitions that are specific to the study of subgroups of the rational numbers under addition. Firstly, any use of the word "subgroups" in this paper will be referring to subgroups of the rationals under addition. Secondly, if a subgroup contains the unity (1), we will call this group unitary. Note that this definition should not be confused with the definition of a unitary ring; we are referring, here, to the multiplicative identity, although the group's operation is addition. The multiplication of a rational number $\frac{m}{n}$ by an integer z will be defined as $\underbrace{\dfrac{m}{n} + \dfrac{m}{n} + .., + \dfrac{m}{n}}_{z \text{ times}}$. Usually, but not always, subgroups will be assigned to the variables A or B, functions will be assigned to the variables h or k, and integers or natural numbers will be given the names m, n, p, or q.

3 Subgroup and Homomorphism Structure

In this section, we will examine some of the general structural nature of subgroups of the rational numbers. Many of the questions we will address are very typical of group theory. For example, what do homomorphisms and isomorphisms between two subgroups look like? These studies will prove very valuable in studying subgroup isomorphism.

The first step towards our goal is to briefly examine the nature of elemental inclusion in subgroups of the rationals. What are some testable conditions under which a rational number is an element of a specific group? It turns out that this question is fairly easily answered. Before we begin, however, it must be stated that the subgroups in the proofs within this section will always be assumed to be unitary.

Theorem 1.1: For relatively prime m, n, $\frac{m}{n} \in A$ if and only if $\frac{1}{n} \in A$.

Proof: First, assume that $\frac{m}{n} \in A$. Since m and n are relatively prime, there are $x, y \in \mathbb{Z}$ such that $1 = xm + yn \to \frac{1}{n} = \frac{xm}{n} + y$. $\frac{m}{n}$ is in A. Additionally, $x, y \in A$ since A contains 1, and, inductively, all of the integers. Thus, $\frac{xm}{n} + y = \frac{1}{n} \in A$.

Now, assume that $\frac{1}{n} \in A$. Then $\frac{m}{n} = \underbrace{\frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}}_{\text{m times}} \in A$. Thus, $\frac{m}{n} \in A$ if and only if $\frac{1}{n} \in A$.$\square$

Theorem 1.2: For relatively prime m and n, the rational number $\frac{1}{mn} \in A$ if and only if $\frac{1}{m} \in A$ and $\frac{1}{n} \in A$.

Proof: First, assume that $\frac{1}{mn} \in A$. So then $\frac{1}{n} = \frac{m}{mn} = \underbrace{\frac{1}{mn} + \frac{1}{mn} + \ldots + \frac{1}{mn}}_{\text{m times}} \in A$. Similarly, $\frac{1}{m} = \frac{n}{mn} = \underbrace{\frac{1}{mn} + \frac{1}{mn} + \ldots + \frac{1}{mn}}_{\text{n times}} \in A$.

Now, assume that $\frac{1}{m} \in A$ and $\frac{1}{n} \in A$. Then $\frac{1}{m} + \frac{1}{n} \in A$. So $\frac{1}{m} + \frac{1}{n} = \frac{n}{mn} + \frac{m}{mn} = \frac{m+n}{mn} \in A$. Since m and n are relatively prime, $m + n$ and mn are relatively prime. Thus, by Theorem 1.1, $\frac{1}{mn} \in A$. And so $\frac{1}{mn} \in A$ if and only if $\frac{1}{m} \in A$ and $\frac{1}{n} \in A$.□

These two theorems give us some tools to deduce what certain elements of a subgroup will look like given one element. Theorem 1.1 emphasizes that it is the denominators of elements that are definitive of a subgroup's structure. This will be important when we attempt to completely characterize a group in a simple way. Theorem 1.2 is useful because it allows us to focus on the elements of a group based on the primes that make up the denominators of the elements. Decomposing the denominator of an element into it's prime factorization puts the element into a form that we can apply Theorem 1.2 to, and subsequently look at elements whose denominators are primes. As an example, let A be a unitary subgroup of $\mathbb{Q}$. If $\frac{3}{10} \in A$, then $\frac{1}{10} \in A$, and furthermore $\frac{1}{2} \in A$ and $\frac{1}{5} \in A$.

In general, for a unitary subgroup A, if $\frac{m}{n} \in A$ with m and n relatively prime, and $n = p_1^{e_1} * \ldots * p_j^{e_j}$ is the prime factorization of n, then each of the $\frac{1}{p_1^{e_1}}, \ldots, \frac{1}{p_j}^{e_j}$ are in A.

Having given some thought to the form and nature of subgroup elements, we will now turn our attention to the homomorphisms and, in turn, isomorphisms of subgroups of the rationals.

Theorem 1.3: For $q \in \mathbb{Q}$, $f(x) = qx$ is a homomorphism.

Proof: Let $q \in \mathbb{Q}$ and let $f(x) = qx$. For any $a, b \in A$, $f(a+b) = q(a+b) = qa + qb = f(a) + f(b)$. So f is a homomorphism.□

Related to Theorem 1.3,

Theorem 1.4: If f is any homomorphism from a subgroup A to $\mathbb{Q}$ (A need not be unitary), then there is a unique q such that $f(x) = qx$.

Proof: Let $f : A \to \mathbb{Q}$ be a homomorphism. For $\frac{p}{q} \in A$, let $f(\frac{p}{q}) = \frac{m}{n}$. Now, we multiply both sides of $a = \frac{p}{q}$ by mq, so that $amq = pm$. Now, multiplying both sides of $f(\frac{p}{q}) = \frac{m}{n}$ by pn, we get that $f(\frac{p}{q})pn = mp = amq$ so then $f(\frac{p}{q}) = \frac{mq}{pn} * a$. Let $g(x) = \frac{mq}{pn} * x$. We want to show that $f = g$, but we need first to prove a quick lemma.

Lemma 1.4.1: If $f, g : A \to \mathbb{Q}$ are homomorphisms and for some non-zero $a \in A$ we have that $f(a) = g(a)$. Then $f(x) = g(x)$.

Proof: First, let $h : A \to \mathbb{Q}$ be a homomorphism such that $h = f - g$. Let $a = \frac{m}{n}$. Let $b = \frac{p}{q} \in A$ be non-zero. Note that $h(a) = 0$. We can yield the equations $anp = mp$ and $bqm = pm$, implying that $anp = bqm$. So $h(b)h(qm) = h(bqm) = h(anp) = h(a)h(np) = 0$. So $h(b) = 0$ since $q \neq 0 \neq m$, and thus $h(x) = 0$. Now, returning to the homomorphisms f and g, we first assume that $f = g$. Then it is obvious that $f(a) = g(a)$. Suppose, on the other hand, that $f(a) = g(a)$. Then $f(a) - g(a) = (f - g)(a) = 0$ so $(f - g)(x) = 0$, implying that $f(x) = g(x)$.□

Now, returning to the proof of Theorem 1.3, the way forward is obvious. Since f and g agree at one point, namely a, it follows that $f = g$, and so $f(x) = \frac{mq}{pn} * x$, which, by Theorem 1.3, is a homomorphism.□

Thus, not only are functions of the form $f(x) = qx$ homomorphisms, but, even more powerfully, *every* homomorphism can be expressed this way. It should be noted that, if $q \neq 0$ and if the range of f is reduced from $\mathbb{Q}$ to the image of all of A under f, it can easily be shown that f is an isomorphism. This simple corollary will be useful in the proof of our next theorem.

Theorem 1.5: If A and A' are non-trivial subgroups of $\mathbb{Q}$, then A is isomorphic to A' if and only if there is a $q \in \mathbb{Q}$ such that $A' = q * A = \{q * a : a \in A\}$.

Proof: A is isomorphic to A' if and only if there exists an isomorphism $f : A \to A'$. This isomorphism can be expressed, by Theorem 1.4, as $f(x) = qx$ for some $q \in \mathbb{Q}$. But f is an isomorphism if and only if, $f(A) = qA = \{qa : a \in A\} = A'$, and we have proved that A is isomorphic to A' if and only if there is a $q \in \mathbb{Q}$ such that $A' = qA = \{qa : a \in A\}$.$\Box$

This last result is what will justify our inclusion of 1 in the groups for the proof of Theorems 1.1 and 1.2 (note that this condition was not needed in proving Theorems 1.3, 1.4, or 1.5).

Theorem 1.6: Every subgroup A of $\mathbb{Q}$ is isomorphic to at least 1 subgroup of $\mathbb{Q}$ that contains 1.

Proof: Let $a = \frac{m}{n} \in A$. Then let $f(x) = \frac{n}{m} * x$. The image of A under f, or $f(A)$, is a group that is isomorphic to A. Since $f(a) = 1$, $1 \in f(A)$, and we have proved the desired result.$\Box$

Because of theorem 1.6, it is reasonable in many cases to begin general proofs by assuming that subgroups are unitary.

4 The Height Function

In Section 3, we looked at some basic theorems regarding the structure of additive subgroups of the rationals. In Section 4, we look at a function that aptly describes subgroups. This function, known as the height function, describes the denominators of elements in a subgroup in terms of their prime decompositions.

Definition 2.1: Let ∞ be the number such that $\infty + a = \infty$ for any integer a.

Definition 2.2: Let $\mathcal{F}$ be the set $\mathcal{F} = \{f : \mathcal{P} \to \mathbb{N} \cup \{\infty\}\}$.

Definition 2.3: For a unitary subgroup A of the rational numbers, $h_A \in \mathcal{F}$ is defined as follows for each prime p: $h_A(p) = \max\{e : \frac{1}{p^e} \in A\}$.

The first thing that we would like to do is show that the height function does, indeed, give us a good description of when two subgroups are equal.

Theorem 2.1: For two subgroups A, A' of $\mathbb{Q}$, $A = A'$ if and only if $h_A(p) = h_{A'}(p)$ for all primes.

Proof: Begin by assuming that $A = A'$. It is a triviality that $h_A(p) = h_{A'}(p)$.

Now, assume that $h_A(p) = h_{A'}(p) = e_3$. If $q \in A$, where $q = \frac{1}{p^j}$, Then $j \leq h_A(p) = h_{A'}(p)$, so $q \in A'$. Now, if $\frac{m}{n} \in A$, where $n = p_1^{e_1} * \cdots * p_j^{e_j}$ is the prime factorization of n, then we know from Theorem 1.1 and Theorem 1.2 that each $\frac{1}{p_1^{e_1}}, \ldots, \frac{1}{p_j^{e_j}} \in A$. From above, they are all also in A'. Thus, taking their product and adding this product to itself m times, we have that $\frac{m}{n} \in A'$.$\square$

This conclusion shows us that the our definition of the height function is logical. Much of the rest of this paper will deal with subgroups' height functions.

Now, we will show some examples in order to familiarize the reader with the height function. First, consider the subgroup of the rationals that is described by the height function that is 1 everywhere. First, this subgroups contains elements like $\frac{1}{3}$, $\frac{1}{5}$, $\frac{-1}{15} = \frac{-1}{3*5}$, and even $\frac{1}{210} = \frac{1}{2*3*5*7}$. By Theorem 1.1, this subgroup also contains elements such as $\frac{3}{5}$, $\frac{-4}{15}$, and $\frac{7}{15}$. However, the maximum power that any prime can be raised to in the denominators of elements of this group is 1, so numbers such as $\frac{1}{9} = \frac{1}{3^2}$ and $\frac{3}{12} = \frac{3}{2^2*3}$ are not in the subgroup. In short, this subgroup contains elements whose denominators have prime factors that appear only to the first degree in the prime factorization.

Next, we will look at two very important and well-known subgroups of $\mathbb{Q}$. First, consider the integers, $\mathbb{Z}$, which is indeed a subgroup of $\mathbb{Q}$. When an integer x is represented as a rational number, it is assumed to be $\frac{x}{1}$. That is, the integers are all of the rational numbers whose denominators are 1 (or, in actuality, the rationals whose denominators are 1 when expressed with relatively prime numerator and denominator). Thus, $h_{\mathbb{Z}}(p) = 0$ for all primes. Now, consider a trivial subgroup of the rationals, namely, all of the rationals. Since there is no restriction on what primes a rational number may contain as a factor of the denominator, it is clear that $h_{\mathbb{Q}}(p) = \infty$ for all primes.

Finally, we will look at the height function of a contrived subgroup. What is the height function of $\langle \frac{1}{2*3^2*11^{20}} \rangle$. By adding $\frac{1}{2*3^2*11^{20}}$ to itself any number of times, the degrees of the primes in the denominator of the sum is never increased. Thus, this subgroup is described by the height function where $h(2) = 1$, $h(3) = 2$, $h(11) = 20$, and $h(p) = 0$ for all other primes.

Now, we will look at a certain type of height function that is very useful in certain applications. For a specific prime p, consider the height function h_A where

$$h_A(q) = \begin{cases} 0 & : q = p \\ \infty & : q \neq p \end{cases}$$

In other words, this height function allows all primes except for one to divide denominators of elements of the group it describes. The group A is known as "the integers localized at the prime p". Furthermore, we can construct a similar height function for a finite set of primes. This construction is very useful because it allows us to focus on a finite number of primes at a time. Thus, it is useful in proofs by induction where each step requires a proof about a different prime.

We would like to define some operations on height functions. The first such operation is a way of changing a height function at a finite number of primes by a finite amount.

Definition 2.4: For a positive integer $x = p_1^{e_1} \dots p_n^{e_n}$, where each p_i is a prime, and a height function h, define $xh(p)$ as

$$xh(p) = \begin{cases} h(p) + e_i & : p = p_i \\ h(p) & : \text{otherwise} \end{cases}$$

To explain in words, xh differs from h only at the primes that are factors of x; at such primes, xh differs from h by the degree to which the prime appears in the prime factorization of x.

Having defined a sort of scalar multiplication on $\mathcal{F}$, we now define an equivalence relation. This relation will be invaluable in determining when two subgroups of the rationals are isomorphic.

Definition 2.5: For two height functions f and h, we write $f \sim h$ if and only if, for some non-negative integers m and n, $mf = nh$.

Claim: The relation $\sim$ is an equivalence relation.

Proof:

1. Reflexivity: Let h be a height function. Then $1 * h = 1 * h$, so $h \sim h$.

2. Symmetry: Let h and k be height functions such that $h \sim k$. Then for some non-negative integers m and n, $mh(p) = nk(p)$ for all primes p, and so $nk(p) = mh(p)$ for all p. Thus, $k \sim h$.

3. Transitivity: Let f, h, and k be height functions such that $f \sim h$ and $h \sim k$. Then for some non-negative integers m, n, i, and j, we have that $mf = nh$ and $ih = jk$. Then we have that $(im)f = i(mf) = i(nh) = (in)h = (ni)h = n(ih) = n(jk) = (nj)k$. Thus, $f \sim k$. (Note: It can be shown that for some integers m and n and a height function f, we $(mn)f = m(nf)$, but the proof is omitted).$\square$

While this definition of $\sim$ is useful in many applications, it is also sometimes hard to apply. Essentially, this definition says that two height functions are related if and only if they differ at a finite number of primes and if, at these primes, they only differ by a finite amount. For example, the height functions $h_{\mathbb{Z}}$ and $h_{\mathbb{Q}}$ are clearly not related by $\sim$, since they do not agree at any point (and, for that matter, they disagree by an infinite amount everywhere). However, for $A = \langle \frac{1}{2^2 * 3^2 * 5^2 * \ldots * 167^2 * 173^2} \rangle$, $h_{\mathbb{Z}}$ and h_A are related because they differ only at the first twenty primes, and neither function takes on an infinite value anywhere.

5 Subgroup Isomorphism and Types

In this section, we show how two groups whose height functions are related are similar in structure.

Theorem 3.1: If A and B are unitary subgroups of the rationals, then $A \cong B$ if and only if $h_A \sim h_B$.

Proof: We begin the proof with a lemma.

Lemma 3.1.1: For some integer x and a unitary subgroup C, $D = \frac{1}{x}C$ is also unitary, and $h_D(p) = xh_C(p)$.

Proof: Clearly, D is unitary, since $x \in C$. Now, note that the height functions of C and D will differ only at primes that divide x. Consider such a prime, say q. The height function of D at q will be increased by exactly the highest power of q that divides x, say e. Thus, $h_D(p) = \left(\prod_{q|x} e_i\right) h_C(p) = xh_C(p)$.□

Now, we return to our proof of the theorem, and begin by assuming that $A \cong B$. Then there is a bijective homomorphism $f : A \to B$. We know that for some rational number $\frac{m}{n}$, $f(x) = \frac{m}{n}x$. Thus, $\frac{1}{m}A = \frac{1}{n}B$. By the lemma, $mh_A(p) = nh_B(p)$, and $h_A \sim h_B$. The proof of the reverse direction is almost exactly the same.□

This is a very significant classification of the subgroups of the rationals. Revisiting some examples from before, note that the height function for the integers and that for the rationals are not related by $\sim$, so the integers are not isomorphic to the rationals. However, we showed that when $A = \langle \frac{1}{2^2 * 3^2 * 5^2 * \ldots * 167^2 * 173^2} \rangle$, $h_A \sim h_{\mathbb{Z}}$, so $A \sim \mathbb{Z}$.

We will now show several more examples to clarify. Consider the integers localized at 3 and the integers localized at 5. Since $h_{\mathbb{Q}_{(3)}}(3) = 0$ and $h_{\mathbb{Q}_{(5)}}(3) = \infty$, these two height functions are not related. Thus, $\mathbb{Q}_{(3)} \ncong \mathbb{Q}_{(5)}$. Next, consider the height function that is everywhere equal to 1 and the height function that is everywhere equal to 2. While these two height functions differ by a finite amount at each prime, they are different at an infinite number of points, so they are not related by $\sim$. Accordingly, their corresponding groups are not isomorphic.

Next, we will introduce some terminology that will simplify working with this new notion of our equivalence relationship's interaction with group isomorphism.

Definition 3.1: The type of a subgroup A, denoted type(A), is defined as the equivalence class of h_A under $\sim$.

We will now spend some time examining certain examples. As an example of two types that are equal, see that type$(\langle \frac{1}{2^2*3^2*5^2*\ldots*167^2*173^2} \rangle) =$ type$(\mathbb{Z})$, because, as stated previously, the height functions for these two groups are related. The infinite type, denoted $\bar{\infty}$, is the set of all height functions that are related to the infinite height function. Since any height function disagreeing with the infinite function at at least one point takes on a finite value at such points, it cannot be related to the infinite function. Thus, the only height function in the equivalence class of the infinite function is the infinite function itself. To write this with our new terminology, type$(\mathbb{Q}) = [h_{\mathbb{Q}}]$. The type of the integers, however, contains more than one element; in fact, it contains an infinite number of functions. The height function of the integers is 0 at all primes. Thus, it's type contains every height function that is non-zero at a finite number of points, and at these points takes on a finite value.

6 Combining Subgroups

We have now classified subgroup isomorphism based on our equivalence relation $\sim$ between two height functions. We now focus on different ways of combining subgroups of the rationals. Specifically, we would like to show how combining subgroups in certain ways to make new subgroups affects the height function of the result.

To begin, we will examine taking the intersection of two subgroups. It is easily shown that the intersection of two subgroups is a subgroup. We will use an example to attempt to gain an intuitive grasp of what the resulting height function might be. Let $A = \langle \frac{1}{2^3 * 3 * 11^5} \rangle$ and $B = \langle \frac{1}{2 * 3 * 11 * 13^2} \rangle$. Looking only at the prime 2, we see that in order for an rational number to be in A, 2^4 is the smallest power of 2 that cannot divide its denominator. In order for a rational number to be in B, 2^2 is the smallest power of 2 that cannot divide its denominator. Thus, in order to be in both subgroups, 2^2 cannot divide the denominator of a rational number. This would indicate that the resulting subgroup has a height function that is the minimum of the two subgroups.

We need to begin by showing that taking the minimum of two height functions preserves $\sim$.

Theorem 4.1: For height functions $h_1 \sim h_2$ and $k_1 \sim k_2$, we have that $h_1 \wedge k_1 \sim h_2 \wedge k_2$.

Proof: The most important step is showing that for $r \in \mathbb{P}$ and $h, k \in \mathcal{F}$, that $rh \wedge k \sim h \wedge k$. The only place where $rh \wedge k$ and $h \wedge k$ might differ is at the prime r. At this point, there are two cases. In one case, $h \wedge k$ takes on an infinite value. In this case, both h and k must take on an infinite value, in which case h and $rh(r)$ are equal, and it is clear that $rh \wedge k \sim h \wedge k$. In the other case, $h \wedge k$ is not infinite, and so either h or k is finite. Whichever is finite, however, $rh \wedge k$ differs from $h \wedge k$ by only a finite amount, and we have the desired result.

If r is not a prime, then, by induction, we still have that $rh \wedge k \sim h \wedge k$. In more detail, if $r = p_1^{e_1} \dots p_j^{e_j}$, we know that $p_1 h \wedge k \sim h \wedge k$. Thus, we also have that $p_1^2 h \wedge k \sim h \wedge k$. Continuing for every prime that divides r, we have the desired result. Now, given height functions $h_1 \sim h_2$ and $k_1 \sim k_2$, we know that for some $m, n \in \mathbb{N}$ that $mh_1 = nh_2$ and for some $i, j \in \mathbb{N}$ that $ik_1 = jk_2$. Then $h_1(p) \wedge k_1(p) \sim mh_1 \wedge k_1 \sim mh_1 \wedge ik_1 = nh_2 \wedge jk_2 \sim h_2 \wedge k_2$ (note that the commutativity of taking the minimum of two functions is used at the end of this proof).$\square$

This result justifies our definition of the minimum of two types.

Definition 4.1: For height functions h and k, $[h] \wedge [k] = [h \wedge k]$.

Now that we have shown that taking the minimum of two types is well-defined with respect to $\sim$, we set out to prove the result that we hypothesized with regard to the height functions of the intersection of two subgroups.

Theorem 4.2: For non-zero subgroups A and B, $h_{A\cap B}(p) = h_A(p) \wedge h_B(p)$.

Proof: Let k be an integer. Then $k \leq h_{A\cap B}$ iff $\frac{1}{p^k} \in A \cap B$ iff $\frac{1}{p^k} \in A$ and $\frac{1}{p^k} \in B$ iff $k \leq h_A(p)$ and $k \leq h_B(p)$ iff $k \leq h_A(p) \wedge h_B(p)$. This shows that $h_{A\cap B}(p) = h_A(p) \wedge h_B(p)$.$\square$

Finally, we will show how this result pertains to subgroup isomorphism. Specifically, we would like to show that taking the intersection of two subgroups preserves isomorphism.

Theorem 4.3: For non-zero subgroups $A \cong A'$ and $B \cong B'$, we have that $A \cap B \cong A' \cap B'$.

Proof: We know that $\text{type}(A) = \text{type}(A')$ and $\text{type}(B) = \text{type}(B')$. Thus, $\text{type}(A \cap B) = \text{type}(A) \wedge \text{type}(B) = \text{type}(A') \wedge \text{type}(B') = \text{type}(A' \cap B')$.$\square$

We would like to examine the sum of two subgroups now; for two subgroups

A and B, $A+B = \{a+b : a \in A, b \in B\}$. First, we will verify that $A+B$ is a subgroup.

Claim: For subgroups A, B, $A+B$ is also a subgroup.

Proof: First, note that the identity is in $A+B$ since $0 = 0+0$. Next, we show that $A+B$ is closed under addition. Let $x, y \in A+B$. Then, for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$ we have $x = a_1 + b_1$ and $y = a_2 + b_2$. Thus, $x + y = a_1 + b_1 + a_2 + b_2 = (a_1 + a_2) + (b_1 + b_2)$, so $x + y \in A + B$. Finally, we need to show that $A+B$ is closed under inverses. For $a + b \in A + B$, $-(a+b) = -a - b \in A + B$ since $-a \in A$ and $-b \in B$.$\square$

Again, we will begin by using an example to search for an intuitive way to describe the height function of the sum of two subgroups. Let $A = \langle \frac{1}{2^3 * 3 * 11^5} \rangle$ and $B = \langle \frac{1}{2*3*11*13^2} \rangle$. Again, looking at the prime 2, we know that $\frac{1}{2^3} \in A$ and $0 \in B$. Thus, $\frac{1}{2^3} + 0 = \frac{1}{2^3} \in A + B$. Thus, one might suspect that the height function of $A+B$ is the maximum of the height functions of A and B.

Similar to before, we first need to show that taking the maximum of height functions preserves $\sim$. The argument that for $h, k \in \mathcal{F}$, and $r \in \mathbb{N}$ that $rh \vee k \sim h \vee k$ is analogous to the proof of Theorem 4.1, shown above, but the "ands" are replaced with "ors".

Theorem 4.4: For height functions $h_1 \sim h_2$ and $k_1 \sim k_2$, we have that $h_1 \vee k_1 \sim h_2 \vee k_2$.

Proof: First, we show that for $r \in \mathbb{P}$ and $h, k \in \mathcal{F}$, that $rh \vee k \sim h \vee k$. The only place where $rh \vee k$ and $h \vee k$ might differ is at the prime r. At this point, there are two cases. In one case, $h \vee k$ takes on an infinite value. In this case, either h or k must take on an infinite value, in which case it is clear that $rh \vee k$ is infinite and so $rh \vee k \sim h \vee k$. In the other case, $h \vee k$ is not infinite, and so neither h nor k is infinite. Thus, $rh(r)$ is also finite, and so $rh \vee k$ is finite and differs from $h \vee k$ by only a finite amount, and we have the desired result. If r is not a prime, then, by induction, we still have that $rh \vee k \sim h \vee k$ by an analogous argument to that used for the minimum of two height functions. Now, given height functions $h_1 \sim h_2$ and $k_1 \sim k_2$, we know that for some $m, n \in \mathbb{N}$ that $mh_1 = nh_2$ and for some $i, j \in \mathbb{N}$ that $ik_1 = jk_2$. Then $h_1(p) \vee k_1(p) \sim mh_1 \vee k_1 \sim mh_1 \vee ik_1 \sim nh_2 \vee jk_2 \sim h_2 \vee k_2$.$\square$

This result justifies our definition of the maximum of two types.

Definition 4.2: For height functions h and k, $[h] \vee [k] = [h \vee k]$.

Theorem 4.5: For two subgroups A and B, $h_{A+B}(p) = h_A(p) \vee h_B(p)$.

Proof: Let k be an integer. Then $k \leq h_{A+B}(p)$ iff $\frac{1}{p^k} \in A + B$ iff $\frac{1}{p^k} \in A$ or $\frac{1}{p^k} \in B$ iff $k \leq h_A(p)$ or $k \leq h_B(p)$ iff $k \leq h_A(p) \vee h_B(p)$. This shows that $h_{A+B}(p) = h_A(p) \vee h_B(p)$$\square$

As was the case with the intersection of two subgroups, we end by showing that taking the sum of two subgroups preserves isomorphism.

Theorem 4.6: For subgroups $A \cong A'$ and $B \cong B'$, that $A + B \cong A' + B'$.

Proof: First, note that $[h_A] = [h_{A'}]$ and $[h_B] = [h_{B'}]$. Then $[h_{A+B}] = [h_A] \vee [h_B] = [h_{A'}] \vee [h_{B'}] = [h_{A'+B'}]$. Thus, $A + B \cong A' + B'$.$\square$

Definition 4.3: For two subgroups A and B, define the product $AB = \{ab : a \in A, b \in B\}$.

Next, we will define the product of two height functions and show how it relates to the product of two subgroups.

Definition 4.4: For $h, k \in \mathcal{F}$ let

$$hk(p) = \begin{cases} \infty & : h(p) = \infty \text{ or } k(p) = \infty \\ h(p) + k(p) & : \text{otherwise.} \end{cases}$$

Theorem 4.6: If $h_1 \sim h_2$ and $k_1 \sim k_2$ then $h_1k_1 \sim h_2k_2$.

Proof: We need to show that for a prime p and height functions h and k, that $(ph)(k) \sim (h)(k)$. If hk is infinite, then h or k is infinite, so either ph or k is infinite, so $(ph)(k)$ is infinite. If hk is finite, then neither h nor k is infinite, so neither ph nor k is infinite and so $(ph)(k)$ is finite. The proof is now easily completed by induction.$\Box$

As before, this justifies the definition $[h][k] = [hk]$.

Definition 4.4: For height functions h and k, $[h][k] = [hk]$.

Theorem 4.7: Suppose that A and B are non-zero subgroups of the rationals. Then $AB = \{ab : a \in A, b \in B\}$ is a subgroup of the rationals and type(AB) =type(A)type(B).

Proof: First, note that the 0 is in AB because $0 * 0 = 0$. Now, we need to show that AB is closed under the operation of addition. Let $x, y \in AB$. Note that, for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$, we have that $x = a_1b_1$ and $y = a_2b_2$. First, consider $< a_1 >$ and $< a_2 >$. These subgroups are cyclic, and are thus isomorphic to the integers. Therefor, they are in the same class as the integers, namely the zero class. We now have that type$(< a_1 >) =$ type$(< a_2 >) = \bar{0}$. Thus, type$(< a_1 > + < a_2 >) = \bar{0}$, and so this group is cyclic, generated by some element $a_3 \in A$. Clearly, $a_1 \in < a_1 > + < a_2 >$ and $a_2 \in < a_1 > + < a_2 >$. Thus, for some integers m and n we can write $x+y = a_1b_1+a_2b_2 = ma_3b_1+na_3b_2 = a_3(mb_1+nb_2) \in AB$. Our goal now is to show that $h_{AB}(p) = h_A(p) + h_B(p)$. Let $k \in \mathbb{Z}$. Then $k \leq h_{AB}(p)$ iff $\frac{1}{p^k} \in AB$ iff there exist $k_1, k_2 \in \mathbb{N}$ such that $k_1 + k_2 = k$ and $\frac{1}{p^{k_1}} \in A$ and $\frac{1}{p^{k_2}} \in B$ iff $k = k_1 + k_2 \leq h_A(p) + h_B(p)$. This proves that $h_{AB}(p) = h_A(p) + h_B(p)$.$\Box$

We will now begin assembling the tools needed to examine the analog to a quotient for two subgroups.

Definition 4.5: If $h, k \in \mathcal{F}$, we will say that $h \leq k$ if and only if for all $p \in \mathcal{P}$ we have that $h(p) \leq k(p)$. For two types τ_1, τ_2, we will say that $\tau_1 \leq \tau_2$ if and only if there is an $h \in \tau_1$ and a $k \in \tau_2$ such that $h \leq k$.

Colloquially, this definition is saying that type(h) is less than or equal to type(k) if and only if h is larger than k only at a finite number of primes, and, furthermore, anywhere that h is infinite, k is also infinite. We would like to now show that it is a partial ordering of $\mathcal{T}$.

Claim: The relation $\leq$ is a partial ordering of $\mathcal{T}$.

Proof:

- Reflexivity: Let $\tau \in \mathcal{T}$. Let $h \in \tau$. Then $h \leq h$ so $\tau \leq \tau$.

- Antisymmetry: Let $\tau_1, \tau_2 \in \mathcal{T}$ such that $\tau_1 \leq \tau_2$ and $\tau_2 \leq \tau_1$. So there exists a $h_1, k_1 \in \tau_1$ and $h_2, k_2 \in \tau_2$ such that $h_1 \leq h_2$ and $k_2 \leq k_1$. Suppose that $f \in \tau_1$. We need to show that f is related to some height function in τ_2. Suppose that $f(p) = \infty$ for a prime p. Because $f \sim h_1$, we know that $h_1(p)\infty$. Because $h_1 \leq h_2$, we now also can conclude that $h_2(p) = \infty$. Now suppose that $f(p)$ is finite. Then we know that $k_1(p)$ is also finite. Since $k_2 \leq k_1$, k_2 is also finite. Finally, since $k_2 \sim h_2$, we know that h_2 is also finite. Thus, anywhere that f is infinite, so is h_2, and anywhere that f is finite, so is h_2. Now, note that f differs from h_1 and k_1 only at a finite number of primes. Additionally, see that $h_2 \leq h_1$ only at a finite number of points and that $k_1 \leq k_2$ only at a finite number of points. Thusly, f is greater that h_2 and less than k_2 only at a finite number of points. Since h_2 and k_2 differ on a finite set, it follows that f differs from h_2 and k_2 only at a finite number of primes. Thus, $f \in \tau_2$. A similar argument shows that a function in τ_2 is also in τ_1.

- Transitivity: Let $\tau_1, \tau_2, \tau_3 \in \mathcal{T}$ such that $\tau_1 \leq \tau_2$ and $\tau_2 \leq \tau_3$. Then there exists a $h_1 \in \tau_1$ and a $h_2 \in \tau_2$ such that $h_1(p) \leq h_2(p)$ and there also exists a $k_1 \in \tau_2$ and a $k_2 \in \tau_3$ such that $k_1(p) \leq k_2(p)$. If $h_1(p) = \infty$, then so does $h_2(p)$. Since $h_2 \sim k_1$, $k_1(p) = \infty$, implying that $k_2(p) = \infty$. So we have that $h_1(p) \leq k_2(p)$. Now if $h_1(p)$ is finite then we have two cases. First, $h_2(p)$ may be infinite. In this case, by the same arguments just made, we know that $k_2(p) = \infty$ and so $h_1(p) \leq k_2(p)$. If $h_2(p)$ is also finite, then $h_2(p) = h_1(p) + m$ for some integer m. Since $h_2 \sim k_1$, we know that $h_2 \sim k_1 + m$ where $(k_1 + m)(p) = k_1(p) + m$. Finally, since $k_1 \leq k_2$, $k_1 + m \leq k_2$. Note that this implies that $h_1(p) \leq k_2(p)$. And so $\tau_1 \leq \tau_2$.
  $\square$

We would like to continue looking at the partial ordering $\leq$, with the ultimate goal being to examine, as we have for intersections, sums, and products, some form of quotient of subgroups. As we have already defined the minimum and maximum of types, we will relate $\leq$ to these notions with the following result.

Theorem 4.8: For two types τ_1 and τ_2, the following are equivalent:

1. $\tau_1 \leq \tau_2$
2. $\tau_1 \vee \tau_2 = \tau_2$
3. $\tau_1 \wedge \tau_2 = \tau_1$

Proof:

1. (1) $\rightarrow$ (2): Since $\tau_1 \leq \tau_2$, there exist height functions $h \in \tau_1, k \in \tau_2$ such that, for all primes, $h(p) \leq k(p)$. Then $\tau_1 \vee \tau_2 = [h] \vee [k] = [h \vee k] = [k] = \tau_2$.
2. (2) $\rightarrow$ (3): Suppose that $\tau_1 = [h]$ and $\tau_2 = [k]$ for height functions h and k. Then $\tau_1 \vee \tau_2 = \tau_2 \rightarrow [h] \vee [k] = [k] \rightarrow [h \vee k] = [k] \rightarrow h \vee k = k \rightarrow$

$h \wedge k = h \to [h \wedge k] = [h] \to \tau_1 \wedge \tau_2 = \tau_1$.

3. (3) $\to$ (1): If $\tau_1 = [h]$ and $\tau_2 = [k]$, $\tau_1 \wedge \tau_2 = \tau_1 \to h \wedge k = h \to h \leq k \to \tau_1 \leq \tau_2$.$\square$

We are now ready to introduce the concept of the quotients of two height functions and, in turn, types.

Definition 4.6: For two height functions $h \leq k$, define

$$\frac{k}{h}(p) = \begin{cases} \infty & : k(p) = \infty \\ k(p) - h(p) & : otherwise \end{cases}$$

We now must show that taking the product of two height functions preserves $\sim$, which, as before, justifies our definition of $\left[\frac{h}{k}\right] = \frac{[h]}{[k]}$.

Theorem 4.9: For height functions $h \sim h'$ and $k \sim k'$ where $k \leq h$ and $k' \leq h'$, $\frac{h}{k} \sim \frac{h'}{k'}$.

Proof: First, we will show that for height functions $g \leq f$ and a natural number x, that $\frac{xf}{g} \sim \frac{f}{g}$. Since $f \leq xf$, it is obvious that $g \leq xf$. Thus, the quotient $\frac{xf}{g}$ is defined. If $f(p) = \infty$, then $xf(p) = \infty$, so wherever $\frac{f}{g}(p)$ is infinite, so is $\frac{xf}{g}$. Furthermore, f and xf differ only at those primes that divide x, which is a finite set, and so $\frac{xf}{g} \sim \frac{f}{g}$.

Next, we show that $\frac{xh}{xk} = \frac{h}{k}$. These two height functions will potentially differ only at those primes that divide x, so we need only show that they agree at these points. Let $p|x$, but $p^n \not| x$ for some n. Then $\frac{xh}{xk} = xh(p) - xk(p)$. Suppose that e is the largest integer such that $p^e|x$. Then $xh(p) - xk(p) = h(p) + e - k(p) - e = h(p) - k(p) = \frac{h}{k}(p)$. If $h(p) = \infty$, then $xh(p) = \infty$, so, again, $\frac{xh}{xk} = \frac{h}{k}(p)$.

Now, we can complete the proof. Note that there exist an $i, j, m, n \in \mathbb{N}$ such that $ih = jh'$ and $mk = nk'$. Thus, $\frac{h}{k} \sim \frac{jmh}{k} = \frac{ijmh}{ik} \sim \frac{ijnh'}{jk'} = \frac{inh'}{k'} \sim \frac{h'}{k'}$.$\square$

Now, we define something similar to a quotient, except for two subgroups.

Definition 4.7: For two subgroups A, B, define $\mathrm{Hom}(A, B) = \{q \in \mathbb{Q} : qA \subset B\}$.

The notation is thus because $\mathrm{Hom}(A, B)$ can also be thought of as the collection of homomorphisms from A to B.

Theorem 4.10: For two unitary subgroups A, B, $\mathrm{Hom}(A, B)$ is a subgroup of $\mathbb{Q}$.

Proof: Clearly, $0 \in \mathrm{Hom}(A, B)$ since $0A = \{0\}$. Now, we need to show that $\mathrm{Hom}(A, B)$ is closed under addition. Let $r, s \in \mathrm{Hom}(A, B)$. So $rA \subset B$ and $sA \subset B$. Thus, for all $a \in A$, $ra \in B$ and $sa \in B$, and so $(ra)+(sa) = (r+s)a \in B$. Therefore, we have that $rs \in \mathrm{Hom}(A, B)$. Next, we have to show closure under inverses. Suppose $t \in \mathrm{Hom}(A, B)$. Then for all $a \in A$, $ta \in B$ and, furthermore, $t(-a) = -ta \in B$. Thus, $\mathrm{Hom}(A, B)$ is closed under inverses.$\square$

Intuitively, $\mathrm{Hom}(A, B)$ can be thought of as the quotient when B is divided by A. We will now prove a result that is analogous to the statement "a rational number a has a non-zero quotient when divided by b if and only if $b \leq a$.

Theorem 4.11: For unitary subgroups A and B, $\mathrm{Hom}(A, B) \neq \{0\}$ if and only if $\mathrm{type}(A) \leq \mathrm{type}(B)$.

Proof: Begin by supposing that $\mathrm{Hom}(A, B) \neq \{0\}$. Then there is a rational number $q \neq 0$ such that $qa \in B$ for all $a \in A$. If a prime p divides the denominator of a, then, for some integer l, $p+l$ divides qa. In other words, $h_A(p)$ and $h_{qA}(p)$ differ by a finite amount. Since $qA \subset B$, $h_{qA} \leq h_B$. By the previous two statements, $\mathrm{type}(A) \leq \mathrm{type}(B)$. The reverse direction of this proof follows from the forward argument.

Now, we show the relationship between the type of $\mathrm{Hom}(A, B)$ and the type of both A and B. Predictably, it involves the notion of the quotient of two types.

Theorem 4.12: For unitary subgroups A, B, where $\mathrm{Hom}(A, B) \neq \{0\}$, we have that $\mathrm{type}(\mathrm{Hom}(A,B)) = \frac{\mathrm{type}(B)}{\mathrm{type}(A)}$.

Proof: For an integer k, $k \le h_{\mathrm{Hom}(A,B)}(p)$ iff there exists a $\frac{m}{n} \in \mathrm{Hom}(A,B)$ such that m and n are relatively prime and $p^k|n$. Note that $\frac{m}{n}A \subseteq B$, so the previous statement is true iff $k + h_A(p) \le h_B(p)$ iff $k \le h_B(p) - h_A(p)$ iff $k \le \frac{h_B}{h_A}(p)$ iff $h_{\mathrm{Hom}(A,B)} = \frac{h_B}{h_A}(p)$. Thus, $\mathrm{type}(\mathrm{Hom}(A,B)) = \frac{\mathrm{type}(h_B)}{\mathrm{type}(h_A)}$.□

Theorem 4.13: For unitary subgroups A, B, where $\mathrm{Hom}(A,B) \neq \{0\}$, we have that $\mathrm{type}(\mathrm{Hom}(A,B)) = \frac{\mathrm{type}(B)}{\mathrm{type}(A)}$.

Proof: Suppose that an integer $k \le h_{\mathrm{Hom}(A,B)}(p)$. This is the case if and only if there exists an element $\frac{m}{n} \in \mathrm{Hom}(A,B), (m,n) = 1$ such that $p^k|n$. Thus, since $\frac{m}{n}A \subset B$, $h_A(p) + k \le h_b(p)$, and so $k \le h_B(p) - h_A(p) = \frac{h_B}{h_A}(p)$. Thus, $h_{\mathrm{Hom}(A,B)}(p) = \frac{h_B}{h_A}(p)$ and $\mathrm{type}(\mathrm{Hom}(A,B)) = \mathrm{type}(B)/\mathrm{type}(A)$.□

Finally, we will complete our study of the Hom operation on two subgroups by showing that it preserves isomorphism.

Theorem 4.14: For unitary subgroups $A \cong A'$ and $B \cong B'$, $\mathrm{Hom}(A,B) \cong \mathrm{Hom}(A',B')$.

Proof: First, note that $[h_A] = [h_{A'}]$ and $[h_B] = [h_{B'}]$. So $[h_{\mathrm{Hom}(A,B)}] = [h_B/h_A] = [h_B]/[h_A] = [h_{B'}]/[h_{A'}] = [h_{B'}/h_{A'}] = [\mathrm{Hom}(A',B')]$. Thus, $\mathrm{Hom}(A,B) \cong \mathrm{Hom}(A',B')$.□

7 Result Relating to General Torsion-Free Groups

We will now prove a theorem that is very elegant. It gives circumstances under which an arbitrary torsion-free group is isomorphic to a subgroup of the rational numbers. This is incredibly useful because it allows us to use the rational numbers to help typify torsion-free groups.

Theorem 5.1: For a torsion free group A, the following statements are equivalent:

1. A is isomorphic to a subgroup of $\mathbb{Q}$.

2. For any two non-zero subgroups X and Y of A, $X \cap Y \neq \{0\}$.

3. For any two non-zero subgroups B and C of A, if B and C are cyclic, then so is $B + C$.

Proof:

- (1) $\rightarrow$ (3) : This proof has essentially been completed elsewhere in this paper. If B and C are cyclic, then their type is $\bar{0}$, so the type of their sum is also $\bar{0}$, and their sum is thusly cyclic.

- (3) $\rightarrow$ (2) : First, let $X' \subset X$ and $Y' \subset Y$ be cyclic subgroups. From our hypothesis, $X' + Y' = S$ is cyclic. If $X' \subset S$ and $Y' \subset S$, then $X' \cap Y' \neq \{0\}$, and thus $X \cap Y \neq \{0\}$

- (2) $\rightarrow$ (1) : Suppose that x is a non-zero element of A. If y is another element of A, then, by hypothesis, $\langle x \rangle \cap \langle y \rangle \neq \{0\}$. Let f be a function such that $f(y) = \frac{k}{l} \in \mathbb{Q}$ where$k, l \in \mathbb{Z}$ such that $kx = ly \neq 0$. We will show that f is an isomorphism from A onto the image of f. First, f is a homomorphism, since, for $a, b \in A$ if j, k, m, n are such that $jx = ka$ and $mx = nb$, then we can multiply both equations by n and k respectively, then add them together, resulting in the equation $jnx + mkx = kna + knb$. Thus, $(jn + mk)x = kn(a + b)$, and so $f(a + b) = \frac{jn+mk}{kn} = \frac{j}{k} + \frac{m}{n} = f(a) + f(b)$. Because we are restricting our attention to the image of f, that f is onto is obvious. Next, f is one-to-one, since, if $f(a) = f(b)$, then $am = xn$ for some m, n, and also $bm = xn$, so $a = b$. Thus, f is an isomorphism, and we have completed our proof.$\square$

8 Conclusion

We have now seen, in detail, the structure of subgroups of the rationals under addition. Furthermore, we have classified them by isomorphism, examined combinations of subgroups, and, finally, proved a significant result regarding the relationship between general torsion-free groups and subgroups of the rationals.

Significantly, the paper is targeted at an undergraduate level, meaning that an undergraduate who studies this paper will have the tools to fully understand it. While the conclusions we have proven are an important first step, the tools we have developed can be used to move much further into the study of the rational-numbers, and, indeed, torsion-free groups generally. The ideas in this paper form an excellent method of studying group theory and number theory. It is the hope of the author that students in the future will study the subjects presented, and use them to further raise the bar of undergraduate mathematical knowledge.

9 Works Cited

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