PERIMETERS OF PRIMITIVE PYTHAGOREAN TRIANGLES

LINDSEY WITCOSKY
ADVISOR: DR. RUSS GORDON

Abstract. This paper examines two methods of determining whether a positive integer can correspond to the semiperimeter of a number of Pythagorean triangles. For all positive integers k, using Bertrand's Postulate, we can find semiperimeters corresponding to k or more isoperimetrical triangles, and using the Prime Number Theorem, we can find exactly k generator pairs which correspond to a semiperimeter.

1. Introduction

Pythagorean triangles are familiar to any high school geometry student. Many could come up with the classic example, a triangle with sides of length 3, 4 and 5. They have likely also come into contact with triangles which have the same perimeter, but different length sides. Less well known are distinct Pythagorean triangles with equal perimeters - and rightly so, since the smallest example of two such Pythagorean triangles share a perimeter of 1716. In this paper, we find methods of determining whether or not a given perimeter can correspond to any Pythagorean triangles, and a way of creating triangles which share their perimeter with an exact number of distinct triangles. The methods are based largely upon the work of A. A. Krishnaswami and Leon Bernstein, from their articles "On Isoperimetrical Pythagorean Triangles"[6] and "On Primitive Pythagorean Triangles With Equal Perimeters,"[2] respectively.

2. Generators for a PPT

A Pythagorean triple consists of three positive integers, (a, b, c) where c is the largest, such that they can represent the sides of a Pythagorean (or right) triangle. The well-known Pythagorean theorem states that for such triples, $a^2 + b^2 = c^2$, where c is the hypotenuse. We call a Pythagorean triple *primitive* if the three numbers are pairwise relatively prime, that is, no two share a common factor greater than 1. To indicate that two integers a and b are relatively prime, we will use the notation $\gcd(a, b) = 1$.

Theorem 1. *If (a, b, c) is a primitive Pythagorean triple then one of a or b is even.*

Proof. It is obvious that a and b cannot both be even for the triple to remain primitive. Assume that both a and b are odd. This means that for some positive integers k and j, $a = 2k - 1$ and $b = 2j - 1$, and therefore

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (2k-1)^2 + (2j-1)^2 &= c^2 \\ 4k^2 - 4k + 4j^2 - 4j - 2 &= c^2 \end{aligned}$$

Because both a and b are odd, c must be even. Let $c = 2i$. This gives us:

$$\begin{aligned} 4k^2 - 4k + 4j^2 - 4j - 2 &= 4i^2 \\ 4k^2 - 4k + 4j^2 - 4j - 4i^2 &= 2 \end{aligned}$$

Now we can factor 4 out of the left side of this equation, but this means that 4 divides 2, which is a contradiction. Therefore a and b cannot both be odd.

Thus exactly one must be even, and c must be odd. □

A natural place to begin a discussion of Pythagorean triangles is a method of generating these triples. In Euclid's *Elements* Book X, Proposition 29, he provides a generator formula for Pythagorean triples. The following formula is a different wording of the same result, and my proof follows the style of Keith Conrad's expository paper, "Pythagorean Triples."[4]

Theorem 2. *The positive integers a, b, and c (where a is even) form a primitive Pythagorean triple if and only if there exist relatively prime positive integers s and t such that $t > s > 0$, exactly one is even,* $\gcd(s,t) = 1$, *and*

$$a = 2st, \qquad b = t^2 - s^2, \qquad c = t^2 + s^2.$$

Proof. Let (a, b, c) where a is even be a primitive Pythagorean triple. We find that

$$a^2 = c^2 - b^2 = (c+b)(c-b).$$

Because both c and b are odd, $c + b$ and $c - b$ must be even. Using simple algebra,

$$\left(\frac{a}{2}\right)^2 = \frac{c+b}{2} \cdot \frac{c-b}{2}$$

Let us prove that $(c+b)/2$ and $(c-b)/2$ are relatively prime. It is a common result in number theory that if an integer divides two numbers, it must also divide their sum and difference. Thus if d is a prime divisor of both of these fractions, then it must divide c and b. By assumption, (a, b, c) is primitive, and so c and b are relatively prime. Thus $d = 1$, and so the fractions $(c+b)/2$ and $(c-b)/2$ are relatively prime. Because by definition c is the hypotenuse, and thus $c > b > 0$, we can see that both $(c + b)/2$ and $(c - b)/2$ are positive. Because they are relatively prime and their product is a square, the Fundamental Theorem of Arithmetic indicates that there exist relatively prime positive integers s and t such that

$$\frac{c+b}{2} = t^2, \qquad \frac{c-b}{2} = s^2.$$

Because we have proved that the fractions equivalent to s^2 and t^2 are relatively prime, it is clear that $\gcd(s,t) = 1$. Algebraically, it follows that $b = t^2 - s^2$, $c = t^2 + s^2$, and thus $a = 2st$.

It still remains to prove that s and t are not of the same parity. If they were of the same parity, then b and c would both be even, which would contradict the fact that (a,b,c) is primitive. Finally, since $b = t^2 - s^2 > 0$, we can conclude that $t > s > 0$.

A proof of the converse is simple. Suppose s and t are positive integers that satisfy $t > s > 0$, $\gcd(s,t) = 1$, and exactly one is even. We can see then that the given formula generates a primitive Pythagorean triple.

$$\begin{aligned} a^2 + b^2 &= (2st)^2 + (t^2 - s^2)^2 \\ &= t^4 + 2s^2t^2 + s^4 \\ &= t^2 + s^2 \\ &= c^2 \end{aligned}$$

Thus we have $a^2 + b^2 = c^2$, but it remains to prove that the triple (a,b,c) is primitive. Assume that it is not primitive, and thus the elements are not relatively prime. Then $\gcd(b,c) = d \neq 1$. Any divisor of b and c must also divide their sum and difference, $2t^2$ and $2s^2$. However, because c is odd, $d \neq 2$. Because s and t are relatively prime, d must equal 1. This is a contradiction. Therefore, the formulas in Theorem 2 generate a primitive Pythagorean triple. □

The next theorem provides a second generating formula which will prove more useful for the purposes of this paper.

Theorem 3. *The positive integers a, b, and c (where a is even) form a primitive Pythagorean triple, taking c to be the hypotenuse, if and only if there exist relatively prime positive integers u and v such that $u < v < 2u$, v is odd, and*

$$a = 2uv - 2u^2, \qquad b = 2uv - v^2, \qquad c = 2u^2 - 2uv + v^2.$$

Proof. Let (a,b,c) be a primitive Pythagorean triple. From Theorem 2, there exist relatively prime integers s and t such that $t > s > 0$ and exactly one is even, such that the formulas in Theorem 2 hold. Let $u = t$ and $v = s + t$. Therefore, we have $s = v - u$, so that

$$\begin{aligned} a &= 2u(v-u) = 2uv - 2u^2 \\ b &= u^2 - (v-u)^2 = u^2 - v^2 + 2uv - u^2 = 2uv - v^2 \\ c &= u^2 + (v-u)^2 = 2u^2 - 2uv + v^2 \end{aligned}$$

Since $s < t$, it is clear that $t < s + t < 2t$, so $u < v < 2u$, and because t and s are of opposite parity, v must be odd. Because $\gcd(s,t) = 1$, from the way we chose u

and v it is clear that $\gcd(u, v) = 1$ as well. Thus we have proven that (u, v) is a generator pair for (a, b, c) with the specified qualities.

Next, beginning with relatively prime positive integers u and v such that $u < v < 2u$ and v is odd, we will prove that the formula in Theorem 3 generates a primitive Pythagorean triple. With some algebraic manipulation, we can see that these values satisfy the Pythagorean Theorem:

$$\begin{aligned}
(2uv - 2u^2)^2 + (2uv - v^2)^2 &= 4u^2v^2 - 8u^3v + 4u^4 + 4u^2v^2 - 4uv^3 + v^4 \\
&= 4u^4 - 8u^3v + 8u^2v^2 - 4uv^3 + v^4 \\
(2u^2 - 2uv + v^2)^2 &= 4u^4 - 8u^3v + 8u^2v^2 - 4uv^3 + v^4
\end{aligned}$$

Similarly to the proof of Theorem 2, assume that the triple (a, b, c) is not primitive. Then there exists some positive integer $d \neq 1$ which divides all three values a, b and c. Thus d must also divide $a + c = v^2$ and $b + c = 2u^2$. However, because u and v are relatively prime, d must equal 1. This is a contradiction, and so the triple (a, b, c) must be primitive. This completes the proof. □

We will call such a u and v a *generator pair.* Each primitive Pythagorean triangle corresponds to a distinct generator pair. This is because the matrix

$$\begin{pmatrix} -2 & 2 & 0 \\ 0 & 2 & -1 \\ 2 & -2 & 1 \end{pmatrix}$$

is invertible, and so

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2 & 2 & 0 \\ 0 & 2 & -1 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} u^2 \\ uv \\ v^2 \end{pmatrix}$$

gives us a one-to-one mapping to (a, b, c). If there were another generator pair (u', v') which satisfied the above equation, so that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2 & 2 & 0 \\ 0 & 2 & -1 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} u'^2 \\ u'v' \\ v'^2 \end{pmatrix}$$

then we will have $u = u'$ and $v = v'$. Thus this generator pair (u, v) is unique, and each primitive Pythagorean triangle is generated by precisely one generator pair.

3. A Discussion of Perimeters

We are interested in the perimeters of right triangles which correspond to primitive Pythagorean triples. Using Theorem 3, the perimeter of such a triangle is equal to

$$a + b + c = 2uv$$

where u and v are positive, relatively prime integers that satisfy $u < v < 2u$ and v is odd. Thus an integer must be even for it to possibly be the perimeter of a primitive right triangle. We will deal primarily with the semiperimeter, denoted s, which is defined as half of the perimeter, so $s = uv$. A positive integer s can be the semiperimeter of a primitive right triangle if and only if it can be factored as $s = uv$, where $(u, v) = 1$, $u < v < 2u$ and v is odd. We will be interested in distinct triangles which share a perimeter - such triangles are called *isoperimetrical.* Because each generator pair corresponds to a distinct primitive right triangle, if we have k generator pairs which correspond to a particular s, then we have k distinct isoperimetrical right triangles with semiperimeter s. It is clear that the conditions $s = uv$ and $u < v < 2u$ are equivalent to $\sqrt{s} < v < \sqrt{2s}$, hence the number of primitive right triangles with semiperimeter s depends on the number of odd divisors v of s which are in the interval $(\sqrt{s}, \sqrt{2s})$. Thus the following definition will be important.

Definition 1. *Let z be a positive integer. A positive integer n is a unitary divisor of z if n divides z and the integers n and z/n are relatively prime. If $z = xy$ and y is an odd unitary divisor of z that satisfies $\sqrt{z} < y < \sqrt{2z}$, then we call this product a P-factorization of z and write $z = x \times y$. Note that the order in which the factors x and y are written is important.*

Counting the number of distinct P-factorizations of s gives the number of primitive right triangles to which s corresponds. The following theorem provides a simple way to check for P-factorizations.

Theorem 4. *Suppose that $z > 1$ is a positive integer and that $z = xy$, where x and y are positive integers. Then $z = \min\{x, y\} \times \max\{x, y\}$ is a P-factorization of z if and only if $\gcd(x, y) = 1$, $\max\{x, y\}$ is odd, and the ratio x/y is between $1/2$ and 2.*

Proof. Assume that $z = \min\{x, y\} \times \max\{x, y\}$ is a P-factorization of z. Thus $\max\{x, y\}$ is an odd unitary divisor of z, and $\gcd(x, y) = 1$ from the definition of a unitary divisor. Because $\max\{x, y\}$ is a unitary divisor of z, the conditions of Theorem 3 apply, and the ratio x/y comes from the stipulation that $u < v < 2u$. When x is the larger of the two, we have $1 < x/y < 2$, and when x is the smaller, we have $1/2 < x/y < 1$. Therefore x/y is between $1/2$ and 2.

Assume that $(x, y) = 1$, $\max(x, y)$ is odd, and the ratio x/y is between $1/2$ and 2, and let $z = \min\{x, y\} \times \max\{x, y\}$. Because $\max\{x, y\}$ divides z and $\gcd(x, y) = 1$, $\max\{x, y\}$ is a unitary divisor of z. It is clear that $1/2 < x/y < 2$ gives us $\min\{x, y\} < \max\{x, y\} < 2\min\{x, y\}$, which is equivalent to $\sqrt{z} < \max\{x, y\} < \sqrt{2z}$. Therefore, $z = \min\{x, y\} \times \max\{x, y\}$ is a P-factorization of z. This completes the proof. □

The following theorem will make use of the greatest integer function, also known as the floor function. Let us define it here.

Definition 2. *For all $x \in \mathbb{R}$, $\lfloor x \rfloor$ is equal to the largest integer less than or equal to x.*

The next theorem concerns integers which can and cannot be semiperimeters of primitive right triangles.

Theorem 5. *The following hold:*

a. *If p is a prime and r is a positive integer, then $s = p^r$ is not the semiperimeter of any primitive right triangle.*
b. *If p is an odd prime, then $s = p(p+2)$ is the semiperimeter of exactly one primitive right triangle.*
c. *If $p > 5$ is an odd prime of the form $3k+2$, then $s = 6p(p+2)$ is the semiperimeter of exactly two primitive right triangles.*
d. *Suppose that p is an odd prime and that f is a positive integer. Then $s = 2^r p^f$ is the semiperimeter of exactly one primitive right triangle if and only if $r = \lfloor \log_2 p^f \rfloor$.*

Proof. The result in part (a) follows from Theorem 3. Because s must be factorizable into the product of two relatively prime integers, it cannot be the power of a single prime. If we factor $s = uv$ as $u = 1$ and $v = p^r$, it is clear that $1 < p^r < 2$ does not hold for any prime p. Therefore, $s = p^r$ does not have any P-factorizations and cannot correspond to the semiperimeter of any primitive right triangles.

For part (b), note that because p is an odd prime, $p > 2$, and $1 < (p+2)/p < 2$. It is clear that $\gcd(p, p+2) = 1$ and $p+2$ is odd. Because $p > 2$,

$$\begin{aligned} 2p < {} & 4p+4 < p^2+4p \\ p^2+2p < {} & p^2+4p+4 < 2p^2+4p \\ \sqrt{s} < {} & p+2 < \sqrt{2s} \end{aligned}$$

Therefore, $s = p \times (p+2)$ satisfies the requirements of a P-factorization of s. We hope to prove that it is the only P-factorization of s. If $p+2$ is prime, then there are simply no other ways to factor s. If $p+2$ is composite, let us choose one factorization, such that $p+2 = ab$ where $3 \le a < b < p$ and $\gcd(a,b) = 1$. Then $s = abp$, and there are 8 unitary divisors of s. By the restriction that $u < v < 2u$, we can see that we cannot factor $s = u \times v$ where v is either a, b, or p, lest we have $p+2 < p$. We also cannot have either $u = 1$ or $v = 1$. Therefore we have three cases to try. Let $u = b$ and $v = pa$, so $b < pa < 2b$. Because $a > 2$, we know that $2b < p+2$ and $p+2 < pa$. This contradicts $b < pa < 2b$, and so we can rule out $s = b \times pa$. Next, choose $u = a$ and $v = pb$, and thus $a < bp < 2a$. However, because $a < b$, this gives us $bp < 2a < 2b$, and therefore $p < 2$. This is a contradiction. Therefore the only P-factorization of s that works is $s = p \times (p+2)$, and so any number of the form $p(p+2)$ can correspond to only one primitive right triangle.

For part (c), we will prove that the only P-factorizations of s are $s = 2p\times 3(p+2)$ and $s = 2(p+2)\times 3p$. Note that these are unitary divisors because p is an odd prime of the form $3k+2$, and so $p+2$ cannot be divisible by 2 or 3. It is also clear that $2p < 3p+6 \le 4p$ and $2p+4 < 3p < 4p+8$, and so by Theorem 4, both are P-factorizations. If $p+2$ is prime, these are the only factorizations of s.

Suppose $p+2$ is composite and $p+2 = xy$, where $5 \le x < y < p$ and $\gcd(x, y) = 1$. Since $s = 2\cdot 3\cdot x\cdot y\cdot p$, there are $2^5 = 32$ unitary divisors of s corresponding to this factorization of $p+2$. Half of these divisors involve two or fewer terms:

$$1, 2, 3, x, y, p, 6, 2x, 2y, 3x, 3y, xy, 2p, 3p, xp, yp$$

By Theorem 4, the ratio of factors in a P-factorization must be between 1/2 and 2. It is clear that $p/(p+2) > 1/2$. First note that

$$\frac{xy}{6p} = \frac{1}{6}\frac{p+2}{p} < \frac{1}{3} < \frac{1}{2}$$

and therefore by Theorem 4, xy cannot be a factor in a P-factorization of s, because it violates the restrictions on the ratio of factors.

All of the terms preceding xy in the list are clearly smaller, and so the ratio of each with its complement in s is smaller than $xy/6p$. Thus none of the terms preceding xy can be factors in a P-factorization of s either. Since

$$\frac{yp}{6x} > \frac{xp}{6y} = \frac{x^2p}{6xy} = \frac{x^2}{6}\frac{p}{p+2} > \frac{x^2}{12} \ge \frac{25}{12} > 2$$

neither xp nor yp can be terms in a P-factorization of s.

Therefore the only remaining options are $s = 2p\times 3(p+2)$ and $s = 2(p+2)\times 3p$, and so there are exactly two P-factorizations of s. Thus for each prime $p = 3j+2$, there are two generator pairs which correspond to the semiperimeter $s = 6p(p+2)$.

For part (d), suppose that $s = 2^r p^f$ is the semiperimeter of a primitive right triangle. Because v must be odd, the only possible P-factorization of s is $u = 2^r$ and $v = p^f$. This means that

$$\begin{aligned} 2^r <\ & p^f < 2^{r+1} \\ r <\ & \log_2 p^f < r+1 \end{aligned}$$

Therefore, $\lfloor \log_2 p^f \rfloor = r$.

Now assume that for an odd prime p and a positive integer f, $s = 2^r p^f$ and $r = \lfloor \log_2 p^f \rfloor$. Using the same, reversible steps above, this means that $2^r < p^f < 2^{r+1}$, and therefore $\sqrt{s} < p^f < \sqrt{2s}$. It is clear that p^f is odd and thus $\gcd(2^r, p^f) = 1$, so p^f is an odd unitary divisor of s. Let us show that there are no other odd unitary divisors of s. Because there are two primes in s, it has 4 unitary divisors: $1, 2^r, p^f, 2^r p^f$. Only two of these are odd. It is clear that $1 < 2^r p^f \not< 2$ for any odd prime p. Therefore there is only one P-factorization of s, and so $s = 2^r p^f$ corresponds to exactly one primitive right triangle. $\square$

The following corollary enumerates several results about the number of integers which can be the semiperimeter of right triangles.

Corollary 1. *The following hold:*

a. *There are an infinite number of positive integers that are not the semiperimeter of any primitive right triangle.*
b. *There are an infinite number of positive integers that are the semiperimeter of exactly one primitive right triangle.*
c. *There are an infinite number of positive integers that are the semiperimeter of exactly two primitive right triangles.*

Proof. By Theorem 5 part (a), all positive integers of the form p^r where p is an odd prime and r is any positive integer cannot correspond to the semiperimeter of any primitive right triangles. Part (b) of Theorem 5 asserts that for any odd prime p, $s = p(p+2)$ corresponds to exactly one primitive right triangle. Because there are infinitely many primes, parts (a) and (b) are proved. For part (c), we will prove that there are infinitely many primes of the form $p = 3j + 2$. Assume there are a finite number of primes $(p_1, p_2, \dots p_n)$ of the form $3j + 2$. Let $q = \prod_{i=1}^{n} p_i^2 + 1$. Because each $p_i^2 = 3(3j_i^2 + 4j_i + 1) + 1$, the expanded product of these squared primes will be a sum of integers. Each term but the final, 1, is divisible by 3. Thus we may say that $\prod_{i=1}^{n} p_i^2 = 3a + 1$ for some positive integer a, and so $q = 3a + 2$. If q is prime, then we have a prime of the form $3j + 2$ which was not on our previous list, contradicting our assumption that there are a finite number of such primes. If q is composite, it is not divisible by 3. Because $q - 1$ is divisible by all primes of the form $3j + 2$, q is not divisible by any of them. Thus it must be a product of primes of the form $3j + 1$. However, we have seen by equations 1 and 2 that such a product is equivalent to $3a + 1$ for some a, which contradicts the fact that $q = 3a + 2$. Therefore, there are infinitely many primes of the form $3j + 2$. Thus, by Theorem 5 part (c), there are infinitely many positive integers corresponding to exactly two primitive right triangles. □

4. Semiperimeters Which Correspond to At Least $k \geq 3$ Triangles

We have proved that there are infinitely many integers which correspond to the semiperimeter of exactly 0, 1, or 2 distinct primitive right triangles. In this section, we will expand this result to semiperimeters corresponding to at least k primitive right triangles for any $k \geq 3$, $k \in \mathbb{Z}^+$.

Bertrand's Postulate states that for a positive integer n, if $n \geq 2$ then there exists at least one prime in the interval $(n, 2n)$. It was first conjectured by Joseph Bertrand in 1845, then proved by Chebyshev [3] in 1850. A proof based on the 1932 proof by Paul Erdős appears in the appendix. Using Bertrand's Postulate, we can prove the following theorem:

Theorem 6. *For each positive integer $k \geq 3$, there exists a positive integer s such that s is the semiperimeter of k or more primitive right triangles.*

Proof. Let s be an odd positive integer containing n distinct odd primes. We have shown that s corresponds to the semiperimeter of primitive right triangles when there are odd divisors of s in the interval $(\sqrt{s}, \sqrt{2s})$. Let b be the highest power of 2 less than s, so that $2^b < s < 2^{b+1}$. We will prove that either s or $2^i s$, where i is a positive integer such that $1 \leq i \leq b+1$, corresponds to at least k primitive right triangles. There are 2^{n-1} unitary divisors of s greater than $\sqrt{s}$. Each divisor must lie in one of the $b+1$ intervals $(\sqrt{s}, \sqrt{2s})$, $(\sqrt{2s}, \sqrt{4s})$, ... $(\sqrt{2^b s}, \sqrt{2^{b+1}s})$. Let r_i, where again $1 \leq i \leq b+1$, denote the number of odd divisors in the interval $(\sqrt{2^{i-1}s}, \sqrt{2^i s})$. Because $2^b < s < 2^{b+1}$, if two divisors of s are in the interval $(\sqrt{2^b s}, \sqrt{2^{b+1}s})$, then there are two complements of these divisors in s which are integer values less than 2. This is impossible, and so no more than one divisor of s can lie in the interval $(\sqrt{2^b s}, \sqrt{2^{b+1}s})$, so

$$r_1 + r_2 + \cdots + r_b = 2^{n-1} - 1.$$

Let the greatest r_i be denoted r_t. It must satisfy $r_t > \dfrac{2^{n-1}-1}{b}$.

Listing the primes in increasing order, let the nth odd prime be denoted p_n. Because Bertrand's Postulate guarantees the existence of a prime in the interval $(n, 2n)$, we can see that

$$\begin{aligned} 3 \cdot 5 \cdot 7 \cdots p_n &< 3 \cdot (3 \cdot 2) \cdot (3 \cdot 2^2) \cdots (3 \cdot 2^{n-1}) \\ &= 3^n \cdot 2^{\frac{n(n-1)}{2}} \\ &< (2^2)^n \cdot 2^{\frac{n(n-1)}{2}} \\ &= 2^{\frac{n(n+3)}{2}} \end{aligned}$$

Suppose that $s = 3 \cdot 5 \cdot 7 \cdots p_n$. Then, using b as defined above, $2^b < s < 2^{\frac{n(n+3)}{2}}$, and therefore $b < \dfrac{n(n+3)}{2}$. Then, taking r_t as defined earlier as the greatest r_i,

$$r_t > \frac{2^{n-1}-1}{b} > \frac{2(2^{n-1}-1)}{n(n+3)} > \frac{2^{n-1}}{n^2+3n}.$$

By choosing a sufficiently large value for n, we can make this larger than any positive integer. Therefore, we can guarantee that there are at least k odd divisors of s in one of the intervals $(\sqrt{2^i s}, \sqrt{2^{i+1}s})$ for $1 \leq i \leq b$. Thus there exists some i within these parameters such that either s or $2^i s$ is a positive integer which corresponds to at least k isoperimetrical right triangles. $\square$

This shows us that for any $k \in \mathbb{Z}^+$, we can choose a large enough integer s - that is, an s comprised of sufficiently many primes - that we can guarantee it corresponds to the semiperimeter of at least k isoperimetrical right triangles. By virtue of this "large enough" s, there must also be infinitely many larger integers

s which also correspond to the semiperimeter of at least k distinct primitive right triangles.

5. Semiperimeters Which Correspond to Exactly k Triangles

Now that we can guarantee that for all $k \in \mathbb{Z}^+$, there exist infinitely many positive integers which correspond to the semiperimeter of at least k distinct right triangles, a natural question is whether it is possible to find integers corresponding to *exactly* k right triangles. Leon Bernstein's article [2] describes a way of generating such numbers. The following proof is based on the method he lays out, but expands upon the details.

We define the prime counting function $\pi(x)$, as equal to the number of primes less than or equal to x for any real number x. For example, $\pi(14) = 6$ and $\pi(14.17) = 6$.

By invoking the much stronger Prime Number Theorem, a generalization of Bertrand's Postulate, we can strengthen our results. The Prime Number Theorem states that

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln x}} = 1.$$

It was proved in 1896 by J. Hadamard [5] and C. de la Vallée Poussin [7] (independently), but the proofs are quite beyond the scope of this paper. As it is quite often used, we will define the function in the denominator as $\ell(x) = x/\ln x$.

Theorem 7. *For each $r > 1$, $\lim_{x\to\infty} (\ell(rx) - \ell(x)) = \infty$*

Proof. Given an $r > 1$ and a number x, the Mean Value Theorem guarantees the existence of a point $c \in (x, rx)$ such that

$$\ell'(c) = \frac{\ell(rx) - \ell(x)}{rx - x}$$

Note $\ell'(x) = (\ln(x) - 1)/(\ln x)^2$. This function's derivative is $\ell''(x) = (2 - \ln x)/(x \ln^3 x)$, which has a critical point at $x = e^2$. On the interval $[e^2, \infty)$, $\ell'(x)$ is decreasing, and so we can say that $\ell'(c) > \ell'(rx)$, assuming $x > e^2$. Note that

$$\ell'(x) = \frac{\ln(x) - 1}{(\ln x)^2} > \frac{\ln x - \frac{1}{2}\ln x}{(\ln x)^2} = \frac{1}{2\ln x}.$$

Therefore, we can see that

$$\begin{aligned} \ell(rx) - \ell(x) &= \ell'(c)(rx - x) \\ &> \ell'(rx)(r-1)x \\ &> \frac{(r-1)x}{2\ln(rx)} \end{aligned}$$

Let us take the limit of this final inequality.

$$\begin{aligned}\lim_{x\to\infty}(r-1)\frac{x}{2\ln(rx)} &= \frac{r-1}{2}\lim_{x\to\infty}\frac{x}{\ln rx}\\ &= \frac{r-1}{2}\lim_{x\to\infty}\frac{1}{\frac{1}{x}}=\infty\end{aligned}$$

Note that we have used L'Hôpital's Rule here, because the limit was an indeterminate form. Because $\ell(rx)-\ell(x) > ((r-1)x)/(2\ln(rx))$, if the limit as x approaches infinity of the right side of the inequality is ∞, the limit of the left must also be ∞. □

Theorem 8. *For all $k \in \mathbb{Z}^+$ and for each $r > 1$, there exists a positive integer $N(r,k)$ such that for each $x \geq N(r,k)$ there exist k prime numbers between x and rx.*

Proof. To prove this theorem, we examine $\lim_{x\to\infty}\frac{\pi(rx)-\pi(x)}{\ell(rx)-\ell(x)}$. First, note the following:

$$\begin{aligned}\lim_{x\to\infty}\frac{\ell(rx)}{\ell(x)} &= \lim_{x\to\infty}\frac{\frac{rx}{\ln rx}}{\frac{x}{\ln x}}\\ &= r\lim_{x\to\infty}\frac{\ln x}{\ln r+\ln x}\\ &= r\end{aligned}$$

If $\lim_{x\to\infty}\frac{\pi(rx)-\pi(x)}{\ell(rx)-\ell(x)}$ is equal to 1, we will be able to approximate $\pi(rx)-\pi(x)$ by the function $\ell(rx)-\ell(x)$.

$$\lim_{x\to\infty}\frac{\pi(rx)-\pi(x)}{\ell(rx)-\ell(x)} = \lim_{x\to\infty}\frac{\frac{\pi(rx)}{\ell(rx)}\cdot\frac{\ell(rx)}{\ell(x)}-\frac{\pi(x)}{\ell(x)}}{\frac{\ell(rx)}{\ell(x)}-1}$$

By the Prime Number Theorem, we know that $\lim_{x\to\infty}\pi(x)/\ell(x)=1$. Thus using this result and our previous limit, we can see that

$$\lim_{x\to\infty}\frac{\pi(rx)-\pi(x)}{\ell(rx)-\ell(x)}=\frac{r-1}{r-1}=1.$$

Now it is clear that Theorem 8 follows from Theorem 7. Because $\lim_{x\to\infty}(\ell(rx)-\ell(x))=\infty$, it follows that $\lim_{x\to\infty}\pi(rx)-\pi(x)=\infty$ as well. Thus by the definition of the limit, for any $k\in\mathbb{Z}^+$ and for each $r>1$, there exists $N(r,k)>0$ such that $\pi(rx)-\pi(x)>k$ for all $x\geq N(r,k)$. Equivalently, there exist k primes in the interval (x,rx). □

This is a generalization of Bertrand's Postulate, which states that $N(2,1)=2$ - that is, there is one prime in the interval $(x,2x)$ for all $x\geq 2$.

The following method for choosing an s which corresponds to exactly k primes comes from Bernstein's article. He provides a way to build s such that there are exactly k possible generator pairs, which he explicitly describes.

Theorem 9. *For each positive integer $k \geq 3$, there exists a positive integer M_k with the following property: for each $n \geq M_k$ there exist primes $p_1, p_2, \ldots, p_k$ and q such that*

$$n < p_1 < p_2 < \cdots < p_k < \frac{4}{3}n, \qquad \frac{1}{p_1^2}\prod_{i=1}^{k} p_i < q < \frac{2}{p_k^2}\prod_{i=1}^{k} p_i.$$

Proof. From Theorem 8, if we choose $r = \frac{4}{3}$ and $x = n$, then we are ensured that for some $N(4/3, k)$, there exist k primes in the interval $(n, \frac{4}{3}n)$ for all $n \geq N(4/3, k)$.

Similarly, to guarantee the existence of q, we can newly apply Theorem 8 for new n and r values. Because we have already chosen our k primes, we let $x = \dfrac{p_2 p_3 \cdots p_k}{p_1}$. By Theorem 8, for every real number $r > 1$, there is a corresponding integer $N(r, k)$ such that for every $x \geq N(r, k)$, there exists a prime in the interval (x, rx), or in our case,

$$\left(\frac{p_2 p_3 \cdots p_k}{p_1}, r\frac{p_2 p_3 \cdots p_k}{p_1}\right).$$

If we can show that $r(p_2 p_3 \cdots p_k)/p_1 < 2(p_1 p_2 \cdots p_{k-1})/p_k$, then we see that our interval $\left(\dfrac{p_2 p_3 \cdots p_k}{p_1}, r\dfrac{p_2 p_3 \cdots p_k}{p_1}\right)$ is a subinterval of the desired interval, and so there exists a prime q such that

$$\frac{p_2 p_3 \cdots p_k}{p_1} < q < 2\frac{p_1 p_2 \cdots p_{k-1}}{p_k}.$$

Choose an appropriate r such that $1 < r < 4/3$ - we will use $r = \sqrt{3/2}$. We will assume our desired conclusion to be true and then follow reversible steps to something we can prove to be true, thus proving that (x, rx) for the x and r values we have specified is a subinterval of the desired interval.

$$\begin{aligned} \sqrt{\frac{4}{3}}\frac{p_2 p_3 \cdots p_k}{p_1} &< 2\frac{p_1 p_2 \cdots p_{k-1}}{p_k} \\ p_k^2 &< \sqrt{\frac{3}{2}}p_1^2 < 2p_1^2 \end{aligned}$$

We can confirm this to be true by our initial choice of the primes. From $p_k < \frac{4}{3}n$ and $n < p_1$, we have $p_k^2 < \frac{16}{9}n^2 < \frac{16}{9}p_1^2 < 2p_1^2$, and so the inequality holds. Because each step is reversible, we have proven that there exists a prime q such that for $1 < r < 4/3$

$$\frac{p_2 p_3 \cdots p_k}{p_1} < q < 2\frac{p_1 p_2 \cdots p_{k-1}}{p_k},$$

as desired. □

Theorem 10. *Let $p_1 < p_2 < \cdots < p_k$ be k consecutive odd primes, such that $p_k < \frac{4}{3}p_1$. Let $A = \prod_{i=1}^{k} p_i$ and let $u_t = \dfrac{A}{p_t}$ for $t \in \mathbb{Z}^+$, $1 \le t \le k$. Assume there exists a prime q such that*

$$\frac{p_2 p_3 \ldots p_k}{p_1} < q < 2\frac{p_1 p_2 \ldots p_{k-1}}{p_k}.$$

If $v_t = qp_t$, then the pairs (u_t, v_t) are generators of k primitive right triangles with equal semiperimeters $s = qA$. Moreover, any (u, v) which generates a primitive right triangle with semiperimeter $s = qA$ is of the form (u_t, v_t).

Proof. We know that such a q exists by Theorem 9. Note that by the definitions of u_t and v_t, $u_t < v_t < 2u_t$ for all positive integers $1 \le t \le k$. We can also see that $\gcd(u_t, v_t) = 1$ is true because q is a prime larger than all of the primes $p_1, p_2, \ldots p_k$. It is also clear that v_t is odd, so each (u_t, v_t) satisfy the conditions on generators as defined Theorem 3. Because the semiperimeter corresponding to (u_t, v_t) is $u_t v_t = qA$, which does not depend on t, the k primitive Pythagorean triangles generated by (u_t, v_t) have equal semiperimeters.

To prove that these are in fact the only generator pairs for $s = qA$, let $s = p_1 p_2 \cdots p_k q = u \times v$ be a P-factorization of s. Assume that q divides u. If $u = q$, then

$$\frac{v}{u} = \frac{v}{q} < \frac{p_1 p_2 \cdots p_k}{\frac{p_2 p_3 \cdots p_k}{p_1}} < p_1^2.$$

However, we know that $\frac{1}{2} < \frac{v}{u} < 2$, and so this cannot be true. Because $u < v$, it is clear that u must not contain all of the k primes if it also contains q. However, if it contains m primes, where $1 \le m < k$, then v contains at least 1 but not k of the primes. This means that

$$\begin{aligned} uv &= p_1 p_2 \cdots p_k q \\ &> \frac{(p_1 p_2 \cdots p_k)^2}{p_1^2} = p_2^2 p_3^2 \cdots p_k^2 \\ &\ge v^2 \\ u &> v \end{aligned}$$

which is a contradiction, and so q cannot divide u. Therefore it must divide v.

We can see that if u contains none of the primes p_i, v will be much too large to satisfy $u < v < 2u$. If u contains all of the primes, then

$$v = q < 2\frac{p_1 p_2 \cdots p_{k-1}}{p_k} < p_1 p_2 \cdots p_k = u,$$

which cannot be true. So u can neither contain all nor none of the primes. The question is, how many of the p_i divide u? Assume that u contains $m > 1$ such primes, where m is a positive integer such that $1 < m < k$. Let x_i denote some p_i, not necessarily in order. Then $u = x_1x_2\cdots x_m$ and $v = (x_{m+1}x_{m+2}\cdots x_k)q$.

$$\begin{aligned}
(x_{m+1}x_{m+2}\cdots x_k)q = v &< 2u \\
(x_{m+1}x_{m+2}\cdots x_k)\frac{p_1p_2\cdots p_k}{p_1^2} &< 2u = 2(x_1x_2\cdots x_m) \\
\frac{(x_1x_2\cdots x_m)(x_{m+1}x_{m+2}\cdots x_k)^2}{p_1^2} &< 2(x_1x_2\cdots x_m) \\
\frac{(x_1x_2\cdots x_m)^2}{{p_1}^2} &< 2
\end{aligned}$$

This cannot be true, because even if $(x_1x_2\cdots x_m)$ was as small as it could possibly be, that is, the first m primes, then

$$\frac{(x_1x_2\cdots x_m)^2}{{p_1}^2} = (p_2p_3\cdots p_m)^2.$$

For $m > 1$, this is always an integer greater than 2, because each p_i is an odd prime. Thus u must contain exactly one prime p_i, since it neither contain more than 1 nor 0, and so all generator pairs corresponding to a triangle of semiperimeter $s = p_1p_2\cdots p_kq$ must be of the form (u_t, v_t) as described in Theorem 10

Because all generators are of the form (u_t, v_t) and there are k primitive Pythagorean triangles of semiperimeter s which are generated by these pairs, there are infinitely many perimeters which correspond to exactly k primitive Pythagorean triangles. □

Theorem 11. *Any (u, v) which generates a primitive right triangle with semiperimeter $s = qA$ is of the form (u_t, v_t).*

Proof. □

6. Appendix 1: A Proof of Bertrand's Postulate

Bertrand's Postulate states that if $n \geq 2$ is an integer, there is always at least one prime in the interval $(n, 2n)$. The following proof is similar to that of Paul Erdős.[1] In order to prove this statement, several lemmas will be necessary.

Lemma 1. *For each $x \geq 1$, $x \in \mathbb{R}^+$, $\pi(x) \leq \frac{1}{2}(x+1)$.*

Proof. We will prove this result for $x \in \mathbb{Z}^+$. The result easily generalizes to all positive real numbers, because for any x which is not an integer, $\pi(x) = \pi(\lfloor x \rfloor)$. Take x to be even. Every alternate integer in the interval $[1, x]$ is even, and there are x integers in the interval. Because 2 is the only even prime and 1 is not prime, we can swap the two, and include 1 in our count of "even" - that is, composite - integers in the interval, and count 2 as a possible prime. Thus there are $\frac{1}{2}x$ integers which, by virtue of being even or 1, cannot be prime. On the other hand, if x is odd, $\pi(x) \leq \pi(x-1)+1$, where $x-1$ is even. Thus $\pi(x) \leq \frac{1}{2}(x-1)+1 = \frac{1}{2}(x+1)$. □

Lemma 2. *For any real number $x \geq 1$, we have*

$$\lfloor 2x \rfloor - 2\lfloor x \rfloor = \begin{cases} 0, & \text{if } 0 \leq x - \lfloor x \rfloor < \frac{1}{2}; \\ 1, & \text{if } \frac{1}{2} \leq x - \lfloor x \rfloor < 1. \end{cases}$$

Proof. Let us first consider the case when $0 \leq x - \lfloor x \rfloor < \frac{1}{2}$.

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + \frac{1}{2}$$
$$2\lfloor x \rfloor \leq 2x < 2\lfloor x \rfloor + 1$$

Therefore $2\lfloor x \rfloor = \lfloor 2x \rfloor$, and so $\lfloor 2x \rfloor - 2\lfloor x \rfloor = 0$.

Similarly,

$$\frac{1}{2} \leq \quad x - \lfloor x \rfloor \quad < 1$$
$$2\lfloor x \rfloor + 1 \leq \quad 2x \quad < 2\lfloor x \rfloor + 2$$

Therefore $\lfloor 2x \rfloor - 2\lfloor x \rfloor = 1$, and thus we have proven the lemma. □

Lemma 3. *The function g defined by $g(x) = x^{-\frac{1}{2}} \ln(x)$ is decreasing on the interval $[e^2, \infty)$.*

Proof. Note that $g'(x) = x^{-\frac{3}{2}} \left(1 - \dfrac{\ln(x)}{2}\right)$. The function g has a critical point at $x = e^2$, and on the interval (e^2, ∞), g' is negative. Thus, g is decreasing on the interval $[e^2, \infty)$. □

Lemma 4. *For any integer $n > 1$, we have $\dfrac{4^n}{2n} < \dbinom{2n}{n}$.*

Proof. Using the Binomial Theorem,

$$4^n = (1+1)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k}.$$

The first and last terms of this sum are the smallest, with each equal to 1. We can sum these and, the resulting term will still be the smallest, because $2 < \binom{2n}{1}$ for all $n > 1$. Thus we now have a sum of $2n$ terms, in which the largest term is $\binom{2n}{n}$, so replacing every term with the largest gives:

$$4^n < (2n)\binom{2n}{n}.$$

Therefore $\dfrac{4^n}{2n} < \dbinom{2n}{n}$, as desired. □

Lemma 5. *For each real number $x \geq 2$, the inequality $\prod_{1<p\leq x} p < 4^x$ holds.*

Proof. We will prove this result for $x \geq 2$, $x \in \mathbb{Z}^+$. The result extends easily to all real numbers, because $\pi(x) = \pi(\lfloor x \rfloor)$. This will be a proof by strong induction. For our base case, let $x = 2$. Then $\prod_{1<p\leq 2} p = 2 < 4^2$. Assume that for a positive integer

m, $\prod_{1<p\le x} p < 4^x$ for all $x < 2m$. Then let us examine $x \in \mathbb{R}$ such that $x = 2m$, the even case. Because x is even, it is not prime, and

$$\prod_{1<p\le 2m} p = \prod_{1<p\le 2m-1} p < 4^{2m-1} < 4^{2m}$$

by our induction hypothesis. If $x = 2m+1$, the odd case, then by the Binomial Theorem, we can simplify by noting that all of the terms in the binomial expansion are positive, so choosing the two largest terms gives something smaller than the entire sum. Thus we have the following inequality:

$$\begin{aligned} 4^m &= \frac{1}{2}(1+1)^{2m+1} = \frac{1}{2}\sum_{k=0}^{2m+1}\binom{2m+1}{k} \\ &> \frac{1}{2}\left(\binom{2m+1}{m} + \binom{2m+1}{m+1}\right) \\ &= \binom{2m+1}{m} \end{aligned}$$

Because $2m+1$ is odd, the mth and the $(m+1)$th terms are the same, by the symmetric nature of Pascal's Triangle.

Because every prime in the interval $(m+1, 2m+1]$ appears in the numerator and not the denominator of $\binom{2m+1}{m}$, each one divides $\binom{2m+1}{m}$, so we have

$$\prod_{m+1<p\le 2m+1} p \le \binom{2m+1}{m}.$$

We have already proved that this product is less than 4^m, so when $x = 2m+1$,

$$\begin{aligned} \prod_{1<p\le 2m+1} p &= \left(\prod_{1<p\le m+1} p\right)\left(\prod_{m+1<p\le 2m+1} p\right) \\ &\le \left(\prod_{1<p\le m+1} p\right) 4^m \\ &\le 4^{m+1}\cdot 4^m = 4^{2m+1} \end{aligned}$$

by our induction hypothesis. This proves the lemma. ☐

Lemma 6. *Let μ_p denote the greatest positive integer such that $p^{\mu_p} | \binom{2n}{n}$. For any prime p, $p^{\mu_p} \le 2n$.*

Proof. The exponent of p in $n!$ is $\sum_{i=1}^{\infty}\left\lfloor \frac{n}{p^i}\right\rfloor$. This comes from counting the multiples of p in the interval $[1, n]$, then counting the multiples of p^2 in the same interval, and so forth. Thus

$$\mu_p = \sum_{i=1}^{\infty}\left\lfloor \frac{2n}{p^i}\right\rfloor - 2\sum_{i=1}^{\infty}\left\lfloor \frac{n}{p^i}\right\rfloor = \sum_{i=1}^{\infty}\left(\left\lfloor \frac{2n}{p^i}\right\rfloor - 2\left\lfloor \frac{n}{p^i}\right\rfloor\right).$$

By Lemma 2, the terms of this final sum are either 0 or 1 depending on the size of x. All of the terms where $i > \log_p(2n)$ are 0, so we are left with $\mu_p \le \log_p(2n)$, and therefore $p^{\mu_p} \le p^{\log_p(2n)} = 2n$. ◻

Lemma 7. *For $p > \sqrt{2n}$,*

$$\mu_p = \left\lfloor \frac{2n}{p} \right\rfloor - 2\left\lfloor \frac{n}{p} \right\rfloor .$$

Proof. Using the sum from the proof of Lemma 6, because $p^2 > 2n$, the first term is the only nonzero term. For $i = 1$, we have $\left\lfloor \frac{2n}{p} \right\rfloor - 2\left\lfloor \frac{n}{p} \right\rfloor$. ◻

Theorem 12. *Bertrand's Postulate: For $n \in \mathbb{Z}^+$, $n > 1$, there exists at least one prime in the interval $(n, 2n)$.*

Proof. This will be a proof by contradiction. For $2 \le n \le 127$, see Table 1 in the appendix to observe that for each, there is a prime in $(n, 2n)$. Assume that $n \ge 128$ is a positive integer for which there are no primes in the interval $(n, 2n)$. Before we begin, note the following important calculations:

We can use simple algebra to show that $\sqrt{2n} < \frac{2n}{3}$ for $n > 5$.

For $\prod_{2n/3 < p \le n} p^{\mu_p}$, we have $p > \sqrt{2n}$, so Lemma 7 applies. Therefore

$$\mu_p = \left\lfloor \frac{2n}{p} \right\rfloor - 2\left\lfloor \frac{n}{p} \right\rfloor .$$

Rearranging the inequality $n \ge p$ gives $\frac{n}{p} \ge 1$ and $\frac{2n}{3} < p$ gives $\frac{2n}{p} < 3$. Thus,

$$\mu_p = \left\lfloor \frac{2n}{p} \right\rfloor - 2\left\lfloor \frac{n}{p} \right\rfloor \le 2 - 2 = 0,$$

and $\prod_{2n/3 < p \le n} p^{\mu_p} = 1$. By our hypothesis, $\prod_{n < p \le 2n} p^{\mu_p} = 1$.

For $p > \sqrt{2n}$, from Lemma 7,

$$\prod_{\sqrt{2n} < p \le 2n/3} p^{\mu_p} = \prod_{\sqrt{2n} < p \le 2n/3} p < \prod_{1 < p \le 2n/3} p.$$

Then

$$\begin{aligned}
\frac{4^n}{2n} &< \binom{2n}{n} && \text{(Lemma 4)}\\
&= \Big(\prod_{1<p\le\sqrt{2n}} p^{\mu_p}\Big)\Big(\prod_{\sqrt{2n}<p\le 2n/3} p^{\mu_p}\Big)\Big(\prod_{2n/3<p\le n} p^{\mu_p}\Big)\Big(\prod_{n<p\le 2n} p^{\mu_p}\Big)\\
&< \Big(\prod_{1<p\le\sqrt{2n}} p^{\mu_p}\Big)\Big(\prod_{\sqrt{2n}<p\le 2n/3} p^{\mu_p}\Big)\cdot 1\cdot 1 && \text{(above)}\\
&\le \Big(\prod_{1<p\le\sqrt{2n}} 2n\Big)\Big(\prod_{\sqrt{2n}<p\le 2n/3} p^{\mu_p}\Big) && \text{(Lemma 6)}\\
&< (2n)^{\frac{1}{2}(1+\sqrt{2n})}\Big(\prod_{\sqrt{2n}<p\le 2n/3} p\Big) && \text{(Lemmas 1 and 7, see above)}\\
&< (2n)^{\frac{1}{2}(1+\sqrt{2n})}4^{\frac{2}{3}n} && \text{(Lemma 5)}
\end{aligned}$$

Now, with a bit of algebra, we see

$$\begin{aligned}
\frac{4^n}{2n} &< (2n)^{\frac{1}{2}(1+\sqrt{2n})}4^{\frac{2}{3}n}\\
4^{\frac{n}{3}} &< (2n)^{\frac{1}{2}(3+\sqrt{2n})} < (2n)^{\frac{2}{3}\sqrt{2n}}\\
2^{\frac{2}{3}n} &< (2n)^{\frac{2}{3}\sqrt{2n}}\\
2^n &< (2n)^{\sqrt{2n}}\\
n &< \sqrt{2n}(\log_2 2n)\\
\sqrt{\frac{n}{2}} &< 1+\log_2 n
\end{aligned}$$

This statement is true for $n = 127$ but fails when $n = 128$, and for larger values of n it fails as well, as $\sqrt{n}$ increases faster that $\log_2 n$. Because this is only true for $n < 2^7$, this is a contradiction. There must be a prime in the interval $(n, 2n)$. □

7. Table 1: Primes in $(x, 2x)$ for values of x between 2 and 127

p	2p	prime in (p,2p)
2	4	3
5	10	7
7	14	13
13	26	23
23	46	43
43	86	83
83	166	163
163	326	307

References

[1] M. Aigner and G. M. Ziegler, *Proofs from the Book, 2nd ed.* New York: Springer-Verlag, 2000.

[2] L. Bernstein, On Primitive Pythagorean Triangles With Equal Perimeters, *Fibonacci Quarterly* **27** (1989) 2-6.

[3] P. Chebyshev, Mémoire sur les nombres premiers, *Mm. Acad. Sci. St. Ptersbourg* **7** (1850) 17-33.

[4] K. Conrad, Pythagorean Triples,
http://www.math.uconn.edu/ kconrad/blurbs/ugradnumthy/pythagtriple.pdf, unpublished

[5] J. Hadamard, Sur la distribution des zéros de la fonction $\zeta(s)$ et ses consquences arithmétiques ('), *Bull. Soc. math. France* **24** (1896) 199-220.

[6] A. A. Krishnaswami, On Isoperimetrical Pythagorean Triangles, *Tôhoku Math J.* **27** (1926) 332-348.

[7] C. de la Vallée Poussin, Recherches analytiques la théorie des nombres premiers, *Ann. Soc. scient. Bruxelles* **20** (1896) 183-256.